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The Multicomponent KP and Fay Trisecant Formula

Andrea Maffei

The n-component KP hierarchy is a system of nonlinear partial differential equations that generalizes the classical equation of Kadometsev-Petviashvili. Kac and van de Leur [7] studied the algebraic aspect of this generalization, giving many equivalent formulations. C. De Concini pointed out to me that one of the equations found by Kac and van de Leur has the same structure as the Fay trisecant formula. From this observation and from the well-known relation between KP equations and the theory of theta functions, a way to prove the trisecant formula is suggested.

In the first part of this paper we complete the work of Kac and van de Leur, showing the relation between the action of a group on a Grassmannian Gr and the solutions of the n-KP. In particular, we prove that a generalized Sato tau function evaluated along the orbit of this action solves the n-KP in the Hirota form. In the second part we generalize Krichever's construction to the case of n points on a Riemann surface, and the relation between the tau function and the theta function defined on the Jacobian of the curve is proved in this case. This relation allows us to prove that the theta function solves the n-component KP. Finally, we give a geometric interpretation of some of these formulas. In particular, using the Plücker equations which appear in the hierarchy, a Fay generalized trisecant formula is proved (see, for a more precise version of this formula, the article of R. C. Gunning [6]).

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1 From the Grassmannian to the n-component KP in the Hirota form

$H^{(n)}$ is the topological vector space on which the Grassmannian will be constructed. If we set the following notation:

$S^2 = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere and z is the coordinate on S^2,

$D_\varepsilon = \{z \in S^2\colon\ |z^{-1}| < \varepsilon\}$,

$D^*_\varepsilon = D_\varepsilon - \{\infty\}$,

the definition of $H^{(n)}$ is given by

$$H^{(n)} = H^{(n)}_+ \oplus H^{(n)}_- = \varinjlim \Gamma(D^*_\varepsilon),$$

$$H^{(n)}_+ = \{f \in \mathcal{O}(\mathbb{C}) \text{ and } f(0) = 0\}^n,$$

$$H^{(n)}_- = \mathbb{C}\{z^{-1}\}^n, \text{ the set of holomorphic germs in } z = \infty.$$

We also set $v^{(j)}_k = (\ldots, z^k, \ldots)$ and v_k in the 1-dimensional case, and we define the linear operator $e^{(ij)}_{kl}$ by $e^{(ij)}_{kl} v^{(h)}_m = \delta_{hj}\delta_{lm} v^{(i)}_k$.

Remark 1.1. $H^{(n)} \simeq H = H^{(1)}$ and the isomorphism maps $(f_1, \ldots, f_n)$ into $z^{1-n} f_1(z^n) + \cdots + f_n(z^n)$. We observe also that this isomorphism preserves the decomposition of the space H: the image of $H^{(n)}_+$ is H_+ and the image of $H^{(n)}_-$ is H_-. Finally, $v^{(i)}_l$ is mapped in $v_{i+n(l-1)}$ and $e^{(ij)}_{kl}$ corresponds to $e_{i+n(k-1),\, j+n(l-1)}$.

Now we can use the results of the 1-dimensional case ([1] or [2]), and we observe that in a standard way the space $H^{(n)}$ turns out to be a locally convex topological vector space as well as the relative spaces of continuous operators $H_{\mu\lambda} = \mathcal{L}(H_\lambda, H_\mu) = \{L\colon\ H_\lambda \to H_\mu$ linear and continuous$\}$ for $\lambda, \mu \in \{+, -\}$. Furthermore, it is possible to define a trace and a determinant on certain ideals of maps. If

$$\mathcal{T}_+ = \{t \in H_{++} \text{ and } t = uv \text{ with } u \in H_{+-} \text{ and } v \in H_{-+}\},$$

$$\mathcal{D}_+ = \{1 + t\colon\ t \in \mathcal{T}_+\},$$

following [1] or [2], we have the following lemma.

Lemma 1.2. Let $a \in \mathcal{D}_+$ and $t = a - 1 \in \mathcal{T}_+$; then:

(1) If F is a Fredholm operator then $F + t$ is Fredholm, too, and $\operatorname{index}(F + t) = \operatorname{index}(F)$.

(2) The trace of $\bigwedge^i t$ is defined for all i.

(3) $\operatorname{Det} a = 1 + \sum_{i=1}^\infty \operatorname{tr}(\bigwedge^i t)$ converges.

(4) $\operatorname{Det} a \neq 0 \Leftrightarrow a$ is an isomorphism.

(5) For $g \in G\mathcal{L}(H_+)$, we have $\operatorname{tr}(gtg^{-1}) = \operatorname{tr} t$ and $\operatorname{Det}(gag^{-1}) = \operatorname{Det} a$.

(6) If t is of finite rank, then Det is the usual determinant.

(7) Finite rank maps are dense in H_{+-}. □

It is possible now to define the Grassmannian Gr:

$$\mathrm{Gr} = \{W\colon\ W \text{ is a closed subspace of } H \text{ s.t. } \pi_+|_W \text{ is Fredholm of index } 0\}$$

where π_+ is the projection of H on H_+ along H_-. Gr turns out to be an holomorphic manifold modeled on H_{-+}. Even though Gr is not compact, its only holomorphic sections are the constant ones. It is worth observing that Gr may be defined also as

$$\mathrm{Gr} \simeq \frac{\mathcal{D}}{G\mathcal{D}_+}$$

where $\mathcal{D} = \{w : H_+ \to H : \pi_+ \circ w : H_+ \mapsto H_+ \in \mathcal{D}_+$ and w is an embedding$\}$ is the set of admissible bases and $G\mathcal{D}_+ = \mathcal{D}_+ \cap G\mathcal{L}$ acts on $\mathcal{D}$ by composition on the right. This characterization allows us to define the line bundle det on Gr, as in the finite-dimensional case.

Definition 1.3. We have

$$\det = \frac{\mathcal{D} \times \mathbb{C}}{\sim}$$

where $\sim$ is the following relation in $\mathcal{D} \times \mathbb{C}$: for $u \in \mathcal{D}$ and $t \in \mathcal{D}_+$, $(ut;\delta) \sim (u;\delta \operatorname{Det} t)$. Thus it follows that

$$\det{}^* = \frac{\mathcal{D} \times \mathbb{C}}{\sim'},$$

$(ut;\delta) \sim' (u;\delta \operatorname{Det} t^{-1})$ for $u \in \mathcal{A}$ and $t \in \mathcal{D}_+$.

To make explicit computations, it is useful to give a complete set of local charts for Gr and det, which will be labelled by the following set:

$$\mathcal{I}_0 = \{I \in \mathbb{Z}^{\mathbb{N}}\colon\ I = (i_1, i_2, \ldots) \text{ and } i_1 < i_2 < \cdots \text{ and } i_l = l \text{ definitively}\}.$$

If $I \in \mathcal{I}_0$ and $H_I = \overline{\langle\{v_j\colon\ j \in I\}\rangle}$, and $H_{I-} = \overline{\langle\{v_j\colon\ j \notin I\}\rangle}$, it is easy to show that $H_I \in \mathrm{Gr}$ and that $H = H_I \bigoplus H_{I-}$. We also establish an isomorphism between H_+ and H_I in the following way:

$$\phi_I^+(v_l) = v_{i_l} \quad \text{for } l > 0;$$

and in the same way we define ϕ_I^- where $\cdots < i_{-1} < i_0$. At this point, and after observing that, given $A \in \mathcal{L}(H_I, H_{I-})$, its graph is in Gr, we can finally introduce the following open subsets of Gr:

$$\mathcal{W}_I = \{W\colon\ W \text{ is the graph of some } A \in \mathcal{L}(H_I, H_{I-})\},$$

and we call $\mathcal{W}_0$ the subset obtained with $I = \mathbb{N}$. We make some remarks and set some notation:

(1) $\mathcal{W}_I = \{W \in \mathrm{Gr}\colon \pi_I : W \to H_I \text{ is an isomorphism}\} = \{W \in \mathrm{Gr}\colon W \cap H_{I-} = \{0\}\}$ where $\pi_I\colon H \to H_I$ is the projection with kernel H_{I-}. Similarly, we also define π_{I-} and $\varpi_I^+ = (\phi_I^+)^{-1} \circ \pi_I, \varpi_I^- = (\phi_I^-)^{-1} \circ \pi_{I-}$.

(2) $\mathcal{W}_I \simeq H_{-+}$, where the isomorphism Φ_I maps $A \in H_{-+}$ in the graph of $\phi_I^- \circ A \circ (\phi_I^+)^{-1}$. We will denote with A_W^I the inverse of this isomorphism. Hence, if for $W \in \mathcal{W}_I$ we set $M_W^I = j_I^+ \circ \phi_I^+ + j_I^- \circ \phi_I^- \circ A_W^I$, where j_I^+ and j_I^- are the inclusions of H_I^+ and H_I^- in H, then M_W^I is a representative element of W in $\mathcal{D}$.

(3) In particular, if $W \in \mathcal{W}_I \cap \mathcal{W}_J$, we have the following crossing maps:

$$M_W^J = M_W^I \circ (\varpi_J^+ \circ M_W^I)^{-1}$$

$$A_W^J = \varpi_J^- \circ (\phi_I^+ + \phi_I^- \circ M_W^I) \circ (\varpi_J^+ \circ (\phi_I^+ + \phi_I^- \circ M_W^I))^{-1}.$$

(4) Local charts for det and $\det^*$ can be given as follows:

$$\chi_I\colon \det|_{\mathcal{W}_I} \longrightarrow H_{-+} \times \mathbb{C} \qquad \text{and} \qquad \chi_I^*\colon \det^*|_{\mathcal{W}_I} \longrightarrow H_{-+} \times \mathbb{C}$$

$$\chi_I(w, \delta) = (A_w^I, \det(\varpi_I^+ \circ w)\delta) \qquad \chi_I^*(w, \delta) = (A_w^I, \det(\varpi_I^+ \circ w)^{-1}\delta).$$

The transition maps are then given by

$$\chi_J \chi_I^{-1}(A_W^I, \delta) = (A_W^J, \det(\varpi_J^+ M_W^I)\delta),$$

$$\chi_J^* {\chi^*}_I^{-1}(A_W^I, \delta) = (A_W^J, \det(\varpi_J^+ M_W^I)^{-1}\delta).$$

With this notation, it is easy to define sections of $\det^*$. We observe that det, as in the finite-dimensional case, has no section.

Definition 1.4. If $I \in \mathcal{I}_0$, we define

$$\sigma_I(w) = (w; \det \varpi_I \circ w) \quad \text{for } w \in \mathcal{D}$$

is a well-given section of $\det^*$ and we define $\sigma_0 = \sigma_{\mathbb{N}}$.

Proposition 1.5. (1) If $\sigma_1, \sigma_2 \in \Gamma(\mathrm{Gr}, \det^*)$ and $\mathrm{Zero}(\sigma_1) = \mathrm{Zero}(\sigma_2)$, then $\sigma_1 = \lambda\sigma_2$ with $\lambda \in \mathbb{C}^*$.

(2) $W \in \mathcal{W}_I \Leftrightarrow \sigma_I(W) \neq 0$.

(3) $\{\sigma_I\}$ is a linearly independent set. □

The action of the linear group

First notice that, through the decomposition $H = H_+ \bigoplus H_-$, given $\varphi \in g\mathcal{L}(H)$, we can think of φ as a two-by-two operators matrix:

$$\varphi = \begin{pmatrix} \varphi_{++} & \varphi_{+-} \\ \varphi_{-+} & \varphi_{--} \end{pmatrix} \quad \text{with } \varphi_{\lambda\mu} \in H_{\lambda\mu}.$$

Definition 1.6. We have

$$G = \{g \in G\mathcal{L}(H)\colon\ g_{++} \text{ is a Fredholm operator of index zero}\}$$

$$G_+ = \{g \in G\colon\ g_{-+} = 0\} \qquad G_- = \{g \in G\colon\ g_{+-} = 0\}$$

$$\Gamma_+^{(n)} = \left\{ e^A \text{ with } A = \begin{pmatrix} f_1 & & 0 \\ & \ddots & \\ 0 & & f_n \end{pmatrix} \text{ and } \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \in H_+^{(n)} \right\} \subset G_+,$$

$$\Gamma_-^{(n)} = \left\{ e^A \text{ with } A = \begin{pmatrix} f_1 & & 0 \\ & \ddots & \\ 0 & & f_n \end{pmatrix} \text{ and } \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \in H_-^{(n)} \right\} \subset G_-.$$

We call these elements e^f. We define also

$$\Lambda = \{\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n\colon\ \lambda_1 + \cdots + \lambda_n = 0\},$$

and we think of Λ as a discrete subgroup of $GL(n, H)$; and for $\lambda \in \Lambda$ we denote

$$z^\lambda = \begin{pmatrix} z^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & z^{\lambda_n} \end{pmatrix},$$

which is easily seen to be in G.

We introduce also the following central extension of G:

$$1 \to \mathbb{C}^* \to \widehat{G} \to G \to 1, \tag{1}$$

where

$$\widehat{G} = \frac{\mathcal{E}}{S\mathcal{D}+},$$

and $S\mathcal{D}_+ = \{t \in G\mathcal{D}_+\colon\ \det t = 1\}$, $\mathcal{E} = \{(g, q) \in G \times G\mathcal{L}(H_+)\colon\ gq^{-1} \in \mathcal{D}\}$, and $S\mathcal{D}_+$ acts on $\mathcal{E}$ on the left by $t(g, q) = (g, tq)$. We observe that G acts transitively on Gr in the natural way, and that $\widehat{G}$ acts on the line bundles det and det* in a compatible way, indeed $(g, q)(\omega, \delta^*) = (g\omega q^{-1}, \delta^*)$. The homomorphism from $\widehat{G}$ to Aut(det*), which is determined by this action, is easily seen to be injective. We can define a section on the subgroups G_+ and G_-, and evidently also on their subgroups $\Gamma_+^{(n)}$ and $\Gamma_-^{(n)}$, to $\widehat{G}$ via $g \mapsto \hat{g} = (g, g_{++})$. As far as Λ, we have to further investigate the possibility of the existence of a section to $\widehat{G}$. In order to do this we need a lemma.

Definition 1.7. For $1 \le i \le n$, let $\delta_i = (0, \dots, 1, \dots, 0)$, and for $1 \le i, j \le n$, $\lambda_{ij} = \delta_i - \delta_j \in \Lambda$.

Proposition 1.8. If $\widehat{z^\nu}, \widehat{z^\mu} \in \widehat{G}$, then

$$\widehat{z^\nu}\widehat{z^\mu} = (-1)^{{}^t\mu\nu}\widehat{z^\mu}\widehat{z^\nu}. \qquad \square$$

Proof. We observe that $\widehat{z^\nu}\widehat{z^\mu}\widehat{z^\nu}^{-1}\widehat{z^\mu}^{-1}$ is an automorphism of det over id, so it must be

$$\widehat{z^\nu}\widehat{z^\mu} = c(\nu, \mu)\widehat{z^\mu}\widehat{z^\nu},$$

and $c(\nu, \mu)$ is independent of the choices we have made of $\widehat{z^\nu}$ over z^ν and of $\widehat{z^\mu}$ over z^μ. If $\widehat{z^\nu} = (z^\nu, q_\nu)$ and $\widehat{z^\mu} = (z^\mu, q_\mu)$, then $c(\nu, \mu) = \det(q_\nu q_\mu q_\nu^{-1} q_\mu^{-1})$. Since c is a symmetric and bimultiplicative function, it is enough to prove that $c(\lambda_{ij}, \lambda_{hk}) = (-1)^{\delta_{ik}+\delta_{ih}+\delta_{jk}+\delta_{jh}}$. Let

$$A = z^\nu = \sum_{l \neq i,j} \sum_{m=-\infty}^{\infty} e_{mm}^{(ll)} + \sum_{m=-\infty}^{\infty} e_{m+1,m}^{(ii)} + \sum_{m=-\infty}^{\infty} e_{m-1,m}^{(jj)}$$

$$A_{++} = \sum_{l \neq i,j} \sum_{m=1}^{\infty} e_{mm}^{(ll)} + \sum_{m=1}^{\infty} e_{m+1,m}^{(ii)} + \sum_{m=2}^{\infty} e_{m-1,m}^{(jj)}$$

$$A_{+-} = e_{1,0}^{(ii)} \qquad A_{-+} = e_{0,1}^{(jj)} \qquad A_0 = e_{11}^{(ij)}$$

$$q_\nu = q = A_{++} + A_0$$

$$B = z^\mu = \sum_{l \neq h,k} \sum_{m=-\infty}^{\infty} e_{mm}^{(ll)} + \sum_{m=-\infty}^{\infty} e_{m+1,m}^{(hh)} + \sum_{m=-\infty}^{\infty} e_{m-1,m}^{(kk)}$$

$$B_{++} = \sum_{ll \neq h,k} \sum_{m=1}^{\infty} e_{mm}^{(ll)} + \sum_{m=1}^{\infty} e_{m+1,m}^{(hh)} + \sum_{m=2}^{\infty} e_{m-1,m}^{(kk)}$$

$$B_{+-} = e_{1,0}^{(hh)} \qquad B_{-+} = e_{0,1}^{(kk)} \qquad B_0 = e_{11}^{(hk)}$$

$$q_\mu = p = B_{++} + B_0.$$

We have

$$A_0 B_0 = \delta_{jh} e_{11}^{(ik)}$$

$$A_0B_{++} = e^{(ij)}_{11}B_{++} = \begin{cases} e^{(ij)}_{11} & \text{if } j \neq h, k \\ 0 & \text{if } j = h \\ e^{(ij)}_{12} & \text{if } j = k \end{cases}$$

$$A_{++}B_0 = A_{++}e^{(hk)}_{11} = \begin{cases} e^{(hk)}_{11} & \text{if } h \neq i, j \\ 0 & \text{if } h = j \\ e^{(hk)}_{21} & \text{if } h = i. \end{cases}$$

We observe that $z^\lambda z^\mu = z^\mu z^\lambda$; hence

$$A_{++}B_{++}\,{}^tA_{++}\,{}^tB_{++} = B_{++}A_{++}\,{}^tA_{++}\,{}^tB_{++} + B_{+-}A_{-+}\,{}^tA_{++}\,{}^tB_{++} - A_{+-}B_{-+}\,{}^tA_{++}\,{}^tB_{++}$$

and $A_{++}\,{}^tA_{++} = I - e^{(ii)}_{11}$ and $B_{++}\,{}^tB_{++} = I - e^{(hh)}_{11}$. Then

$$B_{++}A_{++}\,{}^tA_{++}\,{}^tB_{++} = I - e^{(hh)}_{11} - B_{++}e^{(ii)}_{11}\,{}^tB_{++} = \begin{cases} I - e^{(hh)}_{11} - e^{(ii)}_{11} & \text{if } i \neq h, k \\ I - e^{(hh)}_{11} - e^{(ii)}_{22} & \text{if } i = h \\ I - e^{(hh)}_{11} & \text{if } i = k \end{cases}$$

$$B_{+-}A_{-+}\,{}^tA_{++}\,{}^tB_{++} = e^{(hh)}_{10}e^{(jj)}_{01}\,{}^tA_{++}\,{}^tB_{++} = e^{(hh)}_{10}0\,{}^tB_{++} = 0$$

$$A_{+-}B_{-+}\,{}^tA_{++}\,{}^tB_{++} = e^{(ii)}_{10}e^{(kk)}_{01}\,{}^tA_{++}\,{}^tB_{++} = \delta_{ik}e^{(ii)}_{12}\,{}^tB_{++} = \delta_{ik}e^{(ii)}_{11}.$$

Cases $i = j$, $j = k$, $i = h$, and $j = k$ or $i = k$ and $j = h$ are trivial. We have to study the cases

$i = h$ and $j \neq k$ $\qquad$ $i = k$ and $j \neq h$

$j = k$ and $i \neq h$ $\qquad$ $j = h$ and $i \neq k$.

We complete the calculations only in the second case, noticing that the remaining ones are similar:

$$\begin{aligned} qpq^{-1}p^{-1} &= (A_{++}B_{++} + A_{++}B_0 + A_0B_{++} + A_0B_0) \\ &\quad \cdot ({}^tA_{++}\,{}^tB_{++} + {}^tA_{++}\,{}^tB_0 + {}^tA_0\,{}^tB_{++} + {}^tA_0\,{}^tB_0) \\ &= (A_{++}B_{++} + e^{(hk)}_{11} + e^{(ij)}_{11} + 0)({}^tA_{++}\,{}^tB_{++} + e^{(jh)}_{11} + 0 + 0) \\ &= A_{++}B_{++}\,{}^tA_{++}\,{}^tB_{++} + A_{++}B_{++}e^{(jh)}_{11} + e^{(hk)}_{11}\,{}^tA_{++}\,{}^tB_{++} \\ &\quad + e^{(ij)}_{11}\,{}^tA_{++}\,{}^tB_{++} + e^{(hk)}_{11}e^{(jh)}_{11} + e^{(ij)}_{11}e^{(jh)}_{11} \\ &= I - e^{(hh)}_{11} + A_{++}e^{(jh)}_{11} + 0 + e^{(hk)}_{12}\,{}^tB_{++} + e^{(ih)}_{11} = I - e^{(hh)}_{11} + e^{(hi)}_{11} + e^{(ih)}_{11}, \end{aligned}$$

and then $\det(qpq^{-1}p^{-1}) = -1$. ■

The last proposition shows that there is no group section from Λ to $\widehat{G}$. However, it is useful to introduce an inverse map of the projection of $\widehat{\Lambda}$ to Λ. If $i \neq j$, let

$$\widetilde{z^{\lambda_{ij}}} = (z^{\lambda_{ij}}, (z^{\lambda_{ij}})_{++} + e_{11}^{(ij)}) = (z^{\lambda_{ij}}, q_{ij}) \in \widehat{G}. \tag{2}$$

Definition 1.9. For $\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda$, we define

$$\widehat{z^{\lambda}} = \widetilde{z^{\lambda_{21}}}^{\lambda_2} \cdots \widetilde{z^{\lambda_{n1}}}^{\lambda_n} i_1^{\lambda_1^2 - \lambda_1}.$$

We define also ε to be the bimultiplicative function on $\mathbb{Z}^n \times \mathbb{Z}^n$ such that

$$\varepsilon(\delta_i, \delta_j) = \begin{cases} -1 & \text{if } i > j \\ 1 & \text{if } i \leq j. \end{cases}$$

In particular, we have $\widehat{z^{\lambda_{ij}}} = \varepsilon(\delta_i, \delta_j)\widetilde{z^{\lambda_{ij}}}$ and the following proposition.

Proposition 1.10. If $\nu, \mu \in \Lambda$, then

$$\widehat{z^{\nu+\mu}} = \varepsilon(\nu, \mu)\widehat{z^{\nu}}\widehat{z^{\mu}}. \qquad \square$$

We have the following commutation rules.

Proposition 1.11. If $f \in H_+^{(n)}$ and $\tilde{f} \in H_-^{(n)}$, then as a relation of elements of Aut(det*), we have that

$$\widehat{e^f}\widehat{e^{\tilde{f}}} = e^{S(f,\tilde{f})}\widehat{e^{\tilde{f}}}\widehat{e^f}$$

where

$$S(f, \tilde{f}) = \frac{1}{2\pi i}\int_{|z|=\varepsilon^{-1}} \langle f, \tilde{f}' \rangle dz = \frac{1}{2\pi i}\int_{|z|=\varepsilon^{-1}} \sum_{i=1}^{n} f_i \tilde{f}_i{}' dz. \qquad \square$$

Proof. It follows obviously from the same formula in the $n = 1$ case (see, for example, [10]). ■

Proposition 1.12. Let $f \in H_+$ or $f \in H_-$ and $\lambda \in \Lambda$. Then

$$\widehat{e^f}\widehat{z^{\lambda}} = \widehat{z^{\lambda}}\widehat{e^f}. \qquad \square$$

Proof. The formula can be proved by working as in the proof of the commutation rule for $\widehat{\Lambda}$. ■

We study also the commutation rules between Λ or Γ and the section σ_0 of det*.

Proposition 1.13. (1) If $f \in H_-$, then $\sigma_0(e^f W) = \widehat{e^f}\sigma_0(W)$.

(2) We have

$$\widehat{z^{\lambda_{ij}}} \circ \sigma_0 = \varepsilon(\delta_i, \delta_j)\,(-1)^{(i+1)(\delta_{ij}+1)}\,\sigma_{z^{\lambda_{ij}}H_+} \circ z^{\lambda_{ij}}. \qquad \square$$

Proof. (1) Easy. (2) We prove the formula for $\widetilde{z^{\lambda_{ij}}}$. Let $I \in \mathcal{I}_0$ such that $H_I = z^{\lambda_{ij}}H_+$ and

$$\sigma = \widetilde{z^{\lambda_{ij}}} \circ \sigma_0 \circ z^{-\lambda_{ij}}.$$

σ is a section of $\det^*$ and $\sigma(W) = 0$ if and only if $\sigma_I(W) = 0$. Hence, it must be $\sigma = C\,\sigma_I$. In particular, we can calculate C by evaluating σ and σ_I on H_I. ■

The duality

On the space $H^{(n)}$, it is possible to define a perfect pairing $(\,;\,)$ by the formula

$$(f; g) = \operatorname*{Res}_{z=\infty}(f(z)|g(z))dz^{-1} = \operatorname*{Res}_{z=\infty}\sum_{i=1}^{n} f_i(z)g_i(z)dz^{-1}$$

where $f = (f_1, \dots, f_n)$, $g = (g_1, \dots, g_n)$.

We observe that $(v_h^{(i)}; v_k^{(j)}) = \delta_{ij}\delta_{h+k,1}$ or equivalently $(v_{i+(h-1)n}; v_{j+(k-1)n}) = \delta_{ij}\delta_{h+k,1}$. In particular, $(\,;\,)$ is a perfect pairing between H_+ and H_-, which are totally isotropic subspaces of H. If i^* is the only integer such that $(v_i; v_j) = \delta_{ji^*}$, we can define for $I \in \mathcal{I}_0$

$$I^\perp = \{i^* \in \mathbb{Z}:\ i \notin I\} \in \mathcal{I}_0 \quad \text{in order to have } H_{I^\perp} = (H_I)^\perp,$$

and observe that the duality gives isomorphisms, denoted as F_I^+ and F_I^-, between H_I and $(H_{I^\perp -})^*$ and between H_{I-} and $(H_{I^\perp})^*$. For $I = \mathbb{N}$, let $F_+ = F_{\mathbb{N}}^+$ and $F_- = F_{\mathbb{N}}^-$.

Proposition 1.14. (1) $W \in \mathrm{Gr} :\Rightarrow\ W^\perp \in \mathrm{Gr}$.

(2) $W \in \mathcal{W}_I \Leftrightarrow W^\perp \in \mathcal{W}_{I^\perp}$. □

Proof. Easy. ■

Definition 1.15. By means of the above proposition, we can define a map μ: $\mathrm{Gr} \to \mathrm{Gr}$ by

$$\mu(W) = W^\perp.$$

It is easy to see that this map is an holomorphic one. By considering the charts $\mathcal{W}_I$ and $\mathcal{W}_{I^\perp}$, we have that $\Phi_{I^\perp}^{-1} \circ \mu \circ \Phi_I$ is given by

$$A \longmapsto -(\phi_{I^\perp}^-)^{-1} \circ (F_{I^\perp}^-)^{-1} \circ {}^t(\phi_I^+)^{-1} \circ {}^tA \circ {}^t(\phi_I^-) \circ F_{I^\perp} \circ (\phi_{I^\perp}^+).$$

In order to study the relations between the duality and det, we define a permutation $\rho = \rho_I$ of the set $\{n \le 0\}$: If $I \in \mathcal{I}_0$ with $I = \{i_1 < i_2 < \cdots\}$, $I_- = \{\cdots < i_{-1} < i_0\}$ and $i_l^\perp$ are the relative indices for $I^\perp$, then ρ is defined by

$$i_{\rho_l} = (i^\perp_{l^*})^* \quad \text{for } l \le 0. \tag{3}$$

We observe also that

$$F_{I^\perp} = {}^tF_{I-}, \qquad F_{I^\perp -} = {}^tF_I,$$

$$\varpi^+_{I^\perp} = F_+^{-1} \circ {}^t(j_I^-) \circ F, \qquad \varpi^-_{I^\perp} = F_-^{-1} \circ {}^t(j_I^+) \circ F.$$

Proposition 1.16. There exists exactly one automorphism $\tilde\mu$ of $\det^*$ over μ such that $\widetilde\mu \circ \sigma_0 = \sigma_0 \circ \mu$.

Furthermore, we have $\widetilde\mu \circ \sigma_I = \varepsilon(\rho)(-1)^{\ell(I)}\sigma_{I^\perp} \circ \mu$ where $\ell(I) = \sum_{l\ge 1} l - i_l$. (These numbers are well known in the theory of the Grassmannian, since they are related to the computation of the cohomology.) □

Proof. Following the required commutation rule between σ_0, μ, and $\widetilde\mu$ in the charts $\mathcal{W}_0$, the map $\widetilde\mu$ must be given by the formula $\widetilde\mu\colon (A;\lambda) \longmapsto (-F_-^{-1}\,{}^tAF_+;\lambda)$. Therefore, in the open set $\mathcal{W}_0 \cap \mathcal{W}_I$ the map $\widetilde\mu$ is expressed as follows:

$$\chi_{I^\perp} \circ \widetilde\mu \circ \chi_I^{-1}\colon (A^I_W;\lambda) \longmapsto \left(A^{I^\perp}_{W^\perp}; \frac{\det(\pi_+ M^{I^\perp}_{W^\perp})}{\det(\pi_+ M^I_W)}\lambda\right).$$

If $i_1 < \cdots < i_s < 1 \le i_{s+1} \cdots 1_r = r < \cdots$ and $\cdots < i_{-r} = -r < \cdots < i_{-s} < 1 \le i_{1-s} < \cdots < i_0$, then we have

$$\begin{aligned}\det(\pi_+ M^I_W) &= \det(\pi_+ \circ (j_I^+\phi_I^+ + j_I^-\phi_I^- A^I_W)) \\ &= \det\left(\sum_{l=s+1}^r e_{i_l l} + \sum_{l=1}^r \sum_{h=1-s}^0 a_{hl} e_{i_h l}\right) = (-1)^{\sum_{l=s+1}^r i_l + l} \det(a_{hl})_{\substack{h=0\cdots 1-s\\ l=1\cdots s}}\end{aligned}$$

$$\det(\pi_+ M^{I^\perp}_{W^\perp}) = \det(((\phi_I^-)^{-1}\pi_I^- - A(\phi_I^+)^{-1}(\pi_I^+)) \circ j_-) \det({}^t(F_+)^{-1}\,{}^t(\phi^+_{I^\perp})^t F_{I^\perp}\phi_I^-)$$

and

$$\begin{aligned}\det((\phi_I^-)^{-1}\pi_I^- - A(\phi_I^+)^{-1}(\pi_I^+)) &= \det\left(\sum_{l=-r}^{-s} e_{li_l} - \sum_{l=1}^{s}\sum_{h=-r}^{0} a_{hl} e_{hi_l}\right) \\ &= (-1)^{s + \sum_{l=-r}^{-s} i_l + l} \det(a_{hl})_{\substack{h=0\cdots 1-s\\ l=1\cdots s}}\end{aligned}$$

and

$$B = {}^t(F_+)^{-1}\,{}^t(\phi^+_{I^\perp})^t F_{I^\perp}\phi_I^- = F_-^{-1}(\phi^+_{I^\perp})F_{I-}\phi_I^-.$$

Hence B: $v_{i_{\rho_l}} \mapsto v_{(i^{\perp}_{l^*})^*} \mapsto v_l$ and $\det B = \varepsilon(\rho)$ and

$$\frac{\det(\pi_+ M^{I^\perp}_{W^\perp})}{\det(\pi_+ M^{I}_{W})} = \varepsilon(\rho)(-1)^{s+\sum_{l=-r}^{-s} i_l+l+\sum_{l=s+1}^{r} i_l+l}.$$

It is independent from A. So $\widetilde{\mu}$ is an holomorphic automorphism of det and det*. We remark that $s+\sum_{l=-r}^{-s} i_l + l + \sum_{l=s+1}^{r} i_l + l \equiv \ell(I) \pmod 2$, and the local expression of $\tilde{\mu}$ in the chart $\mathcal{W}_I$ is

$$\widetilde{\mu}\colon\ (A^I_W;\lambda) \longmapsto \left(A^{I^\perp}_{W^\perp};\varepsilon(\rho)(-1)^{\ell(I)}\lambda\right).$$

The assertion on the commutation rule with σ_I is now straightforward. ■

Finally, we study the commutation rules between the duality and $\widehat{G}$.

Proposition 1.17. (1) If $f \in H^{(n)}$, then $(e^f W)^\perp = e^{-f} W^\perp$.
(2) If $\lambda \in \Lambda$, then $(z^\lambda W)^\perp = z^{-\lambda} W^\perp$. □

Proof. Easy. ■

Proposition 1.18. (1) If $f \in H^{(n)}_+$, $\tilde{\mu} \circ \widehat{e^f} = \widehat{e^{-f}} \circ \tilde{\mu}$.
(2) If $\lambda \in \Lambda$, $\tilde{\mu} \circ \widehat{z^\lambda} = \widehat{z^{-\lambda}} \circ \tilde{\mu}$. □

Proof. (1) For the preceding proposition, $\tilde{\mu} \circ \widehat{e^f} \circ \tilde{\mu} \circ \widehat{e^f}$ is an isomorphism of det* over Id. So it is a multiple of the identity. To calculate this constant, we evaluate the two members on $\sigma_0(H_+)$. The claim is thus proved by the following calculations:

$$\tilde{\mu} \circ \widehat{e^f}\, \sigma_0(H_+) = \tilde{\mu} \circ \widehat{e^f} \left(\begin{pmatrix} I \\ 0 \end{pmatrix}, 1 \right) = \tilde{\mu}\sigma_0(H_+) = \sigma_0(H_+),$$

$$\widehat{e^{-f}} \circ \tilde{\mu}\, \sigma_0(H_+) = \widehat{e^{-f}} \sigma_0(H_+) = \sigma_0(H_+).$$

(2) As in the previous case, we have

$$\tilde{\mu} \circ \widehat{z^\lambda} = c(\lambda)\, \widehat{z^{-\lambda}} \circ \tilde{\mu}$$

as actions on det*. Expanding $\tilde{\mu} \circ \widehat{z^{\lambda_1+\lambda_2}}$, we immediately notice that c is a character of Λ, and so it is enough to prove the statement in the case $\lambda = \lambda_{ij}$. Evaluating the two terms of the previous relation on H_+ in this case, we obtain

$$c(\lambda_{ij}) = \varepsilon(\delta_j, \delta_i)\varepsilon(\delta_i, \delta_j)\varepsilon(\rho)(-1)^{i+j}(-1)^{\ell(z^{\lambda_{ij}} H_+)}$$

where $\rho(l) = l$ for $l \le -n$, and in $\{-n+1, \dots, 0\}$ ρ is given by the cycle $\rho = (0\ \ -1\ \ -2\ \dots 2-n\ 1-n)$. Hence $\varepsilon(\rho) = (-1)^{n-1}$ and $\ell(z^{\lambda_{ij}} H_+) = (-1)^{i+j+n}$ and finally $c(\lambda_{ij}) = 1$. ■

The tau function and the n-KP

In this subsection, we introduce an n-component version of the tau function of Sato and of the ψ-Baker function. (The $n = 1$ case is treated, for example, in [10].)

Definition 1.19. Let $\delta^*_W \in \det^*_W$ be a nonzero element of the fiber of the line bundle $\det^*$ over W and $\lambda \in \Lambda$; we define $\tau_{\delta^*_W,\lambda}\colon\ G_+ \longrightarrow \mathbb{C}$ by

$$\tau_{\delta^*_W,\lambda}(g) = \frac{\sigma_0(g\,z^{-\lambda}W)}{\widehat{g z^{-\lambda}}\delta^*_W}$$

and $\tau_{\delta^*_W} = \tau_{\delta^*_W,0}$. In particular, τ is defined on $\Gamma^{(n)}_+$.

Sometimes τ_W will be defined but to a multiplicative constant as a generic τ_{δ_W}.

Proposition 1.20. We have

$$\tau_{\tilde{\mu}\delta^*_W,\lambda}(e^f) = \tau_{\delta^*_W,-\lambda}(e^{-f}). \qquad \square$$

Proof. It automatically follows by Proposition 1.18. ∎

Definition 1.21. If $W \in \mathrm{Gr}$ and $\lambda \in \Lambda$, let us define

$$\Gamma^{(n)}_{+W}(\lambda) = \{g \in \Gamma^{(n)}_+ \colon\ g^{-1}z^{-\lambda}W \in \mathcal{W}_0\} = \Gamma^{(n)}_{+z^{-\lambda}W}.$$

We observe that $g \in \Gamma^{(n)}_{+W}(\lambda) \Leftrightarrow \sigma(g^{-1}z^{-\lambda}W) \neq 0$; and $\Gamma^{(n)}_{+W}(\lambda)$ is an open subset of $\Gamma^{(n)}_+$. We define $\tilde{\Psi}_W(\lambda)$ and $\Psi_W(\lambda)\colon\ \Gamma^{(n)}_{+W}(\lambda) \to \mathfrak{gl}(n, H)$ in the following way: If $g \in \Gamma^{(n)}_{+W}(\lambda)$, $i \in \{1 \cdots n\}$ and

$$\tilde{\Psi}_i(\lambda, g) = \begin{pmatrix} \tilde{\Psi}_{1i}(\lambda, g) \\ \vdots \\ \tilde{\Psi}_{ni}(\lambda, g) \end{pmatrix}$$

is the only element of $g^{-1}z^{-\lambda}W$ such that it has the form $v^{(i)}_1 + w_i$ with $w_i \in H^{(n)}_-$, then let us define

$$\tilde{\Psi}_W(\lambda, g) = (\tilde{\Psi}_{ij}(\lambda, g)) \qquad \text{and} \qquad \Psi_W(\lambda, g) = g z^{\lambda}\tilde{\Psi}_W(\lambda, g).$$

Remark 1.22. We introduce the following conventional notation, which coincides with the ones common in the $n = 1$ case. We observe that each element of $\Gamma^{(n)}_+$ could be expressed in the form $\sum_{i=1}^n \sum_{l=1}^\infty x^{(i)}_l v^{(i)}_l$. Hence, we can think of Ψ_W, $\tilde{\Psi}_W$, and τ_W as functions either in the variable $g \in \Gamma^{(n)}_+$ or in the variables $x^{(i)}_l$, and we will write $\Psi_W(\lambda, x)$ to mean $\Psi_W(\lambda, e^{x\cdot z})$ where

$$e^{x\cdot z} = \begin{pmatrix} e^{\sum_1^\infty x^{(1)}_l z^l} & & 0 \\ & \ddots & \\ 0 & & e^{\sum_1^\infty x^{(n)}_l z^l} \end{pmatrix} \in \Gamma^{(n)}_+.$$

We observe also that $\tilde{\Psi}(\lambda, x, z) = zI + \sum_{m=0}^{\infty} A_m(\lambda, x) z^{-m}$ with $A_m \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ and that $\Psi(\lambda, x, z) = e^{x\cdot z} z^{\lambda} \tilde{\Psi}(\lambda, x, z)$.

Definition 1.23. Let $q_\zeta(z) = 1 - z/\zeta$ and let

$$Q_\zeta^i(z) = \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & q_\zeta(z) & & \\ & & & 1 & \\ 0 & & & & \ddots \end{pmatrix} \in \mathfrak{gl}(n, H_+) \cap G_+.$$

Proposition 1.24. If $W \in \mathrm{Gr}\ \lambda \in \Lambda$ and $g \in \Gamma_{+W}^{(n)}(\lambda)$, then

$$\tilde{\Psi}_{ji}(\lambda, g, \zeta) = \varepsilon(\lambda_{ij}, \lambda)\, \varepsilon(\delta_j, \delta_i) \zeta^{\delta_{ij}} \frac{\tau_{W,\lambda+\lambda_{ij}}(g Q_\zeta^j)}{\tau_{W,\lambda}(g)}. \tag{4}$$

□

Proof. First notice that by choosing $\delta_{z^{-\lambda} g^{-1} W} = \sigma_0(z^{-\lambda} g^{-1} W)$, we have the relation

$$\frac{\tau_{W,\lambda+\lambda_{ij}}(g Q_\zeta^j)}{\tau_{W,\lambda}(g)} = \epsilon(\lambda_{ij}, \lambda) \tau_{g^{-1} z^{-\lambda} W, \lambda_{ij}}(Q_\zeta^j).$$

So it is enough to prove the proposition for $\lambda = 0$ $g = 0$ and $W = \mathrm{Graph}(A) \in \mathcal{W}_0$.

Moreover, if $Q_\zeta^j = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in G_+$ and $m_1 \cdots m_n \geq 0$, then

$$b \begin{pmatrix} z^{-m_1} \\ \vdots \\ z^{-m_n} \end{pmatrix} = \begin{pmatrix} 0 \\ (z^{-m_j} q_\zeta^{-1}(z))_+ \\ 0 \end{pmatrix} \qquad \text{and} \qquad (z^{-m_j} q_\zeta^{-1}(z))_+ = \frac{z}{\zeta^{m_j+1}} q_\zeta^{-1}(z),$$

and finally $a^{-1} b(f_1 \ldots f_n) = ((f_j(\zeta))/\zeta)\, v_1^{(j)}$.

We observe now that

$$\tau_{\sigma_0(W), \lambda_{ij}}(0) = \varepsilon(\delta_j, \delta_i) \det t,$$

where if $i \neq j$, $t = (I - e_{11}^{(jj)} + e_{10}^{(jj)} A^{\,t} q_{ji} + a^{-1} b(e_{01}^{(ij)} + (z^{\lambda_{ji}})_{--} A^{\,t} q_{ji})) = I + B + a^{-1} bC$ and $q_{ji} = (z^{\lambda_{ji}})_{++} + e_{11}^{(ji)}$; while if $i = j$, $t = I + a^{-1} bA$. In the case $i = j$, since the image of $a^{-1} b$ is generated by $v_1^{(j)}$, we have

$$\det t = 1 + \mathrm{tr}(a^{-1} bA),$$

and by following the definition of $\tilde{\Psi}$ we get $A v_1^{(j)} = (\tilde{\Psi}_{1j}(z), \ldots, \tilde{\Psi}_{jj}(z) - z, \ldots, \tilde{\Psi}_{nj}(z))$ and $a^{-1} bA\, v_1^{(j)} = ((\tilde{\Psi}_{jj}(\zeta) - \zeta)/\zeta)\, v_1^{(j)}$. Hence

$$\det t = \frac{\tilde{\Psi}_{jj}(\zeta)}{\zeta}.$$

If $i \neq j$, proceeding as above, we calculate

$$Bv_1^{(j)} = -v_1^{(j)} + e_{10}^{(jj)} A v_1^{(i)} = -v_1^{(j)} + \tilde{\Psi}_{ji}(0) v_1^{(j)}$$

$$\begin{aligned} a^{-1} b C v_1^{(j)} &= a^{-1} b (v_1^{(i)} + (z^{\lambda_{ji}})_{--} A v_1^{(i)}) \\ &= 0 + a^{-1} b (\tilde{\Psi}_{1i}(z), \ldots, z^{-1}\tilde{\Psi}_{ii}(z), \ldots, z\tilde{\Psi}_{ji}(z) - z\tilde{\Psi}_{ji}(0), \ldots, \tilde{\Psi}_{ni}(z)) \\ &= (\tilde{\Psi}_{ji}(\zeta) - \tilde{\Psi}_{ji}(0)) v_1^{(j)}. \end{aligned}$$

Therefore, $(B + a^{-1}bC)v_1^{(j)} = (\tilde{\Psi}_{ji}(\zeta) - 1)v_1^{(j)}$, and finally

$$\operatorname{Det} t = \tilde{\Psi}_{ji}(\zeta).$$ ■

If we adopt the conventions introduced in Remark 1.2.2, we can write the formula (4) in the following way:

$$\tilde{\Psi}_{ji}(\lambda, g, \zeta) = \varepsilon(\lambda_{ij}, \lambda)\, \varepsilon(\delta_j, \delta_i) \zeta^{\delta_{ij}} \frac{\tau_{W,\lambda+\lambda_{ij}}\left(x_l^{(1)}, \ldots, x_l^{(j)} - \dfrac{\zeta^{-l}}{l}, \ldots, x_l^{(n)}\right)}{\tau_{W,\lambda}(x)}.$$

Proposition 1.25. If $W \in \mathrm{Gr}$; $\lambda, \mu \in \Lambda$ and $g \in \Gamma^{(n)}_{+W}(\lambda)$ and $h \in \Gamma^{(n)}_{+W^\perp}(\mu)$, then

$$\operatorname*{Res}_{z=\infty} {}^t\Psi_W(\lambda, g)\, \Psi_{W^\perp}(\mu, h)\, dz^{-1} = 0. \tag{5}$$

□

Proof. Let $\Psi_{Wj}(\lambda, g) = {}^t(\Psi_{1j} \cdots \Psi_{nj})$ and similarly $\Psi_{W^\perp j}$. We observe that

$$\operatorname*{Res}_{z=\infty} s^t\Psi_W(\lambda, g)\, \Psi_{W^\perp}(\mu, h)\, dz^{-1} = \big((\Psi_{Wi}(\lambda, g)\, ;\, \Psi_{W^\perp j}(\mu, h))\big)_{i,j=1\cdots n}.$$

Now the claim follows by observing that $\Psi_{Wi}(\lambda, g) \in W$ and $\Psi_{W^\perp j}(\mu, h) \in W^\perp$. ■

Proposition 1 allows us to write the equation (5) as an equation in τ. In order to make this result more explicit, we use the following simple lemma.

Lemma 1.26. If f is an holomorphic function in the variables $x_1, x_2, \ldots$, then

$$f(x_1 + y_1, x_2 + y_2, \ldots) = (e^{\sum_i y_i(\partial/\partial x_i)} f)(x).$$ □

Following Propositions 1.24 and 1.18, we have

$$\Psi_{Wji}(\lambda, u, z) = \varepsilon(\lambda_{ij}, \lambda)\varepsilon(\delta_j, \delta_i) z^{\delta_{ij}+\lambda_j} e^{\sum_{l=1}^{\infty} u_l^{(j)} z^l} \frac{e^{-\sum_{l=1}^{\infty}(z^{-l}/l)(\partial/\partial u_l^{(j)})} \tau_{W,\lambda+\lambda_{ij}}(u)}{\tau_{W,\lambda}(u)},$$

$$\Psi_{W^{\perp}jh}(-\mu, -\nu, z) = \varepsilon(\lambda_{ij}, \mu)\varepsilon(\delta_j, \delta_h) z^{\delta_{hj}+\mu_j} e^{-\sum_{l=1}^{\infty} \nu_l^{(j)} z^l} \frac{e^{\sum_{l=1}^{\infty}(z^{-l}/l)(\partial/\partial \nu_l^{(j)})} \tau_{W,\mu-\lambda_{hj}}(\nu)}{\tau_{W,\mu}(\nu)}.$$

With the following substitutions: $y = (u-\nu)/2$, $x = (u+\nu)/2$, $\alpha = \lambda + \delta_i$, $\beta = \mu - \delta_h$, and if $S_k(y_1, \ldots) = \sum_{p \text{ multindex and } [p]=k} (1/p!) y^p$ (where $[p] = p_1 + 2p_2 + 3p_3 + \cdots$) is defined by the identity $e^{\sum_{l=1}^{\infty} y_i z^i} = \sum_{l=0}^{\infty} S_l(y) z^l$, then from Proposition 1.25 we obtain the n-component KP in the Hirota form (see [7]): If $\alpha, \beta \in \mathbb{Z}^n$ are such that $(\delta|\alpha) = 1$ and $(\delta|\beta) = -1$ where $(\delta = \delta_1 + \cdots + \delta_n)$ and if $x, y \in H_+^{(n)}$, then

$$\sum_{j=1}^{n} \varepsilon(\delta_j, \alpha+\beta)\, P_{\alpha,\beta,j}\left(y_l^{(h)}, \frac{\partial}{\partial u_l^{(h)}}\right) (\tau_{W,\alpha-\delta_j}(x+u)\, \tau_{W,\beta+\delta_j}(x-u))\Big|_{u=0} = 0 \tag{6}$$

where

$$P_{\alpha,\beta,j}\left(y_l^{(h)}, \frac{\partial}{\partial u_l^{(h)}}\right) = Q_{\alpha,\beta,j}\left(y_l^{(h)}, \frac{\partial}{\partial u_l^{(h)}}\right) \circ e^{\sum_{k=1}^{n} \sum_{l=1}^{\infty} y_l^{(k)} (\partial/\partial u_l^{(k)})}$$

and

$$Q_{\alpha,\beta,j}\left(y_l^{(h)}, \frac{\partial}{\partial u_l^{(h)}}\right) = \sum_{k=0}^{\infty} S_k(2y^{(j)})\, S_{k-1+(\delta_j|\alpha-\beta)}\left(-\frac{1}{l}\frac{\partial}{\partial u_l^{(j)}}\right).$$

In these equations, the y's are indeterminates, so we have an equation for each monomial in y. If we want to obtain pure algebraic equations in τ, we must consider the coefficient of y^0 and choose α and β in such a way that $(\delta_j|\alpha-\beta) \le 1$ for each $j = 1, \ldots, n$. For such α and β, we get

$$\sum_{\substack{j=1 \\ (\delta_j|\alpha-\beta)=1}}^{n} \varepsilon(\delta_j, \alpha-\beta)\, \tau_{W,\alpha-\delta_j}(x)\, \tau_{W,\beta+\delta_j}(x) = 0, \tag{7}$$

which is a class of Plücker relations.

2 The theta function and the n-component KP

In this section, Krichever's construction is generalized to the case of n points on a Riemann surface (for the $n = 1$ case see, for example, [10]). Then we find a generalization of the classical Fay trisecant formula.

Let X be a Riemann surface of genus g, and L a line bundle of Chern class $g-1$ ($c(L) = g-1$) on it. Let $P_1, \dots, P_n$ be distinct points of X, and $z_i\colon U_i \xrightarrow{\sim} S^2$ local charts for X where $P_i \in U_i$, $z_i(P_i) = \infty$ and $\overline{U}_i \cap \overline{U}_j = \emptyset$. To proceed with Krichever's construction, we fix also $\varphi_1, \dots, \varphi_n$ trivializations of L in $U_1, \dots, U_n$, where $\varphi_i\colon L|_{U_i} \to D_\infty \times \mathbb{C}$ is such that $\varphi_i = (z_i \circ \pi, \phi_i)$, where $\pi\colon L \to X$ is the projection of the line bundle on X and ϕ_{iP} is a linear map for all $P \in U_i$. We set also the following notation:

$$X_i = z_i^{-1}(D_\infty) = z_i^{-1}(\overline{D}_1), \qquad X_{i\varepsilon} = z_i^{-1}(\overline{D}_\varepsilon), \qquad C_{i\varepsilon} = \partial X_{i\varepsilon} = z_i^{-1}(\partial \overline{D}_\varepsilon),$$

$$X_0 = X \setminus \cup\{P_i\}, \qquad X_{0\varepsilon} = X \setminus \cup \overset{\circ}{X}_{i\varepsilon}\,.$$

Definition 2.1. $\mathcal{J}$ is the set of data $(X; L; P_1, \dots, P_n; z_1, \dots, z_n; \varphi_1, \dots, \varphi_n)$ described above. We define $\mathcal{K}\colon \mathcal{J} \to \mathrm{Gr}$ as follows: for $f \in H^{(n)}$ defined as a holomorphic function in D_ε^*,

$$f = (f_1, \dots, f_n) \in \mathcal{K}(X; L; P_1, \dots, P_n; z_1, \dots, z_n; \varphi_1, \dots, \varphi_n)$$

if and only if there exists a $g \in \Gamma(X_{0\varepsilon}; L)$ such that $g|_{C_{i\varepsilon}} = \varphi_i^{-1}(\mathrm{Id}; f_i \circ z_i)$ or $\phi_i(g|_{C_{i\varepsilon}}) = f_i \circ z_i$.

The definition is well given by the following proposition.

Proposition 2.2. We have

$$\mathcal{K}(X, L, P_1, \dots, P_n, z_1, \dots, z_n, \varphi_1, \dots, \varphi_n) \in \mathrm{Gr}. \qquad \square$$

Proof. As in the 1-dimensional case, we can see that $W = \mathcal{K}(X, L, P_i, z_i, \varphi_i) \cong H^0(X_0, L)$ $\operatorname{Ker} \pi_+|_W \cong H^0(X, L)$ and $\operatorname{coKer} \pi_+|_W \cong H^1(X, L)$. ■

Some notation for Riemann surfaces

$\wedge\colon H^1(X, \mathbb{C}) \times H^1(X, \mathbb{C}) \to \mathbb{C}$ is the usual extension of the *cap product* defined on $H^1(X, \mathbb{Z})$ by $\omega \wedge \sigma = \int_X \omega \wedge \sigma$.

Let $P_0; \alpha_1 \dots \alpha_g, \beta_1 \cdots \beta_g = \gamma_1, \dots, \gamma_{2g}$ be a *marking* of X, $J \simeq \mathrm{Pic}^0$ the Jacobian and $\Phi\colon X \to J$ the Abel map of base point P_0. Let $(\tilde{X}, p)$ be the universal covering of X, and let $Y \subset \tilde{X}$ be the closure of a connected component of $p^{-1}(X - \bigcup \gamma_i)$; so that $X \simeq Y/\!\sim$ where $\sim$ is the identification along $\Delta = \partial Y = \tilde{\alpha}_1 \tilde{\beta}_1 \tilde{\alpha}_1^{-1} \cdots \tilde{\beta}_g^{-1}$ and $\tilde{\gamma}$ is a lifting of γ to Y. Let also $\tilde{P}_i$ be a point of Y such that $\pi(\tilde{P}_i) = P_i$ and let Y_i be a subset of Y such that $\pi(Y_i) = X_i$.

Furthermore, let $a_1 \cdots a_g, b_1 \cdots b_g = \gamma_1^*, \dots, \gamma_{2g}^* \in H^1(X, \mathbb{Z}) \hookrightarrow H^1_{DR}(X)$ be the dual basis of $\alpha_1 \cdots \beta_g$, and χ the matrix representing the cap product in this basis. We make all the above-mentioned choices, so that $\chi = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ and $\Omega = (I\,\Omega_2)$, where $\Omega\colon \mathbb{Z}^{2g} \to \mathbb{C}^g$ is the matrix related to $\delta\colon H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O})$. We call Ω also the lattice $\Omega(\mathbb{Z}^{2g})$. From

Riemann's relations it follows $\Omega_2 = {}^t\Omega_2$ and $\operatorname{Im}\Omega_2 > 0$. We set also

$$G = P\chi\, {}^t\bar{P} = (\overline{\Omega}_2 - \Omega_2)^{-1}, \text{ where } \begin{pmatrix} P \\ \bar{P} \end{pmatrix} = ({}^t\Omega\, {}^t\bar{\Omega})^{-1}.$$

Let $\theta(x)$ and $\theta[\nu](x)$, defined for $\nu \in (1/2)\mathbb{Z}^g/\mathbb{Z}^g$, be the classical theta functions of first and second order defined on $\mathbb{C}^g \leftrightarrow H^1(X, \mathcal{O})$, and as usual Θ is the associate divisor. Furthermore, $\vec{\theta} = (\theta[\nu])\colon J \to \mathbb{CP}^{2^g-1}$will be the Kummer map.

In order to relate the theta function with the tau function, we introduce a variant of the classical theta function (see, for example, [8]), as proposed by Segal and Wilson [10].

Definition 2.3. Let $h\colon H^1(X, \mathcal{O}) \times H^1(X, \mathcal{O}) \to \mathbb{C}$ be the only Hermitian form such that for all $a, b \in H^1(X, \mathbb{Z})$, it holds that $\operatorname{Im} h(\delta(a), \delta(b)) = a \wedge b$, and let H be the matrix related to h in the given basis.

Let $\eta\colon \mathbb{C}^g \to \mathbb{C}$ be defined by

$$\eta(x) = \sum_{n,m \in \mathbb{Z}^g} (-1)^{{}^t nm} e^{-(1/2)\pi i\, h(n+\Omega_2 m; n+\Omega_2 m+2x)}.$$

I gather together properties of introduced η in the following two propositions.

Proposition 2.4. (1) $H = -2iG = (\operatorname{Im}\Omega_2)^{-1}$.

(2) The definition of η is well given and $\eta(x) = Ce^{-\pi i\, {}^t x G x}\theta(x)$ and $C \in \mathbb{C}^*$. □

Proof. Easy. ■

Proposition 2.5. We have

$$\forall x \in \mathbb{C}^g\ \forall \omega \in \Omega, \qquad \eta(0)\eta(x+\omega) = \eta(x)\eta(\omega)e^{\pi h(\omega,x)}. \tag{8}$$

Conversely, if $\tilde{\eta}\colon \mathbb{C}^g \to \mathbb{C}$ and $\tilde{C} \in \mathbb{C}$ verify $\tilde{\eta}(x+\omega) = \tilde{C}\tilde{\eta}(x)\tilde{\eta}(\omega)e^{\pi h(\omega,x)}$, then

$$\exists \alpha\colon \mathbb{C}^g \to \mathbb{C}, \text{ a } \mathbb{C}\text{-linear map, and } \exists \beta \in \mathbb{C}^g, A \in \mathbb{C}\colon\ \tilde{\eta}(x) = Ae^{\alpha(x)}\eta(x-\beta).$$ □

Proof. Easy. ■

Finally, we make some remarks on a class of line bundles. Let $P \in X$, U a neighbourhood of P, and $z\colon U \to D_\infty$, a local chart such that $z(P) = \infty$. If $f \in H$ is defined on D^*_ε and $f(x) \neq 0$ for all x, we define the line bundle L_f, by means of the associated element of $H^1(\{U_\infty; U_0\}; \mathcal{O}^*)$ where $U_\infty = z^{-1}(D^*_\varepsilon)$ and $U_0 = X - \{P\}$. We set

$$(L_f)_{\infty 0} = f \circ z \in \Gamma(U_\infty \cap U_0, \mathcal{O}^*). \tag{9}$$

Definition 2.6. If $f = (f_1 \cdots f_n) \in H_+^{(n)}$ and $\lambda \in \Lambda$, we define

$$L_{f,\lambda} = L^1_{z^{\lambda_1} e^{f_1}} \otimes \cdots \otimes L^n_{z^{\lambda_n} e^{f_n}},$$

where $L^i_{z^{\lambda_i} e^{f_i}}$ is the line bundle constructed as in (9) with base point P_i and local chart z_i. We define also a map $Z\colon H_+^{(n)} \oplus \Lambda \to H^1(X, \mathcal{O})$ by

$$Z(f, \lambda) = Z - (\lambda \cdot \tilde{P})$$

where $\lambda \cdot \tilde{P} = \lambda_1 \tilde{\Phi}(\tilde{P}_1) + \cdots + \lambda_n \tilde{\Phi}(\tilde{P}_n)$, $\tilde{\Phi}$ is a lifting of Φ to $\tilde{X}$, and Z is given by

$$\{Z_{i0}\} \in H^1(X, \mathcal{O}) \quad \text{with } Z_{i0} = \frac{1}{2\pi i} f_i \circ z_i.$$

Further, let $K_0 = \operatorname{Ker} Z$, $K_{H0} = H_+^{(n)} \cap K_0$, $K = \operatorname{Ker} L$, and $K_H = H_+^{(n)} \cap K$.

Remark 2.7. Let $\varphi_1^{f,\lambda} \cdots \varphi_n^{f,\lambda}$ be standard trivializations of $L_{f,\lambda}$ so that, given $\sigma_i \in \Gamma(U_i, \mathcal{O})$, σ_i will define an element $\sigma \in \Gamma(X, L_{f,\lambda})$ by $\sigma_i \circ z_i = \phi_i^{f,\lambda} \circ \sigma$ if and only if $\sigma_i = z^{\lambda_i} \cdot e^{f_i} \cdot \sigma_0$.

Proposition 2.8. (1) $c(L_{f,\lambda}) = 0$.

(2) $L_{f,\lambda} = L_f \otimes \mathcal{O}(-\sum_{i=1}^n \lambda_i P_i)$.

(3) L and Z are group homomorphisms so that, in the exponential sequence, we have $\exp_{H^1} \circ Z = L$.

(4) $L|_{H_+^{(n)}}$ and $Z|_{H_+^{(n)}}$ are surjective. □

Proof. (1), (2), and (3) are easy; we prove (4): The surjectivity of the map L derives from the fact that $L|_{X-P_1}$ is trivial, because $X - P_1$ is affine. We observe that K_{H0} is a closed subspace of $H_+^{(n)}$ of finite codimension. Let V_0 be a supplementary space for K_{H0}. We observe that $L|_{V_0}$ is a universal covering of J. So $Z|_{V_0}$ must be an isomorphism, and the claim follows. ■

Remark 2.9. We observe that $(f, \lambda) \in K$ if and only if there exist $\varphi \in \Gamma(X_0, \mathcal{O}^*)$ and $f_\infty \in H_-^{(n)}$ such that

$$p z^{\lambda_i} e^{f_i} = (\phi_i \circ \varphi \circ z_i^{-1}) \cdot e^{f_\infty^{(i)}}.$$

In particular, for $(f, \lambda) \in K$ we have that $f_\infty - f_\infty(\infty) \in V = z^{-1} H_-^{(n)}$ is uniquely determined. We define the group homomorphism $a\colon K \longrightarrow V$ by $a(f, \lambda) = f_\infty - f_\infty(\infty)$; and $a_\lambda = a(\cdot, \lambda)$. Noticing that K is a set of generators for $H_+^{(n)}$ as an $\mathbb{R}$-vector space, there exists a unique $\mathbb{R}$-linear extension $a = b + c$ of a to $H_+^{(n)} \oplus \Lambda$ where b is $\mathbb{C}$-linear and c is $\mathbb{C}$-antilinear. We observe also that a_0 is $\mathbb{C}$-linear on K_{H0}.

Finally we describe the isomorphism between K_H / K_{H0} and $H^1(X, \mathbb{Z})$ induced by the map Z. If $f \in K_H$ and $e^f = \varphi e^{f_\infty}$, let $f_0 = f - f_\infty$. We observe that on $\tilde{X}$ we have $\varphi = e^{f_0^{(i)}}$ in

C_i and that for $i = 1 \cdots n$, $1/(2\pi i) \int_{C_i} (d\varphi/\varphi) = 0$. Hence, $\log \varphi = f_0$ is defined on the whole Y_0. We have that

$$n(f, \gamma) = -\int_\gamma df_0 = -\int_\gamma \frac{d\varphi}{\varphi}.$$

By means of this notation, the isomorphism [ ]: $K_H/K_{H0} \to H^1(X, \mathbb{Z})$ induced by Z turns out to be

$$[f] = \sum_{i=1}^{2n} n(k, \gamma_i)\gamma_i^* . \tag{10}$$

The tau and the theta functions

In this subsection, proceeding as in the 1-dimensional case, we study the relations between the tau function and the theta function. The first step is the study of the evolution under the action of $\Gamma_+^{(n)} \times \Lambda$ of an element of the Grassmannian coming from a Riemann surface via the Krichever construction.

Proposition 2.10. Let $(X, L, P_i, z_i, \varphi_i) \in \mathcal{J}$. Then

$$z^\lambda e^f \mathcal{K}(X, L, P_i, z_i, \varphi_i) = \mathcal{K}(X, L \otimes L_{f,\lambda}, P_i, z_i, \varphi_i \otimes \varphi_i^{f,\lambda}). \qquad \square$$

Proof. It is the same as the 1-dimensional case. ■

In order to compare the two functions τ and θ, we have to observe that through the maps L and Z, η can be seen as a K_0-invariant function defined on $\Gamma_+^{(n)} \oplus \Lambda$, which verifies the formula (8). Our strategy is to reduce τ to have the same property.

We first study the case $H^0(X, L) = \{0\}$. This condition is equivalent to $W = \mathcal{K}(X, L, P_i, z_i, \varphi_i) \in \mathcal{W}_0$ so we can choose $\tau_W = \tau_{\sigma_0(W)}$.

Proposition 2.11. If $H^0(X, L) = \{0\}$, $\mu, \lambda \in \Lambda$, $k, f \in H_+^{(n)}$, and $(k, \lambda) \in K$, then

$$\tau_{W,\lambda+\mu}(e^{k+f}) = \varepsilon(\mu, \lambda) e^{-S(f, a_\lambda(k))} \tau_{W,\mu}(e^f) \tau_{W,\lambda}(e^k). \tag{11}$$

$\square$

Proof. We observe that $z^{-\lambda} e^{-k} W = e^{-a_\lambda(k)} W$, and hence that

$$\begin{aligned} \tau_{W,\lambda+\mu}(e^{k+f})\, \widehat{z^{-\lambda-\mu}}\widehat{e^{-f-k}} \sigma_0(W) &= \sigma_0(z^{-\lambda-\mu} e^{-f-k} W) = \widehat{e^{a_\lambda(k)}} \sigma_0(e^{-f} z^{-\mu} W) \\ &= \tau_\mu(f)\, \widehat{e^{a_\lambda(k)}}\, \widehat{e^{-f}}\widehat{z^{-\mu}} \sigma_0(W) = \tau_\mu(f)\, e^{-S(f, a_\lambda(k))} \widehat{e^{-f}}\widehat{z^{-\mu}} \widehat{e^{a_\lambda(k)}} \sigma_0(W) \end{aligned}$$

$$= \tau_\mu(e^f)\tau_\lambda(e^k)e^{-S(f,a_\lambda(k))}\widehat{e^{-f}}\widehat{e^{-k}}\widehat{z^{-\mu}}\widehat{z^{-\lambda}}\sigma_0(W)$$
$$= \varepsilon(\mu,\lambda)\tau_\mu(e^f)\tau_\lambda(e^k)e^{-S(f,a_\lambda(k))}\widehat{e^{-f}}\widehat{e^{-k}}\widehat{z^{-\mu-\lambda}}\sigma_0(W),$$

and the claim follows. ■

This formula has the same structure as (8). In order to give a geometric interpretation of the coefficients, we first study the case $\lambda = 0$ and use the isomorphism [ ] described in (10). Proceeding as in the 1-dimensional case and noticing that

$$\int_{C_j} k_0^{(j)\prime} l_0^{(j)} = \int_{C_j} k_0^{(i)\prime} l_0^{(i)} + 2\pi i n_{ij} \int_{C_j} k_0^{(i)\prime} = \int_{C_j} k_0^{(i)\prime} l_0^{(i)},$$

we obtain the following lemma.

Lemma 2.12. (1) For all $k, l \in K_H$, whatever l_∞ and k_∞ related to l and k,

$$S(k, a(l)) - S(l, a(k)) = S(k, l_\infty) - S(l, k_\infty) = 2\pi i [l] \wedge [k].$$

(2) For all $f, g \in H_+^{(n)}$,

$$S(f, c(g)) = -\pi h(Z(g), Z(f)).$$ □

Thus if, in the case $H^0(X, L) = \{0\}$, we set

$$\tau_1(f) = \tau(e^f)e^{(1/2)S(f,b(k))},$$

we obtain

(1) $\tau_1(f + k) = \tau_1(f)\,\tau_1(k)\,e^{-S(f,c(k))}$ for all $f \in H_+^{(n)}$ and $k \in K_H$,
(2) $\tau_1(f + k) = \tau_1(f)\,\tau_1(k)$ for all $f, k \in K_{H0}$,
(3) $\exists\,\rho\colon H_+^{(n)} \mapsto \mathbb{C}$ a $\mathbb{C}$-linear map such that $\tau_1(k) = e^{\rho(k)}$, for all $k \in K_{H0}$;

hence, if

$$\tau_2(f) = \tau_1(f)e^{-\rho(f)} = \tau(f)e^{(1/2)S(f,b(f))-\rho(f)},$$

we then have that

(1) $\tau_2(f + k) = \tau_2(f)$ for all $k \in K_{H0}$ and for all $f \in H_+^{(n)}$,
(2) $\tau_2(f + k) = \tau_2(f)\,\tau_2(k)\,e^{-S(f,c(k))} = \tau_2(f)\,\tau_2(k)\,e^{\pi h(Z(k),Z(f))}$ for all $f \in H_+^{(n)}$ and $k \in K$,
(3) $\tau_2(0) \neq 0$.

Following the characterization of η, there exist $C \in \mathbb{C}^*$ and a $\mathbb{C}$-linear map $\alpha_W\colon H_+^{(n)} \to \mathbb{C}$ and $\beta \in \mathbb{C}^g$ such that

$$\tau_W(e^f) = C\,e^{\alpha_W(f)-(1/2)S(f,b(f))}\,\eta(Z(f) - \beta). \tag{12}$$

Now we can give the relation between τ and θ.

Theorem 2.13. Let $(X, L, P_i, z_i, \varphi_i) \in \mathcal{J}$ and $W = \mathcal{K}(X, L, P_i, z_i, \varphi_i)$. If $\kappa/2, \ell \in H^1(X, \mathcal{O})$ are so that $\Theta = -W^{g-1} + \exp_{H^1}(\kappa/2)$ and $\exp_{H^1}(\ell) = L - (g-1)P_0 \in J$, then $\exists_1 C \in \mathbb{C}^*$ and $\exists_1 \alpha_{W,\lambda}\ H_+^{(n)} \longrightarrow \mathbb{C}$ a $\mathbb{C}$-linear map such that

$$\tau_{W,\lambda}(e^f) = C_{W,\lambda} e^{\alpha_{W,\lambda}(f) - (1/2)S(f,b(f)) + \mathfrak{g}(Z(f),Z(f))}\, \theta\left(Z(f) - \lambda\cdot\tilde{P} - \ell + \frac{\kappa}{2}\right),$$

where $\mathfrak{g}(u, v) = -\pi i^t u G v$, and $G = P\chi^t P$ is the bilinear form introduced in the previous subsection. □

Proof. First we observe that by Propositions 2.8 and 2.10, for each $V \in \mathcal{K}(\mathcal{J})$, there exists a $g \in H_+^{(n)}$ such that $\sigma_0(e^g V) \neq 0$, and therefore the set of these elements g is an open dense subset of $H_+^{(n)}$. Noticing that $H_+^{(n)}$ is a Baire space, the set of $g \in H_+^{(n)}$ such that $\sigma_0(e^g z^\lambda W) \neq 0$, for all $\lambda \in \Lambda$, is dense in $H_+^{(n)}$. Let g be such an element and set $\delta_W = \widehat{e^{-g}}\sigma_0(e^g W)$. Hence, by formula (12), we have

$$\begin{aligned}
\tau_{\delta_W,\lambda}(e^f) &= \tau_{\sigma_0(z^{-\lambda}e^g W),0}(e^{f+g})\tau_{\sigma_0(e^g W),0}(z^\lambda)\\
&= \tilde{C}_{W,\lambda} e^{\tilde{\alpha}_{W,\lambda}(f)} e^{-(1/2)S(f,b(f))}\eta(Z(f) - \beta)\\
&= C_{W,\lambda} e^{\alpha_{W,\lambda}(f) - (1/2)S(f,b(f)) + \mathfrak{g}(z(f),z(f))}\, \theta(Z(f) - \beta).
\end{aligned}$$

In order to calculate β, we observe that $\tau_{W,\lambda}(f) = 0 \Leftrightarrow \sigma_0(z^{-\lambda}e^{-f}W) = 0 \Leftrightarrow H^0(L \otimes L_{f,\lambda}) \neq \{0\} \Leftrightarrow Z(f) \in -W^{g-1} + \ell + \sum_{i=1}^n \lambda_i P_i$; and that $\eta(Z(f) - \beta) = 0 \Leftrightarrow Z(f) - \beta \in \Theta = -W^{(g-1)} + \kappa/2 \Leftrightarrow Z(f) \in -W^{g-1} + \beta + \kappa/2$.

Hence $-W^{g-1} + \beta + \kappa/2 = -W^{g-1} + \ell + \lambda\cdot\tilde{P}$ and $\beta - \ell - \lambda\cdot\tilde{P} + \kappa/2 \in H^1(X, \mathbb{Z})$. Unicity and thesis follow. ■

The trisecant formula

From the relation between the tau and the theta function of Theorem 2.13 and from formula (6), we obtain a hierarchy of equations for the theta function. In this last part we attempt to interpret some of these formulas by looking at geometric properties of the Kummer variety. These properties are expressed as linear relations in the vectors $\partial_j^{(i)}\vec{\theta}(w + \tilde{P}_j)$, where w range in some subvarieties of J and $\partial_j^{(i)} = Z_*(-(1/2l)(\partial/\partial u_j^{(i)})) = Z_*(D_j^{(i)}) = Z(-(1/2l)v_j^{(i)})$. In the last identity we have identified the derivations on J with $\mathbb{C}^g$, and we have used the linearity of Z. We can compute these derivations explicitly in terms of the Abel map. Let $\vec{\omega}$ be a basis of $H^0(X, \Omega^1)$ and $\tilde{P}_i(\zeta) = \tilde{P}_i + \Phi_i(\zeta) = \tilde{P}_i + \int_{P_i}^{z_i^{-1}(\zeta)} \vec{\omega}$ and let us have Φ_i expressed in Taylor series as $\Phi_i(\zeta) = 2(V_1^{(i)}\zeta^{-1} + V_2^{(i)}\zeta^{-2} + \cdots)$.

We have $L_{z-\zeta_0} = -P + \Phi(\zeta_0)$, and by Abel's lemma we obtain that

$$Z\left(-\sum_{l>0} \frac{1}{l}\frac{1}{\zeta_i^{(0)\,l}} v_l^{(i)}\right) = Z\left(-\sum_{l>0}\frac{1}{l}\left(\frac{\zeta_i}{\zeta_i^{(0)}}\right)^l\right) = \Phi_i(\zeta_0).$$

We have the following lemma.

Lemma 2.14. (1) $\partial_l^{(i)} = V_l^{(i)}$ and $Z(v_l^{(i)}) = -2lV_l^{(i)}$.
(2) $\partial_1^{(i)} \neq 0$. □

Following Theorem 2.13, the KP equation (6) is now given by the following equation:

$$\sum_{j=1}^{n} C_{\alpha,\beta,j} P_{\alpha,\beta,j}\left(y_l^{(h)}, \frac{\partial}{\partial u_l^{(h)}}\right) e^{\alpha_{\alpha-\delta_j}(u)-\alpha_{\beta+\delta_j}(u)-S(u,b(u))} \cdot e^{2\mathfrak{g}(Z(u),Z(u))}\theta(x+Z(u)+P_j-\alpha\cdot\tilde{P})\,\theta(x-Z(u)-P_j-\beta\cdot\tilde{P})\big|_{u=0} = 0 \tag{13}$$

for each $x \in \mathbb{C}^g$ and where $C_{\alpha,\beta,j} = C_{\alpha-\delta_j} C_{\beta+\delta_j}\varepsilon(\delta_j, \alpha+\beta) \neq 0$.

If we apply the Riemann identity, we see that as we claimed above this is a linear equation in the vectors $\partial_{j_1}^{(i_1)} \cdots \partial_{j_s}^{(i_s)}\vec{\theta}(w+P_j)$. We will examine now the equations that appear as coefficients of y^0 in the previous hierarchy of formulas. We observe that it depends only on $\alpha-\beta$, and the most general formula is obtained when $\alpha-\beta = \lambda_1\delta_1 + \cdots + \lambda_r\delta_r - \delta_{r+1} - \cdots - \delta_{r+N-2}$, where $\lambda_i \geq 1$ and $N = \sum_{i=1}^r \lambda_i$. If we apply the Riemann identity, we obtain

$$\sum_{j=1}^{r} C_{\alpha,\beta,j} S_{\lambda_j-1}(2D_l^{(j)}) e^{\alpha_{\alpha-\delta_j}(u)-\alpha_{\beta+\delta_j}(u)-S(u,b(u))} e^{2\mathfrak{g}(Z(u),Z(u))}\vec{\theta}(Z(u)+w+\tilde{P}_j)|_{u=0} = 0 \tag{14}$$

for each $w \in (1/2)(W_{N-2} - \sum_{i=1}^r \lambda_i P_i)$.

In conclusion, we illustrate some examples where classical results are obtained.

If $\lambda_1 = \cdots = \lambda_N = 1$, we obtain a weak form of the generalized trisecant formula obtained by R. C. Gunning in [6]: for each $P_1, \ldots, P_N \in X$ and for each $w \in (1/2)(W_{N-2} - \sum_{i=1}^N P_i)$,

$$\vec{\theta}(w+P_1), \ldots, \vec{\theta}(w+P_N)$$

lie in an $(N-2)$-dimensional space.

If $N = 3$, $\lambda_1 = 2$, and $\lambda_2 = 1$, we obtain the existence of a family of lines tangent to the Kummer variety: for each $P_1, P_2 \in X$ and for each $w \in (1/2)(W_1 - 2P_1 - P_2)$, the line through $\vec{\theta}(w+P_1)$ and $\vec{\theta}(w+P_2)$ is tangent to the Kummer variety in $\vec{\theta}(w+P_1)$.

If $N = 3$ and $\lambda_1 = 3$, we obtain the existence of a family of flexes of the Kummer variety: for each $P \in X$ and for each $w \in (1/2)(W_1 - 3P)$, the line through $\vec{\theta}(w+P)$ with direction $\partial_1^{(1)}\vec{\theta}(w+P)$ is a flex for the Kummer variety.

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Dipartimento di Matematica, Universitá di Roma "La Sapienza," 00185 Roma, Italy; maffei@mat.uniroma1.it