PROJECTIVE NORMALITY OF COMPLETE SYMMETRIC VARIETIES

ROCCO CHIRIVÌ and ANDREA MAFFEI

Abstract

We prove that in characteristic zero the multiplication of sections of line bundles generated by global sections on a complete symmetric variety $X = \overline{G/H}$ is a surjective map. As a consequence, the cone defined by a complete linear system over X or over a closed G-stable subvariety of X is normal. This gives an affirmative answer to a question raised by Faltings in [11]. A crucial point of the proof is a combinatorial property of root systems.

Introduction

Let $\overline{G}$ be an adjoint semisimple algebraic group over an algebraically closed field of characteristic zero, and let G be its algebraic simply connected cover. Given an involutorial automorphism $\sigma : \overline{G} \to \overline{G}$, denote by $\overline{H}$ the subgroup of fixed points of σ. A wonderful compactification X of the symmetric variety $\overline{G}/\overline{H}$ has been constructed by De Concini and Procesi [9]. The main result of our paper can be stated as the following.

THEOREM A

If $\mathscr{L}$ and $\mathscr{L}'$ are line bundles generated by global sections on X, then the multiplication $\Gamma(X, \mathscr{L}) \otimes \Gamma(X, \mathscr{L}') \to \Gamma(X, \mathscr{L} \otimes \mathscr{L}')$ is surjective.

The projective normality of X follows by a standard argument. Hence we give an affirmative answer to a problem raised by Faltings in [11]. Our result has already been proved in [15] by Kannan in the special case of the compactification of a group, in which $\overline{G} = \overline{H} \times \overline{H}$ and the involution exchanges the two copies of $\overline{H}$, by a completely different method that does not apply to this situation. We stress that it is necessary to assume that the line bundles $\mathscr{L}$ and $\mathscr{L}'$ are generated by global sections, as the example after the proof of Theorem A in Section 3 shows.

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Complete symmetric varieties have been constructed also over any field of characteristic different from 2 by De Concini and Springer [10], and the question of Faltings makes sense and is particularly interesting in this case. However, Theorem A is false in general; at the end of Section 3 we give a counterexample that has been explained to us by De Concini.

Now we briefly describe the lines of the proof of Theorem A. We divide the G-modules appearing in $\Gamma(X, \mathscr{L} \otimes \mathscr{L}')$ into three classes, and we use different strategies for each class to show that the G-modules appear in the image of the multiplication. The first class is that of the modules appearing in a product of sections of line bundles that, with respect to the dominant order, are less than $\mathscr{L}$ or $\mathscr{L}'$. These are easily covered by induction on the dominant order on line bundles. The second class is formed by the modules that do not vanish when restricted to some G-stable subvariety Y of X. We recall that Y is a fibration over a partial flag variety with fiber isomorphic to a complete symmetric variety F with $\dim F < \dim X$, and we can suppose that the multiplication map is surjective for F, using induction on dimension. In Proposition 2.9 we show that from the surjectivity of the multiplication map on the fiber we can deduce that the multiplication for Y is surjective. So these kinds of modules also appear in the image. We parametrize the remaining modules, which form the third class, introducing the notion of low triple. Thanks to the result of Section 4, in which we study such triples, we can prove the surjectivity of the multiplication map for this class by a direct argument.

Our classification of low triples is purely combinatorial and makes sense for any root system Φ. Suppose we have chosen a base Δ, and let Λ^+ be the corresponding monoid of dominant weights for Φ.

Definition 1
Given $\lambda, \mu, \nu \in \Lambda^+$, we say that (λ, μ, ν) is a *low triple* if the following conditions hold:

(i) if λ', μ' are dominant weights such that $\lambda' \leq \lambda$, $\mu' \leq \mu$, and $\nu \leq \lambda' + \mu'$, then $\lambda' = \lambda$, $\mu' = \mu$;

(ii) $\nu + \sum_{\alpha \in \Delta} \alpha \leq \lambda + \mu$.

Then, if w_0 is the longest element of the Weyl group of Φ, the result in Section 4 is the following.

THEOREM B
The triple (λ, μ, ν) *of dominant weights is a* low *triple if and only if* λ *and* μ *are minuscule weights,* $\mu = -w_0\lambda$, *and* $\nu = 0$.

The proof of this theorem is somewhat unsatisfactory. Although we succeeded in developing a bit of general treatment, and although T. A. Springer suggested to us a way to further simplify the computations required for the exceptional root systems, in some steps we still have to use a case-by-case analysis.

1. Review of complete symmetric varieties

In this section we collect some preliminary results for the sequel, setting up notation and reviewing the construction of the wonderful compactification of G/H (for details, see [9], [10]).

Let $\mathfrak{g}$ be a semisimple Lie algebra over an algebraically closed field $\Bbbk$ of characteristic zero, and let σ be an involutorial automorphism of $\mathfrak{g}$. Denote by $\mathfrak{h}$ the subalgebra of fixed points of σ in $\mathfrak{g}$. If $\mathfrak{t}$ is a σ-stable toral subalgebra of $\mathfrak{g}$, we can decompose $\mathfrak{t}$ as $\mathfrak{t}_0 \oplus \mathfrak{t}_1$ with $\mathfrak{t}_0$ the $(+1)$-eigenspace of σ and $\mathfrak{t}_1$ the (-1)-eigenspace. We recall that any σ-stable toral subalgebra of $\mathfrak{g}$ is contained in a maximal one that is itself σ-stable. We fix such a σ-stable maximal toral subalgebra $\mathfrak{t}$ for which $\dim \mathfrak{t}_1$ is maximal, and we denote this dimension by ℓ; we call it the *rank* of σ.

Let $\Phi \subset \mathfrak{t}^*$ be the root system of $\mathfrak{g}$, and let $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ be the root space decomposition with respect to the action of $\mathfrak{t}$. Observe that σ acts also on $\mathfrak{t}^*$ and that it preserves Φ and the Killing form $(\cdot,\cdot)$ on $\mathfrak{t}$ and $\mathfrak{t}^*$. Let $\Phi_0 = \{\alpha \in \Phi \mid \sigma(\alpha) = \alpha\}$ and $\Phi_1 = \Phi \smallsetminus \Phi_0$. The choice of a σ-stable toral subalgebra for which $\dim \mathfrak{t}_1$ is maximal is equivalent to the condition $\sigma|_{\mathfrak{g}_\alpha} = \mathrm{id}|_{\mathfrak{g}_\alpha}$ for all $\alpha \in \Phi_0$. Moreover, we can choose the set Φ^+ of positive roots in such a way that $\sigma(\alpha) \in \Phi^-$ for all roots $\alpha \in \Phi^+ \cap \Phi_1$. Let Δ be the base defined by Φ^+, and put $\Delta_0 = \Delta \cap \Phi_0$, $\Delta_1 = \Delta \cap \Phi_1$.

Denote by $\Lambda \subset \mathfrak{t}^*$ the set of integral weights of Φ, and observe that σ preserves Λ. Let Λ^+ be the set of dominant weights with respect to Φ^+, and let ω_α be the fundamental weight dual to the simple coroot $\alpha^\vee$ for $\alpha \in \Delta$. For $\lambda \in \Lambda^+$, let also V_λ be the irreducible representation of $\mathfrak{g}$ of highest weight λ.

We say that $\lambda \in \Lambda^+$ is *spherical* if there exists $h \in V_\lambda \smallsetminus \{0\}$ fixed by $\mathfrak{h}$ (i.e., $\mathfrak{h} \cdot h = 0$); in this case the vector h is also unique up to scalar, and we denote it by h_λ. We denote the set of spherical weights by Ω^+ and the lattice they generate by Ω.

For a root α, define $\widetilde{\alpha} \doteq \alpha - \sigma(\alpha)$, and let $\widetilde{\Phi} = \{\widetilde{\alpha} \mid \alpha \in \Phi_1\}$. This is a (not necessarily reduced) root system, called the *restricted root system*, of rank ℓ with base $\widetilde{\Delta} = \{\widetilde{\alpha} \mid \alpha \in \Delta_1\}$. As a consequence of a result of Helgason [13] (see also [18] or [11] for an algebraic approach), $\Omega \cap \Lambda^+ = \Omega^+$ and Ω can be identified with the lattice of integral weights of the root system $(\widetilde{\Phi}, \Omega \otimes_{\mathbb{Z}} \mathbb{R})$. Given a weight λ, we define its Ω-*support* to be the set $\mathrm{supp}_\Omega(\lambda) = \{\widetilde{\alpha} \in \widetilde{\Delta} \mid (\lambda, \widetilde{\alpha}) \neq 0\}$. We introduce also the lattice $\widetilde{R}$ generated by $\widetilde{\Phi}$ and the monoid $\widetilde{R}^+ = \sum_{\widetilde{\alpha}\in\widetilde{\Delta}} \mathbb{N}\,\widetilde{\alpha}$.

Now we come to the construction of complete symmetric varieties following De Concini and Procesi [9]. Let G be a connected algebraic group over $\Bbbk$ whose Lie algebra is isomorphic to $\mathfrak{g}$. The action of σ on $\mathfrak{g}$ lifts to an automorphism of G, still denoted by σ. Let H be the normalizer in G of the Lie algebra $\mathfrak{h} \subset \mathfrak{g}$. As explained in [9], H is the maximal subgroup having $\mathfrak{h}$ as a Lie algebra. If G is an adjoint group, H coincides with the fixed point set of σ in G. Hence G/H is a symmetric variety. However, since G/H does not depend on the choice of the group G over $\mathfrak{g}$, we prefer to choose G simply connected, so for the rest of the paper *G is a simply connected algebraic group with Lie algebra* $\mathfrak{g}$. We introduce also the torus T (resp., T_0 and T_1), whose Lie algebra is $\mathfrak{t}$ (resp., $\mathfrak{t}_0$ and $\mathfrak{t}_1$), and the parabolic subgroup P of G associated to Δ_0. (In general, to a subset $I \subset \Delta$ we associate the parabolic subgroup whose Lie algebra is given by $\mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi_I \cup \Phi^+} \mathfrak{g}_\alpha$, where Φ_I is the root subsystem of Φ generated by I.)

In our study it is useful to consider also the degenerate case $G = \{e\}$ or, more generally, $\sigma = \mathrm{id}$. In this case, of course, G/H is just a single point.

Let now $\lambda_1, \ldots, \lambda_m$ be spherical weights with disjoint Ω-supports such that $\mathrm{supp}_\Omega(\lambda_1) \cup \cdots \cup \mathrm{supp}_\Omega(\lambda_m) = \widetilde{\Delta}$, and consider the point

$$x_0 = ([h_{\lambda_1}], \ldots, [h_{\lambda_m}]) \in \mathbb{P}(V_{\lambda_1}) \times \cdots \times \mathbb{P}(V_{\lambda_m}).$$

We define the variety $X = X(\sigma)$ as $\overline{Gx_0} \subset \mathbb{P}(V_{\lambda_1}) \times \cdots \times \mathbb{P}(V_{\lambda_m})$. Notice that x_0 is the unique point fixed by H in X and that the map $g \longmapsto gx_0$ induces an embedding $G/H \hookrightarrow X$ which is called the "minimal compactification" of G/H. Moreover, the construction is independent of the choice of the weights $\lambda_1, \ldots, \lambda_m$.

We also need another description of the compactification. Let λ be a spherical weight with $\mathrm{supp}_\Omega(\lambda) = \widetilde{\Delta}$, and consider a finite-dimensional $\mathfrak{g}$-representation of the form $V \doteq V_\lambda \oplus V'$. Take $h = (h_\lambda, h_{V'}) \in V$ to be a vector fixed by $\mathfrak{h}$ such that all T_1 weights of $h_{V'}$ are of the form $\mu = \lambda - \eta$ with $\eta \in \widetilde{R}^+$ and $\mu \neq \lambda$. Then, as proved in [9, §4], the map $G/H \ni gH \longmapsto g[h] \in \mathbb{P}(V)$ extends to an isomorphism $X \longrightarrow \overline{G[h]}$.

The following proposition describes the structure of the compactification.

PROPOSITION 1.1 ([9, Theorem 3.1])
Let $X = X(\sigma)$ be the compactification of G/H described above; then

(i) *X is a smooth projective G-variety;*

(ii) *$X \smallsetminus G{\cdot}x_0$ is a divisor with normal crossing and smooth irreducible components $S_1, \ldots, S_\ell$;*

(iii) *the G-orbits of X correspond to the subsets of the indexes $1, 2, \ldots, \ell$, so that the orbit closures are the intersections $S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_k}$ with $1 \le i_1 < i_2 < \cdots < i_{k-1} < i_k \le \ell$;*

(iv) *the unique closed orbit $Y \doteq \bigcap_{i=1}^{\ell} S_i$ is isomorphic to the partial flag variety G/P.*

We go on to construct some line bundles on the variety X. Let $\lambda \in \Lambda^+$ be such that $\mathbb{P}(V_\lambda)$ contains a point r fixed by H. One can show (see [9], [10]) that the map $G/H \ni gH \mapsto g \cdot r \in \mathbb{P}(V_\lambda)$ extends uniquely to a morphism

$$\psi_\lambda : X \to \mathbb{P}(V_\lambda).$$

We define $\mathscr{L}_\lambda$ as the line bundle $\psi_\lambda^* \mathscr{O}(1)$. In particular, $\mathscr{L}_\lambda$ is generated by its global sections. If we restrict $\mathscr{L}_\lambda$ on $G/P \simeq Y \hookrightarrow X$, we have the usual line bundle $G \times_P \Bbbk_{-\lambda}$ corresponding to λ in the identification of $\mathrm{Pic}(G/P)$ with a sublattice of the weight lattice Λ. Moreover, we have the following.

PROPOSITION 1.2 ([9, Proposition 8.1])
The map $\mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ *induced by the inclusion is injective.*

So we can identify $\mathrm{Pic}(X)$ with a sublattice of the weight lattice. Further, the line bundles constructed above account for all line bundles since we have the following.

PROPOSITION 1.3 ([10, Lemma 4.6])
$\mathrm{Pic}(X)$ *corresponds to the lattice generated by the dominant weights* λ *such that* $\mathbb{P}(V_\lambda)^H$ *is not empty.*

Notice that by construction every line bundle has a natural G-linearization. Moreover, if λ is not dominant, then $H^0(Y, \mathscr{L}_{\lambda|Y}) = \mathrm{Ind}_P^G(\Bbbk_{-\lambda}) = 0$; hence $\mathscr{L}_\lambda$ is generated by global sections if and only if λ is dominant.

Following the literature, we introduce now a particular behavior of a simple root. The action of the involution σ on the set of roots admits the following description. There exists an involutive bijection $\overline{\sigma} : \Delta_1 \to \Delta_1$ such that for every $\alpha \in \Delta_1$ we have

$$\sigma(\alpha) = -\overline{\sigma}(\alpha) - \beta_\alpha,$$

where β_α is a nonnegative linear combination of roots in Δ_0. We say that $\alpha \in \Delta_1$ is an *exceptional* root if $\overline{\sigma}(\alpha) \neq \alpha$ and $(\alpha, \sigma(\alpha)) \neq 0$. Notice that $\overline{\sigma}(\alpha)$ is exceptional if α is. Moreover, the symmetric variety G/H and its compactification X are said to be exceptional if there exist exceptional roots.

PROPOSITION 1.4 ([10, Theorem 4.8])
$\mathrm{Pic}(X)$ *is generated by the spherical weights and the fundamental weights corresponding to the exceptional roots.*

Now we describe the sections of a line bundle $\mathscr{L}$ as a G-module. The first useful remark is that any irreducible G-module appears in $\Gamma(X, \mathscr{L})$ with multiplicity at most one (see [9, Lemma 8.2]).

We analyze first the case of the divisors S_i, $1 \leq i \leq \ell$. Let $\widetilde{\alpha}_1, \dots, \widetilde{\alpha}_\ell$ be the elements of $\widetilde{\Delta}$. Then, up to reindexing the G-stable divisors, we have the following.

PROPOSITION 1.5 ([9, Corollary 8.2])
There exists a unique up to scalar G-invariant section $s_i \in \Gamma(X, \mathscr{L}_{\widetilde{\alpha}_i})$ whose divisor is S_i.

For an element $\nu = \sum_{i=1}^{\ell} n_i \widetilde{\alpha}_i \in \widetilde{R}^+$, the multiplication by $s^\nu \doteq \Pi_i s_i^{n_i}$ gives a linear map

$$\Gamma(X, \mathscr{L}_{\lambda-\nu}) \to \Gamma(X, \mathscr{L}_\lambda).$$

If $\mu \in \operatorname{Pic}(X)$ is dominant, then by construction of $\mathscr{L}_\mu$ we certainly have a submodule of $\Gamma(X, \mathscr{L}_\mu)$ isomorphic to V_μ^* obtained by the pullback of the homogeneous coordinates of $\mathbb{P}(V_\mu)$ to X. Since, as already recalled, the multiplicity of any irreducible submodule is at most one, we can speak of the submodule V_μ^* of $\Gamma(X, \mathscr{L}_\mu)$ without ambiguity. If now $\lambda \in \operatorname{Pic}(X)$ is any element such that $\lambda - \mu \in \widetilde{R}^+$, we can consider the image of V_μ^* under the multiplication by $s^{\lambda-\mu}$ from $\Gamma(X, \mathscr{L}_\mu)$ to $\Gamma(X, \mathscr{L}_\lambda)$. We call this image $s^{\lambda-\mu} V_\mu^*$. We have the following result.

PROPOSITION 1.6 ([9, Theorem 5.10])
Let $\lambda \in \operatorname{Pic}(X)$; then

$$\Gamma(X, \mathscr{L}_\lambda) = \bigoplus_{\mu \in (\lambda - \widetilde{R}^+) \cap \Lambda^+} s^{\lambda-\mu} V_\mu^*.$$

Moreover, for all $\nu = \sum_{i=1}^{\ell} n_i \widetilde{\alpha}_i \in \widetilde{R}^+$, the set of sections vanishing on S_i with multiplicity at least n_i for $i = 1, \dots, \ell$ is the image $s^\nu \Gamma(X, \mathscr{L}_{\lambda-\nu})$ of the multiplication by s^ν in $\Gamma(X, \mathscr{L}_\lambda)$.

2. Stable subvarieties

In this section we study the closures of G-orbits of X, which we call *stable subvarieties*. Following De Concini and Procesi [9], we review the structure of a stable subvariety, recalling in particular that such a variety is a fibration over a partial flag variety with fiber isomorphic to a complete symmetric variety. We use such a result to prove Proposition 2.9, lifting the surjectivity of the multiplication map from the fiber to a stable subvariety. This is used in one inductive step for the proof of Theorem A in Section 3.

We need to introduce some notation related to some special subgroups of G. If I is a subset of Δ containing Δ_0 and such that $\bar{\sigma}(I \cap \Delta_1) \subset I \cap \Delta_1$, let $\Phi_I \subset \Phi$ be the root subsystem of Φ generated by I, and define $G_I \subset G$ as the semisimple group associated to Φ_I (the subgroup whose Lie algebra is generated by $\mathfrak{g}_\alpha$ for $\alpha \in \Phi_I$).

Also, let $P_I = P \cap G_I$ be the parabolic subgroup of G_I associated to Δ_0, and let $T_I = T \cap G_I$, a maximal torus of G_I. We call $\mathfrak{g}_I$ (resp., $\mathfrak{t}_I$) the Lie algebra of G_I (resp., T_I). Observe that $\sigma(\Phi_I) \subset \Phi_I$, as guaranteed by the assumption on I; hence G_I is stable under the action of σ, and so we can define $\sigma_I\colon G_I \longrightarrow G_I$ as the restriction of σ. We denote by $\mathfrak{h}_I$ the intersection $\mathfrak{h} \cap \mathfrak{g}_I$ and by H_I the normalizer of $\mathfrak{h}_I$ in G_I. Observe also that $\sigma|_{\mathfrak{g}_\alpha} = \mathrm{id}|_{\mathfrak{g}_\alpha}$ for all $\alpha \in \Phi_{\Delta_0 \cap I}$; hence $\dim(\mathfrak{t}_I)_1$ is maximal (where $(\mathfrak{t}_I)_1$ is the -1-eigenspace of σ_I on $\mathfrak{t}_I$).

We denote by Λ_I the lattice of integral weights of Φ_I and by $\Lambda_I^+ \subset \Lambda_I$ the monoid of dominant weights with respect to I. Given two subsets I, J of Δ such that $I \subset J$, we have a natural projection $r_I^J : \Lambda_J \longrightarrow \Lambda_I$ induced by the inclusion $\mathfrak{t}_I \hookrightarrow \mathfrak{t}_J$, and a canonical immersion $\iota_J^I : \Lambda_I \hookrightarrow \Lambda_J$ mapping a fundamental weight with respect to Φ_I to the corresponding fundamental weight with respect to Φ_J. If we consider the case $J = \Delta$, we see that Λ_I is identified with the set of characters of T_I; hence also G_I is simply connected. Analogously, as noted after the definition of H in the previous section, H_J is the largest subgroup having $\mathfrak{h}_J$ as Lie algebra, so it is easy to see that $H_J \subset H$.

Now let $\tilde{I}$ be a subset of $\widetilde{\Delta}$, and set $I = \{\alpha \in \Delta_1 \mid \widetilde{\alpha} \in \tilde{I}\} \cup \Delta_0$; notice that such a set I satisfies the condition $\overline{\sigma}(I \cap \Delta_1) \subset I \cap \Delta_1$ above. Choose a one-parameter subgroup $\gamma_{\tilde{I}}\colon \mathbb{k}^* \longrightarrow T_1$ such that

$$\langle \gamma_{\tilde{I}}, \widetilde{\alpha} \rangle = 0 \quad \text{if } \widetilde{\alpha} \in \widetilde{\Delta} \smallsetminus \tilde{I} \qquad \text{and} \qquad \langle \gamma_{\tilde{I}}, \widetilde{\alpha} \rangle < 0 \quad \text{if } \widetilde{\alpha} \in \tilde{I},$$

where the pairing is given by the identification of a one-parameter subgroup of T with an element of $\mathfrak{t}$. We can now define the *stable subvariety* $X_{\tilde{I}}$ corresponding to $\tilde{I} \subset \widetilde{\Delta}$ as the closure $\overline{Gx_{\tilde{I}}}$ of the orbit of the point $x_{\tilde{I}} \doteq \lim_{t \longrightarrow 0} \gamma_{\tilde{I}}(t) x_0$.

PROPOSITION 2.1 ([9, Theorem 3.1, Corollary 8.2])
We have the following:

(i) $X_{\{\widetilde{\alpha}_i\}} = S_i$ *for all* $\widetilde{\alpha}_i \in \widetilde{\Delta}$;

(ii) $X_{\tilde{I} \cup \tilde{J}} = X_{\tilde{I}} \cap X_{\tilde{J}}$, *and in particular,* $X_{\widetilde{\Delta}} = Y$ *is the unique closed* G*-orbit in* X;

(iii) $X_{\tilde{I}}$ *is a projective smooth variety of dimension* $\dim X - |\tilde{I}|$.

In De Concini and Procesi [9, §5], the geometric structure of $X_{\tilde{I}}$ is described. For the convenience of the reader, we review their results below.

Fix $\tilde{I}, I$ as above, and define $\tilde{J} = \widetilde{\Delta} \smallsetminus \tilde{I}$ and $J = (\Delta \smallsetminus I) \cup \Delta_0$. Let also $\widetilde{R}^+_{\tilde{J}} = \sum_{\widetilde{\alpha} \in \tilde{J}} \mathbb{N}\, \widetilde{\alpha}$, and let $\widetilde{R}_{\tilde{J}}$ be the lattice generated by $\widetilde{R}^+_{\tilde{J}}$. If $\lambda \in \Lambda_J^+$, we denote by Z_λ the irreducible representation of G_J of highest weight λ.

Let $\lambda \in \Lambda$ be a spherical weight, and consider a vector h_λ spanning the unique line in V_λ fixed by H. By [9, §2], we know that $h_\lambda = v_\lambda + \sum_{\mu \in \lambda - (\widetilde{R}^+ \smallsetminus \{0\})} u_\mu$, where

v_λ is a highest weight vector and u_μ are eigenvectors of T of weight μ. We define

$$h_\lambda^{\tilde{I}} \doteq v_\lambda + \sum_{\mu \in \lambda - (\widetilde{R}_{\tilde{J}}^{+} \smallsetminus \{0\})} u_\mu. \qquad (*)$$

One can easily see that the point $x_{\tilde{I}}$ defined above is

$$x_{\tilde{I}} = ([h_{\lambda_1}^{\tilde{I}}], \dots, [h_{\lambda_m}^{\tilde{I}}])$$

as a point of $X \subset \mathbb{P}(V_{\lambda_1}) \times \cdots \times \mathbb{P}(V_{\lambda_m})$.

LEMMA 2.2
If λ is a spherical weight, then $[h_\lambda^{\tilde{I}}]$ is fixed by H_J. Further, $h_\lambda^{\tilde{I}} = v_\lambda$ if and only if $\operatorname{supp}_\Omega(\lambda) \subset \tilde{I}$.

Proof
Notice that $H_J \subset H$ certainly fixes $[h_\lambda]$. Observe also that $\gamma_{\tilde{I}}$ commutes with $\mathfrak{g}_J$. Indeed, if $\alpha \in \Phi_J$ and $e_\alpha \in \mathfrak{g}_\alpha$, then

$$[\gamma_{\tilde{I}}, e_\alpha] = \langle \gamma_{\tilde{I}}, \alpha \rangle e_\alpha = 0.$$

Hence $\gamma_{\tilde{I}}$ commutes with G_J and in particular with H_J. The first claim follows.

Suppose that $h_\lambda^{\tilde{I}} = v_\lambda$. Let $\tilde{S} = \widetilde{\Delta} \smallsetminus \operatorname{supp}_\Omega(\lambda)$ and $S = \{\alpha \in \Delta_1 \mid \widetilde{\alpha} \in \tilde{S}\} \cup \Delta_0$. Since $\mathbb{k} h_\lambda^{\tilde{I}}$ is stable under $\mathfrak{h}_J$, we have $\mathfrak{h}_J \subset \operatorname{stab}_{\mathfrak{g}} \mathbb{k} v_\lambda = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi_S \cup \Phi^+} \mathfrak{g}_\alpha$. Now the inclusion $\operatorname{supp}_\Omega(\lambda) \subset \tilde{I}$ follows since, by the particular choice of Φ^+, we have $\mathfrak{h}_J = \mathfrak{t}_J \oplus \bigoplus_{\alpha \in \Phi_0} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \Phi_J \cap \Phi^+} \mathbb{k}(e_\alpha + \sigma(e_\alpha))$.

Now suppose that $\operatorname{supp}_\Omega(\lambda) \subset \tilde{I}$. If μ is a weight such that there exists $\alpha \in I$ with $(\omega_\alpha, \lambda - \mu) \neq 0$, then $\mu \notin \lambda + \widetilde{R}_{\tilde{J}}$. Hence, by the formula $(*)$, it is enough to show that for any weight $\mu \neq \lambda$ with $(V_\lambda)_\mu \neq 0$ there exists $\alpha \in \Delta$ such that $\langle \lambda, \alpha^\vee \rangle \neq 0$ and $(\omega_\alpha, \lambda - \mu) > 0$.

We consider first the case $\mu = w\lambda \neq \lambda$ with w in the Weyl group of Φ. Let $L(\mu)$ be the minimum of the length of w such that $\mu = w\lambda$. If $L(\mu) = 1$, then $\mu = s_\alpha(\lambda) \neq \lambda$ for some α and α satisfies our requests.

If $L(\mu) > 1$, we proceed by induction. Let $\mu = s_\alpha \mu'$, where α is such that $L(\mu') = L(\mu) - 1$. In particular, $\lambda - \mu = \lambda - \mu' + m\alpha$ with $m > 0$. Then if $\beta \in \Delta$ is such that $\langle \lambda, \beta^\vee \rangle \neq 0$ and $(\omega_\beta, \lambda - \mu') > 0$, we have also $(\omega_\beta, \lambda - \mu) > 0$.

Now in the general case, μ is in the convex hull of $\{w\lambda \mid w$ in the Weyl group of $\Phi\}$, and the claim follows. □

In order to describe $X_{\tilde{I}}$ we consider the realization of X as $\overline{G([h_\lambda], [h_\mu])} \subset \mathbb{P}(V_\lambda) \times \mathbb{P}(V_\mu)$, where λ and μ are two spherical weights such that $\operatorname{supp}_\Omega(\lambda) = \tilde{I}$

and $\mathrm{supp}_\Omega(\mu) = \tilde{J}$. (We recall that $\tilde{J} = \widetilde{\Delta} \smallsetminus \tilde{I}$.) Let $\pi : X \longrightarrow \mathbb{P}(V_\lambda)$ be the projection onto the first factor. By Lemma 2.2, we have $\pi(x_{\tilde{I}}) = [v_\lambda]$; hence $\pi(X_{\tilde{I}}) = Gv_\lambda \simeq G/Q$, where Q is the parabolic subgroup of G associated to J. We denote by $\pi_{\tilde{I}} \colon X_{\tilde{I}} \longrightarrow G/Q$ the restriction of π to $X_{\tilde{I}}$ and by F_J the fiber of $\pi_{\tilde{I}}$ over $[v_\lambda]$.

PROPOSITION 2.3 ([9, Theorem 5.3])
The fiber F_J is the closure $\overline{G_J[h_\mu^{\tilde{I}}]}$. Moreover, it is isomorphic to $X(\sigma_J)$, the complete symmetric variety associated to (G_J, σ_J).

We apply this proposition to the computation of $\mathrm{Pic}(X_{\tilde{I}})$. By Lemma 2.2, we have $x_{\widetilde{\Delta}} \in F_J$. The action of G_J on $Y_J \doteq G_J x_{\widetilde{\Delta}}$ induces an identification of G_J/P_J with Y_J sending P_J to $x_{\widetilde{\Delta}}$; in particular, Y_J is the unique closed orbit of F_J. We identify Y with $Gx_{\widetilde{\Delta}}$, and we observe that the inclusion of Y_J in Y induces the natural inclusion of G_J/P_J in G/P sending $P_J = P \cap G_J$ to P. We denote by $\jmath$ the inclusion of the closed orbit $Y \simeq G/P$ in $X_{\tilde{I}}$ and by $\jmath_J$ the inclusion of the closed orbit $Y_J \simeq G_J/P_J$ in F_J. We have the following commutative diagram (whose notation is in force throughout the rest of the paper):

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Pic}(G/Q) & \xrightarrow{\pi_{\tilde{I}}^*} & \mathrm{Pic}(X_{\tilde{I}}) & \xrightarrow{\iota_F^*} & \mathrm{Pic}(F_J) & \longrightarrow & 0 \\
 & & \Big\downarrow & & \Big\downarrow \jmath^* & & \Big\downarrow \jmath_J^* & & \\
 & & \simeq & & \mathrm{Pic}(G/P) & \xrightarrow{\jmath_F^*} & \mathrm{Pic}(G_J/P_J) & & \\
 & & \Big\downarrow & & \Big\downarrow \simeq & & \Big\downarrow \simeq & & \\
0 & \longrightarrow & \Lambda_{\Delta \smallsetminus J} & \xrightarrow{\iota} & \Lambda_{\Delta \smallsetminus \Delta_0} & \xrightarrow{r} & \Lambda_{J \smallsetminus \Delta_0} & \longrightarrow & 0
\end{array}
$$

where we have the following.

(i) We set $\iota = \iota_{\Delta \smallsetminus \Delta_0}^{\Delta \smallsetminus J}$, $r = r_{J \smallsetminus \Delta_0}^{\Delta \smallsetminus \Delta_0}$, and the third row is exact by construction.

(ii) The map ι_F is the inclusion of F_J in $X_{\tilde{I}}$. As in De Concini and Procesi [9], there exists a one-parameter subgroup of G with only isolated fixed points; hence $H^2(X_{\tilde{I}}, \mathbb{Z}) = \mathrm{Pic}(X_{\tilde{I}})$ and odd cohomology vanishes. So by the spectral sequence $H^p(G/Q, R^q \pi_{\tilde{I}*} \mathbb{Z}_{X_{\tilde{I}}}) \Rightarrow H^{p+q}(X_{\tilde{I}}, \mathbb{Z}_{X_{\tilde{I}}})$ given by the fibration, we have that the first row is exact and $\Gamma(G/Q, R^2 \pi_{\tilde{I}*} \mathbb{Z}_{X_{\tilde{I}}}) = H^2(F_J, \mathbb{Z})$ since G/Q is simply connected.

(iii) The map $\jmath_J^*$ is injective by Proposition 1.2.

(iv) The map $\jmath_F^*$ is the pullback of the natural inclusion $G_J/P_J \hookrightarrow G/P$ induced by the inclusion $Y_J \subset Y$, as observed above. Hence the square between the first and second lines is clearly commutative.

(v) The isomorphisms mapping to the third row are the canonical identifications with the weight lattices.

(vi) The square on the left is commutative since $\jmath \circ \pi_{\tilde{I}}$ is the canonical projection induced by $P \subset Q$ and the pullback of the line bundle $G \times_Q \Bbbk_{-\lambda}$ gives the line bundle $G \times_P \Bbbk_{-\lambda}$.

(vii) The square from the second and third rows is commutative since the line bundle $G \times_P \Bbbk_{-\lambda}$ restricted to G_J/P_J gives the line bundle $G_J \times_{P_J} \Bbbk_{-\lambda|_{\mathfrak{t}_J}}$.

So $\jmath^*$ is an injective map. If we identify $\mathrm{Pic}(G/Q) \simeq \Lambda_{\Delta\setminus J}$ and $\mathrm{Pic}(X)$ with a sublattice of $\Lambda_{\Delta\setminus\Delta_0}$ as in Section 1, then $\mathrm{Pic}(X_{\tilde{I}})$ is identified with the sublattice $\mathrm{Pic}(X) + \mathrm{Pic}(G/Q)$ of $\Lambda_{\Delta\setminus\Delta_0}$.

Moreover, observe that the inclusion $\imath_X\colon X_{\tilde{I}} \longrightarrow X$ induces an injective map $\imath_X^*\colon \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X_{\tilde{I}})$ and that, by the characterization of Proposition 1.4, the map $\imath_F^* \circ \imath_X^*$ is surjective. In particular, we see that every line bundle on $X_{\tilde{I}}$ has a natural G-linearization.

PROPOSITION 2.4

Let $\mathscr{L}_\lambda \in \mathrm{Pic}(X_{\tilde{I}})$. Then as a G-module we have

$$\Gamma(X_{\tilde{I}}, \mathscr{L}_\lambda) = \bigoplus_{\mu \in (\lambda - \widetilde{R}^+_{\tilde{J}}) \cap \Lambda^+} s^{\lambda-\mu} V^*_\mu.$$

Proof

Although, for $\tilde{I} \neq \emptyset$, this result is not explicitly claimed in [9], the proof of [9, Theorem 8.3] applies to this case without changes. □

We analyze now the relation between the sections s_i of the complete symmetric variety X and the sections $s_{J,i}$ of the complete symmetric variety F_J.

LEMMA 2.5

Up to rescaling the sections $s_{J,i}$ by a nonzero constant factor, we have $s_i|_{F_J} = s_{J,i}$ for all $\widetilde{\alpha}_i \in \tilde{J}$.

Proof

Observe that $s_i|_{F_J}$ and $s_{J,i}$ are G_J-invariant sections of the line bundle $\mathscr{L}_{r(\widetilde{\alpha}_i)}$. Moreover, $s_i|_{F_J} \neq 0$ by Proposition 1.5 and the claim follows. □

Looking at the decomposition into irreducible modules of the spaces of sections $\Gamma(X_{\tilde{I}}, \mathscr{L}_\lambda)$ and $\Gamma(F_J, \mathscr{L}_{r(\lambda)})$, we see that they are indexed by the same weights. This suggests that one can prove the surjectivity of the multiplication map for sections on $X_{\tilde{I}}$ using that on F_J. In order to make a rigorous proof out of this idea, we analyze the lowest weight vectors as De Concini suggested to us. The following lemmas prepare the work for this proof.

In the remaining part of this section we also make use of the following notation. Set U^- to be the unipotent subgroup of G whose Lie algebra is $\bigoplus_{\alpha\in\Phi^-}\mathfrak{g}_\alpha$, and set $U_J^- = U^-\cap G_J$; moreover, if $\lambda \in \Lambda^+$, let $V_\lambda = \Bbbk v_\lambda \oplus V'_\lambda$ be a T stable decomposition of V_λ with v_λ a highest weight vector.

LEMMA 2.6
If $\lambda \in \Lambda^+_{\Delta_1}$, then the restriction map

$$\jmath_F^* : \Gamma(G/P, \mathscr{L}_\lambda) \longrightarrow \Gamma(G_J/P_J, \mathscr{L}_{r(\lambda)})$$

induces an isomorphism between $\Gamma(G/P, \mathscr{L}_\lambda)^{U^-}$ and $\Gamma(G_J/P_J, \mathscr{L}_{r(\lambda)})^{U_J^-}$.

Proof
The map $\jmath_F^*$ is G_J-equivariant; hence $\jmath_F^*(\Gamma(G/P, \mathscr{L}_\lambda)^{U^-}) \subset \Gamma(G_J/P_J, \mathscr{L}_{r(\lambda)})^{U_J^-}$. Since they are both one-dimensional vector spaces, it is enough to prove that $\jmath_F^*(\Gamma(G/P, \mathscr{L}_\lambda)^{U^-}) \neq \{0\}$.

We recall that the line bundle $\mathscr{L}_\lambda$ on G/P can be constructed in the following way. The stabilizer of $[v_\lambda] \in \mathbb{P}(V_\lambda)$ contains P; hence we have a map $\psi\colon G/P \longrightarrow \mathbb{P}(V_\lambda)$ and $\mathscr{L}_\lambda = \psi^*\mathscr{O}(1)$. In particular, $\Gamma(G/P, \mathscr{L}_\lambda) = V_\lambda^*$ can be realized as the pullback through the map ψ of the space V_λ^* of coordinate functions on $\mathbb{P}(V_\lambda)$.

Let $\varphi \in V_\lambda^*$ be such that $\varphi(v_\lambda) = 1$ and $\varphi = 0$ on V'_λ. Then φ is a lowest weight vector in V_λ^*; hence $\Gamma(G/P, \mathscr{L}_\lambda)^{U^-} = \Bbbk\varphi$. Observe that $\psi(\jmath(P_J)) = \psi(P) = v_\lambda$; hence $v_\lambda \in \psi(\jmath_F(G_J/P_J))$ and $\jmath_F^*(\varphi) \neq 0$. □

In the lemma below we make use of the following straightforward consequence of the definition: the elements $r(\widetilde{\alpha})$ with $\widetilde{\alpha} \in \tilde{J}$ form a base of the restricted root system of (G_J, σ_J).

LEMMA 2.7
For all $\lambda \in \mathrm{Pic}(X_{\tilde{I}})$, the restriction map

$$\imath_F^* : \Gamma(X_{\tilde{I}}, \mathscr{L}_\lambda) \longrightarrow \Gamma(F_J, \mathscr{L}_{r(\lambda)})$$

induces an isomorphism between $\Gamma(X_{\tilde{I}}, \mathscr{L}_\lambda)^{U^-}$ and $\Gamma(F_J, \mathscr{L}_{r(\lambda)})^{U_J^-}$.

Proof
The map $\imath_F^*$ is G_J-equivariant; hence $\imath_F(\Gamma(X_{\tilde{I}}, \mathscr{L}_\lambda)^{U^-}) \subset \Gamma(F_J, \mathscr{L}_{r(\lambda)})^{U_J^-}$.

Suppose now that λ is dominant, and consider $V_\lambda^* \subset \Gamma(X_{\tilde{I}}, \mathscr{L}_\lambda)$ and $Z_{r(\lambda)} \subset \Gamma(F_J, \mathscr{L}_{r(\lambda)})$. Let $\varphi \in V_\lambda^*$ be a lowest weight vector, and notice that $\imath_F^*(\varphi) \in$

$\Gamma(F_J, \mathscr{L}_{r(\lambda)})$, if it is nonzero, is a vector of weight $-r(\lambda)$; hence it spans $(Z^*_{r(\lambda)})^{U_J^-} \subset \Gamma(F_J, \mathscr{L}_{r(\lambda)})^{U_J^-}$. So it is enough to prove that $\varphi|_{G_J/P_J} \neq 0$.

By the description of the sections of $\mathscr{L}_\lambda$ on $X_{\tilde{I}}$ (Proposition 1.6), we have $\varphi|_{G/P} \neq 0$; hence by Lemma 2.6, we have $\varphi|_{G_J/P_J} \neq 0$. In particular, $\varphi|_{F_J} \neq 0$.

Consider now the general case. Let $M = (\lambda - \widetilde{R}^+_{\tilde{J}}) \cap \Lambda^+$ and $N = \big(r(\lambda) - \sum_{\widetilde{\alpha} \in \tilde{J}} \mathbb{N}\, r(\widetilde{\alpha})\big) \cap \Lambda^+_J$, and observe that there is a bijection between the two sets given by $M \ni \mu \longmapsto r(\mu) \in N$ (whose inverse is $N \ni r(\lambda) - \sum n_{\widetilde{\alpha}} r(\widetilde{\alpha}) \longmapsto \lambda - \sum n_{\widetilde{\alpha}} \widetilde{\alpha} \in M$). Also, $\Gamma(X_{\tilde{I}}, \mathscr{L}_\lambda) = \bigoplus_{\mu \in M} s^{\lambda-\mu} V^*_\mu$; hence

$$\Gamma(X_{\tilde{I}}, \mathscr{L}_\lambda)^{U^-} = \bigoplus_{\mu \in M} \Bbbk s^{\lambda-\mu} \varphi_\mu,$$

where $\varphi_\mu \in V^*_\mu \subset \Gamma(X_{\tilde{I}}, \mathscr{L}_\mu)$ is a lowest weight vector. Hence, by the discussion above, $\psi_{r(\mu)} \doteq \iota^*_F(\varphi_\mu) \neq 0$ is a lowest weight vector in $Z^*_{r(\mu)} \subset \Gamma(F_J, \mathscr{L}_{r(\mu)})$ and, by Lemma 2.5, $\iota^*_F(s^{\lambda-\mu}\varphi_\mu) = s_{\tilde{J}}^{r(\lambda-\mu)} \psi_{r(\mu)}$ up to a nonzero scalar factor. Finally,

$$\iota^*_F\big(\Gamma(X_{\tilde{I}}, \mathscr{L}_\lambda)^{U^-}\big) = \bigoplus_{\mu \in M} \Bbbk s_{\tilde{J}}^{r(\lambda-\mu)} \psi_{r(\mu)} = \bigoplus_{\nu \in N} \Bbbk s_{\tilde{J}}^{r(\lambda)-\nu} \psi_\nu = \Gamma(F_J, \mathscr{L}_{r(\lambda)})^{U_J^-},$$

as claimed. □

If $\mathfrak{l}$ is a Lie algebra, we denote by $\mathbf{U}(\mathfrak{l})$ its universal enveloping algebra, and if $\mathfrak{l}$ is a subalgebra of $\mathfrak{m}$, we consider $\mathbf{U}(\mathfrak{l})$ as a subalgebra of $\mathbf{U}(\mathfrak{m})$.

We introduce also the Lie algebra $\mathfrak{u}_J = \bigoplus_{\alpha \in \Phi_J \cap \Phi^+} \mathfrak{g}_\alpha$.

LEMMA 2.8

Let V, V' be two finite-dimensional representations of G, and let Z, Z' be two finite-dimensional representations of G_J. Let $\phi\colon V \longrightarrow Z$ (resp., $\phi'\colon V' \longrightarrow Z'$) be a G_J-equivariant map such that $\phi|_{V^{U^-}}$ (resp., $\phi'|_{V'^{U^-}}$) is an isomorphism between V^{U^-} and $Z^{U_J^-}$ (resp., V'^{U^-} and $Z'^{U_J^-}$). Then $\phi \otimes \phi'((V \otimes V')^{U^-}) = (Z \otimes Z')^{U_J^-}$.

Proof

Since the map is G_J-equivariant, the left-hand side is contained in the right-hand one. Notice also that it is enough to study the case in which V, V', Z, and Z' are irreducible.

Choose φ and φ' to be two lowest weight vectors of V and V', respectively. Observe that, since G_J is linearly reductive, we can consider Z and Z' as the G_J-submodules generated by φ and φ', and the maps ϕ and ϕ' are just the projections along the G_J-invariant complements of Z and Z' in V and V'. So we need to show that $(Z \otimes Z')^{U_J^-} \subset (V \otimes V')^{U^-}$.

Let $x \in (Z \otimes Z')^{U_J^-}$, and write it as $x = \sum_m e_m\varphi \otimes e'_m\varphi'$, where $e_m, e'_m \in \mathbf{U}(\mathfrak{u}_J)$. To check $x \in (V \otimes V')^{U^-}$, we verify $\mathfrak{g}_{-\alpha} \cdot x = 0$ for all $\alpha \in \Delta$. If $f_\alpha \in \mathfrak{g}_{-\alpha}$, we have the following:

(i) if $\alpha \in J$, then $\mathfrak{g}_{-\alpha} \subset \mathfrak{g}_J$; hence $f_\alpha x = 0$ since $x \in (Z \otimes Z')^{U_J^-}$;

(ii) if $\alpha \notin J$, then f_α commutes with e_m and e'_m; hence $f_\alpha x = \sum_m e_m f_\alpha \varphi \otimes e'_m\varphi' + \sum_m e_m\varphi \otimes e'_m f_\alpha \varphi' = 0$. □

We come now to the main result of this section.

PROPOSITION 2.9
Let $\lambda, \mu \in \mathrm{Pic}(X_{\tilde{I}})$. *If the multiplication map*

$$\Gamma(F_J, \mathscr{L}_{r(\lambda)}) \otimes \Gamma(F_J, \mathscr{L}_{r(\mu)}) \longrightarrow \Gamma(F_J, \mathscr{L}_{r(\lambda+\mu)})$$

is surjective, then the multiplication map

$$\Gamma(X_{\tilde{I}}, \mathscr{L}_\lambda) \otimes \Gamma(X_{\tilde{I}}, \mathscr{L}_\mu) \longrightarrow \Gamma(X_{\tilde{I}}, \mathscr{L}_{\lambda+\mu})$$

is also surjective.

Proof
Consider the following commutative diagram, whose horizontal maps are given by multiplication and whose vertical maps are given by restriction:

$$\begin{array}{ccc}
\Gamma(X_{\tilde{I}}, \mathscr{L}_\lambda) \otimes \Gamma(X_{\tilde{I}}, \mathscr{L}_\mu) & \longrightarrow & \Gamma(X_{\tilde{I}}, \mathscr{L}_{\lambda+\mu}) \\
\downarrow & & \downarrow \\
\Gamma(F_J, \mathscr{L}_{r(\lambda)}) \otimes \Gamma(F_J, \mathscr{L}_{r(\mu)}) & \longrightarrow & \Gamma(F_J, \mathscr{L}_{r(\lambda+\mu)})
\end{array}$$

If we look at U^- and U_J^- invariants, we obtain

$$\begin{array}{ccc}
\big(\Gamma(X_{\tilde{I}}, \mathscr{L}_\lambda) \otimes \Gamma(X_{\tilde{I}}, \mathscr{L}_\mu)\big)^{U^-} & \longrightarrow & \Gamma(X_{\tilde{I}}, \mathscr{L}_{\lambda+\mu})^{U^-} \\
\downarrow & & \downarrow \simeq \\
\big(\Gamma(F_J, \mathscr{L}_{r(\lambda)}) \otimes \Gamma(F_J, \mathscr{L}_{r(\mu)})\big)^{U_J^-} & \twoheadrightarrow & \Gamma(F_J, \mathscr{L}_{r(\lambda+\mu)})^{U_J^-}
\end{array}$$

where, by Lemma 2.7, the vertical map on the right is an isomorphism; by Lemma 2.7 and Lemma 2.8, the vertical map on the left is surjective; and by the assumption on the multiplication on F_J, G_J being a linearly reductive group, the horizontal map on the bottom is surjective.

Hence also the horizontal map on the top has to be surjective. Further, since $\Gamma(X_{\tilde{I}}, \mathscr{L}_{\lambda+\mu})^{U^-}$ generates $\Gamma(X_{\tilde{I}}, \mathscr{L}_{\lambda+\mu})$ as a G-module, we deduce that the multiplication map on $X_{\tilde{I}}$ is surjective too. □

3. Projective normality

In this section we prove the surjectivity of the multiplication map for line bundles generated by global sections, that is, for line bundles $\mathscr{L}_\lambda$ with $\lambda \in \mathrm{Pic}^+(X) \doteq \mathrm{Pic}(X) \cap \Lambda^+$. The main ingredients are Proposition 2.9, which allows us to set up an induction on the dimension of the symmetric variety, and Theorem B (proved in Section 4).

We denote by W the Weyl group of Φ and by w_0 the longest element of W. Given two weights $\lambda, \mu \in \mathrm{Pic}(X)$, we call $m_{\lambda,\mu}$ the multiplication map from $\Gamma(X, \mathscr{L}_\lambda) \otimes \Gamma(X, \mathscr{L}_\mu)$ to $\Gamma(X, \mathscr{L}_{\lambda+\mu})$. The following lemma deals with a very special case of Theorem A.

LEMMA 3.1
Given a weight $\lambda \in \mathrm{Pic}^+(X)$*, set* $\mu \doteq -w_0\lambda$*. Then* $\mu \in \mathrm{Pic}^+(X)$ *and* $s^{\lambda+\mu} V_0^* \subset \mathrm{Im}\, m_{\lambda,\mu}$.

Proof
We can identify V_λ^* with V_μ and V_μ^* with V_λ. By Proposition 1.3, there exists $h_\lambda \in V_\lambda \setminus \{0\}$ such that $[h_\lambda] \in \mathbb{P}(V_\lambda)^H$. Hence there exists a one-dimensional H-submodule χ of V_λ. H being reductive, we have that χ^* is a submodule of V_μ. So there exists $h_\mu \in V_\mu \setminus \{0\}$ such that $[h_\mu] \in \mathbb{P}(V_\mu)^H$ and $\langle h_\lambda, h_\mu \rangle = 1$. By Proposition 1.3, we deduce that $\mu \in \mathrm{Pic}(X)$. Clearly $\mu \in \Lambda^+$; hence $\mu \in \mathrm{Pic}^+(X)$.

Now complete the vectors $h_\mu \in V_\lambda^*$ and $h_\lambda \in V_\mu^*$ to dual bases $h_\mu, v_1, \ldots, v_n$ and $h_\lambda, u_1, \ldots, u_n$. Consider the following element of $V_\lambda^* \otimes V_\mu^*$:

$$F = h_\mu \otimes h_\lambda + \sum_{i=1}^{n} v_i \otimes u_i.$$

If we identify $V_\lambda^* \otimes V_\mu^*$ with $\mathrm{End}(V_\lambda)$, the element F corresponds to the identity map; in particular, it is a G-invariant vector. Hence $f = m_{\lambda,\mu}(F) \in \Gamma(X, \mathscr{L}_{\lambda+\mu})$ is G-invariant. We claim that $f \neq 0$, proving the lemma. Indeed, consider the morphism $\psi : X \to \mathbb{P}(V_\lambda) \times \mathbb{P}(V_\mu)$ defined by $\psi(x) = (\psi_\lambda(x), \psi_\mu(x))$ (see Section 1 for the definition of ψ_λ, ψ_μ). Notice that

$$f(h_\lambda, h_\mu) = h_\mu(h_\lambda) h_\lambda(h_\mu) + \sum_{i=1}^{n} v_i(h_\lambda) u_i(h_\mu) = 1$$

since we have chosen dual bases. Hence the section f does not vanish on $([h_\lambda], [h_\mu]) \in \mathrm{Im}\, \psi$. □

We recall that a complete symmetric variety $X = X(\sigma)$ is said to be *simple* if $\mathfrak{g}$ has no σ stable proper ideal. It is known (see, e.g., [14, Table VI, Chapter X]) that a simple complete symmetric variety corresponds to an irreducible root system $\widetilde{\Phi}$ and that either G is simple or X is the compactification of a simple group. Further, any complete symmetric variety is the product of simple complete symmetric varieties.

We need to make some preliminary remarks on the various lattices and on the relations between the Weyl group W of Φ and the Weyl group $\widetilde{W}$ of $\widetilde{\Phi}$. As recalled in Section 1, the lattice Ω generated by the spherical weights is identified with the lattice of integral weights of the restricted root system $\widetilde{\Phi}$. Further, the set Ω^+ of spherical weights is equal to $\Lambda^+ \cap \Omega$ and corresponds to the dominant chamber defined by $\widetilde{\Delta}$. So the longest element $\widetilde{w}_0$ of the Weyl group $\widetilde{W}$ and the longest element w_0 of the Weyl group W act in the same way on Ω.

Before giving the proof of the surjectivity of $m_{\lambda,\mu}$, we introduce some notation and make some remarks to treat the exceptional case. If X is an exceptional simple complete symmetric variety, then $\widetilde{\Phi}$ is of type BC_ℓ and there exist exactly two (simple) exceptional roots that we denote by α and β. Also, we denote by $\widetilde{\alpha}_\ell$ the unique simple root in $\widetilde{\Delta}$ such that $2\widetilde{\alpha}_\ell \in \widetilde{\Phi}$ and by $\widetilde{\omega}_\ell$ the fundamental weight dual to $(2\widetilde{\alpha}_\ell)^\vee$ (see Section 4.2). We have $\mathrm{Pic}(X) = \Omega \oplus \mathbb{Z}\omega_\alpha$, $\mathrm{Pic}^+(X) = \Omega^+ + \mathbb{N}\,\omega_\alpha + \mathbb{N}\,\omega_\beta$, $\omega_\alpha + \omega_\beta = \widetilde{\omega}_\ell$, and $-w_0(\omega_\alpha) = \omega_\beta$ (see [6]). We recall also that if X is nonexceptional, then $\mathrm{Pic}(X) = \Omega$ and $\mathrm{Pic}^+(X) = \Omega^+$.

Finally, we notice that, given two weights $\lambda, \mu \in \mathrm{Pic}(X)$, we have $\mu \in (\lambda - \widetilde{R}^+)$ if and only if $\mu \leq \lambda$ with respect to the dominant order of $\widetilde{\Phi}$ (see Section 4.3 for the definitions in the exceptional case).

Now we come to the main result of the paper.

THEOREM A
Let λ, μ be two weights in $\mathrm{Pic}^+(X)$. Then the multiplication map

$$m_{\lambda,\mu} : \Gamma(X, \mathscr{L}_\lambda) \otimes \Gamma(X, \mathscr{L}_\mu) \to \Gamma(X, \mathscr{L}_{\lambda+\mu})$$

is surjective.

Proof
We prove first the case in which $\lambda, \mu \in \Omega^+$. We proceed by induction lexicographically on $\dim X$ and on the dominant order on λ and μ with respect to the restricted root system $\widetilde{\Phi}$. If $\dim X = 0$, that is, if X is a point, then $\lambda = \mu = 0$ and the claim is obvious. Also, if X is not simple, say, $X = X_1 \times X_2$ with $\dim X_1, \dim X_2 > 0$, then the claim follows by induction on the dimension using the description of the sections of the line bundles on X in Proposition 1.6. So we can assume that X is simple and hence that $\widetilde{\Phi}$ is irreducible.

We fix notation. Given a weight $\eta \in \mathrm{Pic}^+(X)$, set $\Lambda(\eta) = \Lambda^+ \cap (\eta - \widetilde{R}^+)$. Notice that clearly $\Gamma(X, \mathscr{L}_\eta) = \oplus s^{\eta-\nu} V_\nu^*$ where the sum runs over $\nu \in \Lambda(\eta)$. Our thesis is that $s^{\lambda+\mu-\nu}V_\nu^* \subset \operatorname{Im} m_{\lambda,\mu}$ for all $\nu \in \Lambda(\lambda+\mu)$. We divide the set of $\nu \in \Lambda(\lambda+\mu)$ into three different classes.

First class. This is the class of weights $\nu \in \Lambda(\lambda+\mu)$ such that the following condition is fulfilled: There exist $\lambda', \mu' \in \Omega^+$ such that $(\lambda', \mu') \neq (\lambda, \mu)$, $\lambda' \in \Lambda(\lambda)$, $\mu' \in \Lambda(\mu)$, and $\nu \in \Lambda(\lambda'+\mu')$. Consider the commutative diagram

$$\begin{array}{ccc} \Gamma(X, \mathscr{L}_{\lambda'}) \otimes \Gamma(X, \mathscr{L}_{\mu'}) & \xrightarrow{m_{\lambda',\mu'}} & \Gamma(X, \mathscr{L}_{\lambda'+\mu'}) \\ \downarrow{\scriptstyle s^{\lambda-\lambda'} \otimes s^{\mu-\mu'}} & & \downarrow{\scriptstyle s^{\lambda-\lambda'+\mu-\mu'}} \\ \Gamma(X, \mathscr{L}_{\lambda}) \otimes \Gamma(X, \mathscr{L}_{\mu}) & \xrightarrow{m_{\lambda,\mu}} & \Gamma(X, \mathscr{L}_{\lambda+\mu}) \end{array}$$

Notice that $m_{\lambda',\mu'}$ is surjective by induction on the dominant order on λ and μ. Also, $s^{\lambda+\mu-\nu}V_\nu^*$ is contained in the image of the right vertical map since $s^{\lambda'+\mu'-\nu}V_\nu^* \subset \Gamma(X, \mathscr{L}_{\lambda'+\mu'})$. So $s^{\lambda+\mu-\nu}V_\nu^*$ is contained in $\operatorname{Im} m_{\lambda,\mu}$.

Second class. This class is formed by the weights $\nu \in \Lambda(\lambda+\mu)$ such that $\lambda+\mu-\nu = \sum_{i=1}^{\ell} c_i \widetilde{\alpha}_i$ with an index i such that $c_i = 0$.

If we consider the restriction of sections to the stable subvariety $X_{\{\widetilde{\alpha}_i\}}$, we have the following commutative diagram:

$$\begin{array}{ccc} \Gamma(X, \mathscr{L}_{\lambda}) \otimes \Gamma(X, \mathscr{L}_{\mu}) & \xrightarrow{m_{\lambda,\mu}} & \Gamma(X, \mathscr{L}_{\lambda+\mu}) \\ \downarrow & & \downarrow \\ \Gamma(X_{\{\widetilde{\alpha}_i\}}, \mathscr{L}_{\lambda}) \otimes \Gamma(X_{\{\widetilde{\alpha}_i\}}, \mathscr{L}_{\mu}) & \xrightarrow{m_{\lambda,\mu}} & \Gamma(X_{\{\widetilde{\alpha}_i\}}, \mathscr{L}_{\lambda+\mu}) \end{array}$$

Notice that the bottom horizontal map is surjective by Proposition 2.9 and induction on dimension since $\dim(F_J) < \dim X$, where $J = \{\alpha \in \Delta_1 \mid \widetilde{\alpha} \neq \widetilde{\alpha}_i\} \cup \Delta_0$. Also notice that the left vertical map is surjective by Lemma 2.7. Finally, observe that $s^{\lambda+\mu-\nu}V_\nu^*$ appears in the decomposition of $\Gamma(X_{\{\widetilde{\alpha}_i\}}, \mathscr{L}_{\lambda+\mu})$ described in Proposition 2.4 since $c_i = 0$. Hence $s^{\lambda+\mu-\nu}V_\nu^* \subset \operatorname{Im} m_{\lambda,\mu}$ since G is reductive and all modules appear with multiplicity at most one.

Third class. This is the set of the remaining $\nu \in \Lambda(\lambda+\mu)$; what is left is described by the triples of weights (λ, μ, ν) of Ω^+ such that

(i) $\lambda' \in \Lambda(\lambda)$, $\mu' \in \Lambda(\mu)$, $\nu \in \Lambda(\lambda'+\mu')$ implies $\lambda' = \lambda$ and $\mu' = \mu$ and

(ii) $\lambda+\mu-\nu = \sum_{i=1}^{n} c_i \widetilde{\alpha}_i$ with $c_i \geq 1$ for all $i = 1, \ldots \ell$.

So (λ, μ, ν) is a low triple for the root system $\widetilde{\Phi}$.

By Theorem B applied to the root system $\widetilde{\Phi}$, we have $\nu = 0$ and $\mu = -w_0\lambda$. So we conclude using Lemma 3.1.

This finishes the proof for $\lambda, \mu \in \Omega^+$ and in particular the case of X nonexceptional. So we are reduced to studying the case of X exceptional and, using what has already been done, we can assume that the theorem is true for all $\lambda, \mu \in \Omega^+$. We proceed again by induction on $\dim X$ and on the dominant order on $\mathrm{Pic}^+(X)$ defined by $\widetilde{\Phi}$. As in the case of Ω^+, we can assume X simple.

We consider first a particular case. Let $\lambda = \omega_\alpha$, $\mu = \mu' + h\omega_\alpha$ for some $h \geq 0$ and $\mu' \in \Omega^+$. By Proposition 1.6 and Lemma 4.9, we have the decompositions

$$\begin{aligned} \Gamma(X, \mathscr{L}_{\omega_\alpha}) &= V^*_{\omega_\alpha}, \\ \Gamma(X, \mathscr{L}_{\mu}) &= \oplus_{\nu \in \Lambda(\mu)} s^{\mu - \nu} V^*_{\nu}, \\ \Gamma(X, \mathscr{L}_{\omega_\alpha + \mu}) &= \oplus_{\nu \in \Lambda(\mu)} s^{\mu + \omega_\alpha - \nu} V^*_{\omega_\alpha + \nu}. \end{aligned}$$

Denote by φ_η a lowest weight vector of the module V^*_η. So to prove the surjectivity of $m_{\omega_\alpha, \mu}$ in this situation, we observe that $\varphi_{\omega_\alpha} \otimes s^{\mu-\nu}\varphi_\nu$ is a nonzero lowest weight vector of weight $\omega_\alpha + \mu$ and hence a multiple of $s^{\mu-\nu}\varphi_\nu$.

Now we proceed as in the case of Ω^+. The arguments given in the “first class” and in the “second class” above hold without any change, and so we are reduced again to studying the low triples of weights (λ, μ, ν) of $\mathrm{Pic}^+(X)$ with respect to the dominant order defined by $\widetilde{\Phi}$. Hence, by Corollary 4.11, we have, up to symmetry, $\lambda = a\omega_\alpha$ and $\mu = b\omega_\beta$ for some integers $a \geq b \geq 1$.

We analyze first the particular case $a = b = 1$. In this case, by Corollary 4.11, we have $\nu = 0$. Hence we can apply Lemma 3.1 since $\omega_\beta = -w_0\omega_\alpha$ and conclude that $s^{\widetilde{\omega}_\ell} V^*_0$ is contained in the image of $m_{\omega_\alpha, \omega_\beta}$, as claimed.

Now we deduce the surjectivity in the remaining cases of $\lambda = a\omega_\alpha$ and $\mu = b\omega_\beta$ with $a \geq b \geq 1$ using the associativity of multiplication. We consider the commutative diagram

$$\begin{array}{ccc}
\Gamma(X, \mathscr{L}_{\omega_\alpha})^{\otimes(a-b)} \otimes \big(\Gamma(X, \mathscr{L}_{\omega_\alpha}) \otimes \Gamma(X, \mathscr{L}_{\omega_\beta})\big)^{\otimes b} & =\!= & \Gamma(X, \mathscr{L}_{\omega_\alpha})^{\otimes a} \otimes \Gamma(X, \mathscr{L}_{\omega_\beta})^{\otimes b} \\
\downarrow{\scriptstyle m_1} & & \downarrow \\
\Gamma(X, \mathscr{L}_{\omega_\alpha})^{\otimes(a-b)} \otimes \big(\Gamma(X, \mathscr{L}_{\widetilde{\omega}_\ell})\big)^{\otimes b} & & \Gamma(X, \mathscr{L}_{a\omega_\alpha}) \otimes \Gamma(X, \mathscr{L}_{b\omega_\beta}) \\
\downarrow{\scriptstyle m_2} & & \downarrow{\scriptstyle m_{a\omega_\alpha, b\omega_\beta}} \\
\Gamma(X, \mathscr{L}_{\omega_\alpha})^{\otimes(a-b)} \otimes \Gamma(X, \mathscr{L}_{b\widetilde{\omega}_\ell}) & \xrightarrow{\ m_3\ } & \Gamma(X, \mathscr{L}_{(a-b)\omega_\alpha + b\widetilde{\omega}_\ell})
\end{array}$$

where

(i) $m_1 = \mathrm{id} \otimes m^{\otimes b}_{\omega_\alpha, \omega_\beta}$ is surjective using what has been proved in the particular case $a = b = 1$;

(ii) m_2 is the multiplication of the sections in $\Gamma(X, \mathscr{L}_{\widetilde{\omega}_\ell})^{\otimes b}$ which is surjective using what has been proved for weights in Ω^+;

(iii) m_3 is the multiplication of sections in $\Gamma(X, \mathscr{L}_{\omega_\alpha})^{\otimes(a-b)}$ and $\Gamma(X, \mathscr{L}_{b\widetilde{\omega}_\ell})$ which is surjective using recursively what has been proved in the case $\lambda = \omega_\alpha$ and $\mu = \mu' + h\omega_\alpha$ analyzed above with $h = 1, 2, \dots, a-b-1$.

Hence $m_{a\omega_\alpha, b\omega_\beta}$ is also surjective. □

COROLLARY 3.2

For all $\tilde{I} \subset \widetilde{\Delta}$ and for all $\lambda, \mu \in \mathrm{Pic}^+(X_{\tilde{I}}) = \Lambda^+ \cap \mathrm{Pic}(X_{\tilde{I}})$, we have that the multiplication map

$$\Gamma(X_{\tilde{I}}, \mathscr{L}_\lambda) \otimes \Gamma(X_{\tilde{I}}, \mathscr{L}_\mu) \to \Gamma(X_{\tilde{I}}, \mathscr{L}_{\lambda+\mu})$$

is surjective.

Proof

This follows at once by Theorem A and Proposition 2.9. □

As a consequence (see, e.g., Hartshorne [12, Exercise II.5.14]), we have the following.

COROLLARY 3.3

Let $\tilde{I}$ be a subset of $\widetilde{\Delta}$, and let $\mathscr{L}_\lambda \in \mathrm{Pic}^+(X_{\tilde{I}})$ be a dominant line bundle. Consider the map $X_{\tilde{I}} \to \mathbb{P}(\Gamma(X_{\tilde{I}}, \mathscr{L}_\lambda)^*)$ defined by the line bundle $\mathscr{L}_\lambda$. Then the cone over the image of $X_{\tilde{I}}$ is normal. In particular, this applies to X.

We see an example showing the necessity to assume that the line bundles are dominant for the surjectivity of the multiplication map. Simply take a complete symmetric variety with $\widetilde{\Phi}$ of type A_2, and let $\lambda = \widetilde{\alpha}_1$, $\mu = \widetilde{\alpha}_2$. Then $\mathrm{Im}\, m_{\lambda,\mu} = s_1 s_2 V_0^*$, whereas $\Gamma(X, \mathscr{L}_{\lambda+\mu}) = V^*_{\widetilde{\omega}_1+\widetilde{\omega}_2} \oplus s_1 s_2 V_0^*$. As another example with $\lambda = \mu$, consider the case of the compactification of the group of type C_ℓ, and let $\lambda = \mu = -\omega_{\ell-1} + \omega_\ell$. Then $\Gamma(X, \mathscr{L}_\lambda) = \Gamma(X, \mathscr{L}_\mu) = 0$, while $\Gamma(X, \mathscr{L}_{\lambda+\mu}) = s_\ell V_0^*$, as one can easily see by Proposition 1.6.

We finish this section with some remarks about the positive characteristic case. The wonderful compatification of De Concini and Procesi has been constructed by De Concini and Springer also over $\mathbb{Z}[1/2]$ in [10]. In particular, it can be defined over any field of characteristic different from 2. Although the description of the boundary, of the Picard group, and of the sections of a line bundle does not change substantially, Theorem A is false in general. The following counterexample has appeared for the first time (in a characteristic 2 version) in [8] and appears also in [7]. We report it here for the convenience of the reader.

Let X be the wonderful compactification of $\mathbb{P}\mathrm{SL}(6)$ over a field $\Bbbk$ of characteristic 3; this corresponds to the case where $G = \mathrm{SL}(6) \times \mathrm{SL}(6)$ and $\sigma(x, y) = (y, x)$.

Observe that any line bundle on X trivializes on the cover SL(6) of the open subset $\mathbb{P}$SL(6) of X. Hence we may identify $\Gamma(X, \mathscr{L}_\lambda)$ with a submodule of $\Bbbk$[SL(6)], and the multiplication of sections corresponds to the usual multiplication of functions over SL(6) (see Kannan [15] for details on this identification).

Given two sequences $1 \le i_1 < i_2 < \cdots < i_h \le 6$ and $1 \le j_1 < j_2 < \cdots < j_h \le 6$, we denote by $[i_1, i_2, \ldots, i_h | j_1, j_2, \ldots, j_h]$ the function on SL(6) given by the determinant of the minor formed by the rows $i_1, i_2, \ldots, i_h$ and the columns $j_1, j_2, \ldots, j_h$; such a minor is said to be of *rank* h. We may identify the space $\Gamma(X, \mathscr{L}_{\omega_3})$ with the subspace of $\Bbbk$[SL(6)] spanned by the determinants of the minors of rank 3 and $\Gamma(X, \mathscr{L}_{2\omega_3})$ with the subspace spanned by the products of the determinant of a minor of rank $3-h$ and the determinant of a minor of rank $3+h$ with $0 \le h \le 3$.

So the function $[1, 2|1, 2][3, 4, 5, 6|3, 4, 5, 6]$ is an element of $\Gamma(X, \mathscr{L}_{2\omega_3})$. But, using the computer program Macaulay (see [1]), it is easy to see that such a function does not belong to the vector space generated by the products of the determinants of two minors of rank 3. Hence the multiplication map $\Gamma(X, \mathscr{L}_{\omega_3})^{\otimes 2} \to \Gamma(X, \mathscr{L}_{2\omega_3})$ is not surjective.

By the construction of De Concini and Springer and by semicontinuity, it is clear that Theorem A holds for almost all primes once the type of involution has been fixed. It should be interesting, but probably difficult, to understand for which primes Theorem A is true. Results in this direction for the compactification of $\mathbb{P}$SL are implicit in [5], [3], and [4].

4. Low triples

In this section we introduce and study low triples for an irreducible root system. As seen in Section 3, these are the triples of dominant weights that furnish the base step for the inductive proof of the surjectivity of the multiplication map. In the first paragraph we consider only reduced root systems, developing a bit of general theory as far as we are able to. Then we consider the case of a nonreduced root system that presents no extra difficulty using what has already been done in the first paragraph. Finally, we give a little more general and technical result concerning low triples for an enlarged weight lattice for type BC; this is needed for exceptional complete symmetric varieties.

4.1. Reduced root system

Let Φ be an irreducible reduced root system with base Δ, let Φ^+ be the associated set of positive roots, and let Λ be the lattice of integral weights for Φ. For a simple root α, ω_α is the fundamental weight dual to $\alpha^\vee$. The lattice of dominant weights is denoted by Λ^+, and we order such a set with the usual dominant order: $\mu \le \lambda$ if and only if $\lambda - \mu \in R^+$, where R^+ is the monoid generated by the simple roots and R is the lattice generated by the simple roots. If w_0 is the longest element of the Weyl group of Φ, we say that two dominant weights λ, μ are *dual* to each other if $\mu = -w_0\lambda$. We number the simple roots $\alpha_1, \ldots, \alpha_\ell$ as the tables in [2, pages 250–275] and $\omega_1, \ldots, \omega_\ell$ are the corresponding fundamental weights. Also, we set $\zeta = \alpha_1 + \alpha_2 + \cdots + \alpha_\ell$.

Given a weight $\lambda = \sum_{i=1}^{\ell} a_i\omega_i$, we define its *support* as the set $\mathrm{supp}(\lambda)$ of fundamental weights ω_i such that $a_i \neq 0$, and we define $\mathrm{supp}^+(\lambda)$ as the set of fundamental weights ω_i with $a_i > 0$. Sometimes we identify the elements of $\mathrm{supp}^+(\lambda)$ with the corresponding vertices of the Dynkin diagram of Φ.

As we may see by Definition 1 given in the introduction, a low triple (λ, μ, ν) is of a very special kind; indeed, ν must be very close to $\lambda + \mu$ by Definition 1(i), whereas ν must be quite far from $\lambda + \mu$ by Definition 1(ii). The rest of this section is devoted to giving precise meaning to this idea, proving the following.

THEOREM B
The triple (λ, μ, ν) is a low *triple if and only if λ and μ are minuscule weights dual to each other and $\nu = 0$.*

For $I \subset \Delta$, let Φ_I denote the root subsystem generated by the simple roots α for $\alpha \in I$. Given a subset $I \subset \Delta$, we say that I is irreducible of type A, B, … if Φ_I is, and we identify I with its type if this does not raise any confusion.

If Φ_I is irreducible, we denote by θ_I the highest short root (we notice that in [17] the root θ_I is called the *local short dominant root* relative to I), considering all roots short if Φ_I has just one root length. In particular, $\theta \doteq \theta_\Delta$ denotes the highest short root of Φ.

Given two weights $\lambda, \mu \in \Lambda^+$, we say that λ *covers* μ if

(i) $\mu < \lambda$ and

(ii) $\mu \leq \eta \leq \lambda$, $\eta \in \Lambda^+$, imply $\eta = \mu$ or $\eta = \lambda$.

Of particular importance for the sequel is the following characterization of the covering relation for the dominant order $\leq$ restricted to the dominant weights (see [17] for details).

PROPOSITION 4.1 ([17, Theorem 2.6])
If λ covers μ, then either $\lambda - \mu = \theta_I$ for an irreducible $I \subset \Delta$, or $\Phi \simeq \mathsf{G}_2$ and $\lambda - \mu = \alpha_1 + \alpha_2$.

We explicitly write down the highest short root for each type of root system. This is an easy computation using the tables in [2]. We make the following remarks reading out of Table 1. If $I \subset \Delta$ is irreducible and not of type A, then there exists a unique fundamental weight in Φ_I, denoted by ω_I, such that $\theta_I = \omega_I - \eta$ with η a dominant

Table 1

type of Φ	*highest short root*
A_ℓ	$\alpha_1 + \cdots + \alpha_\ell = \omega_1 + \omega_\ell$
B_ℓ	$\alpha_1 + \cdots + \alpha_\ell = \omega_1$
C_ℓ	$\alpha_1 + 2(\alpha_2 + \cdots + \alpha_{\ell-1}) + \alpha_\ell = \omega_2$
D_ℓ	$\alpha_1 + 2(\alpha_2 + \cdots + \alpha_{\ell-2}) + \alpha_{\ell-1} + \alpha_\ell = \omega_2$
E_6	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 = \omega_2$
E_7	$2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 = \omega_1$
E_8	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8 = \omega_8$
F_4	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 = \omega_4$
G_2	$2\alpha_1 + \alpha_2 = \omega_1$

weight with support in $\Delta \setminus I$. We stress that such a weight ω_I is defined only for Φ_I of type *not* A.

Another remark is the following: If I and J are subsets of Δ not of type A, then $\theta_I \le \theta_J$ if and only if $I \subset J$.

Our first simple observation about low triples is the following.

LEMMA 4.2

Let (λ, μ, ν) *be a* low *triple, and let* $I \subset \Delta$ *with* $I \neq \emptyset$. *Then we have the following:*

(i) $\lambda - \sum_{i \in I} \alpha_i \notin \Lambda^+$ *and* $\mu - \sum_{i \in I} \alpha_i \notin \Lambda^+$;

(ii) *if* I *is irreducible and* $\lambda + \mu - \nu \ge \theta_I$, *then* $\lambda - \theta_I \notin \Lambda^+$ *and* $\mu - \theta_I \notin \Lambda^+$.

Proof

For the first claim, set $\lambda' \doteq \lambda - \sum_{i \in I} \alpha_i$, and suppose $\lambda' \in \Lambda^+$. Then $\lambda' + \mu = \lambda - \sum_{i \in I} \alpha_i + \mu \ge \nu + (\zeta - \sum_{i \in I} \alpha_i) \ge \nu$ against the first condition for a low triple, and similarly for μ.

For the second claim, set $\lambda' \doteq \lambda - \theta_I$, and suppose $\lambda' \in \Lambda^+$. Our hypothesis gives $\lambda' + \mu \ge \nu$, and we conclude as above. □

If (λ, μ, ν) is a low triple, we consider the weight $\beta \doteq \lambda + \mu - \nu$. Our strategy is to find out some crucial properties of such a weight β (Lemma 4.3) and then to study all weights with such properties (Proposition 4.4).

We write $\beta = \sum_{i=1}^{\ell} b_i \alpha_i = \sum_{i=1}^{\ell} a_i \omega_i$. For a weight expressed in this form, we define the *positive height* as

$$\mathrm{ht}^+(\beta) \doteq \max_I \sum_{i \in \mathrm{supp}^+(\beta) \cap I} a_i,$$

where I ranges over all connected simply laced subsystems of Δ.

The property of β we investigate is given in the following lemma.

LEMMA 4.3
Given a low triple (λ, μ, ν), set $\beta \doteq \lambda + \mu - \nu$. Then the weights λ and μ are either fundamental or the sum of two fundamental weights dual to simple roots of different lengths. In particular, $\mathrm{ht}^+(\beta) \le 2$.

Proof
If we suppose the contrary of the first claim, we have two possibilities up to symmetry: either $\lambda = a_\gamma \omega_\gamma + \eta$ for some $\gamma \in \Delta$, $\eta \in \Lambda^+$, and $a_\gamma \ge 2$, or $\lambda = a_\gamma \omega_\gamma + a_\delta \omega_\delta + \eta$ for some $\gamma, \delta \in \Delta$ with γ, δ of the same length, $\eta \in \Lambda^+$, and $a_\gamma, a_\delta \ge 1$. In the first case, $\lambda - \gamma \in \Lambda^+$, and by Lemma 4.2, this is impossible, (λ, μ, ν) being a low triple.

In the second case, notice that γ and δ are joined by a type A segment I in the Dynkin diagram of Φ. Then $\lambda - \theta_I = \lambda - \omega_\gamma - \omega_\delta + \eta'$ for some $\eta' \in \Lambda^+$; hence $\lambda - \theta_I \in \Lambda^+$. Further, $\beta \ge \theta_I = \sum_{i \in I} \alpha_i$ by the second property of low triples. By Lemma 4.2, this is impossible.

The claimed property of β follows now by $\mathrm{ht}^+(\beta) \le \mathrm{ht}^+(\lambda) + \mathrm{ht}^+(\mu)$. □

We can now make the main step toward the proof of Theorem B.

PROPOSITION 4.4
Suppose that Φ is not of type G_2. If a weight $\beta \ge \zeta$ is such that $\mathrm{ht}^+(\beta) \le 2$*, then either $\beta \ge \theta$ or there exists an irreducible, not of type A subset I of Δ such that $\omega_I \in \mathrm{supp}^+(\beta)$ and $\beta \ge \theta_I$.*

Proof for type not E
We perform a case-by-case check.

Type A_ℓ *and* B_ℓ. For these types we have $\beta \ge \zeta = \theta$ by the definition of low triple.

Type C_ℓ. We may assume $\beta \not\ge \theta$. Notice that $\beta \ge \theta_{\mathsf{C}_2}$ by $\beta \ge \zeta$. So let h be maximal such that $\beta \ge \theta_{\mathsf{C}_h}$ and $\beta \not\ge \theta_{\mathsf{C}_{h+1}}$, and set $t \doteq \ell + 1 - h$. Notice that we clearly have $\beta \ge \theta_{\mathsf{C}_k}$ for all $k = 2, 3, \ldots, h$. Further, $\beta \not\ge \theta_{\mathsf{C}_{h+1}}$ gives $b_t = 1$; also, $b_{t+1} \ge 2$ if $h \ge 3$. Hence

$$a_{t+1} + a_{t+2} + \cdots + a_\ell = \begin{cases} -1 + b_{t+1} & \text{if } h \ge 3, \\ -1 + 2b_\ell & \text{if } h = 2. \end{cases}$$

In any case, this sum is positive; hence there exists a $t + 1 \le j \le \ell$ such that $a_j > 0$. So $\omega_j = \omega_{\mathsf{C}_{\ell+2-j}} \in \mathrm{supp}^+ \beta$ and $\beta \ge \theta_{\mathsf{C}_h} \ge \theta_{\mathsf{C}_{\ell+1-j}}$.

Type D_ℓ. We may suppose $\beta \not\geq \theta$. First assume also $\beta \geq \theta_{\mathsf{D}_4}$. So let $h \geq 4$ be maximal such that $\beta \geq \theta_{\mathsf{D}_h}$ and $\beta \not\geq \theta_{\mathsf{D}_{h+1}}$. Setting $t \doteq \ell+1-h$, we have $b_t = 1$ and $b_{t+1} \geq 2$. Also, $\beta \geq \theta_{\mathsf{D}_k}$ for $k = 4, 5, \ldots, h$.

Suppose that $a_{\ell-1}+a_\ell = 2(b_\ell+b_{\ell-1}-b_{\ell-2})$ is not positive. Then $a_{t+1}+a_{t+2}+\cdots+a_{\ell-2} = -b_t + b_{t+1} + b_{\ell-2} - b_{\ell-1} - b_\ell \geq 1$, and we conclude as in type C_ℓ.

On the other hand, if $a_{\ell-1}+a_\ell$ is positive, it is at least 2. But $a_1+a_2+\cdots+a_{t-1} = b_1 + b_{t-1} - 1 \geq 1$; that now is impossible since $\mathrm{ht}^+(\beta) \leq 2$.

If $\beta \not\geq \theta_{\mathsf{D}_4}$, we have $b_{\ell-2} = 1$; hence $a_{\ell-1} + a_\ell \geq 2$ and we finish as above with $t = \ell - 2$.

Type F_4. We first observe that $\beta \geq \zeta > \theta_{\mathsf{B}_3} > \theta_{\mathsf{B}_2}$; hence we may assume that $a_1, a_2 \leq 0$ since $\omega_1 = \omega_{\mathsf{B}_3}$ and $\omega_2 = \omega_{\mathsf{B}_2}$.

If $\beta \not\geq \theta_{\mathsf{C}_3}$, we have $b_3 = 1$, and so $a_3 = 2-b_4-2b_2 \leq 0$. Otherwise, if $\beta \geq \theta_{\mathsf{C}_3}$, we may assume as well $a_3 \leq 0$ since $\omega_3 = \omega_{\mathsf{C}_3}$. In any case, $\beta = a_4\omega_4 - \eta$ for some $a_4 \leq 2$ by $\mathrm{ht}^+(\beta) \leq 2$, and $\eta \in \Lambda^+$ orthogonal to α_4. Notice also that $\eta \in R^+$ since $\omega_4 = \theta \in R$. Using $\beta \geq \zeta$, we find

$$\eta \leq (a_4 - 1)\alpha_1 + (2a_4 - 1)\alpha_2 + (3a_4 - 1)\alpha_3 + (2a_4 - 1)\alpha_4.$$

Then in η the coefficient of α_1 is at most 1. Using the tables in [2, pages 250–275], this forces $\eta = 0$, η being orthogonal to α_4. Hence $\beta = a_4\omega_4 \geq \theta$. □

Although the same line of proof may be used for type E, we prefer to follow a different strategy using a suggestion by T. A. Springer.

We begin by introducing some notation for the two lemmas below. Notice that a Dynkin diagram of type E is a tree with three branches and one central node. When we fix a branch, we number its vertices from the endpoints to the node as $1, 2, \ldots, n$, where n is the length of the given branch; we accordingly number the roots, the fundamental weights, the coefficients b_i of β with respect to the simple root α_i in the given branch, and the coefficients a_i of β with respect to the fundamental weights ω_i of the given branch.

LEMMA 4.5

Fix a branch of the Dynkin diagram of Φ.

(i) *Assume that no element of* $\mathrm{supp}^+(\beta)$ *lies on the given branch. Then* $b_i \geq i$ *for* $1 \leq i \leq n$.

(ii) *Assume that only the vertex* h *of the given branch is in* $\mathrm{supp}^+(\beta)$*, and assume that* $a_h = 1$*. Then* $b_i \geq i$ *for* $i \leq h$ *and* $b_i \geq h$ *for* $h \leq i \leq n$.

(iii) *Assume that* $\mathrm{supp}^+(\beta)$ *intersects the given branch in* h *and* k *with* $1 \leq h < k \leq n$*, and that* $a_h = a_k = 1$*. Assume further that there exists* $h \leq t < k$ *such that* $b_t < b_{t+1}$*; then* $b_h \leq b_{k-1} < b_k \leq b_{k+1} \leq b_{k+2} \leq \cdots \leq b_n$.

Proof
Notice that $a_i = 2b_i - b_{i-1} - b_{i+1}$ for $i = 1, \ldots, n-1$, setting $b_0 = 0$. The hypotheses are
(i) $a_i \leq 0$ for all $i = 1, \ldots, n-1$;
(ii) $a_i \leq 0$ for all $i = 1, \ldots, n-1$ but $i = h$, in which case $a_h = 1$;
(iii) $a_i \leq 0$ for all $i = 1, \ldots, n-1$ but $i = h$ and $i = k$, in which case $a_h = a_k = 1$.
Now the claims follow easily by induction. □

The following intermediate result is due to Springer.

LEMMA 4.6
Suppose that Φ *is of type* E *and that a weight* $\beta \geq \zeta$ *is such that* $\mathrm{ht}^+(\beta) \leq 2$. *Then either there exists a fundamental weight of the form* ω_I *with* $\beta \geq \theta_I$, *or* $\mathrm{supp}^+(\beta)$ *is formed by endpoints of the Dynkin diagram and* $\mathrm{ht}^+(\beta) = 2$.

Proof
If $\mathrm{ht}^+(\beta) = 1$, then $\beta = \omega_\alpha - \eta$ with $\eta \in \Lambda^+$ orthogonal to α. Notice that $\omega_\alpha - \beta \in \Lambda^+$ and $\omega_\alpha > \omega_\alpha - \beta$ since $\beta \in R^+ \setminus \{0\}$. By Proposition 4.1, there exists $I \subset \Delta$ such that $\omega_\alpha > \omega_\alpha - \theta_I \geq \omega_\alpha - \beta$ and $\omega_\alpha - \theta_I \in \Lambda^+$. We get $\beta \geq \theta_I$; also, I is not of type A and $\omega_I = \omega_\alpha$ because otherwise $\omega_\alpha - \theta_I \notin \Lambda^+$.

For the general case, observe that any fundamental weight that is not an endpoint is of the form ω_I for a suitable subset $I \subset \Delta$ of type D. We now study the various configurations of the elements of $\mathrm{supp}^+(\beta)$ with respect to the three branches.

First notice that there exists a branch that does not intersect $\mathrm{supp}^+(\beta)$ since $\mathrm{ht}^+(\beta) \leq 2$. So, by Lemma 4.5(i) and $\beta \geq \zeta$, we have $\beta \geq \theta_{\mathsf{D}_4}$. So if the node is contained in $\mathrm{supp}^+(\beta)$, we have finished since ω_{D_4} is the node.

Assume that we have a branch intersecting $\mathrm{supp}^+(\beta)$ only in h, with $a_h = 1$, and assume that h is not an endpoint, so $h \geq 2$. Then $b_i \geq 2$ for $i \geq 2$ by Lemma 4.5(ii). This gives $\beta \geq \theta_I$ for I the type D subset comprising the roots $\alpha_{h-1}, \alpha_h, \ldots, \alpha_n$ and the other two roots adjacent to α_n in the Dynkin diagram. Also, $\omega_h = \omega_I$, and the lemma is proved.

So we may assume that $\mathrm{supp}^+(\beta)$ is contained in a branch with $\beta = \omega_h + \omega_k - \eta$ for some $\eta \in \Lambda^+$ orthogonal to α_h, α_k and $1 \leq h \leq k \leq n$. Also, if $h = k = 1$, then $\mathrm{supp}^+(\beta)$ is contained in the endpoints, so we can suppose $k > 1$. Further, we can suppose $k < n$ since we have already disposed of the node. We claim that in this situation $\beta \geq \theta_I$ with $\omega_I = \omega_k$.

By Lemma 4.5, we have $1 \leq b_1 < b_2 < \cdots < b_h \leq \cdots \leq b_k$. We want to show that $b_k \geq 2$. Indeed, supposing otherwise, we have $b_{k-1} = 1$, and using $a_k \geq 1$, we get $b_{k+1} \leq 0$, contrary to $\beta \geq \zeta$.

Assume now that there exists $h \le t < k$ satisfying the hypotheses of part (iii) of Lemma 4.5. We find $b_i \ge 2$ for $i = k, k+1, \dots, n$, and so $\beta \ge \theta_I$ with I the type D subset comprising $\alpha_{k-1}, \alpha_k, \dots, \alpha_n$ and the other two roots adjacent to α_n. Hence $\omega_k = \omega_I$, and the claim follows.

So we may suppose that $b_h = b_{h+1} = \cdots = b_k \ge 2$. Let $\beta' = \beta - (\alpha_h + \cdots + \alpha_k)$. Then $\beta' \ge \zeta$, and it is such that $\mathrm{ht}^+(\beta') \le 2$. By descending induction on k, whose base step $k = n$ (the node) has already been proved, we may assume $\beta' \ge \theta_J$ with $\omega_J = \omega_{k+1}$. But then $\beta \ge \theta_J + \alpha_h + \cdots + \alpha_k \ge \theta_I$ with $\omega_I = \omega_k$, and the claim is proved. □

We return to the global numbering of the simple roots, and we introduce other notation. If λ covers μ in Λ^+ and $\lambda - \mu = \theta_I$, we write $\lambda \xrightarrow{I} \mu$ (Proposition 4.1). A *covering diagram* for a weight $\lambda \in \Lambda^+$ is a direct graph whose vertexes are dominant weights $\eta \le \lambda$ and whose arrows are the covering relations. If a covering diagram contains all weights $\eta \le \lambda$, then we say it is *complete*. We give an example. Let Φ be of type B_3, and let $\lambda = 2\omega_3$; then its complete covering diagram is

$$2\omega_3 \xrightarrow{\alpha_\ell} \omega_2 \xrightarrow{\mathsf{B}_2} \omega_1 \xrightarrow{\mathsf{B}_3} 0$$

since $2\omega_3 - \omega_2 = \alpha_\ell$, $\omega_2 - \omega_1 = \theta_{\{\alpha_2, \alpha_3\}}$, and $\omega_1 = \theta$. Also notice that it is easy to write down the complete covering diagram of a given dominant weight; use Proposition 4.1 and the remark after the list of highest short roots. Finally, given a linearly ordered covering diagram

$$\lambda_1 \xrightarrow{I_1} \lambda_2 \xrightarrow{I_2} \cdots \xrightarrow{I_r} \lambda_{r+1},$$

we have $\lambda_1 - \lambda_{r+1} \ge \zeta$ if and only if $\bigcup_{i=1}^r I_i = \Delta$.

Proof of Proposition 4.4 for type E
By Lemma 4.6, we may assume that $\mathrm{supp}^+(\beta)$ is formed by endpoints of the Dynkin diagram and that $\mathrm{ht}^+(\beta) = 2$. In the sequel we write $\beta = 2\omega_h - \eta$ or $\beta = \omega_h + \omega_k - \eta$ if, respectively, $\mathrm{supp}^+(\beta) = \{h\}$ or $\mathrm{supp}^+(\beta) = \{h, k\}$, with $\eta \in \Lambda^+$, $\mathrm{supp}(\eta) \subset \Delta \setminus \mathrm{supp}^+(\beta)$.

Suppose $\beta = 2\omega_1 - \eta$. First notice that $\eta \le 2\omega_1$ since $\beta \in R^+$. If $\beta \ge \theta_{\mathsf{E}_7}$, we have $\omega_{\mathsf{E}_7} = \omega_1 \in \mathrm{supp}^+(\beta)$, proving the proposition. So we may assume $\beta \not\ge \theta_{\mathsf{E}_7}$; we want to show that this is not possible. Indeed, the weight η should verify

(i) $\eta \le 2\omega_1 - \zeta$ since $\beta \ge \zeta$, and

(ii) $\eta \not\le 2\omega_1 - \theta_{\mathsf{E}_7}$ since $\beta \not\ge \theta_{\mathsf{E}_7}$.

For type E_8, the covering diagram of $2\omega_1$ begins as

$$2\omega_1 \xrightarrow{\alpha_1} \omega_3 \xrightarrow{\mathsf{D}_5} \omega_6 \xrightarrow{\mathsf{D}_6} \omega_1 + \omega_8,$$

and in $\rho \doteq \omega_1+\omega_8$ we have already $2\omega_1-\rho = \alpha_1+\theta_{\mathsf{D}_5}+\theta_{\mathsf{D}_6} = \theta_{\mathsf{E}_7}$, but $2\omega_1-\rho \not\geq \zeta$. So there is no weight η with the properties (i) and (ii) above. For type E_7, the same proof with $\omega_8 = 0$ in the diagram above works. For type E_6, the covering diagram stops in the minuscule weight ω_6, and so no weight η verifies (i); this disposes also of the case $\mathrm{supp}^+(\beta) = \{\omega_6\}$ by symmetry.

If $\mathrm{supp}^+(\beta) = \{\omega_1, \omega_7\}$ or $\mathrm{supp}^+(\beta) = \{\omega_1, \omega_8\}$, we conclude in the same way. Further, if $\omega_2 \in \mathrm{supp}^+(\beta)$, we may argue as above with θ_{E_6} instead of θ_{E_7} in (ii). Also, the case $\mathrm{supp}^+(\beta) = \{\omega_8\}$ is similar with θ_{E_8} in (ii).

So we are left with two cases, for which we claim that $\beta \geq \theta$. The first is $\mathrm{supp}^+(\beta) = \{\omega_7\}$ for E_7. Using the same strategy as above with only property (i), we consider the complete covering diagram

$$2\omega_7 \xrightarrow{\alpha_7} \omega_6 \xrightarrow{\mathsf{D}_6} \omega_1 \xrightarrow{\mathsf{E}_7} 0.$$

So the unique weight $\eta \leq 2\omega_7 - \zeta$ is 0; hence $\beta = 2\omega_7 \geq \theta$.

Finally, for type E_6 and $\mathrm{supp}^+(\beta) = \{\omega_1, \omega_6\}$, we have the complete covering diagram

$$\omega_1 + \omega_6 \xrightarrow{\mathsf{A}_5} \omega_2 \xrightarrow{\mathsf{E}_6} 0.$$

So also in this case, $\eta = 0$ is the unique weight with $\eta \leq \omega_1 + \omega_6 - \zeta$, and we get $\beta = \omega_1 + \omega_6 \geq \theta$. □

We return to a generic reduced root system.

COROLLARY 4.7
Let (λ, μ, ν) be a low triple, and set $\beta \doteq \lambda+\mu-\nu$. If Φ is not of type G_2, then $\beta \geq \theta$.

Proof
If in the conclusion of Proposition 4.4 we have an irreducible, not of type A subset I such that $\omega_I \in \mathrm{supp}^+(\beta)$, then either $\omega_I \in \mathrm{supp}^+(\lambda)$ or $\omega_I \in \mathrm{supp}^+(\mu)$. By symmetry, we can suppose $\omega_I \in \mathrm{supp}^+(\lambda)$. Hence $\lambda - \theta_I \in \Lambda^+$; further, $\beta = \lambda + \mu - \nu \geq \theta_I$, and this is impossible by Lemma 4.2. □

We can now prove the main result of this section.

Proof of Theorem B
We show first that any triple of the form $(\lambda, -w_0\lambda, 0)$ with λ minuscule is a low triple. Since λ is minuscule, $-w_0\lambda$ is also minuscule and the property (i) in Definition 1 is obvious. We have also $w_0\lambda < \lambda$ since λ is dominant. Hence $\lambda - w_0\lambda = \beta$, where β is some nonnegative linear combination of simple roots, say, $\beta = \sum_{\alpha\in\Delta} b_\alpha \alpha$. We want to show that $b_\alpha > 0$ for all simple roots α, proving property (ii) in Definition 1.

Suppose that this is not the case, and let γ be a simple root with $b_\gamma = 0$ and with γ adjacent in the Dynkin diagram to a simple root δ with $b_\delta > 0$. Then in $\sum_{\alpha\in\Delta} b_\alpha \alpha$ the fundamental weight ω_γ, dual to $\gamma^\vee$, appears with a negative coefficient; this is not possible since λ and $-w_0\lambda$ are dominant.

Let us see the other implication. First we prove that λ and μ are minuscule weights if Φ is not of type G_2. If we suppose that λ is not minuscule, by Proposition 4.1, there exists an irreducible I with $\lambda - \theta_I \in \Lambda^+$. Also, by Corollary 4.7, we have $\beta \geq \theta \geq \theta_I$; that is impossible by Lemma 4.2.

We perform now a case-by-case analysis. We see the details only for types A, B, and G_2 since the other types are similar.

Type A_ℓ. In this case, λ and μ are fundamental weights, say, $\lambda = \omega_i$, $\mu = \omega_j$ for some $1 \leq i, j \leq \ell$. Since $\lambda + \mu - \nu \geq \zeta$, the complete covering diagram of $\lambda + \mu$ must be of the form

$$\omega_i + \omega_j \xrightarrow{\mathsf{A}(i,j)} \omega_{i-1} + \omega_{j+1} \xrightarrow{\mathsf{A}(i-1,j+1)} \cdots \xrightarrow{\mathsf{A}(2,\ell-1)} \omega_1 + \omega_\ell \xrightarrow{\mathsf{A}(1,\ell)} 0,$$

where $\mathsf{A}(h,k) = \{\alpha_h, \alpha_{h+1}, \ldots, \alpha_k\}$ and $\nu = 0$. So $i + j = \ell + 1$, proving our claim.

Type B_ℓ. In this case we have $\lambda = \mu = \omega_\ell$, which is the unique minuscule weight. We have the complete covering diagram

$$2\omega_\ell \xrightarrow{\alpha_\ell} \omega_{\ell-1} \xrightarrow{\mathsf{B}_2} \omega_{\ell-2} \xrightarrow{\mathsf{B}_3} \cdots \xrightarrow{\mathsf{B}_{\ell-1}} \omega_1 \xrightarrow{\mathsf{B}_\ell} 0.$$

Since $\lambda + \mu - \nu \geq \zeta$, we have $\nu = 0$.

Type G_2. We do not know that λ and μ are minuscule weights. Instead, notice that $\alpha_1 + \alpha_2 = -\omega_1 + \omega_2$; so we apply Lemma 4.2 with $I = \{\alpha_1, \alpha_2\}$ and $I = \{\alpha_1\}$ to obtain $\lambda = \mu = \omega_1$. So we have the complete covering diagram

$$2\omega_1 \xrightarrow{\alpha_1} \omega_2 \xrightarrow{\zeta} \omega_1 \xrightarrow{\mathsf{G}_2} 0.$$

Since $\nu \leq \lambda + \mu - \zeta$, we find $\nu = \omega_1$ or $\nu = 0$. In both cases, setting $\lambda' = 0$, we have $\nu \leq \lambda' + \mu$ and $\lambda' < \lambda$ against the first condition for a low triple. □

4.2. Nonreduced root system

We still need to treat the case of the irreducible nonreduced root system, that is, Φ of type BC_ℓ. We begin with a word of explanation about this type. We think of Φ as the union of B_ℓ with square root lengths $1, 2$ and C_ℓ with square root lengths $2, 4$. The base $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ for Φ is the same for B_ℓ with indexing from [2], so α_ℓ is the unique simple root such that $2\alpha_\ell \in \Phi$. One can define the set Λ of integral weights λ requiring that $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}$ for all $\alpha \in \Phi$. As one can easily see, this condition is equivalent to $\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}$ for $i = 1, \ldots, \ell - 1$ and $\langle \lambda, (2\alpha_\ell)^\vee \rangle \in \mathbb{Z}$. Hence the fundamental weights $\omega_1, \ldots, \omega_\ell$ are those of C_ℓ. Further, we order Λ as usual, defining $\mu \leq \lambda$ if and only if $\lambda - \mu \in R^+$. So the same definition of low triples seen for reduced root systems applies to this situation.

Notice that for this type there are no minuscule weights; that is, the weight lattice coincides with the root lattice. In the following proposition we see that Theorem B is valid for nonreduced root systems too.

PROPOSITION 4.8
If Φ is an irreducible nonreduced root system, then there is no low triple.

Proof
Suppose that (λ, μ, ν) is a low triple. We proceed as in the proof of Lemma 4.2. We have $\sum_{i=h}^{\ell} \alpha_i = -\omega_{h-1} + \omega_h$ for $1 \leq h \leq \ell$. (Notice that $\alpha_\ell = -\omega_{\ell-1} + \omega_\ell$ since the fundamental weights are those of C_ℓ and the simple roots are those of B_ℓ.) Suppose $h \in \operatorname{supp} \lambda$. Using $\lambda + \mu \geq \nu + \zeta$, we find $\left(\lambda - \sum_{i=h}^{\ell} \alpha_i\right) + \mu \geq \nu$. This is impossible since $\lambda - \sum_{i=h}^{\ell} \alpha_i = \lambda - \omega_{h-1} + \omega_h$ is a dominant weight and (λ, μ, ν) is a low triple. This forces $\lambda = 0$. By symmetry, $\mu = 0$ too; hence (λ, μ, ν) cannot be a low triple. □

4.3. Enlarged weight lattice
In order to treat the exceptional complete symmetric varieties, we need a slight generalization of Theorem B for Φ of type BC_ℓ.

Fix a base Δ of Φ, and consider a realization of this root system in a Euclidean space E. Suppose that $E \hookrightarrow F$ is an immersion of Euclidean spaces with $\operatorname{codim}_F E = 1$. Let $\omega_1, \ldots, \omega_\ell$ be the fundamental weights with respect to Δ, and choose two vectors $u, v \in F \setminus E$ such that $u + v = \omega_\ell$.

Consider the monoids $\Lambda^+ = \langle \omega_1, \ldots, \omega_\ell \rangle_{\mathbb{N}}$, $P^+ = \langle \omega_1, \ldots, \omega_{\ell-1}, u, v \rangle_{\mathbb{N}}$ and the related lattices $\Lambda = \Lambda^+_{\mathbb{Z}}$ and $P = P^+_{\mathbb{Z}}$. We order P by declaring $\mu \leq \lambda$ if and only if $\lambda - \mu \in R^+$, where $R^+ \subset E$ is the monoid generated by Δ. We may define low triples for "weights" in P^+ using Definition 1.

LEMMA 4.9
Let $\nu \in P^+$ and $\lambda = \lambda' + au \in P^+$ for some $\lambda' \in \Lambda^+$ and $a \in \mathbb{N}$. Then $\nu \leq \lambda$ if and only if $\nu = \nu' + au$ for some $\Lambda^+ \ni \nu' \leq \lambda'$. The same holds for $\lambda = \lambda' + av$.

Proof
This is clear from the definition of the order since $R^+ \subset \Lambda$. □

We want to show the following.

PROPOSITION 4.10

Let λ, μ, ν be three weights in P^+. If (λ, μ, ν) is a low triple, then up to symmetry, $\lambda = au$, $\mu = bv$ with $a, b \geq 1$. If $a \geq b$, then $\nu = \nu' + (a-b)u$ with $\Lambda^+ \ni \nu' \leq b\omega_\ell$. Moreover, if $\lambda = u$, $\mu = v$, then $\nu = 0$.

Proof

Notice that $\sum_{i=h}^{\ell} \alpha_i = -\omega_{h-1} + \omega_h$ for $h = 1, \ldots, \ell$, as already seen for type BC_ℓ. This shows that $\lambda = a_1 u + a_2 v$, $\mu = b_1 u + b_2 v$. Also, $a_1 a_2 = 0$ since otherwise $\lambda = \omega_\ell + \lambda'$ for some $\lambda' \in P^+$ and this is not possible, as one can see in the proof of Proposition 4.8. This shows that $\lambda, \mu \in \mathbb{N}u \cup \mathbb{N}v$. Further, if $\lambda = au$, $\mu = bu$, then using Lemma 4.9, $\lambda + \mu = (a+b)u$ is a minimal element in P^+, against $P^+ \ni \nu < \lambda + \mu$. So we must have $\lambda = au$, $\mu = bv$ up to symmetry.

Now suppose that $a \geq b$. Then $\lambda + \mu = au + bv = b\omega_\ell + (a-b)u$, and the stated form of ν follows by Lemma 4.9. Finally, notice that (u, v, ν) is a low triple in P^+ if and only if $(2\omega_\ell, 2\omega_\ell, \nu)$ is a low triple for type B_ℓ since we are using the simple roots of B_ℓ and $2\omega_\ell$ is the fundamental weight dual to $\alpha_\ell^\vee$. So $\nu = 0$ by Theorem B. □

For the sake of readability, we finish this section with a link to the actual application of Proposition 4.10 in the proof of Theorem A. So let (G, σ) be a simple exceptional symmetric variety of rank ℓ, choose a base Δ of the root system of G as in Section 1, and let α, β be the two exceptional roots. The restricted root system $\widetilde{\Phi}$ is of type $\mathsf{B}C_\ell$; also, the Picard group of the wonderful compactification X of (G, σ) is generated by the set Ω^+ of spherical weights and by $\omega_\alpha, \omega_\beta$. We recall that the lattice Ω generated by Ω^+ is naturally identified with the lattice of integral weights of $\widetilde{\Phi}$. Further, if we number the base of the restricted root system in a suitable way, $\widetilde{\alpha}_\ell$ is the unique simple root such that $2\widetilde{\alpha}_\ell \in \widetilde{\Phi}$ and, for such numbering, $\omega_\alpha + \omega_\beta = \widetilde{\omega}_\ell$. So we are in a position to apply Proposition 4.10 to $\widetilde{\Phi}$.

Let $E \subset \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ be the Euclidean space spanned by $\widetilde{\Phi}$, and let F be the span of E and of $\omega_\alpha, \omega_\beta$. Taking $u = \omega_\alpha$, $v = \omega_\beta$ in Proposition 4.10, we have the following.

COROLLARY 4.11

Let X be the wonderful compactification of a simple exceptional symmetric variety, and let $\lambda, \mu, \nu \in \mathrm{Pic}^+(X)$. If (λ, μ, ν) is a low triple, then up to symmetry, $\lambda = a\omega_\alpha$, $\mu = b\omega_\beta$ for some $a, b \geq 1$. If $a \geq b$, then $\nu = \nu' + (a-b)\omega_\alpha$ with $\Omega^+ \ni \nu' \leq b\widetilde{\omega}_\ell$. Moreover, if $\lambda = \omega_\alpha$, $\mu = \omega_\beta$, then $\nu = 0$.

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Chirivì

Dipartimento di Matematica "Leonida Tonelli," Università di Pisa, Via Buonarroti 2, I-56127 Pisa, Italy; chirivi@dm.unipi.it

Maffei

Dipartimento di Matematica "Guido Castelnuovo," Università di Roma "La Sapienza," Piazzale Aldo Moro 5, I-00185 Roma, Italy; amaffei@mat.uniroma1.it