Area of subsets in nilpotent groups

Conference on "Analysis on metric spaces"

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Outline

1. Area formula and Hausdorff measure in metric spaces
2. Area of smooth submanifolds in homogeneous groups
3. Area of nonsmooth algebraic graphs
4. Area of nonsmooth subsets through “rough equations”

The notion of area is important both in Analysis and Geometry. It is even more important when the aim is to develop a “natural” Geometric Measure Theory in non-Riemannian metric spaces.

Hausdorff measure in Riemannian manifolds

If M is an n-dimensional Riemannian manifold with metric g, (U, ψ) is a local chart of M, $\psi : A \to U$, $A \subset \mathbb{R}^n$ open set, then we have

$$\mathcal{H}^n_\rho(U) = \int_A \sqrt{\det(g_{ij})}\,dx,$$

where ρ is the Riemannian distance associated to g.

Hausdorff measure in metric spaces

More generally, if $f : A \to X$ is a Lipschitz mapping taking values in a metric space, then

$$\mathcal{H}^k(f(A)) = \int_A Jf(x)dx$$

where $A \subset \mathbb{R}^k$ is a Lebesgue measure subset and $k \leq n$.
This result was proved in [B. Kirchheim, Proceeding of AMS, 1994].

Definition (Rectifiable sets)

A subset $E \subset X$ in a metric space X is called rectifiable *if there exists a Lipschitz mapping $f : A \to E$, where $A \subset \mathbb{R}^k$ such that $E = f(A)$.*

As soon as a subset is *rectifiable*, then its are can be computed.

Kirchheim's result relies on the a.e. *metric differentiability* of any metric space valued Lipschitz mapping $f : A \to X$, with $A \subset \mathbb{R}^k$.

Theorem (B. Kirchheim, V. M., 2003)

There exist a Lipschitz map $f : \mathbb{H} \to X$ *that is nowhere metrically differentiable, where* $\mathbb{H}$ *is the first Heisenberg group.*

Definition (Metric differentiability on groups)

A mapping $f : A \to X$*, with* A *contained in a* stratified (or Carnot) group $\mathbb{G}$*, is* metrically differentiable *at* $x \in A$ *if there holds*

$$\rho_X(f(x), f(xh)) - s(h) = o(d_{\mathbb{G}}(h, 0)) \quad \textit{as} \quad xh \in A \textit{ and } h \to 0,$$

where $s : \mathbb{G} \to [0, +\infty)$ *is a* homogeneous seminorm.

Theorem (V. M., T. Rajala, 2014)

If $A \subset \mathbb{G}$*,* Q *is the Hausdorff dimension of* $\mathbb{G}$ *and* $f : A \to X$ *is Lipschitz and a.e. metrically differentiable, then*

$$\mathcal{H}^Q(f(A)) = \int_A Jf(x)dx$$

where metric Jacobian $Jf(x)$ at x *is* $\mathcal{H}^Q_{s_x}(B_1)/\mathcal{H}^Q_d(B_1)$ *and* s_x *is the metric differential of* f *at* x*.*

The question of area in sub-Riemannian manifolds

Let $\Sigma \subset \mathbb{M}$ be a smooth submanifold of an *equiregular sub-Riemannian manifold* M and fix $p \in M$.
If $\mathcal{D}$ is the horizontal distribution of M, we define the following flag

$$T_p^1 M = \mathcal{D}_p, \quad T_p^2 M = \mathcal{D}_p + [\mathcal{D},\mathcal{D}]_p, \quad \ldots,$$
$$\ldots \quad T_p^\iota M = \mathcal{D}_p + [\mathcal{D},\mathcal{D}]_p + \cdots + [[\cdots[\mathcal{D},\mathcal{D}],\mathcal{D}],\ldots,\mathcal{D}]_p$$

The induced flag on $T_p\Sigma$ defined by $T_p^j\Sigma = T_p\Sigma \cap T_p^j M$ for each $j = 1,\ldots,\iota$, gives the "pointwise Hausdorff dimension"

$$D'(p) = \sum_{j=0}^{\iota} j \dim \left(T_p^j\Sigma / T_p^{j-1}\Sigma \right).$$

M. Gromov 1996

Smooth submanifolds $\Sigma \subset \mathbb{M}$ have generically Hausdorff dimension $D_H(\Sigma) = \max_{p\in\Sigma} D'(p)$.

Absence of Lipschitz parametrizations in sub-Riemannian manifolds

The formula for the Hausdorff dimension $D_H(\Sigma)$ shows that $D_H(\Sigma) > \dim_{top}\Sigma$ whenever Σ is not everywhere tangent to $\mathcal{D}$.

As a consequence, Σ *cannot be rectifiable of dimension* $\dim_{top}\Sigma$, otherwise the standard property of Lipschitz mappings would imply the opposite inequality

$$D_H(\Sigma) \le \dim_{top}\Sigma.$$

All previous area formulae cannot be applied to "almost all" smooth submanifolds in of a sub-Riemannian manifolds or of a stratified group.

This conflicts with the fact that we we have a natural notion of area fpr ay smooth submanifold $\Sigma \subset \mathbb{G}$ of Hausdorff dimension N, that is

$$\mathcal{H}_\rho^{\mathrm{N}} \llcorner \Sigma \quad \text{or} \quad \mathcal{S}_\rho^{\mathrm{N}} \llcorner \Sigma,$$

where ρ here is the sub-Riemannian distance.

- Which conditions ensure that these measures are finite and positive?
- Can we compute these measures?

Open question

When Σ is any smooth surface, the formula for $\mathcal{H}^{\mathrm{N}}(\Sigma)$ is not yet unknown, even in the Heisenberg group.

We will consider the spherical measure $\mathcal{S}^{\mathrm{N}}$ in *homogeneous groups*, that are not necessarily stratified groups.

Homogeneous groups

A *homogeneous group* $\mathbb{G}$ can be seen as graded finite dimensional real nilpotent Lie algebra that is the direct sum of subspaces

$$\mathbb{G} = H^1 \oplus \cdots \oplus H^\iota$$

with $H^{i+j} \subset [H^i, H^j]$ for every $i, j \geq 1$ and $H^j = \{0\}$ whenever $j > \iota$.
There are *intrinsic dilations* $\delta_r : \mathbb{G} \to \mathbb{G}$, $\delta_r x = r^j x$, whenever $x \in H^j$ and $j = 1, \ldots, \iota$, that are Lie group homomorphisms.
A distance function $d : \mathbb{G} \times \mathbb{G} \to [0, +\infty)$ which satisfies

$$d(xz, xw) = d(z, w) \quad \text{and} \quad d(\delta_r z, \delta_r w) = r\, d(z, w)$$

for each $x, z, w \in \mathbb{G}$ and $r > 0$ is a *homogeneous distance*.

When the first layer H^1 Lie spans all of $\mathbb{G}$ the group is called *stratified*. Only in this case the *sub-Riemannian distance* can be also introduced and $\mathbb{G}$ with this distance is called *Carnot group*.

Example (Codimension one)

The Heisenberg group $\mathbb{H} = H^1 \oplus H^2 \approx \mathbb{R}^3$ has Hausdorff dimension 4 and all 2-dimensional surfaces have Hausdorff dimension three. In addition one can compute their spherical measure

$$\mathcal{S}^3(\Sigma) = \int_\Sigma |\nu_H| \, d\sigma.$$

Here ν_H is the orthogonal projection of a normal vector field ν to Σ onto the left invariant horizontal distribution $\mathcal{D}$ of $\mathbb{H}$.

Naive approach to the proof

One considers a Riemannian measure σ on a surface $\Sigma \subset \mathbb{H}$ and tries to compute the blow-up limit

$$\frac{\sigma(B(x,r))}{r^3} \to \frac{\omega(d,x)}{|\nu_H(x)|} \quad \text{as} \quad r \to 0^+. \tag{1}$$

Then the $\mathcal{S}^3$ negligibility of characteristic points x with $\nu_H(x) = 0$, joined with some Federer's differentiation theorems, leads to (1).

Warning

If we consider the sub-Riemannian distance ρ in the first Heisenberg group, then the geometric constant $\omega(\rho, x)$ must be corrected, see [V. M., to appear 2017]. We need new differentiation theorems.

By these new tools, it is also possible to show for instance that a well known unrectifiable subset $\Sigma \subset \mathbb{H}$ constructed by [B. Kirchheim and F. Serra Cassano, 2004] has 3-density $\mathcal{S}^3$-a.e. equal to one

$$\lim_{r \to 0^+} \frac{\mathcal{S}^3(\Sigma \cap B(x,r))}{\omega_d r^3} = 1,$$

whenever we choose a proper distance. Replacing the spherical measure by the Hausdorff measure would answer an open concerning the extension of the D. Preiss rectifiability theorem into metric spaces.

Area in higher codimension

These new differentiation theorems are also important to understand the geometry of higher codimensional submanifolds in stratified groups, that is much more complicated than that of hypersurfaces.

Measure theoretic area formula

If $\Sigma \subset X$ is a Borel set, μ is a Borel regular measure concentrated on Σ, that is absolutely continuous with respect to $\mathcal{S}^\alpha \llcorner \Sigma$, then

$$\mu = \theta^\alpha(\mu, \cdot)\, \mathcal{S}^\alpha \llcorner \Sigma, \tag{2}$$

see [V. M., Proc. Royal. Soc. Ed., 2015].
Weak assumptions are needed on the diameter function $\mathrm{diam}(\mathbb{B}(x, r))$, such as continuity with respect to $r >$, and for every $x \in X$.
The key of (2) is the explict representation of the density θ^α, namely the `spherical Federer` α`-density`:

$$\theta^\alpha(\mu, x) = \inf_{\varepsilon>0} \sup \left\{ \frac{\mu(\mathbb{B})}{c_\alpha (2r)^\alpha} : \mathbb{B} \in \mathcal{F}_b,\ x \in \mathbb{B},\ \mathrm{diam}\mathbb{B} \leq \varepsilon \right\}.$$

that is $\mathcal{S}^\alpha$ measurable.

The general method

We fix an auxiliary measure μ on Σ, that could be for instance its Riemannian or its Euclidean area. Then we seek for a distance such that $\theta^\alpha(\mu,\cdot)$ becomes the ratio

$$\frac{\omega_0}{\delta_\mu(\cdot)}$$

where ω_0 plays the role of a geometric constant that can be included in the definition of $\mathcal{S}^\alpha$, therefore

$$\mathcal{S}^\alpha_X(E) := \omega_0 \mathcal{S}^\alpha(E) = \int_E \delta_\mu(x)\, d\mu(x)$$

for every $\mathcal{S}^\alpha$-measurable set E.

Making the suitable choice of μ can give $\delta_\mu \equiv 1$, as it is the case for the surface measure in Euclidean spaces.

Intrinsic measure of submanifolds

For stratified groups the measure μ on a manifold Σ for which $\delta_\mu = 1$ is the *intrinsic measure* of Σ. Then this measure provides a way to compute the Hausdorff measure of submanifolds.

We fix any auxiliary Riemannian metric $\tilde{g}$ on $\mathbb{G}$, such that τ_Σ is a *unit* tangent n-vector with respect to $\tilde{g}$. Let N be the Hausdorff dimension of Σ (more precisely the algebraic degree).
Then the *intrinsic measure* of Σ is given by

$$\mu_\Sigma = |\pi_{\mathrm{N}}(\tau_\Sigma)| \, vol_{\tilde{g}} \llcorner \Sigma,$$

see [V. M. and D. Vittone, J. Reine Ang. Math., 2008].

All directions belonging to the j-th stratum $\mathcal{V}_j$ of the Lie algebra

$$\mathrm{Lie}(\mathbb{G}) = \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_\iota,$$

are assumed to have degree j. The grading of $\mathrm{Lie}(\mathbb{G})$ is associated to the graded structure of $\mathbb{G} = H^1 \oplus \cdots \oplus H^\iota$.

More generally if $Z_1, \ldots, Z_k \in \mathrm{Lie}(\mathbb{G})$ have degrees $n_1, \ldots, n_k$, respectively, then we define

$$\deg(Z_1 \wedge \cdots \wedge Z_k) = n_1 + \cdots + n_k.$$

If $X_I = X_{i_1} \wedge X_{i_2} \wedge \cdots \wedge X_{i_k}$ with multiindex $I = \{i_1, \ldots, i_k\}$ we define the N-*projection* as follows

$$\pi_{\mathrm{N}}\Big(\sum_{I \in \mathcal{I}_k} c_I X_I\Big) = \sum_{\substack{I \in \mathcal{I}_k \\ d(X_I) = \mathrm{N}}} c_I X_I \in \Lambda_k\left(\mathrm{Lie}(\mathbb{G})\right)$$

where $c_I \in \mathbb{R}$.

Degree of submanifolds

If $\Sigma \subset \mathbb{G}$ is an n-dimensional smooth submanifold of $\mathbb{G}$ and $p \in \Sigma$, then we define a tangent n-vector of Σ at p as follows

$$\tau_\Sigma(p) = t_1 \wedge \cdots \wedge t_{\mathrm{n}},$$

where $(t_1, \ldots, t_{\mathrm{n}})$ is a basis of $T_p\Sigma$.
We define the *degree* $d_\Sigma(p)$ of Σ at p as the following integer

$$d_\Sigma(p) = \max\left\{j \in \mathbb{N} : \ j \le Q, \ \pi_j(\tau_\Sigma(p)) \neq 0\right\}.$$

The *degree* of Σ is $d(\Sigma) = \max\{d_\Sigma(p) : p \in \Sigma\}$.

Blow-up

Those points p of a C^2 smooth submanifold $\Sigma \subset \mathbb{G}$ such that $d_\Sigma(p) = d(\Sigma)$, namely have *maximum degree*, have a blow-up limit

$$\delta_{1/r}(p^{-1}\Sigma) \to S_p\Sigma \quad \text{as} \quad r \to 0^+.$$

Moreover $S_p\Sigma$ is a subgroup of $\mathbb{G}$.

Is it possible to define an “intrinsic” tangent group before we know the existence of a blow-up?

V. M. 2017, homogeneous tangent space

Let $\Sigma \subset \mathbb{G}$ be a C^1 smooth submanifold of dimension n and let $d_\Sigma(p) = \mathrm{N}$, where $p \in \Sigma$. We consider the unique left invariant n-vector field ξ such that $\xi(p) = \tau_\Sigma(p)$ and $\tau_\Sigma(p)$ is a tangent n-vector of Σ at p. We project ξ on the space of left invariant n-vector fields ξ with degree N, that is

$$\xi_\mathrm{N} = \pi_\mathrm{N}(\xi) \in \Lambda_\mathrm{n}\left(\mathrm{Lie}(\mathbb{G})\right)$$

and we consider the *Lie homogeneous tangent space*

$$\mathcal{A}_p\Sigma = \left\{X \in \mathrm{Lie}(\mathbb{G}) : X \wedge \xi_\mathrm{N} = 0\right\}.$$

The *homogeneous tangent space of* Σ *at* p as the subspace

$$A_p\Sigma = \left\{X(p) \in T_p\mathbb{G} : X \in \mathcal{A}_p\Sigma\right\}.$$

Definition (Algebraic regularity)

We say that $p \in \Sigma$ *is* algebraically regular*if the Lie homogeneous tangent space* $\mathcal{A}_p\Sigma$ *is a Lie subalgebra of* $\mathrm{Lie}(\mathbb{G})$.

Example

Let us consider the first Heisenberg group $\mathbb{H} \approx \mathbb{R}^3$ with vector fields

$$X = \partial_x - y\partial_z \qquad Y = \partial_y + x\partial_z$$

and the submanifold $\Sigma = \left\{(x, y, x^2 + y^2) \in \mathbb{R}^3 : x, y \in \mathbb{R}\right\}$.
The tangent space at the origin $T_0\Sigma$ is spanned by $X(0)$ and $Y(0)$, that both belong to the first layer of $\mathrm{Lie}(\mathbb{H})$, so that

$$\pi_2(X \wedge Y) = X \wedge Y \quad \text{and} \quad \mathcal{A}_0\Sigma = \mathrm{span}\,\{X, Y\}\,.$$

Since $\mathcal{A}_0\Sigma$ is not a subalgebra of $\mathrm{Lie}(\mathbb{H})$, the origin $0 \in \Sigma$ is not algebraically regular. Indeed 0 is a characteristic point of $\mathbb{H}$, that behaves like a singular point of the submanifold.

Theorem (V. M. 2017 and previous works)

If Σ is a C^1 smooth submanifold and $p \in \Sigma$ has maximum degree, namely $d_\Sigma(p) = d(\Sigma)$, then

$$\delta_{1/r}(p^{-1}\Sigma) \to S_p\Sigma \quad \text{as} \quad r \to 0^+ \tag{3}$$

if any of the following conditions holds.

1. $\mathcal{A}_p\Sigma$ *is a commutative subalgebra of* $\mathcal{V}_1$ *(V. M. 2017)*
2. $\mathcal{A}_p\Sigma$ *is a subalgebra of* $\mathrm{Lie}(\mathbb{G})$ *and* $\mathbb{G}$ *has step two (V. M. 2017).*
3. $\mathcal{A}_p\Sigma$ *has dimension one (with R. Korte 2012).*
4. $\mathcal{A}_p\Sigma$ *is a subgroup of maximal degree (with J. T. Tyson and D. Vittone 2015)*

In all of these cases p is algebraically regular *and* $S_p\Sigma = \exp \mathcal{A}_p\Sigma$.

Conjecture

If Σ is C^1 smooth, $p \in \Sigma$ has maximum degree and it is algebraically regular, then there exist

$$\lim_{r \to 0^+} \delta_{1/r}(p^{-1}\Sigma) = \exp \mathcal{A}_p \Sigma$$

Stronger conjecture

If Σ is C^1 smooth and $p \in \Sigma$ has maximum degree, then it is algebraically regular and there exists

$$\lim_{r \to 0^+} \delta_{1/r}(p^{-1}\Sigma) = \exp \mathcal{A}_p \Sigma$$

At these points p we can compute the spherical Federer density

$$\theta^{\mathrm{N}}(\mu_\Sigma, p) = \omega_0$$

for distance with special symmetries, then $\mu_\Sigma \llcorner \Sigma_0 = \mathcal{S}^{\mathrm{N}} \llcorner \Sigma_0$.
The hope is that at those points of $\Sigma \setminus \Sigma_0$ where the blow-up does not exist are $\mathcal{S}^{\mathrm{N}}$ negligible.

The *singular set* or *characteristic set* of Σ is the following subset

$$C(\Sigma) = \{p \in \Sigma : d_\Sigma(p) < d(\Sigma)\}.$$

Negligibility problem

Find the minimal regularity of Σ, such that

$$\mathcal{S}^{\mathrm{N}}(C(\Sigma)) = 0.$$

Theorem (M., J. T. Tyson and D. Vittone, JAM, 2015)

If Σ is a C^1 smooth transversal submanifold of degree N, namely

$$\mathrm{N} = \max\{d(\Sigma) : \Sigma \textit{ is an } \mathrm{n}\textit{-dimensional submanifold of } \mathbb{G}\}$$

then the following statements hold

1. $\mathcal{S}^{\mathrm{N}}(C(\Sigma)) = 0$,
2. *for each $p \in \Sigma$, with $d_\Sigma(p) = \mathrm{N}$*
   - $\delta_{1/r}(p^{-1}\Sigma) \to S_p\Sigma$ *as* $r \to 0^+$,
   - $S_p\Sigma$ *is a subgroup of* $\mathbb{G}$.

Theorem (M. 2017 and previous work.)

If d is a suitable distance *and Σ is a C^1 smooth submanifold of degree* N*, then we have*

$$\mathcal{S}^{\mathrm{N}}(\Sigma) = \int_{\Sigma} |\pi_{\mathrm{N}}(\tau_{\Sigma})| \, dvol \qquad (4)$$

in each of the following cases.

1. *Σ is a smooth Legendrian submanifold (V. M. 2017)*
2. *Σ is a transversal submanifold (with J. T. Tyson and D. Vittone 2015)*
3. *Σ is a curve (with R. Korte 2012)*

Question

Extending the SR area formula (4) to all C^1 submanifolds is still a *largely open question*.

Remark

All of these results hold in general *homogeneous groups*, namely we do not use any Lie bracket generating condition as it is the case for stratified, or Carnot groups.

Sets of finite perimeter

We say that a measurable set $E \subset G$ has finite perimeter if there exists a Radon measure $|\partial_H E|$ and a $|\partial_H E|$-measurable section ν_E of $H\mathbb{G}$ such that

$$\int_E \mathrm{div}\phi = \int_{\mathcal{F}_H E} \langle \phi, \nu_E \rangle d|\partial_H E| \tag{5}$$

for each smooth horizontal vector field

$$\phi = \sum_{j=1}^{m} \phi_j X_j(\phi)$$

with compact support, where $(X_1, \ldots, X_m)$ is a basis of $\mathcal{V}_1$.

The concentration of $|\partial_H E|$ on the *h-reduced boundary* $\mathcal{F}_H E$, according to (5), is a consequence of a general differentiation theorem by L. Ambrosio 2001.

Works on sets of finite perimeter are by L. Ambrosio, E. Le Donne, B. Kleiner, B. Franchi, R. Serapioni, F. Serra Cassano and M. Marchi.

$\mathbb{G}$-regular set

A hypersurface $\Sigma \subset \mathbb{G}$ is $\mathbb{G}$-regular if at each of its points $p \in \Sigma$ there exists an open set Ω containing p and a function $f : \Omega \to \mathbb{R}$ such that the *horizontal gradient*

$$(X_1 f, \dots, X_m f) \neq 0$$

and it is continuous in Ω, with

$$\Omega \cap \Sigma = f^{-1}(0).$$

It was shown in a 2004 paper by B. Kirchheim and F. Serra Cassano that $\mathbb{G}$-regular sets might be a kind of special fractals, *whose Euclidean area is infinite*.

$\mathbb{G}$-rectifiable set

We say that $S \subset \mathbb{G}$ is $\mathbb{G}$-rectifiable if there exists a countable family $\{\Sigma_j\}_{j\ge 0}$ of $\mathbb{G}$-regular sets such that $\mathcal{S}^{Q-1}(S \setminus \bigcup \Sigma_j) = 0$.

Open question

If $\mathbb{G}$ is a stratified group of step higher than two it is not known whether any set $E \subset \mathbb{G}$ of h-finite perimeter has reduced boundary $\mathbb{G}$-rectifiable.

By the measure theoretic area formula we can find a general relationship between perimeter measure and spherical measure

$$|\partial_H E| = \mathcal{S}^{Q-1} \llcorner \mathcal{F}_H E$$

for a general class of distances with some special symmetries and assuming that $\mathcal{F}_H E$ is $\mathbb{G}$-rectifiable, [V. M., IUMJ 2017].

A key point is that $\mathbb{G}$-rectifiable sets can be reduced to $\mathbb{G}$-regular sets and for the latter a special *implicit function theorem* holds, that allows us to represent locally the $\mathbb{G}$-regular set as algebraic graph

$$\{n\varphi(n) \in \mathbb{G} : n \in U\}.$$

Higher codimensional $\mathbb{G}$-regular sets

In the n-th Heisenberg group $\mathbb{H}^n \approx \mathbb{R}^{2n+1}$ we may define more general low codimensional $\mathbb{H}$-regular sets by considering defining mappings $f : \Omega \to \mathbb{R}^k$ such that

$$D_H f = \begin{pmatrix} X_1 f_1 & X_2 f_1 & \cdots & X_{2n} f_1 \\ X_1 f_2 & X_2 f_2 & \cdots & X_{2n} f_2 \\ \vdots & \vdots & \ddots & \vdots \\ X_1 f_k & X_2 f_k & \cdots & X_{2n} f_k \end{pmatrix}$$

is everywhere of maximal rank and continuous in Ω, with

$$\Omega \cap \Sigma = f^{-1}(0).$$

In the case $k \leq n$ there is an *algebraic splitting*

$$\mathbb{H}^n = N \cdot \mathbb{V}, \tag{6}$$

that is: *for every* $N = \ker D_H f$ *there exists a horizontal subgroup* $\mathbb{V}$ *such that* (6) *holds*.

Developments I: area of algebraic graphs in homogeneous groups

For more general couples of groups $\mathbb{G}$ and $\mathbb{M}$, the validity of an *algebraic splitting* allows us to write level sets as algebraic graphs

$$\Sigma = f^{-1}(0) \cap \mathbb{G} = \{n\varphi(n) : n \in O \subset N\} \tag{7}$$

where $f : \Omega \to \mathbb{M}$ has nonsingular h-differential. The validity of (4) precisely amounts to a *general implicit function theorem*, [V. M., 2013].

In Heisenberg groups the simplified version of the previous implicit function theorem had lead to higher codimensional area formulae, [F. Franchi, S. Serapioni and F. Serra Cassano, 2007].

It is natural to expect that also the general algebraic graph parametrization of level sets (7) leads to general area-type formulae.

Developments II: rough level sets

The simplest case where the splitting assumption does not hold is that of mapping $f : \mathbb{H} \to \mathbb{R}^2$. Here the kernel

$$\ker D_H f = N = \left\{(0,0,t) \in \mathbb{R}^3 : t \in \mathbb{R}\right\}$$

has no complementary 2-dimensional horizontal subgroups.

There is no way to represent the level set as an algebraic graph.

Existence of nontrivial level sets as continuous curves has been established in [G. P. Leonardi, V. M., 2010] and [A. Kozhevnikov, 2011]

[V. M., E. Stepanov and D. Trevisan, 2016]

If $F : \mathbb{H} \to \mathbb{R}^2$ is of class $C_H^{1,\alpha}$ and $D_H f(p)$ has maximal rank then $F^{-1}(F(p)) \cap B(p, r)$ is the image of a Hölder continuous curve $\gamma : I \to F^{-1}(F(p))$ such that

$$\gamma_\sharp(\mathcal{L}^1 \llcorner I)) = \mathcal{S}_d^2 \llcorner \gamma(I).$$

Moreover γ is genuinely 1/2-Hölder continuous

$$d(\gamma_s, \gamma_t) \approx |t - s|^{1/2}$$

and satisfies a special “LSDE”.

A naive idea of the method

We pretend there exists a parametrization $t \to \gamma_t$ such that

$$F(\gamma_t) = F(p).$$

However the $C_H^{1,\alpha}$ regularity, namely the α-Hölder continuity of XF and YF does not allow us to differentiate this equality.

We follow some ideas behind the theory of "rough differential equations" introduced by T. Lyons, 1990's, precisely the approach developped by M. Gubinelli 2004 for "controlled rough differential equations"

$$dY_t = F(Y_t)dX_t.$$

Instead of taking derivatives, we consider only finite increments with suitable estimates of errors.

Consider the remainder

$$R(p,\gamma_t) = F(\gamma_t) - F(p) - D_H f(p)(p^{-1}\gamma_t)$$
$$= -D_H f(p)(p^{-1}\gamma_t)$$

and exploiting the maximal rank of $D_H f(p)$ take the difference

$$\pi_H(\gamma_s^{-1}\gamma_t) = \nabla_H f(p)^{-1}(R(p,\gamma_t) - R(p,\gamma_s).$$

Assume the curve has "unit velocity" formally imposing

$$\theta(\dot\gamma) = 1,$$

where θ is the contact form. Then

$$1 = \dot\gamma_t^3 - \gamma_t^1\dot\gamma_t^2 + \gamma_t^2\dot\gamma_t^1.$$

This last equality does not make any sense for Hölder continuous curves!

A heuristic argument shows that we should seek solutions in the space of $(1+\alpha)/2$-Hölder continuous. This suffices to integrate in the Young sense the previous equality, getting

$$t - s = \gamma_t^3 - \gamma_s^3 - \int_s^t \gamma^1 d\gamma^2 + \int_s^t \gamma^2 d\gamma^1.$$

We can reduce the equation to differences adding an error

$$\begin{aligned}\int_s^t \gamma^1 d\gamma^2 - \int_s^t \gamma^2 d\gamma^1 &= \gamma_s^1(\gamma_t^2 - \gamma_s^2) - \gamma_s^2(\gamma_t^1 - \gamma_s^1) + E_{st} \\ &= \gamma_s^1\gamma_t^2 - \gamma_s^2\gamma_t^1 + E_{st}.\end{aligned}$$

Level set differential equations

We eventually reach a system of “difference equations”, that we call *Level Set Differential Equations*.

$$(LSDE)\begin{cases}\pi_H(\gamma_s^{-1}\gamma_t)=-\nabla_H F(p)^{-1}\big(R(p,\gamma_t)-R(p,\gamma_s)\big)\\(\gamma_s^{-1}\gamma_t)^3=t-s+E_{st}\\|E_{st}|\le C|s-t|^{1+\alpha}\end{cases}$$

For this system of LSDE we prove existence, uniqueness and stability of solutions, that are automatically level sets of F from the way the system has been constructed.

Although the solutions to (LSDE) are Hölder continuous, these equations allow us to study the intrinsic blow-up in order to compute precisely their spherical measure $\mathcal{S}^2$.

The existence of solutions is established by the Schauder's fixed point theorem, constructing the "solution mapping":

1. we first integrate, *in the sense of Young*, the "vertical equation"

$$(\gamma_s^{-1}\gamma_t)^3 = t - s + E_{st}$$

2. then we plug this solution into the "horizontal equation"

$$\pi_H(\gamma_s^{-1}\gamma_t) = -\nabla_H F(p)^{-1}\left(R(p,\gamma_t) - R(p,\gamma_s)\right)$$

   and perform a second Young integration, finally getting the solution mapping.

The modern way to perform these integrations is given by the following elementary, but important result.

Sewing lemma

Let $0 < \alpha < 1$ and let $(s,t) \to A_{st}$ continuous on $\Delta = \{(s,t) \in \mathbb{R}^2 : a \le s \le t \le b\}$. If there holds

$$|A_{st} - A_{su} - A_{ut}| \le C|t-s|^{1+\alpha} \quad \text{for } s < u < t$$

and $y \in \mathbb{R}$, then there exists $\kappa > 1$ and a unique $I : [a,b] \to \mathbb{R}$ such that $I(a) = \xi$ and

$$|I_t - I_s - A_{st}| \le \kappa C|t-s|^{1+\alpha}$$

Example

Considering $A_{st} = f_t(g_t - g_s)$ with $f \in C^\alpha$, $g \in C^\beta$ and $\alpha + \beta > 1$ yields a “germ” A_{st} that satisfies the hypothese of the sewing lemma. Then the mapping $s \to I_s$ is the Young integral

$$I_t = y + \int_a^t f dg.$$

Natural open questions

1. Can we reach the case $\alpha = 0$ through this method, perhaps by some compactness as $\alpha \to 0^+$?
2. Could we use other notions of “integral” for the case $\alpha = 0$?
3. Can we extend this method to higher dimensional level sets?