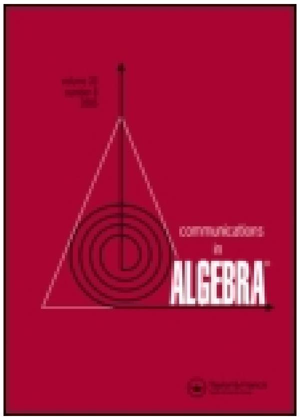

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A NOTE ON NORMALITY OF CONES OVER SYMMETRIC VARIETIES

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Let G be a semisimple and simply connected algebraic group, and let H^0 be the subgroup of points fixed by an involution of G. Let V be an irreducible representation of G with a nonzero vector v fixed by H^0. In this article, we prove a property of the normalization of the coordinate ring of the closure of $G \cdot [v]$ in $\mathbb{P}(V)$.

Key Words: Complete symmetric variety; Projective normality.

2000 Mathematics Subject Classification: Primary 14M17; Secondary 14L30.

INTRODUCTION

Let G be a semisimple and simply connected algebraic group over an algebraically closed field $\Bbbk$ of characteristic zero. Let σ be an involution of G, $H^o = G^\sigma$ the set of points fixed by σ, and H the normalizer of H^o in G. We denote by X the wonderful compactification of G/H introduced by De Concini and Procesi in [4].

If V is an irreducible representation of G, we say that it is *spherical* if $V^{H^o} \neq \{0\}$. In this case, let h_V be a nonzero vector fixed by H^o. The map $g \mapsto g \cdot [h_V]$ from G to $\mathbb{P}(V)$ determines a map q_V from X to $\mathbb{P}(V)$, and we denote its image by Z_V. Let also $\mathscr{L}_V$ be the pullback of the line bundle $\mathscr{O}_{\mathbb{P}(V)}(1)$ through the map q_V. Let A_V be the ring $\bigoplus_{n\geq 0} \Gamma(X, \mathscr{L}_V^n)$ and B_V the projective coordinate ring of $Z_V \subset \mathbb{P}(V)$.

In [2] we have proved that A_V is the integral closure of B_V; in that article, following an argument of Brion, this result was deduced by a general argument using the results contained in [3]. In this note we present a new proof which is quite longer than the one given by Brion, but which gives a slightly more precise result; this result was needed in a first version of [6].

NOTATIONS

We need to introduce a certain number of objects; for most of them, we use standard notations. For further details about the results given in this section one may look at [3] and the references there.

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Let G, σ, H^0, H, and $\Bbbk$ be as in the introduction and choose a maximal torus T of G that is σ stable and such that the dimension of the subtorus S given by the identity component of $\{t \in T : \sigma(t) = t^{-1}\}$ is the maximal possible. Choose also a Borel subgroup B containing T and such that the dimension of $\sigma(B) \cap B$ is minimal possible. Let Λ be the set of characters of T, $\Phi \subset \Lambda$ be the set of roots, and Φ^+ (resp., Δ) the choice of positive roots, (resp., simple roots) determined by B. Let also Λ^+ be the set of dominant weights with respect to these choices and, for $\lambda \in \Lambda^+$, let V_λ be the irreducible representation of G of highest weight λ.

We denote the Lie algebras of G and H by $\mathfrak{g}$ and $\mathfrak{h}$, respectively; further $\mathfrak{g}_\alpha$ is the root space of weight α and x_α is a nonzero element in $\mathfrak{g}_\alpha$. Let $\mathfrak{t}$ be the Lie algebra of T, we identify $\mathfrak{t}^*$ with $\Lambda \otimes_{\mathbb{Z}} \Bbbk$ and $\mathfrak{t}$ with $\mathrm{Hom}(\Bbbk^*, T) \otimes_{\mathbb{Z}} \Bbbk$. If $\alpha \in \Phi$, we denote by $\alpha^\vee \in \mathfrak{t}$ the corresponding coroot. More generally if $x \in \mathfrak{t}^*$ is not zero, we denote by $x^\vee$ the element of $\mathfrak{t}$ such that $\langle x^\vee, y\rangle = \frac{2\kappa(x,y)}{\kappa(x,x)}$ for all $y \in \mathfrak{t}$, where κ is the dual Killing form on $\mathfrak{t}^*$.

If $\alpha \in \Phi$ we define the associated *restricted root* as $\tilde{\alpha} = \alpha - \sigma(\alpha)$, and we set $\widetilde{\Phi}$ to be the set of nonzero restricted roots. This is a (possibly not reduced) root system, and $\widetilde{\Delta} = \{\tilde{\alpha} \neq 0 : \alpha \in \Delta\}$ is a simple basis for this system.

There are two possible dominant orders on the set of weights, the dominant order defined by Δ and the one defined by $\widetilde{\Delta}$; if $\lambda, \mu \in \Lambda$; we write $\lambda \leq \mu$ if $\mu - \lambda \in \mathbb{N}[\Delta]$ and $\lambda \leq_\sigma \mu$ if $\mu - \lambda \in \mathbb{N}[\widetilde{\Delta}]$.

Let also $\Phi_0 = \{\alpha \in \Phi : \sigma(\alpha) = \alpha\}$, $\Phi_1 = \Phi \setminus \Phi_0$, $\Delta_0 = \{\alpha \in \Delta : \sigma(\alpha) = \alpha\}$ and $\Delta_1 = \Delta \setminus \Delta_0$. If $\Delta' \subset \Delta$, we denote by $\Phi_{\Delta'} \subset \Phi$ the root subsystem generated by Δ'. We also denote by w_Δ the longest element of the Weyl group of Φ.

As recalled in the introduction, an irreducible representation is said to be *spherical* if it has a nonzero weight vector fixed by H^0. In particular, we say that $\lambda \in \Lambda^+$ is spherical, if V_λ is spherical and we denote by Ω^+ the set of spherical weights and by Ω the lattice spanned by Ω^+. Similarly, we say that a dominant weight is *quasi spherical* if $\mathbb{P}(V_\lambda)^H$ is not empty, in this case this set is just a single point that we denote by x_λ (see [5]). We denote by Π^+ the set of quasi spherical weights and by Π the sublattice generated by Π^+ in Λ. We have $\Omega \cap \Lambda^+ = \Omega^+$, $\Pi \cap \Lambda^+ = \Pi^+$ and, by a result of Helgason,

$$\Omega = \{\lambda \in \Lambda : \sigma(\lambda) = -\lambda \text{ and } \langle \tilde{\alpha}^\vee, \lambda\rangle \in \mathbb{Z} \text{ for all } \alpha \in \Phi\}.$$

In particular, Ω is the set of weights of $\widetilde{\Phi}$.

A wonderful compactification X of the symmetric variety G/H has been constructed in [4] in characteristic zero. The following theorem describes some of the main properties of X.

Theorem 1 (Theorem 3.1 in [4]).

(i) *X is a smooth projective G-variety.*
(ii) *$X \setminus G/H$ is a divisor with normal crossings and smooth irreducible components $\{X_{\tilde{\alpha}} : \tilde{\alpha} \in \widetilde{\Delta}\}$ parametrized in a canonical way by the simple roots of the restricted root system.*
(iii) *The closures of G-orbits are given by the subvarieties $X_J = \bigcap_{\tilde{\alpha} \in J} X_{\tilde{\alpha}}$ for J any subset of $\widetilde{\Delta}$, in particular there exists only one closed orbit namely $X_{\widetilde{\Delta}}$.*

For each $\lambda \in \Pi^+$, the map $G/H \longrightarrow \mathbb{P}(V_\lambda)$ defined by $gH \mapsto g \cdot x_\lambda$ extends to a morphism from X to $\mathbb{P}(V_\lambda)$, and we denote by $\mathcal{L}_\lambda$ the inverse image of $\mathcal{O}_{\mathbb{P}(V_\lambda)}(1)$ through this morphism. These line bundles generate the Picard group of X, which, in particular, we will identify with the lattice Π ([5], Proposition 8.1). The divisors $X_{\tilde{\alpha}}$ can be parametrized in such a way that $\mathcal{O}(X_{\tilde{\alpha}}) \simeq \mathcal{L}_{\tilde{\alpha}}$. There exists a G-invariant section $s_{\tilde{\alpha}} \in \Gamma(X, \mathcal{L}_{\tilde{\alpha}})$ whose divisor is $X_{\tilde{\alpha}}$.

If $\mu \in \Pi^+$, then the module V_μ^* appears with multiplicity 1 in $\Gamma(X, \mathcal{L}_\mu)$. For an element $\nu = \sum_{\tilde{\alpha} \in \tilde{\Delta}} n_{\tilde{\alpha}} \tilde{\alpha} \geq_\sigma 0$ the multiplication by $s^\nu \doteq \Pi_{\tilde{\alpha}} s_{\tilde{\alpha}}^{n_{\tilde{\alpha}}}$ injects $\Gamma(X, \mathcal{L}_{\lambda-\nu})$ in $\Gamma(X, \mathcal{L}_\lambda)$. If $\lambda - \mu \geq_\sigma 0$, we denote by $s^{\lambda-\mu} V_\mu^* \subset \Gamma(X, \mathcal{L}_\lambda)$ the image of V_μ^* under the multiplication by $s^{\lambda-\mu}$. We have the following theorem.

Theorem 2 (Theorem 5.10 [4]). *Let $\lambda \in \Pi$ then $\Gamma(X, \mathcal{L}_\lambda) = \bigoplus_{\mu \leq_\sigma \lambda, \mu \in \Pi^+} s^{\lambda-\mu} V_\mu^*$.*

PRELIMINARIES AND STATEMENT OF THE THEOREM

The main objects of this article are the following two rings: Given $\lambda \in \Omega^+$ and a natural number n, let $A_n(\lambda) = \Gamma(X, \mathcal{L}_{n\lambda})$, and define $A(\lambda)$ as the graded ring $\bigoplus_{n \in \mathbb{N}} A_n(\lambda)$ and $B(\lambda)$ as the subring of $A(\lambda)$ generated by the module $V_\lambda^* \subset A_1(\lambda)$. Further, denote by $B_n(\lambda)$ the homogeneous component $B(\lambda) \cap A_n(\lambda)$ of $B(\lambda)$.

In this article, we prove the following result.

Theorem 3. *Let $\lambda, \mu \in \Omega^+$ and $\mu \leq_\sigma \lambda$. Then there exists a positive integer n such that $s^{n(\lambda-\mu)} V_{n\mu}^* \subset B_n(\lambda)$.*

The following corollary, which is an immediate consequence, was needed in [6].

Corollary 4. *Let $\lambda, \mu \in \Omega^+$ and $\mu \leq_\sigma \lambda$. Let f be a highest weight vector in $s^{\lambda-\mu} V_\mu^* \subset A_1(\lambda)$. Then there exists a positive integer n such that $f^n \in B(\lambda)$.*

Now we can use this result to prove the following proposition.

Proposition 5. *$A(\lambda)$ is the integral closure of $B(\lambda)$.*

This result was already stated in [2]. Its proof is in the following three steps: (1) $A(\lambda)$ is integral over $B(\lambda)$, (2) $A(\lambda)$ is integrally closed, and (3) the two rings have the same quotient field. The last step was proved in [2], but the proof there contains a gap that we fill now.

Proof. We know that $A(\lambda)$ is generated in degree one (see [3]), hence the highest weight vectors of $A_1(\lambda)$ generate $A(\lambda)$ as a G–algebra. Since these vectors are integral over $B(\lambda)$ by Corollary 4 and Theorem 2, $A(\lambda)$ is integral over $B(\lambda)$. On the other hand, $A(\lambda)$ is integrally closed as proved in [3], so we just need to show that $A(\lambda)$ and $B(\lambda)$ have the same quotient field.

Using again that $A(\lambda)$ is generated in degree one, we have that $Y_A = \operatorname{Spec} A(\lambda) \subset A_1(\lambda)^*$ is the cone over $\operatorname{Proj} A(\lambda) \subset \mathbb{P}(A_1(\lambda)^*)$. Recall that, by definition, $Y_B = \operatorname{Spec} B(\lambda) \subset V_\lambda$ is the cone over $Z_{V_\lambda} \subset \mathbb{P}(V_\lambda)$. The map $\varphi : Y_A \longrightarrow Y_B$ determined by the inclusion $B(\lambda) \subset A(\lambda)$ is given by the restriction to Y_A of the G–equivariant projection from $A_1(\lambda)^*$ to V_λ.

Let $a \in Y_A$ be a nonzero vector in the line fixed by H, and let $b = \varphi(a)$ be a nonzero multiple of the vector h_{V_λ}.

The group $K = G \times \mathbb{k}^*$ acts on Y_A and Y_B with the second factor acting with multiplication by scalars. The map φ is K equivariant, and the orbits $K \cdot a$ and $K \cdot b$ are dense in Y_A and Y_B, respectively.

We now prove that the points a and b have the same stabilizers; in particular this proves that φ is birational and completes the proof that $A(\lambda)$ and $B(\lambda)$ have the same quotient field.

We argue by induction on the dimension of G. We assume first that $G = G_1 \times G_2$ with G_1 and G_2 nontrivial σ–stable connected subgroups. Let $K_i = G_i \times \mathbb{k}^*$ for $= 1, 2$ so that $K = G \times \mathbb{k}^* = (K_1 \times K_2) \cap (G_1 \times G_2 \times \Delta\mathbb{k}^*)$. In this case V_λ is the tensor product $V_{\lambda_1} \otimes V_{\lambda_2}$ of two irreducible representations of G_1 and G_2, respectively. Also by the description of the sections, we have that $A_1(\lambda) = A_1(\lambda_1) \otimes A_1(\lambda_2)$. So $a = a_1 \otimes a_2$ and $b = b_1 \otimes b_2$, where a_i, b_i are points in the line fixed by H in $A_1(\lambda_i)^*$ and V_{λ_i}. By induction, we have $\mathrm{Stab}_{K_i} a_i = \mathrm{Stab}_{K_i} b_i$ for $i = 1, 2$, so

$$\begin{aligned} \mathrm{Stab}_K a &= (\mathrm{Stab}_{K_1} a_1) \times (\mathrm{Stab}_{K_2} a_2) \cap (G \times \Delta\mathbb{k}^*) \\ &= (\mathrm{Stab}_{K_1} b_1) \times (\mathrm{Stab}_{K_2} b_2) \cap (G \times \Delta\mathbb{k}^*) \\ &= \mathrm{Stab}_K b. \end{aligned}$$

So we can assume that G cannot be written as the product of two groups as above. In this case, we say that the involution is *simple*.

If λ is zero, the statement is trivial. If λ is not zero, in Lemma 2.3 in [2], using a simple dimension argument, it is proved that the points $[a] \in \mathbb{P}(A_1(\lambda)^*)$ and $[b] \in \mathbb{P}(V_\lambda)$ have the same stabilizer in G and this is equal to H. (In [2] we conclude from this result that Y_A and Y_B are birational without any further explanation; we give the complete argument here.)

In particular, the stabilizers of a and b are contained in $H \times \mathbb{k}^*$. More precisely if χ_a is the character given by the action of H on the line $\mathbb{k}\, a$ and χ_b is the character given by the action of H on the line $\mathbb{k}\, b$, then $\mathrm{Stab}_K(a) = \{(h, \chi_a(h)^{-1}) \in H \times \mathbb{k}^*\}$ and $\mathrm{Stab}_K(b) = \{(h, \chi_b(h)^{-1}) \in H \times \mathbb{k}^*\}$. Now the thesis follows since $\chi_a(h) b = \varphi(\chi_a(h) a) = \varphi(h \cdot a) = h \cdot b = \chi_b(h)\, b$ and hence $\chi_a = \chi_b$. □

The proof of Theorem 3 will be by induction on the dimension of the variety X. However, in any dimension, it will remain to analyze some particular cases. To deal with these cases, we need a sharper version of Lemma 3.1 in [3]. If $\lambda, \mu \in \Omega$ let

$$m_{\lambda,\mu} : \Gamma(X, \mathscr{L}_\lambda) \otimes \Gamma(X, \mathscr{L}_\mu) \to \Gamma(X, \mathscr{L}_{\lambda+\mu})$$

be the multiplication map. The proof of Lemma 3.1 in [3] gives the following result.

Lemma 6. *Let $\lambda \in \Omega^+$ and let $\mu = -w_\Delta \lambda$. Consider the modules $V_\lambda^* \subset \Gamma(X, \mathscr{L}_\lambda)$ and $V_\mu^* \subset \Gamma(X, \mathscr{L}_\mu)$. Then $s^{\lambda+\mu} V_0^* \subset m_{\lambda,\mu}(V_\lambda^* \otimes V_\mu^*)$.*

The induction will be performed using the closure of G orbits X_I. We recall some basic facts about these varieties. Let $I \subset \widetilde{\Delta}$, set $J = \widetilde{\Delta} \setminus I$, $\Delta(J) = \Delta_0 \cup \{\alpha \in \Delta_1 \mid \tilde{\alpha} \in J\}$, denote by G_J the semisimple part of the Levi associated to $\Phi_{\Delta(J)}$, and let

P_J be the parabolic of G containing B whose Levi factor is G_J. We denote by σ_J the restriction of σ to the subgroup G_J. Let H_J be the normaliser of the subgroup of fixed points of σ_J in G_J, and let $X(J)$ be the wonderful compactification of the symmetric variety G_J/H_J. Notice that the center of G_J acts trivially on $X(J)$, so also P_J acts on this variety through its adjoint semisimple quotient. By [4] §5, we have an equivariant isomorphism

$$X_I \simeq G \times_{P_J} X(J).$$

In particular, we denote the subset $1 \cdot X(J) \subset X_I$ by F_J and the inclusion of F_J in X by $\jmath_J$.

We want to describe some properties of the inclusion $\jmath_J$ proved in [3] §2. Let Λ_J be the lattice of integral weights of $\Phi_{\Delta(J)}$, and denote by $\Lambda_J^+ \subset \Lambda_J$ the monoid of dominant weights with respect to $\Delta(J)$. For $\lambda \in \Lambda_J^+$, we denote by $V_\lambda^{(J)}$ the irreducible representation of G_J of highest weight λ. Observe that the inclusion $T_J \hookrightarrow T$ induces a map $r_J : \Lambda \longrightarrow \Lambda_J$. Let $\Omega_J \subset \Lambda_J$ be the sublattice generated by spherical weights whit respect to (G_J, σ_J), and notice that by definition $r_J(\Omega) \subset \Omega_J$.

We identify $\mathrm{Pic}(F_J)$ with the sublattice Π_J of Λ_J. This sublattice contains Ω_J, and we denote by $\mathscr{L}_\lambda$ a line bundle of F_J associated to the weight $\lambda \in \Lambda_J$.

For $\tilde{\alpha} \in J$, we choose $s_{J,\tilde{\alpha}}$ to be a nonzero G_J invariant section of $\Gamma(F_J, \mathscr{L}_{\tilde{\alpha}})$. Finally, we recall that we have $\Gamma(F_J, \mathscr{L}_\lambda) = \bigoplus_{\mu \in \Lambda_J^+, \mu \leq_{\sigma_J} \lambda} s_J^{\lambda-\mu}\big(V_\mu^{(J)}\big)^*$ for any $\lambda \in \Pi_J$.

Proposition 7 (Lemma 2.5 and Lemma 2.7 in [3]).

(i) *If* $\lambda \in \Pi \subset \Lambda$, *then* $\jmath_J^*(\mathscr{L}_\lambda) \simeq \mathscr{L}_{r_J(\lambda)}$.

(ii) *Up to rescaling the sections* $s_{J,\tilde{\alpha}}$ *by nonzero constant factors we have* $\jmath_J^*(s_{\tilde{\alpha}}) = s_{J,\tilde{\alpha}}$ *for all* $\tilde{\alpha} \in J$.

(iii) *Let* $\lambda \in \Pi$, $\mu \in \Lambda^+$ *with* $\mu \leq_{\sigma_J} \lambda$, *and let* φ *be a (nonzero) lowest weight vector in* $s^{\lambda-\mu}V_\mu^*$. *Then* $\jmath_J^*(\varphi)$ *is a (nonzero) lowest weight vector in* $s_J^{\lambda-\mu}\big(V_{r_J(\mu)}^{(J)}\big)^* \subset \Gamma(F_J, \mathscr{L}_{r_J(\lambda)})$.

1. PROOF OF THEOREM 3

We now need to introduce some notations and to recall some results on the dominant order by Stembridge [7]. Given two dominant weights $\lambda, \mu \in \Lambda^+$, we write $\lambda \stackrel{\sigma}{\longrightarrow} \mu$ if λ *covers* μ with respect to $\leq_\sigma$; this means that $\mu \leq_\sigma \lambda$ and that if $\mu \leq_\sigma \eta \leq_\sigma \lambda$, then either $\eta = \lambda$ or $\eta = \mu$. Recall that the set Ω^+ of spherical weights is identified with the set of dominant weights of the restricted root system $\widetilde{\Phi}$. Also the longest element $\tilde{w}_{\widetilde{\Delta}}$ of the Weyl group of $\widetilde{\Phi}$ and the longest element w_Δ of the Weyl group of Φ act in the same way on $\Omega_{\mathbb{R}}^+$.

In the following, supposing $\widetilde{\Phi}$ to be irreducible, we will denote by $\tilde{\theta}$ the unique short dominant root. Since $\widetilde{\Phi}$ may be not reduced, i.e., of type BC_ℓ, we want to add some explanation about this type. We think $\widetilde{\Phi}$ of type BC_ℓ as the union of a type B_ℓ root system with square root lengths 1, 2 and of a type C_ℓ root system with square root lengths 2, 4; the base $\widetilde{\Delta} = \{\tilde{\alpha}_1, \ldots \tilde{\alpha}_\ell\}$ is that of B_ℓ with $\tilde{\alpha}_\ell$ the unique simple root such that $2\tilde{\alpha}_\ell \in \widetilde{\Phi}$, while the fundamental weights $\tilde{\omega}_1, \ldots \tilde{\omega}_\ell$ are those of C_ℓ; finally, $\tilde{\theta} = \tilde{\omega}_1$ is the shortest dominant root.

Given a weight $\eta \in \Omega$, we let $\mathrm{supp}_{\tilde\Delta}(\eta)$ be the set of restricted simple roots $\tilde\alpha$ such that $\eta(\tilde\alpha)$ is nonzero.

Lemma 8. *Suppose that $\widetilde\Phi$ is irreducible, and let $\lambda, \mu \in \Omega^+$ be such that $\lambda \xrightarrow{\sigma} \mu$ with $\mathrm{supp}_{\tilde\Delta}(\lambda - \mu) = \tilde\Delta$. Then we have the following possibilities (the simple roots and the fundamental weights are numbered as in Bourbaki [1]):*

(1) *$\widetilde\Phi$ is of type A_1 and $\lambda = m\tilde\omega_1$, $\mu = (m-2)\tilde\omega_1$, $m \geq 2$;*
(2) *$\lambda = \tilde\theta$ (with $\tilde\theta$ the short dominant root of $\widetilde\Phi$) and $\mu = 0$; further, this is the unique possibility if $\widetilde\Phi$ is of type BC_ℓ with $\ell \geq 2$;*
(3) *$\widetilde\Phi$ is of type B_ℓ and $\lambda = \tilde\omega_1 + \tilde\omega_\ell$, $\mu = \tilde\omega_\ell$;*
(4) *$\widetilde\Phi$ is of type G_2 and either $\lambda = \tilde\omega_2$, $\mu = \tilde\omega_1$ or $\lambda = \tilde\omega_1 + \tilde\omega_2$, $\mu = 2\tilde\omega_1$;*
(5) *$\widetilde\Phi$ is of type BC_1 and $\lambda = m\tilde\omega_1$, $\mu = (m-1)\tilde\omega_1$ with $m \geq 1$.*

Proof. For reduced root systems, the cases (1), (2), (3), and (4) are, respectively, consequence of cases (a), (b), (c), and (d) of Theorem 2.8 in [7].

So suppose $\widetilde\Phi$ of type BC_ℓ, and suppose also $\ell \geq 2$. Recall our convention about $\widetilde\Phi$ being the union of B_ℓ and of C_ℓ. In the sequel of this proof, we will add a B to denote the corresponding object of B_ℓ. For example, we have $\tilde\omega_\ell = 2\tilde\omega_\ell^B$.

By $\mathrm{supp}_{\tilde\Delta}(\lambda-\mu) = \tilde\Delta$ and $\lambda, \mu \subset \Omega^+ \subset \Lambda_B$, we have that λ and μ are two dominant weights of the root system B_ℓ satisfying the hypothesis of the lemma. So there are two possibilities corresponding to (2) and (3). In case (2), we have $\lambda = \tilde\theta^B = \tilde\alpha_1 + \cdots + \tilde\alpha_\ell = \tilde\omega_1^B = \tilde\omega_1$ and $\mu = 0$; this is our claim about type BC_ℓ in (2). In case (3), we have $\lambda = \tilde\omega_1^B + \tilde\omega_\ell^B$, $\mu = \tilde\omega_\ell^B$; but this is impossible since $\tilde\omega_\ell^B \notin \Omega$.

For $\ell = 1$, the claim in (5) is trivial using $\tilde\alpha_1 = \tilde\omega_1$. □

We can now prove Theorem 3.

Proof of Theorem 3. We proceed by induction on $\dim X$. If X is a point, there is nothing to prove. Also if $\widetilde\Phi$ is not simple, we can write $G = G_1 \times G_2$, G_1 and G_2 being proper subgroups, and there exist two involutions $\sigma_i : G_i \to G_i$, $i = 1, 2$, in such a way that $\sigma = \sigma_1 \times \sigma_2$ and $X = X(\sigma_1) \times X(\sigma_2)$; in this case $\mathrm{Pic}(X) = \mathrm{Pic}(X(\sigma_1)) \oplus \mathrm{Pic}(X(\sigma_2))$ and, given $\mathscr{L} = (\mathscr{L}_1, \mathscr{L}_2) \in \mathrm{Pic}(X)$, we have $\Gamma(X, \mathscr{L}) = \Gamma(X(\sigma_1), \mathscr{L}_1) \otimes \Gamma(X(\sigma_2), \mathscr{L}_2)$. So our claim follows by induction on the dimension. Hence we may assume that X is simple (so $\widetilde\Phi$ is irreducible) and the claim true for lower dimensional complete symmetric varieties. In what follows, given a weight $\eta \in \Omega^+$, we choose a lowest weight vector φ_η in V_η^*.

We proceed in three steps.

First Step. Here we prove our claim assuming also $\mathrm{supp}_{\tilde\Delta}(\lambda-\mu) \neq \tilde\Delta$. We use induction on dimension. Let $I = \mathrm{supp}_{\tilde\Delta}(\lambda-\mu)$, $J = \tilde\Delta \setminus I$. Consider the variety X_I and the fibration $\pi_I : X_I \to G/P_J$ with fiber F_J. Recall that F_J is the wonderful compactification of the symmetric variety associated to (G_J, σ_J). Given $\eta \in \Omega^+$, let $\psi_{r(\eta)}$ be a lowest weight vector in $V_{r(\eta)}^{(J)}$.

Clearly, $\dim F_J < \dim X$, hence the claim is true for F_J. So there exists $n > 0$ such that

$$s^{n(r(\lambda)-r(\mu))}\big(V_{nr(\mu)}^{(J)}\big)^* \subset B_n(F_J, r(\lambda)),$$

where $r = r_J$ and $B_n(F_J, r(\lambda))$ is the part of degree n of the subring $B(F_J, r(\lambda))$ of $\bigoplus_{n\geq 0} \Gamma(F_J, \mathscr{L}_{nr(\lambda)})$ generated by $\big(V^{(J)}_{r(\lambda)}\big)^* \subset \Gamma(F_J, \mathscr{L}_{r(\lambda)})$. In particular, if ψ is a nonzero lowest weight vector of $s^{r(\lambda)-r(\mu)}\big(V^{(J)}_{r(\mu)}\big)^*$, then $\psi^n \in B(F_J, r(\lambda))$.

Consider the multiplication map

$$m^J : (\Gamma(F_J, \mathscr{L}_{r(\lambda)}))^{\otimes n} \supset \big((V^{(J)}_{r(\lambda)})^*\big)^{\otimes n} \to B_n(F_J, r(\lambda)).$$

Since G_J is linearly reductive and m^J is G_J equivariant, there exists a lowest weight vector $\psi \in \big((V^{(J)}_{r(\lambda)})^*\big)^{\otimes n}$ such that $m^J(\psi) = (s^{r(\lambda)-r(\mu)}\psi_{r(\mu)})^n$. We can write

$$\psi = \sum_{h=1}^{N} x_{h,1}\psi_{r(\lambda)} \otimes \cdots \otimes x_{h,n}\psi_{r(\lambda)}$$

for some $x_{h,k} \in \mathbf{U}(\mathfrak{u}^+_J) \subset \mathbf{U}(\mathfrak{g}_J) \subset \mathbf{U}(\mathfrak{g})$, the universal enveloping algebra of the positive unipotent part $\mathfrak{u}^+_J$ of the Lie algebra $\mathfrak{g}_J$ of G_J. Consider now

$$\varphi = \sum_{h=1}^{N} x_{h,1}\varphi_\lambda \otimes \cdots \otimes x_{h,n}\varphi_\lambda.$$

One can show, as in the proof of Lemma 2.8 in [3], that φ is a lowest weight vector in $(V^*_\lambda)^{\otimes n}$ of weight $-n\mu$. Let m be the multiplication map $\Gamma(X, \mathscr{L}_\lambda)^{\otimes n} \supset (V^*_\lambda)^{\otimes n} \to B_n(\lambda)$. Notice that $m(\varphi)$ is a lowest weight vector of weight $-n\mu$ provided it is different from zero. So if we show $m(\varphi) \neq 0$, we have finished. But

$$\begin{aligned} j^*_J(m(\varphi)) &= m^J(j^*_J(\varphi)) \\ &= m^J\bigg(\sum_{h=1}^{N} j^*_J(x_{h,1}\varphi_\lambda) \otimes \cdots \otimes j^*_J(x_{h,n}\varphi_\lambda)\bigg) \\ &= m^J\bigg(\sum_{h=1}^{N} x_{h,1}j^*_J(\varphi_\lambda) \otimes \cdots \otimes x_{h,n}j^*_J(\varphi_\lambda)\bigg) \\ &= m^J(\psi) \\ &\neq 0, \end{aligned}$$

where the last equality follows since $j^*_J(\varphi_\lambda) = \psi_{r(\lambda)}$ by Proposition 7.

Second Step. Now suppose $\lambda = m\lambda'$ and $\mu = m\mu'$ for a positive integer m, and suppose also that λ', μ' are such that $\lambda' \xrightarrow{\sigma} \mu'$ with $\mathrm{supp}_{\widetilde{\Delta}}(\lambda' - \mu') = \widetilde{\Delta}$.

By Lemma 8 this happens in few situations and, for such values of λ' and μ', we explicitly find the integer n satisfying the claim. We will prove, more precisely, that a power of the lowest weight vector in $s^{m\lambda'-m\mu'}V^*_{m\mu'}$ lies in $B_n(m\lambda')$. Our proof relies on Lemma 6 and on the result obtained in the first step. The different possibilities are the following (as above the numbering of simple roots and fundamental weights is as in Bourbaki [1]). In what follows we compare powers of weight vectors, any equation of this sort is intended up to nonzero scalar factor.

(1) $\widetilde{\Phi}$ is of type A_1, $\lambda' = k\tilde{\omega}_1$, $\mu' = (k-2)\tilde{\omega}_1$, $k \geq 2$.

In this case we can take $n = k$ (independently on m). Indeed we have the following identities:

$$(s^{\lambda-\mu}\varphi_\mu)^k = s^{2km\tilde{\omega}_1}\varphi_{m\tilde{\omega}_1}^{(k-2)k} = (s^{2km\tilde{\omega}_1}\varphi_0)(\varphi_\lambda)^{k-2}.$$

Now notice that $s^{2km\tilde{\omega}_1}\varphi_0 \in m_{\lambda,\lambda}(V_\lambda^* \otimes V_\lambda^*)$ by Lemma 6, hence our claim.

(2) $\widetilde{\Phi}$ is of type BC_1, $\lambda' = k\tilde{\omega}_1$ and $\mu' = (k-1)\tilde{\omega}_1$.
Proceeding as in the previous case we can show that we can take $n = 2k$.

(3) $\lambda' = \tilde{\theta}$ and $\mu' = 0$.
In this case we can take $n = 2$. Indeed notice that $-w_\Delta\lambda = -\tilde{w}_{\widetilde{\Delta}}\lambda = \lambda$ being $\tilde{\theta}$ the unique short (or shortest for BC_ℓ) dominant root, so we find $s^{2\lambda}V_0^* \subset m_{\lambda,\lambda}(V_\lambda^* \otimes V_\lambda^*)$ by Lemma 6. Hence $(s^{\lambda-\mu}\varphi_\mu)^2 = (s^\lambda\varphi_0)^2 \in B_2(\lambda)$.

(4) The following three cases are still left out:

(i) $\widetilde{\Phi}$ of type B_ℓ, $\lambda' = \tilde{\omega}_1 + \tilde{\omega}_\ell$, $\mu' = \tilde{\omega}_\ell$;
(ii) $\widetilde{\Phi}$ of type G_2, $\lambda' = \tilde{\omega}_2$, $\mu' = \tilde{\omega}_1$,
(iii) $\widetilde{\Phi}$ of type G_2, $\lambda' = \tilde{\omega}_1 + \tilde{\omega}_2$ and $\mu' = 2\tilde{\omega}_1$.

In all these cases we notice that there exist natural numbers $k > h > 0$ such that $h\lambda' \geq_\sigma k\mu'$ and $\mathrm{supp}_{\widetilde{\Delta}}(h\lambda' - k\mu') \neq \widetilde{\Delta}$. Indeed we can choose $h = \ell$ and $k = \ell + 2$ in the first case, $h = 2$ and $k = 3$ in the second case, and $h = 4$ and $k = 5$ in the third case.
Then by what we have proved in the first step, there exists $n > 0$ such that $s^{n(h\lambda-k\mu)}V_{nk\mu}^* \subset B_n(h\lambda) \subset B_{nh}(\lambda)$.
Notice also that $-w_\Delta(\lambda) = \lambda$ so by Lemma 6 we have $s^{2\lambda}V_0^* \subset B_2(\lambda)$.
Hence

$$s^{2nk\lambda-2nk\mu}\varphi_\mu^{2nk} = \left(s^{n(h\lambda-k\mu)}\varphi_{nk\mu}\right)^2\left(s^{2\lambda}\varphi_0\right)^{n(k-h)} \in B_{2nk}(\lambda).$$

Third Step. Conclusion. Let $\lambda = \lambda_0 \xrightarrow{\sigma} \lambda_1 \cdots \xrightarrow{\sigma} \lambda_m = \mu$ be a sequence of covers from λ to μ. We argue by induction on m. If $m = 1$, then the claim is contained in the result of the first step in the case $\mathrm{supp}_{\widetilde{\Delta}}(\lambda - \mu) \neq \widetilde{\Delta}$ and in the result of the second step in the case $\mathrm{supp}_{\widetilde{\Delta}}(\lambda - \mu) = \widetilde{\Delta}$. So assume $m > 1$ and that the statement is true for $m - 1$. Let $\nu = \lambda_{m-1}$, then by induction there exists n_1 such that $s^{n_1(\lambda-\nu)}V_{n_1\nu}^* \subset B_{n_1}(\lambda)$. Hence the ring generated by this submodule is contained in $B(\lambda)$ or more explicitly $s^{n(n_1\lambda-n_1\nu)}B_n(n_1\nu) \subset B_{nn_1}(\lambda)$ for any natural n. Now we consider the cover $\nu \xrightarrow{\sigma} \mu$. By using what we proved in first step, and in the second step, we have that there exists $n_2 > 0$ such that $s^{n_2(n_1\nu-n_1\mu)}V_{n_2n_1\mu}^* \subset B_{n_2}(n_1\nu)$ and multiplying by $s^{n_2(n_1\lambda-n_1\nu)}$ and using the previous inclusion we obtain $s^{n(\lambda-\mu)}V_{n\mu}^* \subset B_n(\lambda)$ where we have set $n = n_1n_2$. □

As a final remark we notice that one can easily extend Theorem 3 from weights in Ω^+ (i.e., spherical weights) to weights in Π^+ (i.e., quasi spherical weights). For example a line of proof may be adapted from [3] (see the end of proof of Theorem A starting in the first paragraph of p. 109 in [3]).

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