Transformation Groups, Vol. 14, No. 1, 2009, pp. 183–194 

# ORBITS IN DEGENERATE COMPACTIFICATIONS OF SYMMETRIC VARIETIES

ANDREA MAFFEI

Dipartimento di Matematica
"G. Castelnuovo"
Università di Roma "La Sapienza"
00185 Roma, Italy
amaffei@mat.uniroma1.it

**Abstract.** Let $G$ be a simply connected semisimple algebraic group and let $H^0$ be the subgroup of points fixed under an involution of $G$. If $V$ is an irreducible representation with a line $L$ of vectors fixed by $H^0$ we consider the closure of the $G$-orbit of $L$ in $\mathbb{P}(V)$. We describe the $G$-orbits of this closure and we prove that the normalization of this variety is homeomorphic to the variety itself.

## Introduction

Let $G$ be a semisimple and simply connected group over an algebraically closed field $\Bbbk$ of characteristic 0 and $\sigma$ an involution of $G$. Let $H^0$ be the subgroup of points fixed by $\sigma$ and $H$ its normalizer. A wonderful compactification of the space $G/H$ has been introduced in [4] by De Concini and Procesi and, more generally, normal compactifications of this space have been studied by Brion, Luna and Vust (see the paper by Vust for an application of their general theory of compactifications of spherical varieties to symmetric varieties [8]). In this paper we study the geometry of the $G$-orbits of the most degenerate compactifications which do not even need to be normal and of their normalization. In this study we make use of the wonderful compactification of $G/H$. This generalizes part of a recent work of Bruns and Conca [1] in the case of the compactifications of $\mathrm{PGL}(n)$.

An irreducible representation $V$ of $G$ is said to be spherical if there exists a nonzero vector $v$ fixed by $H^0$. In this case this vector is unique up to a scalar so that the line in $\mathbb{P}(V)$ through $v$ is fixed by $H$. We can construct a compactification $Z_V$ of $G/H$ by taking the closure of the $G$-orbit of $\Bbbk \cdot v$ in $\mathbb{P}(V)$. These are the smallest possible $G$-equivariant compactifications of $G/H$. The normalization $r_V : Y_V \to Z_V$ can be described (see [2]) in terms of the wonderful compactifications of $G/H$.

In Theorem 9 we describe the set of $G$-orbits of $Z_V$ and $Y_V$ and in Theorem 14 we describe the geometry of these orbits. As a consequence we have that $r_V$ is one to one (Corollary 15). The classification of orbits could have been obtained in another way. As pointed out to me by Brion for the normal variety $Y_V$ the

DOI: 10.1007/s00031-008-9040-y
Received June 26, 2007. Accepted November 27, 2007. Published online November 20, 2008.

classification of $G$-orbits could also be deduced from the paper by Vust [8] and the general theory of spherical embeddings. In the nonnormal case Timashev has proved that that if $Z$ is a $G$ variety with a dense orbit of a Borel subgroup of $G$ and $r\colon Y \to Z$ is its normalization then $r$ induces a bijection at the level of $G$ orbits (see [7]).

Finally let us notice that part of these results generalize to positive characteristic when some basic facts are known (see Remarks 8, 10, 12).

*Acknowledgements.* I want to thank Dmitri Timashev for a number of corrections. In particular, the actual proof of Lemma 3 is due to him. The proof I gave in the first version of the paper, as was noticed by Timashev was wrong. A proof of this lemma was given by Rocco Chirivì and the author some time ago (see [3]). The present proof which is more elegant and simpler deduces the claim of the lemma from the results in [2] and in [6].

## 1. Preliminaries

In this section we introduce some notations and we recall some results about the wonderful completion of a symmetric variety introduced by De Concini and Procesi in [4].

### 1.1. Notations

Let $G, \sigma, H^0, H$ and $\mathbb{k}$ be as in the Introduction. Choose a maximal torus $T$ of $G$ that is $\sigma$-stable and such that the dimension of the subtorus $S$ given by the identity component of $\{t \in T \mid \sigma(t) = t^{-1}\}$ is the maximal possible. Choose also a Borel subgroup $P_0$ containing $T$ and such that the dimension of $\sigma(P_0) \cap P_0$ is the minimal possible. Let $\Lambda$ be the set of characters of $T$, let $\Phi \subset \Lambda$ be the set of roots, and let $\Phi^+$ (resp., $\Delta$) the choice of positive roots determined by $P_0$ (resp., simple roots). Let also $\Lambda^+$ be the set of dominant weights with respect to these choices and for $\lambda \in \Lambda^+$ let $V_\lambda$ be the irreducible representation of $G$ of highest weight $\lambda$.

If $\alpha \in \Phi$ we define the associated restricted root as $\tilde{\alpha} = \alpha - \sigma(\alpha)$ and we set $\tilde{\Phi}$ to be the set of nonzero restricted roots. This is a (possibly not reduced) root system and $\tilde{\Delta} = \{\tilde{\alpha} \mid \alpha \in \Delta\}$ is a simple basis for this system.

On the set of weights there are two possible dominant orders. The dominant order defined by $\Delta$ and the one defined by $\tilde{\Delta}$. If $\lambda, \mu \in \Lambda$ we write $\lambda \leqslant \mu$ if $\mu - \lambda \in \mathbb{N}[\Delta]$ and $\lambda \leqslant_\sigma \mu$ if $\mu - \lambda \in \mathbb{N}[\tilde{\Delta}]$.

Let also $\Phi_0 = \{\alpha \in \Phi \mid \sigma(\alpha) = \alpha\}$, $\Phi_1 = \Phi \smallsetminus \Phi_0$, $\Delta_0 = \{\alpha \in \Delta \mid \sigma(\alpha) = \alpha\}$ and $\Delta_1 = \Delta \smallsetminus \Delta_0$. The involution $\sigma$ induces an involution $\bar{\sigma}$ of $\Delta_1$ such that $\sigma(\alpha) + \bar{\sigma}(\alpha)$ is in the span of $\Delta_0$ and we have that if $\beta = \bar{\sigma}(\alpha)$ then $\tilde{\beta} = \tilde{\alpha}$. We denote the Lie algebras of $G$ and $H$ by $\mathfrak{g}$ and $\mathfrak{h}$ respectively, with $\mathfrak{g}_\alpha$ the root space of weight $\alpha$ and with $x_\alpha$ a nonzero element in $\mathfrak{g}_\alpha$. We denote by $\kappa$ the Killing form. Let $\mathfrak{t}$ be the Lie algebra of $T$ and we identify $\mathfrak{t}^*$ with $\Lambda \otimes_{\mathbb{Z}} \mathbb{k}$ and $\mathfrak{t}$ with $\mathrm{Hom}(\mathbb{k}^*, T) \otimes_{\mathbb{Z}} \mathbb{k}$. If $\alpha \in \Phi$ we denote by $\alpha^\vee \in \mathfrak{t}$ the corresponding coroot. More generally, if $x \in \mathfrak{t}^*$ is anisotropic with respect to $\kappa$ we denote by $x^\vee$ the element of $\mathfrak{t}$ such that $\langle x^\vee, y\rangle = 2\kappa(x,y)/\kappa(x,x)$ for all $y \in \mathfrak{t}$.

If $\Delta' \subset \Delta$ we denote by $\Phi_{\Delta'} \subset \Phi$ the parabolic root subsystem generated by $\Delta'$ and by $P^{\Delta'}$ the parabolic subgroup of $G$ containing $P_0$ and such that its

Levi factor has $\Phi_{\Delta'}$ as the root system. We also denote by $W$ the Weyl group of $\Phi$, with $W_{\Delta'} \subset W$ the parabolic subgroup associated to $\Phi_{\Delta'}$ and with $w_{\Delta'}$ the longest element in $W_{\Delta'}$. Recall that the action of $\sigma$ on the simple roots is given by $\sigma(\alpha) = \alpha$ if $\alpha \in \Delta_0$ and $\sigma(\alpha) = -w_{\Delta_0}\bar{\sigma}(\alpha)$ if $\alpha \in \Delta_1$.

Finally if $P$ is a parabolic containing $P_0$ we denote by $\Lambda_P \subset \Lambda$ the set of characters of $P$. We identify $\mathrm{Pic}(G/P)$ with $\Lambda_P$ by considering the opposite of the action of $P$ on the fibre over the point $\mathrm{id}P \in G/P$ and we denote by $\mathcal{L}_\lambda$ the line bundle corresponding to $\lambda \in \Lambda_P$ so that if $\lambda \in \Lambda_P \cap \Lambda^+$ we have $\Gamma(G/P, \mathcal{L}_\lambda) \simeq V_\lambda^*$.

### 1.2. Spherical weights

As recalled in the introduction an irreducible representation is said to be spherical if it has a nonzero weight vector fixed by $H^0$. In particular we say that $\lambda \in \Lambda^+$ is spherical if $V_\lambda$ is spherical and we denote by $\Omega^+$ the set of spherical weights and by $\Omega$ the lattice spanned by $\Omega^+$. Also we say that a weight is special if $\sigma(\lambda) = -\lambda$ and we denote by $\Lambda^s$ the set of special weights. We have that $\Omega \cap \Lambda^+ = \Omega^+$ and by a result of Helgason we have that

$$\Omega = \{\lambda \in \Lambda \mid \sigma(\lambda) = -\lambda \text{ and } \langle \tilde{\alpha}^\vee, \lambda \rangle \in \mathbb{Z} \text{ for all } \alpha \in \Phi\}.$$

In particular $\Omega$ is the set of weights of $\tilde{\Phi}$. Let $\tilde{\omega}_{\tilde{\alpha}}$ for $\tilde{\alpha} \in \tilde{\Delta}$ be the fundamental weights dual to the simple basis $\tilde{\Delta}$.

### 1.3. Boundaries and supports

We identify $\tilde{\Delta}$ and $\Delta$ with the vertices of the associated Dynkin diagrams, so that we will speak of connected subsets or connected components of $\tilde{\Delta}$ and $\Delta$ or linked simple roots. If $A \subset \tilde{\Delta}$ we define the boundary of $A$ as

$$\partial A = \{\alpha \in \tilde{\Delta} \mid \alpha \text{ is linked to an element of } A\} \smallsetminus A.$$

and similarly for a subset of $\Delta$.

For $\lambda = \sum_{\alpha \in \Delta} c_\alpha \alpha \in \Lambda$ and $\mu = \sum_{\tilde{\alpha} \in \tilde{\Delta}} c_{\tilde{\alpha}} \tilde{\alpha} \in \Omega$ with $c_\alpha, c_{\tilde{\alpha}} \in \mathbb{Q}$ we also define

$$\operatorname{supp} \lambda = \{\alpha \in \Delta \mid \kappa(\alpha, \lambda) \neq 0\}; \qquad \operatorname{supp}_\Omega \mu = \{\tilde{\alpha} \in \tilde{\Delta} \mid \kappa(\tilde{\alpha}, \mu) \neq 0\};$$
$$\operatorname{supp}_\Delta \lambda = \{\alpha \in \Delta \mid c_\alpha \neq 0\}; \qquad \operatorname{supp}_{\tilde{\Delta}} \mu = \{\tilde{\alpha} \in \tilde{\Delta} \mid c_{\tilde{\alpha}} \neq 0\}.$$

Also if $B \subset \Delta$ we define $\operatorname{supp}_B \lambda = B \cap \operatorname{supp} \lambda$.

### 1.4. The wonderful compactification

A wonderful compactification $X$ of the symmetric variety $G/H$ has been constructed in [4] in characteristic 0 and in characteristic different from 2 in [5]. Recall that a weight $\lambda$ is called quasi-spherical if it is in the lattice generated by dominant weights $\lambda$ such that $\mathbb{P}(V_\lambda)^H$ is not empty. The following theorem describes some of the main properties of $X$.

**Theorem 1 (**Theorem 3.1 in [4] and Proposition 8.1 in [5]**).**

(i) *$X$ is a smooth projective $G$-variety;*
(ii) *$X \smallsetminus G/H$ is a divisor with normal crossings and smooth irreducible components $\{X_{\tilde\alpha} \mid \tilde\alpha \in \tilde\Delta\}$ parametrized in a canonical way by the simple roots of the restricted root system;*
(iii) *the closures of $G$-orbits are given by the subvarieties $X_J = \bigcap_{\tilde\alpha\in J} X_{\tilde\alpha}$ for $J$ any subset of $\tilde\Delta$, in particular there exists only one closed orbit namely $X_{\tilde\Delta}$;*
(iv) *the restriction of line bundles from $X$ to $X_{\tilde\Delta}$ determines an injective map from the Picard group $\mathrm{Pic}(X)$ of $X$ to the Picard group $\mathrm{Pic}(X_{\tilde\Delta})$ of $X_{\tilde\Delta}$;*
(v) *$X_{\tilde\Delta}$ is isomorphic to $G/P^{\Delta_0}$ and the image of $\mathrm{Pic}(X)$ in $\mathrm{Pic}(X_{\tilde\Delta}) = \Lambda_{P^{\Delta_0}}$ is given by the lattice $\Pi$ of quasi-spherical weights.*

For each $\lambda \in \Pi$ we choose a line bundle on $X$ such that its restriction to $X_{\tilde\Delta}$ is isomorphic to $\mathcal{L}_\lambda$ and we denote this line bundle still by the symbol $\mathcal{L}_\lambda$ and we denote its global sections by $\Gamma_\lambda$. All line bundles on $X$ have a $G$-linearization which is unique.

The divisors $X_{\tilde\alpha}$ can be parametrized in such a way that $\mathcal{O}(X_{\tilde\alpha}) \simeq \mathcal{L}_{\tilde\alpha}$. There exists a $G$-invariant section $s_{\tilde\alpha} \in \Gamma_{\tilde\alpha}$ whose divisor is $X_{\tilde\alpha}$. For an element $\nu = \sum_{\tilde\alpha\in\tilde\Delta} n_{\tilde\alpha}\tilde\alpha \geqslant_\sigma 0$ the multiplication by $s^\nu \doteq \prod_{\tilde\alpha} s_{\tilde\alpha}^{n_{\tilde\alpha}}$ injects $\Gamma_{\lambda-\nu}$ in $\Gamma_\lambda$.

If $\mu \in \Pi^+$ then the module $V_\mu^*$ appears with multiplicity 1 in $\Gamma_\mu$. If $\lambda - \mu \geqslant_\sigma 0$ we denote by $s^{\lambda-\mu}V_\mu^*$ the image of $V_\mu^*$ under the multiplication by $s^{\lambda-\mu}$. We have the following theorem.

**Theorem 2** (Theorem 5.10 [4])**.** *Let $\lambda \in \Pi$ then $\Gamma_\lambda = \bigoplus_{\mu\leqslant_\sigma\lambda \text{ and } \mu\in\Pi^+} s^{\lambda-\mu}V_\mu^*$.*

### 1.5. The degenerate compactifications $Y_\lambda$ and $Z_\lambda$

Let $\lambda \in \Omega^+$ and let $h \in V_\lambda$ be a nonzero vector fixed by $H^0$. Then we can define a map from $G/H$ to $\mathbb{P}(V_\lambda)$ by $gH \mapsto g[h]$. This map extends to $X$ and we denote by $Z_\lambda$ its image and by $q_\lambda\colon X \longrightarrow Z_\lambda$ the associated map. The normalization of $Z_\lambda$ can be constructed in the following way (see [2] for the details). Let $\mathcal{L}_\lambda = q_\lambda^*(\mathcal{O}_{\mathbb{P}(V_\lambda)}(1))$ and define $A(\lambda)$ as the ring $\bigoplus_{n\geqslant 0}\Gamma_{n\lambda}$, and $Y_\lambda$ as $\mathrm{Proj}\,A(\lambda)$. The line bundle $\mathcal{L}_\lambda$ is generated by global sections so the associated map $p_\lambda\colon X \to Y_\lambda$ is well defined. The ring $A(\lambda)$ is generated in degree 1 and since $X$ is smooth, it is normal. So $Y_\lambda \subset \mathbb{P}(\Gamma_\lambda^*)$ is a projectively normal subvariety. Notice also that by construction $V_\lambda^* \subset \Gamma_\lambda$ and that the subring $B(\lambda)$ of $A(\lambda)$ generated by $V_\lambda^* \subset \Gamma_\lambda$ is isomorphic to the projective coordinate ring of $Z_\lambda \subset \mathbb{P}(V_\lambda)$. We have a projection $r_\lambda\colon Y_\lambda \to Z_\lambda$ and we have $q_\lambda = r_\lambda \circ p_\lambda$. By [2] the ring $A(\lambda)$ is the integral closure of $B(\lambda)$ so that $r_\lambda$ is the normalization of $Z_\lambda$. However we need a result which is slightly more precise. The proof I give here is due to Dmitri Timashev.

**Lemma 3.** *Let $\lambda, \mu \in \Pi^+$ and $\mu \leqslant_\sigma \lambda$. Let $\varphi = s^{\lambda-\mu}\varphi_\mu$ be a highest weight vector in $s^{\lambda-\mu}V_\mu^* \subset \Gamma_\lambda$. Then there exists a positive integer $n$ such that $\varphi^n \in B(\lambda)$.*

*Proof.* Let $A = A(\lambda)$, $B = B(\lambda)$ and let $U_0$ be the maximal unipotent subgroup of $P_0$. We prove first that $A^{U_0}$ is a finite extension of $B^{U_0}$. By [6, Theorem 9.4] both $A^{U_0}$ and $B^{U_0}$ are finitely generated $\Bbbk$-algebras. Let now $I = \{b \in B \mid bA \subset B\}$. Since $A$ is a finite extension of $B$ the ideal $I$ is not zero. Then there exists a nonzero element $b$ in $I$ fixed by $U_0$. So $A^{U_0} \simeq bA^{U_0}$ as a $B^{U_0}$-module and $bA^{U_0}$ is an ideal in $B^{U_0}$, hence it is finitely generated.

Hence there exists a nontrivial polynomial $p(t) = t^m + b_1 t^{m-1} + b_2 t^{m-2} + \cdots + b_{m-1} t + b_m \in B^{U_0}[t]$ such that $p(\varphi) = 0$. It is clear that we can assume $p$ to be $T$-homogeneous and homogeneous with respect to the graduation of the ring $B$. In particular $b_i \in \Gamma_{i\lambda}$ and it is of weight $i\mu^*$ where $\mu^*$ is the highest weight of $V_\mu^*$.

Since $A$ is a domain there exists $i$ such that $b_i \neq 0$. Both $b_i, \varphi^i$ are two $U_0$-invariant vectors in $\Gamma_{i\lambda}$ of weight $i\mu^*$, so they are both highest weight vectors of $s^{i(\lambda-\mu)} V_{i\mu}^* \subset \Gamma_{i\lambda}$. Hence $\varphi^i$ is a multiple of $b_i$ and in particular $\varphi^i \in B(\lambda)$. □

## 2. The $G$-orbits of the degenerate compactifications $Z_\lambda$ and $Y_\lambda$

We recall first the description of $G$-orbits in $X$. For each $J \subset \tilde{\Delta}$ let $\gamma_J \colon \mathbb{k}^* \longrightarrow S$ be a group homomorphism such that $\langle \gamma_J, \tilde{\alpha} \rangle \geqslant 0$ for all $\tilde{\alpha} \in \tilde{\Delta}$ and it is equal to 0 if and only if $\tilde{\alpha} \notin J$. Let now $x_0 = x_\varnothing = \mathrm{id}H \in G/H \subset X$ and define

$$x_J = \lim_{t \to \infty} \gamma_J(t) x_0 \qquad \text{and} \qquad O_J = G\, x_J.$$

Then the closure of $O_J$ is equal to $X_J$.

We describe now the $G$-orbits in the degenerate compactifications $Z_\lambda$ and $Y_\lambda$. These orbits are the images of the $G$-orbits in $X$, so we need to understand which of these orbits have the same image.

We start by some remarks on spherical weights. If $I \subset \tilde{\Delta}$ define

$$I_\Delta = \{\alpha \in \Delta_1 \mid \tilde{\alpha} \in I\};$$
$$\overline{I_\Delta} = I_\Delta \cup \bigcup \{C \mid C \text{ is a connected component of } \Delta_0 \text{ and } C \cap \partial I_\Delta \neq \varnothing\};$$
$$I^\Delta = I_\Delta \cup \Delta_0.$$

Then we have the following lemma.

**Lemma 4.**

(i) *Let* $\lambda \in \Omega$ *and* $\lambda \geqslant_\sigma 0$ *and set* $I = \mathrm{supp}_{\tilde{\Delta}} \lambda$ *and* $J = \mathrm{supp}_\Omega \lambda$. *Then* $\mathrm{supp}_\Delta \lambda = \overline{I_\Delta}$ *and* $\mathrm{supp}\, \lambda = J_\Delta$*;*

(ii) *for every* $I \subset \tilde{\Delta}$ *the parabolic root subsystems* $\Phi_{\overline{I_\Delta}}$ *and* $\Phi_{I^\Delta}$ *are* $\sigma$*-stable;*

(iii) *Let* $B, C \subset \tilde{\Delta}$ *be two disjoint subsets such that* $\partial B \cap C = \varnothing$. *Then* $\overline{B_\Delta} \cap \overline{C_\Delta} = \varnothing$ *and* $\partial \overline{B_\Delta} \cap \overline{C_\Delta} = \varnothing$.

*Proof.* The second part of (i) follows from the fact that if $\lambda \in \Omega$ then $\sigma(\lambda) = -\lambda$ and $\kappa(\tilde{\alpha}, \lambda) = 2\kappa(\alpha, \lambda)$. So $\alpha \in \mathrm{supp}\, \lambda$ if and only if $\tilde{\alpha} \in \mathrm{supp}_\Omega \lambda$.

To prove the first part of (i) we can assume $\lambda = \tilde{\alpha}$ with $\alpha \in \Delta_1$. Let $\beta = \bar{\sigma}(\alpha)$ and recall that $\sigma(\beta) = -w_{\Delta_0}(\alpha)$ and that $\tilde{\alpha} = \tilde{\beta}$ so that $\beta \in I_\Delta$ and, moreover, $I_\Delta = \{\alpha, \beta\}$.

Let $\alpha = x - y$ where $x$ is orthogonal to $\Delta_0$ and $y$ is orthogonal to $\Delta_1$. Notice that, since $\alpha \in \Delta_1$, the vector $y$ is dominant with respect to $\Delta_0$. Let also $\{S_i\}$ be the connected components of $\Delta_0$ that are linked to $\alpha$ or equivalently that are not orthogonal to $y$. Write $y = \sum y_i$ where $y_i$ is orthogonal to $(\Delta \smallsetminus S_i)$, so that $y_i$ is a nonzero dominant element.

We have $\tilde{\alpha} = \tilde{\beta} = \beta + w_{\Delta_0}(\alpha) = \beta + \alpha + y - w_{\Delta_0}(y) = \beta + \alpha + \sum_i (y_i - w_{S_i}(y_i))$. So it is enough to prove that $\mathrm{supp}_{S_i}(y_i - w_{S_i}(y_i)) = S_i$ and this follows from the fact that any dominant weight for an irreducible root system is a sum of simple roots with strictly positive coefficients.

Part (ii) follows immediately from (i), we now prove (iii). We can assume $B$ and $C$ to be connected. Let $\alpha \in \overline{B_\Delta} \cap \overline{C_\Delta}$. Then $\alpha \in \Delta_0$ and let $S$ be the connected component of $\Delta_0$ containing $\alpha$. Then there exist $\beta \in B_\Delta$ and $\gamma \in C_\Delta$ linked to $S$. Notice that $\tilde\beta$ and $\tilde\gamma$ are linked. Indeed $\kappa(\tilde\beta, \tilde\gamma) = 2\kappa(\tilde\beta, \gamma)$ and by (i) we have $\gamma \notin \operatorname{supp}_\Delta \tilde\beta$ and $S \subset \operatorname{supp}_\Delta \tilde\beta$ so that $\kappa(\tilde\beta, \gamma) < 0$.

Let now $\alpha \in \partial\overline{B_\Delta} \cap \overline{C_\Delta}$. Then $\alpha \in \Delta_1$ so $\tilde\alpha \in C$. Now there exists $\beta \in \overline{B_\Delta}$ linked to $\alpha$. If $\beta \in \Delta_1$ then $\tilde\beta \in B$ and $\langle \tilde\beta^\vee, \tilde\alpha \rangle \neq 0$ against $\partial B \cap C = \varnothing$. So $\beta \in \Delta_0$ so there exists $\beta' \in B_\Delta$ and a connected component of $\Delta_0$ containing $\beta$ and linked to $\beta'$ and we conclude as above. □

If $I, J$ are subsets of $\tilde\Delta$ define

$$\mathcal{A}_I = \{A \subset \tilde\Delta \mid \text{ every connected component of } A \text{ contains an element of } I\};$$
$$\Delta_I(J) = \bigcup\{\text{connected components of } \tilde\Delta \smallsetminus J \text{ which intersect } I\}.$$

We have the following simple property of the set $\Delta_I(J)$.

**Lemma 5.** *Let $\lambda, \mu \in \Omega^+$ with $\lambda \geqslant_\sigma \mu$ and set $I = \operatorname{supp}_\Omega \lambda$. Then for any $J \subset \tilde\Delta$, $\operatorname{supp}_{\tilde\Delta}(\lambda - \mu) \cap J = \varnothing$ if and only if $\operatorname{supp}_{\tilde\Delta}(\lambda - \mu) \subset \Delta_I(J)$.*

*Proof.* If $\operatorname{supp}_{\tilde\Delta}(\lambda - \mu) \subset \Delta_I(J)$ then it is clear that $\operatorname{supp}_{\tilde\Delta}(\lambda - \mu) \cap J = \varnothing$. Conversely let $\operatorname{supp}_{\tilde\Delta}(\lambda - \mu) \cap J = \varnothing$ and let $\{S_i\}$ be the connected components of $\operatorname{supp}_{\tilde\Delta}(\lambda - \mu)$ and let $\eta_i \in \mathbb{N}[S_i]$ be such that $\lambda - \mu = \sum_i \eta_i$. Now, $\eta_i$ being a sum with nonnegative coefficients of positive roots it cannot be antidominant, in particular, for each $i$ there exists $\tilde\alpha_i \in S_i$ such that $\langle \tilde\alpha_i^\vee, \eta_i \rangle > 0$. As $\mu$ is dominant, the $S_i$ are disconnected from each other and $\mu = \lambda - \sum_i \eta_i$, we must have that $\tilde\alpha_i \in I$. Hence for each $i$ we have $S_i \cap I \neq \varnothing$ or, equivalently, $S_i \subset \Delta_I(J)$. □

If $A \subset \tilde\Delta$ and $\tilde\alpha \in \tilde\Delta$ then we define

$$\operatorname{dist}_A(\tilde\alpha, I) = \min\{n \mid \exists \text{ a connected subset } C \text{ of } A \text{ of } n \text{ elements such that } C \cup \{\tilde\alpha\} \text{ is connected and intersects } I\}.$$

The empty set is considered to be connected in the last definition. If $d = \operatorname{dist}_A(\tilde\alpha, I)$ is finite and $C$ is a connected subset of $A$ of $d$ elements such that $C \cup \{\tilde\alpha\}$ is connected and intersects $I$ we say that $C$ is a minimal path from $\tilde\alpha$ to $I$ in $A$. Notice also that if $A \in \mathcal{A}_I$ then this number is finite if and only if $\tilde\alpha \in A \cup \partial A \cup I$.

**Lemma 6.** *Let $\lambda \in \Omega^+$ and $h \in V_\lambda^{H^0}$ be different from 0. Let also $h = \sum v_\nu$ be the decomposition of $h$ in $T$-weight vectors so that $v_\nu$ has weight $\nu$. Let also $I = \operatorname{supp}_\Omega \lambda$, let $J$ be subsets of $\tilde\Delta$ and set $A = \Delta_I(J)$. Then the following properties hold:*

(i) $v_\lambda \neq 0$*;*
(ii) $v_\nu \neq 0$ *implies* $\lambda - \nu \in \mathbb{N}[\tilde\Delta]$*;*
(iii) $v_{\lambda - \tilde\alpha} \neq 0$ *for all* $\tilde\alpha \in \operatorname{supp}_\Omega \lambda$*;*

(iv) $h_J = \lim_{t\to\infty} t^{-\langle\lambda,\gamma_J\rangle}\gamma_J(t)h$ *is well defined and is equal to*

$$h_J = \sum_{\nu|\mathrm{supp}_{\tilde\Delta}(\lambda-\nu)\subset A} v_\nu;$$

(v) *for all* $\tilde\alpha \in (A\cup\partial A\cup I)$ *and for all* $C$ *minimal paths from* $\tilde\alpha$ *to* $I$ *in* $A$ *there exists a weight* $\nu$ *such that* $v_\nu \neq 0$ *and* $C = \mathrm{supp}_{\tilde\Delta}(\lambda-\nu)$. *In particular* $\langle\tilde\alpha^\vee,\nu\rangle > 0$.

*Proof.* We prove first that if $v_\nu \neq 0$ and $\nu \neq \lambda$ then there exists $\tilde\alpha \in \tilde\Delta$ such that $v_{\nu+\tilde\alpha} \neq 0$. This implies (i) and (ii).

If $\alpha \in \Delta_0$ then $x_\alpha \in \mathfrak{h}$ so $x_\alpha \cdot h = 0$ from which we deduce $x_\alpha \cdot v_\nu = 0$ for all $\nu$. Now let $\nu \neq \lambda$ and $v_\nu \neq 0$ then there exists $\alpha \in \Delta_1$ such that $x_\alpha \cdot v_\nu \neq 0$ and set $y = x_\alpha + \sigma(x_\alpha)$. The element $y$ is in $\mathfrak{h}$ so we have $y \cdot h = 0$ from which we deduce that there exists $\eta$ such that $x_\alpha \cdot v_\nu = -\sigma(x_\alpha)\cdot v_\eta$. This in particular implies that $v_\eta \neq 0$ and $\eta = \nu + \tilde\alpha$.

Now let $\alpha \in \Delta$ such that $\tilde\alpha \in I$. Then $\kappa(\tilde\alpha,\lambda) = 2\kappa(\alpha,\lambda)$ so $\langle\alpha^\vee,\lambda\rangle \neq 0$. In particular, $x_{-\alpha}\cdot v_\lambda \neq 0$. Then we can argue as above and prove that $v_{\lambda-\tilde\alpha} \neq 0$. This proves (iii).

We now prove (iv). Notice that

$$\lim_{t\to\infty} t^{-\langle\lambda,\gamma_J\rangle}\gamma_J(t)\,h = \sum_\nu t^{-\langle\lambda-\nu,\gamma_J\rangle} v_\nu$$

and that $-\langle\lambda-\nu,\gamma_J\rangle \leqslant 0$ and is equal to zero if and only if $\mathrm{supp}_{\tilde\Delta}(\lambda-\nu)\cap J = \varnothing$ so the limit exists and it is equal to $\sum_{\nu|\mathrm{supp}_{\tilde\Delta}(\lambda-\nu)\cap J=\varnothing} v_\nu$. Now by Lemma 5 the condition $\mathrm{supp}_{\tilde\Delta}(\lambda-\nu)\cap J = \varnothing$ is equivalent to $\mathrm{supp}_{\tilde\Delta}(\lambda-\nu) \subset A = \Delta_I(J)$ proving (iv).

We prove (v) by induction on $d = \mathrm{dist}_A(\tilde\alpha, I)$. For $d = 0$ we have $C = \varnothing$ and we can choose $\nu = \lambda$. Let now $d > 0$ and let $\tilde\beta \in C$ linked to $\tilde\alpha$. Set $C' = C \smallsetminus \{\tilde\beta\}$ and notice that we can apply the induction hypothesis to $C', \tilde\beta$. So there exists $\nu'$ such that $v_{\nu'} \neq 0$, $C' = \mathrm{supp}_{\tilde\Delta}(\lambda-\nu')$ and in particular $\langle\nu',\beta^\vee\rangle > 0$ which implies $x_{-\beta}\cdot v_{\nu'} \neq 0$. Now considering as above $y = x_{-\beta} + \sigma(x_{-\beta})$ we prove $v_\nu \neq 0$ for $\nu = \nu' - \tilde\beta$ proving the claim. □

Now let $\lambda$ be a dominant spherical weight with support equal to $I$. We want to compute $p_\lambda(x_J)$.

**Lemma 7.** *If* $\Delta_I(J) = \Delta_I(K)$ *then* $p_\lambda(x_J) = p_\lambda(x_K)$ *and* $q_\lambda(x_J) = q_\lambda(x_K)$.

*Proof.* Let $p_\lambda(x_0) = [h]$ and write it as a sum $h = \sum h_\mu$ where $h_\mu \in V_\mu^{H^0} \subset \Gamma_\lambda^*$ in particular, if $h_\mu \neq 0$ then $\mu \leqslant_\sigma \lambda$. Define $h_{\mu,J} = \lim_{t\to\infty} t^{-\langle\mu,\gamma_J\rangle}\gamma_J(t)h_\mu$ and by part (i) in Lemma 6 notice that $h_\mu \neq 0$ implies $h_{\mu,J} \neq 0$. Notice also that if $\mu \leqslant_\sigma \lambda$ then $\lim_{t\to\infty} t^{-\langle\lambda-\mu,\gamma_J\rangle}$ is zero if and only if $\mathrm{supp}_{\tilde\Delta}(\lambda-\mu)\cap J \neq \varnothing$ and is 1 otherwise. So we have

$$p_\lambda(x_J) = \lim_{t\to\infty}[\gamma_J(t)h] = \lim_{t\to\infty}\left[t^{\langle\lambda,\gamma_J\rangle}\Big(\sum t^{-\langle\lambda-\mu,\gamma_J\rangle}t^{-\langle\mu,\gamma_J\rangle}\gamma_J(t)h_\mu\Big)\right] =$$

$$= \left[\sum_{\substack{\mu\in\Omega^+|\mu\leqslant_\sigma\lambda \\ \text{and } \mathrm{supp}_{\tilde\Delta}(\lambda-\mu)\cap J=\varnothing}} h_{\mu,J}\right]. \tag{1}$$

Now by Lemma 5 the condition $\mathrm{supp}_{\tilde\Delta}(\lambda-\mu)\cap J=\varnothing$ is equivalent to $\mathrm{supp}_{\tilde\Delta}(\lambda-\mu)\subset\Delta_I(J)$. So the set of weights $\mu$ such that $h_{\mu,J}\neq 0$ in this sum depends only on $\lambda$ and $\Delta_I(J)$ (and not on $J$).

Now we need to prove that if $\Delta_I(J)=\Delta_I(K)$ and $\mu\in\Omega^+$ is such that $\mathrm{supp}_\Omega(\lambda-\mu)\subset\Delta_I(J)$ then $h_{\mu,J}=h_{\mu,K}$ and by Lemma 6(iv) it is enough to prove that $\Delta_{\mathrm{supp}_\Omega\mu}(J)=\Delta_{\mathrm{supp}_\Omega\mu}(K)$.

Notice that since $\mathrm{supp}_{\tilde\Delta}(\lambda-\mu)\subset\Delta_I(J)$ then $\mathrm{supp}_\Omega\mu\subset\Delta_I(J)\cup J$ so if $C$ is a connected component of $\tilde\Delta\smallsetminus J$ intersecting $\mathrm{supp}_\Omega\mu$ then it also intersects $I$. So $\Delta_{\mathrm{supp}_\Omega\mu}(J)=\Delta_{\mathrm{supp}_\Omega\mu}(K)$ is the union of the components of $\Delta_I(J)=\Delta_I(K)$ which intersects $\mathrm{supp}_\Omega\mu$.

The second part of the statement follows from $q_\lambda=r_\lambda\circ p_\lambda$. □

*Remark 8.* The same proof shows that if $\Delta_I(J)=\Delta_I(K)$ then $q_\lambda(x_J)=q_\lambda(x_K)$ in arbitrary characteristic.

Now we can describe the set of orbits of $X_\lambda$ and $Y_\lambda$.

**Theorem 9.** *Let $I,J,K$ be subsets of $\tilde\Delta$ and let $\lambda$ be a dominant spherical weight with support equal to $I$ then:*

(i) $p_\lambda(O_J)=p_\lambda(O_K)$ *if and only if* $\Delta_I(J)=\Delta_I(K)$*;*
(ii) $q_\lambda(O_J)=q_\lambda(O_K)$ *if and only if* $\Delta_I(J)=\Delta_I(K)$*.*

*In particular, the $G$-orbits in $Y_\lambda$ and $Z_\lambda$ are classified by the set $\mathcal{A}_I$.*

*Proof.* Assume that $q_\lambda(O_J)=q_\lambda(O_K)$. We prove first that $I\cap J=I\cap K$. Let $\tilde\alpha\in I\cap J$. Then $2\lambda-\tilde\alpha$ is dominant so that $W=s_{\tilde\alpha}\Gamma_{2\lambda-\tilde\alpha}\subset\Gamma_{2\lambda}$ is a nonzero subspace of sections that are zero on $X_J$ and let $f\in W$ be a highest weight vector. Now by Lemma 3 (here we use characteristic 0) there exists $n$ such that $f^n\in\Gamma_{2n\lambda}$ is contained in the subring $B(\lambda)$ of $A(\lambda)$ generated by $V_\lambda^*\subset\Gamma_\lambda$ which is the coordinate ring of $Z_\lambda$. Now, by $q_\lambda(O_J)=q_\lambda(O_K)$ and $s_{\tilde\alpha}=0$ on $X_J$, we have that $f^n=0$ on $X_K$ or equivalently that $\tilde\alpha$ is in $K$.

Now let $C$ be a connected component of $\Delta_I(J)$ containing an element of $I$. Notice that there exists an element $\theta$ in the root lattice such that its $\tilde\Delta$-support is equal to $C$ and such that $m\lambda-\theta$ is dominant for some positive integer $m$. As above choose a highest weight vector $f$ in $s^\theta\Gamma_{m\lambda-\theta}\subset\Gamma_{m\lambda}$. Then $f\neq 0$ on $X_J$. As above a power of $f$ is in the ring generated by $V_\lambda^*$ and we can deduce that $f\neq 0$ on $X_K$. Hence $s_{\tilde\alpha}\neq 0$ on $X_K$ for all $\tilde\alpha\in C$ or equivalently $C\subset\tilde\Delta\smallsetminus K$. This proves that $C$ is contained in a connected component of $\tilde\Delta\smallsetminus K$ containing an element of $I$ or, equivalently, $C\subset\Delta_I(K)$. So $\Delta_I(J)\subset\Delta_I(K)$ and by exchanging the role of $J$ and $K$ we conclude that $\Delta_I(J)=\Delta_I(K)$.

The converse statement is a consequence of Lemma 7. □

If $I=\mathrm{supp}_\Omega\lambda$ and $A\in\mathcal{A}_I$ we will denote by $O^Z_{\lambda,A}$ (resp., $O^Y_{\lambda,A}$) the image $q_\lambda(O_J)$ (resp., $p_\lambda(O_J)$) for a $J$ such that $A=\Delta_I(J)$.

*Remark 10.* The same arguments prove that in arbitrary characteristic if $p_\lambda(O_J)=p_\lambda(O_K)$ then $\Delta_I(J)=\Delta_I(K)$. In particular, the theorem is true in arbitrary characteristic if $\lambda$ is $\sigma$-minuscule (see [2]) or equivalently if $\Gamma_\lambda=V_\lambda^*$. In particular this covers the cases studied by Bruns and Conca in [1].

*Remark 11.* Notice that as a consequence of Lemma 7 we have that the conditions in the Theorem are also equivalent to $p_\lambda(x_J) = p_\lambda(x_K)$.

*Remark 12.* Dmitri Timashev has pointed out to me that it is possible to prove that there is always a bijection between the set of $G$-orbits of an irreducible $G$-variety equivariantly embedded in a projective space and with a dense orbit of a Borel subgroup and the set of $G$-orbits of its normalization (see [7]). Moreover this result holds in any characeristic. This should give a proof of the previous theorem in arbitrary characteristic.

Now we want to describe the geometry of these orbits. If $B \subset \tilde{\Delta}$ define $L_B$ to be the Levi of $G$ containing $T$ and with the root system equal to $\Phi_{\overline{B_\Delta}}$ and with $P_B$ being the parabolic subgroup of $G$ containing $P_0$ and with $L_B$ as a Levi factor. Let also $G_{B,\mathrm{ad}} = L_B/Z(L_B)$ and denote by $\pi_B$ the projection from $P_B$ to $G_{B,\mathrm{ad}}$. Notice that the involution $\sigma$ stabilizes $L_B$ so it defines an involution of $G_{B,\mathrm{ad}}$. Let $G_B \subset L_B$ be its semisimple part. Notice that the characters of $T_B = T \cap G_B$ are the restriction to $T_B$ of the characters of $T$. In particular, since $G$ is simply connected, the restriction of the characters $\omega_\alpha$ for $\alpha \in \overline{B_\Delta}$ are the fundamental weights with respect to $\overline{B_\Delta}$. So $G_B$ is simply connected and let $H_B \subset G_B$ be the normalizer of $G_B^\sigma$ in $G_B$ so that $G_B/H_B \simeq G_{B,\mathrm{ad}}/G^\sigma_{B,\mathrm{ad}}$. Finally denote by $\mathfrak{l}_B$, $\mathfrak{p}_B$ and $\mathfrak{g}_B$ the Lie algebras of the groups $L_B$, $P_B$ and $G_B$. If $V$ is an irreducible highest weight representation of $G$ let $V^{(B)}$ be the $G_B$-submodule generated by the highest weight vector. Notice that if $B$ is the disjoint union of two subsets $C$ and $D$ that are not linked to each other then by Lemma 4(iii) we have $G_{B,\mathrm{ad}} = G_{C,\mathrm{ad}} \times G_{D,\mathrm{ad}}$ and $G_B = G_C \times G_D$, in particular, we can define $\pi^B_C \colon P_B \longrightarrow G_{C,\mathrm{ad}}$ by applying $\pi_B$ and then projecting onto the first factor.

From [4] it follows that for all $J \subset \tilde{\Delta}$ we have

$$O_J \simeq G \times_{P_{J^c}} G_{J^c}/H_{J^c}$$

where $J^c = \tilde{\Delta} \smallsetminus J$ and, in particular, $\mathrm{Stab}_G(x_J) = \{g \in P_{J^c} \mid \pi_{J^c}(g) \in G^\sigma_{J^c,\mathrm{ad}}\}$. Indeed in [4] it is proved that $O_J \simeq G \times_{\tilde{P}_{J^c}} \tilde{G}_{J^c}/\tilde{H}_{J^c}$ where $\tilde{G}_B$, $\tilde{P}_B$, $\tilde{H}_B$ are defined similarly to $G_B$, $P_B$, $H_B$ but using the root system $\Phi_{B^\Delta}$ instead of the root system $\Phi_{\overline{B_\Delta}}$. The equivalence of the two descriptions of $O_J$ follows from the fact that the complement $B^\sharp$ of $\overline{B_\Delta}$ in $B^\Delta$ is contained in $\Delta_0$ and is not connected to $\overline{B_\Delta}$. Hence if $G^\sharp_B$ is the semisimple part of the Levi containing $T$ and with root system equal to $\Phi_{B^\sharp}$ then we have $\tilde{G}_B = G_B \times G^\sharp_B$ and $\tilde{H}_B = H_B \times G^\sharp_B$.

Now we want to compute the stabilizer of $p_\lambda(x_J)$. When $J = \varnothing$ it is not difficult to prove that the stabilizer is equal to $H$ (see, e.g., [2, Lemma 2.2]). We will use this result to treat the general case.

First of all we notice that if $A \in \mathcal{A}_I$ then between the sets $J$ such that $\Delta_I(J) = A$ there is a maximum and a minimum:

$$J_{\min}(A) = \partial A \cup \{I \smallsetminus A\} \qquad \text{and} \qquad J_{\max}(A) = \tilde{\Delta} \smallsetminus A.$$

**Lemma 13.** *Let* $\lambda \in \Omega^+$, $I = \mathrm{supp}_\Omega \lambda$, $A \in \mathcal{A}_I$ *and set* $J_{\min} = J_{\min}(A)$. *Let* $h \in V_\lambda^{H^0}$ *a spherical vector and* $h_{J_{\min}} = \lim_{t\to\infty} t^{-\langle \lambda, \gamma_{J_{\min}}\rangle} \gamma_{J_{\min}}(t) h$. *Then:*

(i) $h_{J_{\min}} \in V_\lambda^{(A)}$;

(ii) $\operatorname{Stab}_{P_{J^c_{\min}}} q_\lambda(x_{J_{\min}}) = \{g \in P_{J^c_{\min}} \mid \pi_A^{J^c_{\min}}(g) \in G^\sigma_{A,\mathrm{ad}}\}$;

(iii) $\operatorname{Stab}_{P_{J^c_{\min}}} p_\lambda(x_{J_{\min}}) = \{g \in P_{J^c_{\min}} \mid \pi_A^{J^c_{\min}}(g) \in G^\sigma_{A,\mathrm{ad}}\}$.

*Proof.* The first assertion follows immediately from Lemma 6(iv). We now prove (ii). First we notice that by $G$-equivariance and by $q_\lambda(x_{J_{\min}}) = [h_{J_{\min}}]$ we have that $\operatorname{Stab}_G[h_{J_{\min}}] \supset \operatorname{Stab}_G(x_{J_{\min}}) = \{g \in P_{J^c_{\min}} \mid \pi_{J^c_{\min}}(g) \in G^\sigma_{J^c_{\min}}\} \supset H_{J^c_{\min}} \cdot \operatorname{Rad}_{\mathrm{ris}} P_{J^c_{\min}}$ where $\operatorname{Rad}_{\mathrm{ris}}$ denotes the solvable radical. In particular, $\operatorname{Stab}_G[h_{J_{\min}}]$ contains $\operatorname{Rad}_{\mathrm{ris}} P_{J^c_{\min}}$. Now we look at the action of $G_{J^c_{\min}}$. The set $J^c_{\min}$ is the disjoint union of the two subsets $A$ and $B = \Delta \smallsetminus \{I \cup A \cup \partial A\}$ that are not linked to each other since at least they are separated by $\partial A$ so $G_{J^c_{\min}} = G_A \times G_B$. Now $B \cap I = \varnothing$ so $G_B$ acts trivially on the irreducible $G_A$-module $V_\lambda^{(A)}$ so we have to compute $\operatorname{Stab}_{G_A}(q_\lambda(x_{J_{\min}}))$. Notice that the identity connected component $H^0_A$ of $H_A$ is contained in the identity component $H^0$ of $H$ and that it commutes with $\gamma_{J_{\min}}$ so $H^0_A$ fixes $h_{J_{\min}}$. Hence $h_{J_{\min}}$ is a spherical vector for $G_A$ in the irreducible $G_A$-module $V_\lambda^{(A)}$. So $\operatorname{Stab}_{G_A}[h_{J_{\min}}] = H_A$ by the result of Lemma 2.3 in [2] recalled above, applied to the group $G_A$.

The proof in the case of the stabilizer of $p_\lambda(x_{J_{\min}})$ is similar. First notice that $\operatorname{Stab}_G x_{J_{\min}} \subset \operatorname{Stab}_G p_\lambda(x_{J_{\min}}) \subset \operatorname{Stab}_G q_\lambda(x_{J_{\min}})$, in particular, $H_A \times \operatorname{Rad}_{\mathrm{ris}} P_{J^c_{\min}} \subset H_{J^c_{\min}} \times \operatorname{Rad}_{\mathrm{ris}} P_{J^c_{\min}} \subset \operatorname{Stab}_G p_\lambda(x_{J_{\min}})$. Then it remains to analyze the action of $G_B$. Let $k \in \Gamma^*_\lambda$ be such that $[k] = p_\lambda(x_0)$ and write it as a sum $k = \sum k_\mu$ with $\mu \in \Omega^+$, $\mu \leqslant_\sigma \lambda$ and $k_\mu \in V_\mu \subset \Gamma^*_\lambda$. Define $k_{\mu,J_{\min}} = \lim_{t\to\infty} t^{-\langle \mu, \gamma_{J_{\min}}\rangle} \gamma_{J_{\min}}(t) k_\mu$ and as in the proof of Lemma 7 after formula (1) we have

$$p_\lambda(x_{J_{\min}}) = \Bigg[ \sum_{\operatorname{supp}_{\tilde\Delta}(\lambda-\mu) \subset A} k_{\mu,J_{\min}} \Bigg].$$

Now we have that $I_\mu = \operatorname{supp}_\Omega \mu \subset I \cup A \cup \partial A$, in particular, it does not intersect $B$. Moreover, $A_\mu = \Delta_{I_\mu}(J_{\min}) \subset A$ and by part (i) we have $k_{\mu,J_{\min}} \in V_\mu^{(A_\mu)} \subset V_\mu^{(A)}$ on which $G_B$ acts trivially. □

We now want to prove that these stabilizers coincide with the complete stabilizers of these points in $G$.

**Theorem 14.** *Let $\lambda \in \Omega^+$, $I = \operatorname{supp}_\Omega \lambda$, $A \in \mathcal{A}_I$ and set $J_{\min} = J_{\min}(A)$. Then $\operatorname{Stab}_G q_\lambda(x_{J_{\min}}) = \operatorname{Stab}_G p_\lambda(x_{J_{\min}}) = \{g \in P_{J^c_{\min}} \mid \pi_A^{J^c_{\min}}(g) \in G^\sigma_{A,\mathrm{ad}}\}$. In particular we have the following description of the corresponding orbits:*

$$O^Z_{\lambda,A} \simeq O^Y_{\lambda,A} \simeq G \times_{P_{J^c_{\min}}} G_A/H_A.$$

*Proof.* Let $M = \operatorname{Stab}_G p_\lambda(x_{J_{\min}})$ and $N = \operatorname{Stab}_G q_\lambda(x_{J_{\min}})$. If we prove that $M$ and $N$ are contained in $P_{J^c_{\min}}$ then the theorem would follow from the previous lemma. We analyze first the case of $M$.

Consider the restriction $\lambda\big|_{\mathfrak{t}_A}$ of $\lambda$ to the vector space $\mathfrak{t}_A$ spanned by $\{\tilde\alpha^\vee \mid \tilde\alpha \in A\}$. We can write $\lambda\big|_{\mathfrak{t}_A} = (1/n) \sum_{\tilde\alpha \in A} c_{\tilde\alpha} \tilde\alpha\big|_{\mathfrak{t}_A}$ where $n, c_{\tilde\alpha}$ are integers. Moreover, since $\lambda$ is dominant and each connected component of $A$ intersects $\operatorname{supp}_\Omega \lambda =$

$I$ the $c_{\tilde\alpha}$ are strictly positive. Let $\beta = \sum_{\tilde\alpha\in A} c_{\tilde\alpha}\tilde\alpha$. Hence $\operatorname{supp}_{\tilde\Delta}\beta = A$ and $\operatorname{supp}_\Omega(n\lambda-\beta) = (I\smallsetminus A)\cup\partial A = J_{\min}$. Let $\mu = n\lambda-\beta$ and consider $s^\beta V_\mu^* \subset \Gamma_{n\lambda}$. Now $\Gamma_{n\lambda}$ is by the definition of $Y_\lambda$ in its coordinate ring and since $\operatorname{supp}\beta = A$ the section $s^\beta$ is never zero on $O_{J_{\min}}$ and, more generally, on $O_B$ for all $B\subset J_{\min}$. So we have a well defined map $\phi$ from $p_\lambda(\tilde O)$ to $\mathbb{P}(V_\mu)$ where $\tilde O$ is the union of the orbits $O_B$ for $B\subset J_{\min}$. We have $\phi(p_\lambda(x_{J_{\min}})) = \lim_{t\to\infty}\gamma_{J_{\min}}(t)\phi(p_\lambda(x_\varnothing))$ and $\phi(p_\lambda(x_\varnothing))$ is $H$ invariant so it must coincide with $[h]$ where $h$ is a nonzero invariant vector in $V_\mu$. So by Lemma 6(iv) we have that $\phi(p_\lambda(x_{J_{\min}}))$ is the highest weight line in $V_\mu$ and its stabilizer is equal to $P_{J^c_{\min}}$. In particular, the stabilizer of $p_\lambda(x_{J_{\min}})$ is contained in $P_{J^c_{\min}}$ as claimed.

Now we want to prove the same statement for $N$. We know that $r_\lambda$ is a finite map so also its restriction to $p_\lambda(O_{J_{\min}})$ is a finite map, in particular the Lie algebras of $M$ and $N$ coincide. Let $\mathfrak{m}$ be this Lie algebra and by the previous analysis we have that $\mathfrak{m}$ is a semidirect product of $\operatorname{rad}_{\mathrm{ris}}\mathfrak{p}_{J^c_{\min}}$ and $\mathfrak{g}_B\oplus\mathfrak{h}^\sigma_A$. In particular, $\operatorname{rad}_{\mathrm{nil}}\mathfrak{m} = \operatorname{rad}_{\mathrm{nil}}(\operatorname{rad}_{\mathrm{ris}}\mathfrak{p}_{J^c_{\min}})$ since the other factor is reductive. Now the adjoint action of $N$ on its Lie algebra must preserve its nilpotent radical. Now notice that $P_{J^c_{\min}}$ also preserves this nilpotent radical, so the normalizer of $\operatorname{rad}_{\mathrm{nil}}\mathfrak{m}$ in $G$ is a parabolic containing $P_{J^c_{\min}}$. But a parabolic which strictly contains $P_{J^c_{\min}}$ has a strictly smaller nilpotent radical so the normalizer of $\operatorname{rad}_{\mathrm{nil}}\mathfrak{m}$ is equal to $P_{J^c_{\min}}$. In particular, $N\subset P_{J^c_{\min}}$ as claimed. $\square$

We can also describe the map from $O_J$ to $p_\lambda(O_J)$ for a general $J$. Let $I = \operatorname{supp}_\Omega\lambda$, $A = \Delta_I(J)$ and set $J_{\min} = J_{\min}(A)$ we have $P_{J^c}\subset P_{J^c_{\min}}$. If $B = J^c\smallsetminus A$ by the lemma notice that $G_{J^c} = G_A\times G_B$ so we have a projection from $G_{J^c}$ to $G_A$. The map from $O_J = G\times_{P_{J^c}} G_{J^c}/H_{J^c}$ to $p_\lambda(O_J) = G\times_{P_{J^c_{\min}}} G_A/H_A$ is induced by the inclusion $P_{J^c}\subset P_{J^c_{\min}}$ and the projection from $G_{J^c}$ to $G_A$. In particular, its fibre is isomorphic to

$$P_{J^c_{\min}}\times_{P_{J^c}} G_B/H_B.$$

In particular, for $J = J_{\min}$ the fibre is affine and isomorphic to a symmetric variety and for $J = J_{\max}(A) = \tilde\Delta\smallsetminus A$ the fibre is compact and isomorphic to $P_{J^c_{\min}}/P_{J^c_{\max}}$.

A simple corollary of the previous theorem is the following.

**Corollary 15.** *For all $\lambda\in\Omega^+$ the normalization $r_\lambda\colon Y_\lambda\to Z_\lambda$ is bijective.*

## References


[1] W. Bruns, A. Conca, *The variety of exterior powers of linear maps*, J. Algebra, to appear.

[2] R. Chirivì, C. De Concini, A. Maffei, *Normality of cones over symmetric varieties*, Tohoku Math. J. **58** (2006), 599–616.

[3] R. Chirivì, A. Maffei, *A note on normality of cones over symmetric varieties*, Preprint.

[4] C. De Concini, C. Procesi, *Complete symmetric varieties*, in: *Invariant Theory* (Montecatini, 1982), Lecture Notes in Mathematics, Vol. 996, Springer-Verlag, Berlin, 1983, pp. 1–44.

[5] C. De Concini, T. Springer, *Compactification of symmetric varieties*, Transform. Groups **4** (1999), no. 2–3, 273–300.

[6] F. Grosshans, *Algebraic Homogeneous Spaces and Invariant Theory*, Lecture Notes in Mathematics, Vol. 1673, Springer-Verlag, Berlin, 1997.

[7] Д. Тимашёв, *Эквивариантные компактификации редуктивных групп*, Мат. сб. **194** (2003), no. 4, 119–146. English transl.: D. Timashëv, *Equivariant compactifications of reductive groups*, Sb. Math. **194** (2003), no. 3–4, 589–616.

[8] T. Vust, *Plongements d'espaces symétriques algébriques: Une classification*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (4) **17** (1990), no. 2, 65–195.