Equations defining symmetric varieties and affine Grassmannian I and II

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(joint work with Rocco Chirivì)

The talks are reports on joint work [1] with Rocco Chirivì (Pisa, Italy).
Let G be a connected semisimple algebraic group over the complex numbers, let σ be an involution of G and let H be the subgroup of points fixed by σ. We assume σ to be simple, this means that the action of $G \rtimes \{\mathrm{id}, \sigma\}$ on the Lie algebra of G is irreducible. Let $\bar{H}$ be the normalizer of H in G and let X be the wonderful compactification of $G/\bar{H}$ constructed by De Concini and Procesi [5]. We have a G equivariant map $\pi : G/H \longrightarrow X$ factoring through the quotient $G/\bar{H}$.

We are interested in the study of the coordinate ring of the affine variety G/H and in the coordinate rings given by projective immersions of X; they are strictly related through the map π.

Let Ω be the set $\{\mathcal{L} \in \mathrm{Pic}(X) : \pi^*\mathcal{L}$ is isomorphic to the trivial line bundle$\}$; it is a free lattice and any line bundle on Ω has a G linearization. So the vector space $\Gamma_X = \bigoplus_{\mathcal{L}\in\mathrm{Pic}(X)} \Gamma(X, \mathcal{L})$ is a G algebra and we have an equivariant morphism of algebras $\pi^* : \Gamma_X \longrightarrow \mathbb{C}[G/H]$. The complement of $G/\bar{H}$ in X is the union of ℓ smooth divisors $X_1, \dots, X_\ell$ which intersect transversally; and for each divisor there exists a G invariant section s_i of $\Gamma(X, \mathcal{O}(X_i))$ whose associated divisors is equal to X_i. These sections can be normalized in such a way that $\pi^*(s_i) = 1$.

Making use of the results of De Concini and Procesi [5] and Helgason and Vust [6, 14], it is easy to check from the decomposition of Γ_X and of $\mathbb{C}[G/H]$ into G modules that π^* induces an isomorphism

$$\frac{\Gamma_X}{(s_i - 1 \,:\, i = 1, \dots, \ell)} \simeq \mathbb{C}[G/H].$$

Let Δ be the subset of Ω given by the isomorphism classes of $\mathcal{O}(X_1), \dots, \mathcal{O}(X_\ell)$. By [12] it is known that Δ is a simple basis of an irreducible root system Φ and, by [6, 14], it is known that Ω is a set of possible weights of Φ (i.e. Ω is a lattice containing the root lattice and contained in the weight lattice). In particular the submonoid Ω^+ given by line bundes generated by global sections corresponds to the set of dominant weights in Ω w.r.t. Δ.

When Ω^+ is a free monoide, Γ_X and $\mathbb{C}[G/H]$ have a natural choice of generators which correspond through the map π^*; let us denote by $\mathbb{V}^*$ the vector space spanned by such generators. In [2] a SMT in these generators has been constructed. The relation among these generators have not been computed in [2] but there it is proved that such relations can be written in a certain form. Using this rough description one may easily prove the following result:

Proposition. *If Φ is of type A, BC or C and G is simply connected or if Φ is of type B and G is adjoint, then Ω^+ is a free monoide and the relations between the generators of $\mathbb{C}[G/H]$ are quadratic.*

The aim of the paper [1] is to make a further step and give a precise description of the relations among these generators in the cases of the Proposition above by introducing some new simmetry into the problem. More precisely we introduce a group L containing G as the semisimple part of a maximal Levi of G, and we show that the relations in the generators of $\mathbb{C}[G/H]$ may be deduced by the Plücker relations of a grassmannian of L. In particular the relations are determined by the representation theory of L.

The construction of this extended group L is uniform and goes as follows. Fix a suitable spherical dominant weight ϵ, add a node n_0 to the Dynkin diagram of G and, for all simple root α, join n_0 with the node n_α of the simple root α by $\epsilon(\alpha^\vee)$ lines, further put an arrow in the direction of n_α if $\epsilon(\alpha^\vee) \geq 2$. In the cases of the Proposition above this extended diagram is of finite or affine type.

Then one takes $\mathcal{L}$ to be the ample generator of the Picard group of the Grassmann variety $\mathcal{G}r = L/P$, where P is the maximal parabolic subgroup corresponding to the new node n_0. We show that in this Grassmann variety there exists a Richardson variety $\mathcal{R}$ such that $\oplus_{n\geq 0} H^0(\mathcal{R}, \mathcal{L}^n) = \mathbb{C}[G/H]$; in particular $H^0(\mathcal{R}, \mathcal{L}) \simeq \mathbb{V}^*$.

We need to recall a few facts about the generalized Plücker relations. In [8], a basis $\mathbb{F} \subset \Gamma(\mathcal{G}r, \mathcal{L})$ has been constructed together with a partial order "$\geq$", such that the monomials $\mathbb{F}^2 = \{ff' \mid f, f' \in \mathbb{F}, f \leq f'\} \subset \Gamma(\mathcal{G}r, \mathcal{L}^{\otimes 2})$ form a basis. For a pair $f, f' \in \mathbb{F}$ of non comparable elements, let $R_{f,f'} \in \mathsf{S}^2(\Gamma(\mathcal{G}r, \mathcal{L}))$ be the relation expressing the product ff' as a linear combination of elements in $\mathbb{F}^2$. It was shown in [7] that the $R_{f,f'}$'s generate the defining ideal of $\mathcal{G}r \hookrightarrow \mathbb{P}(\Gamma(\mathcal{G}r, \mathcal{L})^*)$; moreover such basis and relations are comparable with Richardson varieties. So in particular there exists a (finite) set $\mathbb{F}_0$ of $\mathbb{F}$ such that the subvariety $\mathcal{R}$ of $\mathcal{G}r$ is defined by the vanishing of all the elements of $\mathbb{F} \setminus \mathbb{F}_0$.

In order to analyse the structure of $\mathbb{C}[G/H]$ we construct a G-equivariant ring homomorphism $\varphi : \Gamma_{\mathcal{G}r} \longrightarrow \mathbb{C}[G/H]$. If Φ is of finite type, then the morphism φ is just the pull back of a canonical G equivariant map $G/H \to \mathcal{G}r$. In the general case, the underlying idea is the same, but the construction is more involved.

Furthermore we can define a G equivariant injection $i : \mathbb{V}^* \hookrightarrow \Gamma(\mathcal{G}r, \mathcal{L})$ such that $\varphi \circ i : \mathbb{V}^* \to \mathbb{C}[G/H]$ is an isomorphism onto the image and $i(\mathbb{V}^*) = \mathbb{F}_0$.

Notice, however, that the relations $R_{f,f'}$ for $f, f' \in \mathbb{F}_0$ involve also elements in $\mathbb{F} - \mathbb{F}_0$. Let $\mathbb{F}_1 \sqcup \mathbb{F}_0$ be the (finite) set of functions appearing in some polynomial $R_{f,f'}$ for $f, f' \in \mathbb{F}_0$. Denote by $\hat{R}_{f,f'} \in \mathsf{S}^2(\mathbb{V}^*)$ the relation obtained from $R_{f,f'}$ by replacing a generator $h \in \mathbb{F}_0$ by $g_h \in \mathbb{G}$ and a generator $h \in \mathbb{F}_1$ by the function $F_h = \varphi(h)$ of $\mathbb{G}$.

Theorem (1)**.** *The relations* $\{\hat{R}_{f,f'} | f, f' \in \mathbb{F}_0$ *not comparable*$\}$ *generate the ideal Rel of the relations among the generators* $\mathbb{G}$ *of* $\mathbb{C}[G/H]$.

Theorem (2)**.** *Consider* $\mathbb{G} = \{g_f \mid f \in \mathbb{F}_0\}$ *as a partially ordered set with the same partial order as on* $\mathbb{F}$. *Then* $\mathbb{G}$ *is a basis of* $\mathbb{V}^* \subset \mathbb{C}[G/H]$, *the set* $\mathbb{SM}_0$ *of ordered monomials in* $\mathbb{G}$ *realizes a standard monomial theory for* $\mathbb{C}[G/H]$ *and the relations* $\hat{R}_{f,f'}$ *for the non standard* ff' *are a set of straightening relations.*

If L is of finite type (or, equivalently, the restricted root system is of type A) we can show that $\mathbb{F}_1$ is given by just two elements f_0, f_1 and that

$$F_{f_0} = F_{f_1} = 1.$$

In particular, in these cases the explicit relations may be summarized in the following description of the coordinate ring of the symmetric variety:

$$\mathbb{C}[G/H] \simeq \frac{\Gamma_{\mathcal{G}r}}{(f_0 = f_1 = 1)}.$$

In some special cases a standard monomial theory for $\mathbb{C}[G/H]$ had been developed before

- for $G/H = SL(n)$, corresponding to the involution $(x, y) \mapsto (y, x)$ of the group $SL(n) \times SL(n)$ and whose restricted root system is of type A, here our construction gives the same as the construction of De Concini, Eisenbud and Procesi [4];
- for $G/H =$'symmetric quadrics', corresponding to the involution $x \mapsto (x^{-1})^t$ of the group $SL(n)$ and whose restricted root system is of type A, a theory of standard monomials has been introduced by Strickland [13] and Musili [10, 9]; however, we do not know whether their SMT is equivalent to ours;
- for $G/H = Sp(2\,n)$, corresponding to the involution $(x, y) \mapsto (y, x)$ of the group $Sp(2\,n) \times Sp(2\,n)$ and whose restricted root system is of type C, a theory of standard monomials has been introduced by De Concini in [3]. Also in this case we do not know whether this SMT is equivalent to ours.

The results above cover almost all cases with restricted root system of type A; there are only two families missing whose restricted root system is of type A_1 (and hence they are very simple), the 'symplectic quadrics' and an involution of E_6 which is briefly discussed at the end of [1].

Finally we want to stress that the condition on the restricted root system to be of type A, B, C or BC, while looking strong, is actually fulfilled for many involutions. In the Tables in [11] it holds for 12 families of involutions out of a total of 13 families and in 4 exceptional cases out of a total of 12. Moreover one should add to such list of families the involutions such that $G = H \times H$, H is simple and the involution is given by $(x, y) \mapsto (y, x)$; for these cases $\mathbb{C}[G/H]$ is the coordinate ring of H and our condition is equivalent to H equals to $SL(n)$ or $Sp(2n)$ or $SO(2n + 1)$.

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