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Equations Defining Symmetric Varieties and Affine Grassmannians

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Suppose σ be a simple involution of a semisimple algebraic group G, and suppose H is the subgroup of G of points fixed by σ. If the restricted root system is of type A, C, or BC and G is simply connected, or if the restricted root system is of type B and G is of adjoint type, then we describe a standard monomial theory and the equations for the coordinate ring $\Bbbk[G/H]$ using the standard monomial theory and the Plücker relations of an appropriate (maybe infinite-dimensional) Grassmann variety.

Introduction

The aim of this paper is to describe the coordinate ring of a symmetric variety and certain rings related to its wonderful compactification. The main tool to achieve this goal is a (possibly infinite-dimensional) Grassmann variety associated to a pair consisting of a symmetric space and a spherical representation.

More precisely, let G be a semisimple algebraic group over an algebraically closed field $\Bbbk$ of characteristic 0, and let σ be a simple involution of G (i.e. $G \rtimes \{\mathrm{id}, \sigma\}$ acts irreducibly on the Lie algebra of G). Let $H = G^\sigma$ be the subgroup of fixed points. The quotient G/H is an affine variety, called a *symmetric variety*.

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A simple finite-dimensional G module V is called *spherical* (for H) if $V^H \neq 0$. In view of the results of Helgason [10] and Vust [27], these modules are parametrized by a submonoid Ω^+ of the dominant weights of a suitable root system, called the *restricted root system*. Hence, as a G module, $\Bbbk[G/H]$ is well understood, in the sense that it is the direct sum $\bigoplus_{V\text{spherical}} V^*$.

When Ω^+ is a free monoid and $V_1, \ldots, V_\ell$ are the spherical modules corresponding to the basis of Ω^+, then the space $\mathbb{V}^* = \bigoplus_{j=1}^{\ell} V_j^*$ is a canonical set of generators for $\Bbbk[G/H]$. The aim of the paper is to describe the relations among this set of generators. In Section 1, we introduce a condition (see Definition 3) on the monoid Ω^+ which guarantees that the relations among these generators are quadratic and we believe (see Conjecture 18) that this condition is also necessary. When this condition is fulfilled, we say that Ω^+ is *quadratic*. It is easy to check that this happens precisely when G is simply connected and the restricted root system is of type A, BC or C, or the group G is of adjoint type and the restricted root system is of type B. Our description of the relations in the set of generators $\mathbb{V}^*$ is obtained in these cases.

The main ingredient for such a description is the construction of a Richardson variety whose coordinate ring, as a G module, is isomorphic to the coordinate ring of G/H. Part of this construction can be carried out in a more general setting.

Fix a dominant weight ε in Ω^+. We add a node n_0 to the Dynkin diagram of G and, for all simple roots α, we join n_0 with the node n_α of the simple root α by $\varepsilon(\alpha^\vee)$ lines, and we put an arrow in direction of n_α if $\varepsilon(\alpha^\vee) \geq 2$. In the cases relevant for us, the Kac–Moody group ${}^e\!G$ associated to this extended diagram will be of finite or affine type but in general can be more complicated. Let ${}^e\!P$ be the maximal parabolic subgroup corresponding to the new node and let $\mathcal{G}r = {}^e\!G/{}^e\!P$ be the associated generalized Grassmann variety. The Picard groups of $\mathcal{G}r$ have a unique generator $\mathcal{L}$ generated by global sections. The homogeneous coordinate ring $\Gamma_{\mathcal{G}r} = \bigoplus_{j\geq 0} \Gamma(\mathcal{G}r, \mathcal{L}^j)$ is the quotient of the symmetric algebra $\mathsf{S}(\Gamma(\mathcal{G}r, \mathcal{L}))$ by an ideal generated by quadratic relations, the *generalized Plücker relations*.

A *Richardson variety* in $\mathcal{G}r$ is the closure of the intersection of an orbit of a Borel subgroup with an orbit of the opposite Borel subgroup. In Section 4, we define the Richardson variety $\mathcal{R}$ using particular elements of the Weyl group of the restricted root system. We give here a description of the main properties of $\mathcal{R}$.

The space $Z^* = \Gamma(\mathcal{G}r, \mathcal{L})$ has a natural grading $Z^* = \bigoplus_{n\geqslant 0}(Z^*)_n$ such that $(Z^*)_n$ is finite-dimensional and G-stable. We can order the modules $V_1, \ldots, V_\ell$ in such a way that V_i^* is a G submodule of $(Z^*)_i$ and does not appear in $(Z^*)_j$ for $j < i$. Moreover, V_i^* appears with multiplicity 1 in $(Z^*)_i$.

For $i = 1, \dots, \ell$, choose a G-stable complement C_i of V_i^* in $(Z^*)_i$ and let $C_i = (Z^*)_i$ for $i > \ell$. Then $\mathcal{R}$ is defined by the vanishing of sections in C_i for all i. In particular, we have the following result, which together with Theorem 42 is the main technical point of the paper.

Theorem (Corollary 40). If Ω^+ is quadratic, then $\Gamma(\mathcal{R}, \mathcal{L}|_{\mathcal{R}}) \simeq \mathbb{V}^*$ as G modules. Moreover, if $\Gamma_{\mathcal{R}} = \bigoplus_{n \geqslant 0} \Gamma(\mathcal{R}, \mathcal{L}^n|_{\mathcal{R}})$, we have $\Gamma_{\mathcal{R}} \simeq \Bbbk[G/H]$ as G modules. $\square$

The relation between the geometry of the affine Grassmannian and the coordinate ring of G/H is obtained, in the case Ω^+ is quadratic, by choosing ε to be the weight corresponding to the smallest possible spherical representation. Under this assumption, we construct a G-equivariant homomorphism of rings $\varphi : \Gamma_{\mathcal{Gr}} \longrightarrow \Bbbk[G/H]$ as follows. The Lie algebra of ${}^e\!G$ has a natural grading, such that in degree 0 there is a maximal torus and the Lie algebra of G, and in degree -1 there is the G spherical module V of highest weight ε. In particular, there exists a vector h_{-1} fixed by H in degree -1. If ${}^e\!G$ is of finite type, then we may consider the exponential $e^{h_{-1}}$ and the H-stable point $x = e^{h_{-1}}\,{}^eP$ in the Grassmannian ${}^e\!G/{}^eP$. So in this case, we can define a G-equivariant map from G/H to $\mathcal{Gr}$ using the map $gH \longmapsto gx$. The pullback of such map gives the ring homomorphism $\varphi : \Gamma_{\mathcal{Gr}} \longrightarrow \Bbbk[G/H]$. In fact, the morphism $\varphi : \Gamma_{\mathcal{Gr}} \longrightarrow \Bbbk[G/H]$ can also be defined when ${}^e\!G$ is not of finite type (see Section 5).

We are now in a position to give a first rough formulation of the main result of the paper:

> if Ω^+ is quadratic, then the defining relations for $\Bbbk[G/H]$ can be obtained using the map φ from the defining relations of $\mathcal{Gr}$, and a standard monomial theory for $\Bbbk[G/H]$ can be obtained from the standard monomial theory of $\mathcal{R}$ of $\mathcal{Gr}$.

To formulate this result more precisely, we need to recall a few facts about the generalized Plücker relations. In [22], a basis $\mathbb{F} \subset \Gamma(\mathcal{Gr}, \mathcal{L})$ has been constructed together with a transitive, antisymmetric binary relation "$\leftarrow$", such that the set of monomials $\mathbb{F}^2 = \{ff' \mid f, f' \in \mathbb{F}, f \leftarrow f'\} \subset \Gamma(\mathcal{Gr}, \mathcal{L}^{\otimes 2})$ form a basis. For a pair $f, f' \in \mathbb{F}$ of non $\leftarrow$ comparable elements, we denote by $R_{f,f'} \in \mathsf{S}^2(\Gamma(\mathcal{Gr}, \mathcal{L}))$ the relation expressing the product ff' as a linear combination of elements in $\mathbb{F}^2$. It was shown in [17] that the $R_{f,f'}$ generate the defining ideal of $\mathcal{Gr} \hookrightarrow \mathbb{P}(\Gamma(\mathcal{Gr}, \mathcal{L})^*)$.

The basis $\mathbb{F}$ is compatible with $\mathcal{R}$ in the following sense. Let $\mathbb{F}_0$ be the subset of $\mathbb{F}$ of elements not vanishing on $\mathcal{R}$. Then the restriction of elements in $\mathbb{F}_0$ to $\mathcal{R}$ is a basis of $\Gamma(\mathcal{R},\mathcal{L})$, and the standard monomials of elements in the set $\mathbb{F}_0$ restricted to $\mathcal{R}$ form a basis of $\Gamma_{\mathcal{R}}$.

For $f\in\mathbb{F}$, set $g_f=\varphi(f)$. The set $\mathbb{G}=\{g_f \mid f\in\mathbb{F}_0\}$ is a generating set for $\Bbbk[G/H]$, in particular any function g_f with $f\in\mathbb{F}-\mathbb{F}_0$ can be expressed as a polynomial F_f in terms of elements in $\mathbb{G}$. On $\mathbb{G}$ we consider the same transitive, an antisymmetric binary relation as on $\mathbb{F}$.

The relations $R_{f,f'}$ for $f,f'\in\mathbb{F}_0$ also involve elements in $\mathbb{F}-\mathbb{F}_0$. Let $\mathbb{F}_1\sqcup\mathbb{F}_0$ be the (finite) set of functions appearing in some polynomial $R_{f,f'}$ for $f,f'\in\mathbb{F}_0$. Denote by $\hat{R}_{f,f'}$ the relation obtained from $R_{f,f'}$ by replacing a generator $h\in\mathbb{F}_0$ by $g_h\in\mathbb{G}$ and a generator $h\in\mathbb{F}_1$ by the function F_h. The main result of the paper can be summarized in the following form.

Theorem (Theorems 44 and 47)**.** The set $\mathbb{G}$ is a basis of $\mathbb{V}^*\subset\Bbbk[G/H]$ and relations $\{\hat{R}_{f,f'}\ :\ f,f'\in\mathbb{F}_0$ non $\leftarrow$comparable $\}$ generate the ideal of the relations among the generators $\mathbb{G}$ of $\Bbbk[G/H]$.

Moreover, the set $\mathbb{SM}_0$ of ordered monomials in $\mathbb{G}$ realizes a standard monomial theory for $\Bbbk[G/H]$. □

Indeed, we prove a stronger form of this result by proving that the relations $\hat{R}_{f,f'}$ are straightening relations in the set of generators $\mathbb{G}$.

A key point in the proof of the theorem above is Theorem 42, whose proof in turn uses some results on the product in $\Bbbk[G/H]$ from [4] that are valid only in characteristic zero. We need this hypothesis of course also for the definition of φ. However, we want to point out that in most of the cases where the restricted root system is of type A, it is possible to directly define the point x. If one can check the conclusions of Theorem 42 in these cases, then the corresponding result holds in arbitrary characteristic, since the remaining arguments are characteristic-free.

The standard monomial theory is compatible with the decomposition in G modules in the following sense: there exists a filtration of $\Bbbk[G/H]$ by G modules F_i with simple quotients such that for all i, the set $\mathbb{SM}_0\cap F_i$ is a $\Bbbk$ basis of F_i (Remark 45).

The generalized Plücker relations are completely determined by the representation theory of G, for example, they can be computed using the result of Kostant on the action of the Casimir operator on an irreducible module (see [13]). Hence, the actual computation of the functions F_f depends only on the exponential $e^{h_{-1}}$ and on the representation theory of G (see Remark 46). Such computations may be considered as algorithmic, but it seems very difficult to obtain more explicit formulas; so the relations

$\hat{R}$ cannot be considered as completely explicit. Clearly, it should be interesting to have more information on such formulas as in the following discussion for finite types.

Indeed, if ${}^{e}G$ is of finite type (or, equivalently, the restricted root system is of type A), we can show that $\mathbb{F}_1$ is given by just two elements f_0, f_1, and that

$$F_{f_0} = F_{f_1} = 1.$$

In particular, in these cases the relations may be summarized in the following description of the coordinate ring of the symmetric variety:

$$\mathbb{k}[G/H] \simeq \frac{\Gamma_{\mathcal{G}r}}{(f_0 = f_1 = 1)}.$$

The study of the coordinate rings $\mathbb{k}[G/H]$ is strongly related to the study of the multicone associated to the wonderful compactification of the symmetric varieties of adjoint type. In [7], De Concini and Procesi defined the wonderful compactification $\bar{X}$ of $G/\bar{H}$, where $\bar{H}$ is the normalizer of H. In [3], the total ring of sections $\Gamma = \oplus_{\mathcal{M}\in \mathrm{Pic}(X)}\Gamma(\bar{X}, \mathcal{M})$ and a canonical set of generators for these rings were introduced. The computation of the relations among these generators is equivalent to the computation of the relations in the ring $\mathbb{k}[G/H]$ above.

The idea of a connection between the equations of certain symmetric varieties and the equations of a (generalized) flag variety is entwined with the story of the development of the standard monomial theory for quotients of reductive groups by parabolic subgroups. The story of such development may be roughly told as follows.

A basis of standard monomials was considered for the first time by Hodge in [11] and [12] for the classical Grassmannians of subspaces of a complex vector space. Many years later, a basis with similar properties for the coordinate ring of the space of matrices was studied by Doubilet, Rota, and Stein in [8].

In [6], De Concini and Procesi reproved and generalized this result about the coordinate ring of matrices also for the coordinate ring of symmetric and antisymmetric matrices.

Seshadri began its program of constructing a standard monomial theory for a general flag variety in [25]; in this paper, he studied the case of the quotient of a reductive group by a minuscule parabolic subgroup. The next step of such program was crucial for the connection we are interested in.

Indeed, in [19] Lakshmibai and Seshadri noticed that the case of matrices considered by Doubilet, Rota, Stein and De Concini and Procesi, could be explained as an application of the standard monomial theory constructed by Hodge. Beyond some technical differences, this idea is same as the strategy of our paper; the set of $n \times m$ matrices can be embedded as an open subset in the classical Grassmannian of n-dimensional vector subspaces of an $n+m$-dimensional vector space. They also noticed that the case of antisymmetric matrices considered by De Concini and Procesi may be similarly obtained from the standard monomial theory of the Grassmannian of maximal isotropic subspaces of an even-dimensional vector space; the standard monomial theory for such a Grassmannian was already developed by Seshadri in [25], since it is associated to a minuscule parabolic subgroup.

On the contrary, the remaining case of symmetric matrices was used to construct the standard monomial theory for the Grassmannian of Lagrangian subspaces of a symplectic vector space; this Grassmannian does not correspond to a minuscule parabolic. This led them to their basic conjectures. This conjecture was finally proved (in a further generalized form) by the second author in [22].

From our point of view, the results above cover almost all the cases with restricted root system of type A. The new cases are: two families with restricted root system of type A_1 (hence very simple) and an involution of E_6. At the end of the last section, we discuss briefly the example of the involution of E_6 by showing that the new symmetries introduced into the problem with the construction of the group ${}^{e}G$ allow us to determine easily the equation of the symmetric variety.

In the literature, there is still another case of a standard monomial theory for a symmetric variety. This is the case of the coordinate ring of a symplectic group studied by De Concini in [5] and it corresponds to a restricted root system of type C. The construction given by De Concini is more explicit than ours, since we have to use an affine group.

Finally, we want to stress that while the condition on the restricted root system to be of type A, B, C, or BC appears as a strong condition, it is actually fulfilled by many involutions. In the tables in [23], it holds for 12 families of involutions out of a total of 13 families, and in 4 exceptional cases out of a total of 12. Moreover, one should add to such a list of families the involutions such that $G = H \times H$, H is simple and the involution is given by $(x, y) \mapsto (y, x)$; for these cases, $\mathbb{k}[G/H]$ is the coordinate ring of H and our condition is equivalent to: H is $SL(n)$ or $\mathrm{Sp}(2n)$ or $\mathrm{SO}(2n+1)$.

The article is organized as follows. In the first section below, we introduce the notations and give some preliminary results on the combinatorics of the set of spherical weights.

In Section 2, we review the main properties of the De Concini–Procesi's wonderful compactification of a symmetric variety. We relate the multiplication of sections of line bundles on such compactification and the multiplication of functions on the symmetric variety.

In Section 3, we study some simple properties of the group ${}^e\!G$. In the cases related to our problem stated above, the group ${}^e\!G$ is of finite type if and only if the restricted root system is of type A, and it is of affine type if and only if the restricted root system is of type B, BC, C, or D (see Proposition 23).

In Section 4, we introduce and study a certain module of the extended Lie algebra corresponding to the new node of the extended Dynkin dagram. In the same section, we also study the Richardson variety $\mathcal{R}$.

In Section 5, all results of the previous sections are used to relate the symmetric variety and the Grassmannian $\mathcal{G}r$. Lastly, in Section 6 we study the simpler situation where the Grassmannian $\mathcal{G}r$ is finite-dimensional.

In the appendix, we prove that two standard monomial bases related to the symmetric variety coincide. One of the two bases is the basis considered above, the other is the standard monomial basis one may construct via lifting and pullback from the standard monomial theory of the multicone over the closed orbit in the wonderful compactification.

Finally, for the convenience of the reader, we recall in Appendix B the Satake diagrams of the involutions, together with the additional node relevant for the constructions and other information.

1 The Coordinate Ring of G/H and Quadratic Lattices

In this section, we introduce some notation and we make some remarks on the combinatorics of spherical weights.

Let G be a semisimple simply connected algebraic group over an algebraically closed field of characteristic zero. Let σ be an involution of G and H_{sc} its fixed-point subgroup. Since G is simply connected, H_{sc} is known to be connected (see, for example, [23]).

Let now $q : G \longrightarrow G_q$ be an isogeny and let K_q be the kernel of q. If $\sigma(K_q) = K_q$, then we can consider the involution σ_q of G_q induced by σ and its fixed points $G_q^{\sigma_q}$. We also define H_q as the inverse image of $G_q^{\sigma_q}$ in G. The groups H_q are reductive, so the quotients $X_q = X_q(\sigma) = G_q/G_q^{\sigma_q} = G/H_q$ are affine varieties. These varieties are called *symmetric varieties*. When q is the identity, we use the subscript sc instead of $q = \mathrm{id}$. We also denote by $q = \mathrm{ad}$ the adjoint quotient; in this case, H_{ad} is known to be equal to the normalizer of H_{sc} in G (see [7], § 1).

1.1 Spherical representations

If V is an irreducible representation of G, then we say that it is *q-spherical* (resp. spherical) if there exists a nonzero vector fixed by H_q (resp. H_{sc}). The subspace V^{H_q} consisting of H_q-fixed vectors is then one-dimensional, and hence

$$\Bbbk[X_q] = \Bbbk[G]^{H_q} = \bigoplus_{V\ irr.\ rep.} V^* \otimes V^{H_q} = \bigoplus_{V\ q\text{-spherical}} V^*.$$

We now want to give a more precise description of the set of q-spherical representations.

Let T be a maximally split σ-stable maximal torus of G, that is, a maximal torus of G stable under σ such that the dimension of $\{t \in T : \sigma(t) = t^{-1}\}$ is maximal; such a T is unique up to conjugacy in H. Let S be the identity component of this subgroup. The dimension of S is called the *rank* of the symmetric variety G/H; we denote this dimension by ℓ. Let Λ be the weight lattice of T and let Λ_q be the sublattice of weights trivial on K_q. The Killing form κ defines a positive-definite bilinear form on Λ and on Λ_q. A weight λ is said to be special if $\sigma(\lambda) = -\lambda$, and we denote by Λ^s (resp. Λ_q^s) the sublattice of Λ (resp. Λ_q) of special weights.

Denote by $\Phi \subset \Lambda$ the set of roots. We choose the set of positive roots Φ^+ in such a way that if α is positive, then $\sigma(\alpha)$ is either equal to α or is a negative root (see [7] § 1). We denote by Δ the set of simple roots of Φ defined by the choice of Φ^+. In exactly the same way, let $\Lambda^+ \subset \Lambda$ be the monoid of dominant weights. If $\alpha \in \Phi$ is not fixed by σ, then we define the *restricted root* $\tilde\alpha$ as $\alpha - \sigma(\alpha)$ and the *restricted root system* $\tilde\Phi \subset \Lambda^s$ as the set of all restricted roots. This is a (not necessarily reduced) root system (see [24]) of rank ℓ, and the subset $\tilde\Phi^+$ (resp. $\widetilde{\Delta}$) of restricted roots $\tilde\alpha$ with positive α (resp. simple α) is a choice of positive roots (resp. a basis of simple roots) for $\tilde\Phi$. For $\tilde\alpha \in \tilde\Phi$, we define $\tilde\alpha^\vee \in \mathfrak{t}$ such that $\langle\tilde\alpha^\vee, \lambda\rangle = 2\kappa(\lambda, \tilde\alpha)/\kappa(\tilde\alpha, \tilde\alpha)$ for all $\lambda \in \mathfrak{t}^*$. A special weight $\lambda \in \Lambda^s$ is said to be *spherical* if $\langle\tilde\alpha^\vee, \lambda\rangle \in \mathbb{Z}$ for all $\tilde\alpha \in \tilde\Phi$. The subset $\Omega_{sc} \subset \Lambda^s$ of spherical weights is a weight lattice for $\tilde\Phi$ (with respect to κ), and we observe that if $\lambda \in \Omega_{sc}$, then it is dominant with respect to Φ^+ if and only if it is dominant with respect to $\tilde\Phi^+$. On Ω_{sc} one has two different dominant orders: namely, the one with respect to Φ^+ indicated by $\leq$, and the other with respect to $\tilde\Phi^+$ indicated by $\leq_\sigma$: if $\lambda, \mu \in \Omega_{sc}$, then $\mu \leq_\sigma \lambda$ iff $\lambda - \mu \in \mathbb{N}[\tilde\Phi^+]$.

We can now describe the set of spherical representations. For $\lambda \in \Lambda^+$, let V_λ be the irreducible representation of G of highest weight λ. Define the set

$$\Omega_q^+ := \{\lambda \in \Lambda^+ : V_\lambda \text{ is } q\text{-spherical}\},$$

and let Ω_q be the lattice generated by Ω_q^+. If $\lambda \in \Omega_{sc}^+$, then we denote by $h_\lambda \in V_\lambda$ a nonzero vector fixed by H_{sc}. Given $\lambda, \mu \in \Omega_q$, we can think of V_λ, V_μ as sections of a line bundle over the flag variety of G, and hence the product of the sections $h_\lambda \cdot h_\mu$ is a nonzero vector fixed by H_q in $V_{\lambda+\mu}$. In particular, we see that Ω_q^+ is a monoid. In the simply connected case, this definition of Ω_{sc} coincides with the one given above using the restricted root system (see [10]). In general, the set of q spherical weights has been characterized by Vust [27] as shown in the following theorem.

Theorem 1 [27, Théorème 3]**.** Let $S_q = q(S)$ and let $\Lambda(S_q)$ be the weight lattice of S_q, and let $\lambda \in \Lambda_q^+$. Then $\lambda \in \Omega_q^+$ if and only if $\sigma(\lambda) = -\lambda$ and $\lambda|_{S_q} \in 2\Lambda(S_q)$. □

The following corollary collects some consequences of the characterizations by Helgason and Vust.

Corollary 2.

(i) For every q, we have $\Omega_q^+ = \Omega_q \cap \Lambda^+$;

(ii) for every q, we have $\Omega_q = \{\lambda - \sigma(\lambda) \,:\, \lambda \in \Lambda_q\}$;

(iii) for every q, we have $\Lambda_q \cap \Omega \supset \Omega_q \supset \mathbb{Z}[\tilde{\Phi}]$;

(iv) in the adjoint case, we have $\Omega_{\mathrm{ad}} = \mathbb{Z}[\tilde{\Phi}]$;

(v) if $H_q \subset H_{q'}$, then $H_q = H_{q'}$ if and only if $\Omega_q = \Omega_{q'}$. □

Proof. In the simply connected case, the above assertions form part of the results of Helgason.

The condition in Vust's criterion is linear, so part (i) follows by Vust's characterization. In particular, if $\lambda \in \Lambda_q$, then $\lambda \in \Omega_q$ if and only if $\sigma(\lambda) = -\lambda$ and $\lambda\big|_{S_q} \in 2\Lambda(S_q)$.

Let $L = \{\lambda - \sigma(\lambda) \,:\, \lambda \in \Lambda_q\}$. The inclusion $L \subset \Omega_q$ is evident. To prove the opposite inclusion, note that the restriction map $\rho : \Lambda_q \longrightarrow \Lambda(S_q)$ is surjective, $\rho\big|_{\Lambda^s}$ is injective, and $\rho \circ \sigma = -\rho$. Let now $\mu \in \Omega_q$ and consider $\rho(\mu)$. By the criterion of Vust, there exists $\lambda \in \Lambda_q$ such that $2\rho(\lambda) = \rho(\mu)$. Now $2\rho(\lambda) = \rho(\lambda - \sigma(\lambda))$ and $\mu, \lambda - \sigma(\lambda) \in \Lambda^s$, so that $\mu = \lambda - \sigma(\lambda)$. This proves part (ii).

Point (iv) follows directly from part (ii), and part (iii) follows from parts (iv) and (ii). Finally, (v) is an obvious consequence of the description of the coordinate ring of X_q given above. ■

1.2 Quadratic lattices

As explained in the introductory section, our construction of a standard monomial theory for G/H_q starts with a choice of canonical generators of the coordinate ring. For this reason, we require Ω_q^+ to be freely generated. The following combinatorial conditions will ensure in addition that the relations between these generators will be quadratic.

Definition 3. Let R be a root system with a choice of positive roots R^+, let P be the weight lattice with P^+ as the monoid of dominant weights, and let $Q \subseteq P$ be the root lattice. For a sublattice $L \subseteq P$, set $L^+ = L \cap P^+$. The sublattice L is called *admissible* if

(i) L contains Q;
(ii) the finitely generated commutative monoid L^+ is free.

The (unique) basis $\mathcal{B}$ of the free monoid L^+ (note that $\mathcal{B}$ is also a basis of L) is called the *admissible basis* of L. If $\lambda \in L$ and $\lambda = \sum_{\varepsilon\in\mathcal{B}} a_\varepsilon\, \varepsilon$, then we define $\mathrm{hgt}_{\mathcal{B}}(\lambda) = \sum_{\varepsilon\in\mathcal{B}} \mathrm{a}_\varepsilon$. An admissible lattice L is called *quadratic* if the following additional property holds:

(iii) if $\lambda \in L^+$ is such that $\lambda \leq \varepsilon + \eta$ for some $\varepsilon, \eta \in \mathcal{B}$ (with respect to the dominant order), then $\mathrm{hgt}_{\mathcal{B}}(\lambda) \leq 2$. □

This definition is strongly related to the description of the coordinate ring of G/H_q: take $R = \tilde{\Phi}$ and suppose that Ω_q is admissible. Let $\mathcal{B} = \{\varepsilon_1, \ldots, \varepsilon_\ell\}$, then fixing a basis of $V^*_{\varepsilon_1} \oplus \cdots \oplus V^*_{\varepsilon_\ell}$ is a canonical choice for fixing a generating set of $\mathbb{C}[X_q]$. A rough description of the relations between the generators is given in the next section. By such a description it will be clear that if Ω_q is quadratic, then also the relations in these generators are quadratic (see Corollary 17).

Convention 4. Before the next proposition, we make a convention for the fundamental weights of a root system Φ of type BC_ℓ. Let $\alpha_1, \ldots, \alpha_\ell$ be simple roots of Φ such that $2\alpha_\ell \in \Phi$. Note that $\alpha_\ell^\vee = 2(2\alpha_\ell)^\vee$. We define the fundamental weights $\omega_1, \ldots, \omega_\ell$ as the weights such that $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$ if $i \neq \ell$ or $j \neq \ell$ and $\langle \omega_\ell, \alpha_\ell^\vee \rangle = 2$. With this definition, $\{\omega_1, \ldots, \omega_\ell\}$ is a basis of the weight lattice. □

Now we classify the quadratic lattices for an abstract root system.

Proposition 5. Let R, R^+, P, Q be as in Definition 3 above, with R simple. Then a lattice $L \subset P$ is quadratic only in the following cases:

(i) R is of type A_1 and $L = P$ or $L = Q$;
(ii) R is of type BC_1 and $L = P$;
(iii) R is of type $\mathsf{A}_\ell, \mathsf{C}_\ell$, or BC_ℓ with $\ell \geqslant 2$ and $L = P$;
(iv) R is of type B_ℓ with $\ell \geqslant 2$ and $L = Q$. □

Proof. Let $\omega_1, \ldots, \omega_\ell$ be the fundamental weights, $\alpha_1, \ldots, \alpha_\ell$ the simple roots, and let $n = c\mathrm{ard} P/Q$. For a quadratic lattice L, let $\mathcal{B}$ be as in Definition 3. Note that L is a lattice of rank ℓ and let $\mathcal{B} = \{\varepsilon_1, \ldots, \varepsilon_\ell\}$. By condition (i) we know that $n\,\omega_i \in L^+$ for all i, and hence by condition (ii) the ε_i have to be multiples of the fundamental weights. So, up to renumbering them, we have $\varepsilon_i = c_i\omega_i$ for some $c_i \in \mathbb{N}$.

Thus, given a simple root $\alpha_i = \sum_j c_{ij}\omega_j$, then $c_{ij}\omega_j \in L$. In particular, if R is of type A_ℓ ($\ell \geqslant 2$), C_ℓ ($\ell \geqslant 3$), D_ℓ, or E_ℓ, for every i there exists a j such that $c_{ij} = -1$, and hence $L = P$ in these cases. In the cases BC_ℓ, F_4, and G_2, we have $n = 1$ and so $L = P = Q$.

The condition for L to be quadratic is obviously equivalent to $\mathrm{hgt}_{\mathcal{B}}(\alpha) \geqslant 0$ for all simple roots α. So if $L = P$ and the root system is of type A_ℓ, C_ℓ, BC_ℓ, then the condition is satisfied; for the root systems of type D_ℓ or E_ℓ, if α is the simple root corresponding to the ramification node in the diagram, then we have $\mathrm{hgt}_{\mathcal{B}}(\alpha) < 0$; and for the root systems of type B_ℓ with $\ell \geqslant 3$, G_2, and F_4, if α is the simple long root "near" a short root, then we have $\mathrm{hgt}_{\mathcal{B}}(\alpha) < 0$.

There remains to consider the cases A_1 and B_ℓ with $L = Q$. (Note that in both cases $n = 2$, so the only possibilities are $L = P$ or $L = Q$). For A_1 the proposition is trivially true, and for B_ℓ one has $Q = \langle\omega_1, \ldots, \omega_{l-1}, 2\omega_\ell\rangle$, again the fact that the lattice is quadratic is easily verified. ■

The proof also shows that the only admissible lattices L for which $L \neq P$ are the ones with $R = \mathsf{B}_\ell$ or A_1 and $L = Q$.

Let X_q be a symmetric variety for a simple involution and assume that Ω_q is quadratic. In Section 3, we will construct a group ${}^e G$ with the properties explained briefly in the introduction. In these cases, the restricted root system is always of type A, B, C, or BC. For convenience we introduce the following convention that will be used in the subsequent sections.

Convention 6. Let R be a simple root system of type A_ℓ, B_ℓ, C_ℓ, or BC_ℓ. Note that a simple basis of R is linearly ordered and we number it as in Bourbaki [1]. In particular, note that we number in different ways the bases of B_2 and C_2. Let $\omega_1, \ldots, \omega_\ell$ be the fundamental weights and define

$$\varepsilon_i = \begin{cases} \omega_i & \text{if } i \neq \ell \text{ or } R \text{ is not of type } \mathsf{B}_\ell; \\ 2\omega_\ell & \text{if } i = \ell \text{ and } R \text{ is of type } \mathsf{B}_\ell. \end{cases}$$

We refer to $\varepsilon_1, \ldots, \varepsilon_\ell$ as the *quadratic basis,* since the lattice spanned by $\varepsilon_1, \ldots, \varepsilon_\ell$ is quadratic and all quadratic lattices (with the exception of $L = Q$ and R of type A_1) of simple root systems are of this form. In order to have a uniform notation, we consider root systems of types A_1 and B_1 as different and we choose in the first case $L = P$ and $\varepsilon_1 = \omega_1$ and in the second case $L = Q$ and $\varepsilon_1 = 2\omega_1$. □

We will need later the following combinatorial lemma about bases of quadratic lattices.

Lemma 7. Let R be a simple root system of type A_ℓ, B_ℓ, C_ℓ or BC_ℓ and let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_\ell$ be the quadratic basis according to Convention 6. Then for all $i = 1, \ldots, \ell$ we have

(i) $\varepsilon_i \leq \varepsilon_1 + \varepsilon_{i-1}$;

(ii) if $\lambda, \mu \in P^+$ and $\mu \leq \varepsilon_1$, $\lambda \leq \varepsilon_{i-1}$, $\varepsilon_i \leq \lambda + \mu$ then $\mu = \varepsilon_1$, $\lambda = \varepsilon_{i-1}$. □

Proof. Part (i) follows from $\alpha_1 + \cdots + \alpha_{i-1} = \varepsilon_1 + \varepsilon_{i-1} - \varepsilon_i$ for all i and all types.

Part (ii) is trivial for R of type A. Assuming R to be of type B, we have that

$$0 < \varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_i$$

is a complete list of elements less than or equal to ε_i, for all i. So the statement follows from $\varepsilon_i \not\leq \varepsilon_1 + \varepsilon_{i-2}$. For R of type C we have, for all i, that

$$\cdots \varepsilon_{i-4} < \varepsilon_{i-2} < \varepsilon_i$$

is a complete list of elements less than or equal to ε_i. So the claim follows from $\varepsilon_i \not\leq \varepsilon_1 + \varepsilon_{i-3}$.

Finally, note that for this problem the arguments for R of type BC are the same as those in the case R of type B. ■

2 The Ring of Sections of a Complete Symmetric Variety

In this section, we recall some facts about the wonderful compactification of a symmetric variety of adjoint type defined by De Concini and Procesi in [7]. We describe the relation between the multiplication of sections of line bundles on this compactification and the multiplication of functions on the symmetric variety.

2.1 The wonderful compactification of a symmetric variety

We shall use the notation introduced in the previous section, in particular, ℓ is the rank of the lattice Ω_{sc}. A spherical weight $\lambda \in \Omega^+_{\mathrm{sc}}$ such that $\langle \widetilde{\alpha}^\vee, \lambda \rangle \neq 0$ for all $\widetilde{\alpha} \in \tilde{\Phi}$ is called regular. If λ is regular, then we have an embedding $X_{\mathrm{ad}} = G/H_{\mathrm{ad}} \hookrightarrow \mathbb{P}(V_\lambda)$ given by $[g] \mapsto g[h_\lambda]$. The wonderful compactification of De Concini and Procesi of X_{ad} is defined as the closure of this image and its main properties are listed in the following theorem.

Definition–Theorem 8 [7, Theorem 3.1 and Proposition 8.1]**.** The closure of X_{ad} in $\mathbb{P}(V_\lambda)$ does not depend on the choice of the spherical regular weight λ, up to isomorphism. We call it the *wonderful compactification* of X_{ad} and we denote it by $\bar{X} = \bar{X}(\sigma)$. This variety has the following properties:

(i) $\bar{X}$ is a smooth projective G variety;

(ii) $\bar{X} \smallsetminus X_{\mathrm{ad}}$ is a divisor with normal crossings and smooth irreducible components $S_1, \dots, S_\ell$;

(iii) $\bar{X}$ has a unique closed orbit $Y = Y(\sigma)$ and the restriction of line bundles $\mathrm{Pic}(\bar{X}) \longrightarrow \mathrm{Pic}(Y)$ is injective. In particular, $\mathrm{Pic}(\bar{X})$ is identified with a sublattice of Λ and we denote by $\mathcal{L}_\lambda$ a line bundle corresponding to a weight $\lambda \in \mathrm{Pic}(\bar{X})$ ($\mathcal{L}_\lambda$ is unique up to isomorphism);

(iv) for every $\lambda \in \Omega^+_{\mathrm{sc}}$ (not necessarily regular), the map $[g] \mapsto g[h_\lambda]$ from X_{ad} to $\mathbb{P}(V_\lambda)$ extends to a morphism $\psi_\lambda : \bar{X} \to \mathbb{P}(V_\lambda)$ and $\mathcal{L}_\lambda \simeq \psi_\lambda^* \mathcal{O}(1)$.

Moreover, recall that any line bundle on $\bar{X}$ admits a unique G linearization.

By properties (iii) and (iv), we know that $\Omega_{\mathrm{sc}} \subset \mathrm{Pic}(\bar{X})$. Moreover, the weights $\widetilde{\alpha}_1, \dots, \widetilde{\alpha}_\ell \in \widetilde{\Delta}$ are the weights corresponding to the line bundles $\mathcal{O}(S_1), \dots, \mathcal{O}(S_\ell)$. In particular, there exists a G-invariant section $s_i \in \Gamma(X, \mathcal{L}_{\widetilde{\alpha}_i})$ such that $\mathrm{div}(s_i) = S_i$.

For an element $\nu = \sum_{i=1}^{\ell} n_i \widetilde{\alpha}_i \in \mathbb{N}[\tilde{\Phi}]$, the multiplication by $s^\nu := \Pi_i s_i^{n_i}$ gives a G-equivariant map from $\Gamma(\bar{X}, \mathcal{L}_{\lambda-\nu})$ to $\Gamma(\bar{X}, \mathcal{L}_\lambda)$.

We now can describe the sections of a line bundle as a G module. Observe that every line bundle $\mathcal{L}_\lambda$ with $\lambda \in \Omega_{\mathrm{sc}}$ has a natural G linearization and, since the variety has a dense orbit under the Borel subgroup, any irreducible G module appears in $\Gamma(\bar{X}, \mathcal{L}_\lambda)$ with multiplicity at most 1 [7, Lemma 8.2].

If $\mu \in \Omega^+_{\mathrm{sc}}$, then by the construction of $\mathcal{L}_\mu$ we have a submodule of $\Gamma(\bar{X}, \mathcal{L}_\mu)$ isomorphic to V^*_μ obtained by the pullback of the homogeneous coordinates of $\mathbb{P}(V_\mu)$ to $\bar{X}$. Since the multiplicity of any irreducible submodule is at most 1, we can speak of the submodule V^*_μ of $\Gamma(\bar{X}, \mathcal{L}_\mu)$ without ambiguity. If now $\lambda \in \Omega$ is such that $\mu \leq_\sigma \lambda$, then we can consider the image of $V^*_\mu \subset \Gamma(\bar{X}, \mathcal{L}_\mu)$ under multiplication by $s^{\lambda-\mu}$. We denote this image by $s^{\lambda-\mu} V^*_\mu$. We have the following theorem.

Theorem 9 [7, Theorem 5.10]**.** If $\lambda \in \Omega_{\mathrm{sc}}$, then

$$\Gamma(\bar{X}, \mathcal{L}_\lambda) = \bigoplus_{\mu \in \Omega^+_{sc} \,:\, \mu \leq_\sigma \lambda} s^{\lambda-\mu} V^*_\mu. \qquad \square$$

2.2 Standard monomial theories

We recall the definition of standard monomial theory.

Let A be a commutative $\mathbb{k}$-algebra. Let $\mathbb{A}$ be a finite subset of A and let $\leftarrow$ be a transitive antisymmetric binary relation (t.a.b.r. for short) on $\mathbb{A}$. (Note that $\leftarrow$ is not necessarily reflexive.) If $a_1 \leftarrow a_2 \leftarrow \cdots \leftarrow a_n$, then we say that the monomial $a_1 \cdot a_2 \cdots a_n$ is a *standard monomial*. We denote by $\mathbb{SM}(\mathbb{A})$ the set of all standard monomials. We say that $(\mathbb{A}, \leftarrow)$ is a standard monomial theory (for short, SMT) for A if $\mathbb{SM}(\mathbb{A})$ is a $\mathbb{k}$ basis of A.

The construction of a standard monomial theory comes often together with the description of the straightening relations, i.e. a set of relations in the elements of $\mathbb{A}$ which provides an inductive procedure to rewrite a nonstandard monomial as a linear combination of standard monomials.

Let $(\mathbb{A}, \leftarrow)$ be an SMT for the ring A. In particular, $\mathbb{A}$ generates A and we denote by Rel_A the kernel of the natural morphism from the symmetric algebra $\mathsf{S}(\mathbb{A})$ to A. Let $\mathbb{M}(\mathbb{A}) \subset \mathsf{S}(\mathbb{A})$ be the set of all monomials in the set of generators $\mathbb{A}$, and let $<_t$ be a monomial order which refines the t.a.b.r. on $\mathbb{A}$. (We recall that a monomial order is a total order on the set of monomials such that (1) if m, m', m'' are monomials and $m' <_t m''$, then $mm' <_t mm''$ and (2) $1 <_t m$ for all monomials $m \neq 1$ (see [9, Section 15.2]).) For any two $a, a' \in \mathbb{A}$ that are non $\leftarrow$ comparable, assume that there exists a relation $R_{a,a'} \in \mathrm{Rel}_A$ such that

$$R_{a,a'} = a\,a' - P_{a,a'}$$

and $P_{a,a'}$ is a sum of monomials which are strictly smaller than $a\,a'$ with respect to the order $<_t$. A set of relations satisfying these properties is called a set of *straightening relations*. In this case, we have the following simple lemma.

Lemma 10. Let $(\mathbb{A}, \leftarrow)$ be an SMT for the ring A and let $\mathcal{R} = \{R_{a,a'} : a, a' \in \mathbb{A}$ are non $\leftarrow$ comparable$\}$ be a set of straightening relations. Then $\mathcal{R}$ generates Rel_A. □

Proof. Let I be the ideal generated by $\mathcal{R}$. We have a natural surjective morphism $\varphi : B = \mathsf{S}(\mathbb{A})/I \longrightarrow \mathsf{S}(\mathbb{A})/\mathrm{Rel}_A = A$ induced by $I \subset \mathrm{Rel}_A$. Now we prove that the set of standard monomials generates the ring B as a vector space. This implies that φ is an isomorphism, and hence $I = \mathrm{Rel}_A$.

Let m be any monomial and assume that it is not standard. The monomial can be written in the form $a\,a'\,m'$, where $a, a' \in \mathbb{A}$ are not comparable and m' is a smaller monomial. So $m \equiv P_{a,a'}m'$ (mod I) and each monomial in $m'P_{a,a'}$ is strictly smaller with respect to $<_t$ than m, and we can conclude by induction. ■

Example 11. Consider the conic in $\mathbb{P}^2$, defined by the equation $y^2 - xz$. Let $A = \Bbbk[x, y, z]/(y^2 - xz)$ be the homogeneous coordinate ring of this variety. Let $\mathbb{A} = \{x, y, z\}$ and define the t.a.b.r. as follows:

$$x \leftarrow y \leftarrow z, \quad x \leftarrow x, \quad z \leftarrow z.$$

The standard monomials in A are of the form $x^k z^\ell$ or $x^k yz^\ell$, $k, \ell \geq 0$. Note that y^2 is not standard in A. The total order on the set of monomials in $\Bbbk[x, y, z]$ is the lexicographic ordering induced by the total order $x <_t y <_t z$. □

2.3 Standard monomial theory for flag and Schubert varieties

Let A be the coordinate ring of the cone over a generalized flag variety $\mathcal{F}$ of a symmetrizable Kac–Moody group $\mathcal{G}$. For these type of algebras, a standard monomial theory has been constructed in [22]. We recall the main properties of this SMT.

Fix a maximal torus $\mathcal{T}$ and a Borel subgroup $\mathcal{B}$ in $\mathcal{G}$ such that $\mathcal{T} \subset \mathcal{B}$. Let $\mathcal{L}$ be a line bundle generated by global sections over $\mathcal{F}$ and consider the ring $\Gamma_{\mathcal{L}}(\mathcal{F}) = \bigoplus_{n\geqslant 0} \Gamma(\mathcal{F}, \mathcal{L}^n)$. A basis $\mathbb{F}_{\mathcal{L}}$ of $\Gamma(\mathcal{F}, \mathcal{L})$ has been constructed in [22], together with a t.a.b.r. $\leftarrow$ on this set such that $(\mathbb{F}_{\mathcal{L}}, \leftarrow)$ is an SMT for $\Gamma_{\mathcal{L}}(\mathcal{F})$.

Example 12. The conic in $\mathbb{P}^2$ defined in Example 11 is the highest-weight orbit for $G = SL_2$ in $\mathbb{P}(S^2\Bbbk^2)$. The t.a.b.r. defined in the example is induced by the indexing of the basis vectors by the so-called LS-paths. In the general setting, these paths are weighted linearly ordered subsets of the Weyl group, see Section 7 for a precise definition. In our example, $W = S_2 = \{\mathrm{id}, s\}$, and the set of LS-paths for this embedding is $\pi_1 = (\mathrm{id})$, $\pi_2 = (\mathrm{id}, s; 1/2)$, $\pi_3 = (s)$, $x = f_{\pi_1}$, $y = f_{\pi_2}$, $z = f_{\pi_3}$. The t.a.b.r. $\pi_i \leftarrow \pi_j$ is defined as follows: consider the largest element in the corresponding linearly ordered subset of W in π_i; we say $\pi_i \leftarrow \pi_j$ if this element is less than or equal to (in the Bruhat ordering) the smallest element in the linearly ordered subset of W in π_j. This is the same t.a.b.r. defined in Example 11. □

We denote by $\mathbb{SM}_{\mathcal{L}}(\mathcal{L}^n)$ the set of standard monomials of degree n, by $\mathbb{SM}_{\mathcal{L}}$ the set of all standard monomials, and by $\mathbb{M}(\mathbb{F}_{\mathcal{L}})$ the set of all monomials in the set of generators $\mathbb{F}_{\mathcal{L}}$. For $f, f' \in \mathbb{F}_{\mathcal{L}}$ that are non $\leftarrow$ comparable, the product $f\,f'$ can be expressed as a sum

$P_{f,f'}$ of standard monomials of degree 2. In [22], a total order $<_t$ has been introduced on $\mathbb{M}(\mathbb{F}_{\mathcal{L}})$ with the properties required in the previous discussion of a general SMT, so that the relations $R_{f,f'} = ff' - P_{f,f'}$ form a set of straightening relations. These relations are called Plücker relations, since they generalize the usual Plücker relations for the Grassmannian.

Furthermore, this theory is adapted to Schubert varieties. Let $\mathcal{S} \subset \mathcal{F}$ be a closed $\mathcal{B}$-stable subvariety and set $\Gamma_{\mathcal{L}}(\mathcal{S}) = \bigoplus_{n\geqslant 0} \Gamma(\mathcal{S}, \mathcal{L}^n\big|_{\mathcal{S}})$. Denote by $r : \Gamma_{\mathcal{L}}(\mathcal{F}) \longrightarrow \Gamma_{\mathcal{L}}(\mathcal{S})$ the restriction map, let $I_{\mathcal{S}}$ be its kernel and define $\mathbb{F}_{\mathcal{L}}(\mathcal{S}) = \{a \in \mathbb{F}_{\mathcal{L}} : r(a) \neq 0\}$. Then the set $\{r(a) : a \in \mathbb{F}_{\mathcal{L}}(\mathcal{S})\}$ with the t.a.b.r. induced by the t.a.b.r. $\leftarrow$ on $\mathbb{F}_{\mathcal{L}}$ realizes an SMT for $\Gamma_{\mathcal{L}}(\mathcal{S})$, and the monomials $m \in \mathbb{SM}_{\mathcal{L}}$ that contain elements not in $\mathbb{F}_{\mathcal{L}}(\mathcal{S})$ form a $\Bbbk$ basis of $I_{\mathcal{S}}$. Finally, the restrictions $r(R_{f,f'})$ of relations $R_{f,f'}$ to $\mathcal{S}$ for $f, f' \in \mathbb{F}_{\mathcal{L}}(\mathcal{S})$ which are non $\leftarrow$ comparable form a set of straightening relations. Summarizing, we have the following.

Theorem 13 [22]**.**

(i) $(\mathbb{F}_{\mathcal{L}}, \leftarrow)$ is an SMT for $\Gamma_{\mathcal{L}}(\mathcal{F})$, and the relations $R_{f,f'}$ for $f, f' \in \mathbb{F}_{\mathcal{L}}$ which are non $\leftarrow$ comparable form a set of straightening relations.

(ii) $(\{r(a) \mid a \in \mathbb{F}_{\mathcal{L}}(\mathcal{S})\}, \leftarrow)$ is an SMT for $\Gamma_{\mathcal{L}}(\mathcal{S})$, and the relations $r(R_{f,f'})$ for $f, f' \in \mathbb{F}_{\mathcal{L}}(\mathcal{S})$ which are non $\leftarrow$ comparable form a set of straightening relations. Moreover, the kernel $I_{\mathcal{S}}$ of the restriction map has a basis consisting of the set of all standard monomials that contain elements not in $\mathbb{F}_{\mathcal{L}}(\mathcal{S})$. $\square$

The elements of $\mathbb{F}_{\mathcal{L}}$ are eigenvectors for the action of $\mathcal{T}$ and we denote by weight(f) the weight of $f \in \mathbb{F}_{\mathcal{L}}$ with respect to the action of $\mathcal{T}$. The t.a.b.r. $\leftarrow$ is compatible with the dominant order in the following way: if $f \leftarrow f'$ and $f \neq f'$, then weight$(f) <$ weight(f') with respect to the dominant order. Moreover, $\mathbb{F}_{\mathcal{L}}$ has a minimum f_0 which is the lowest-weight vector.

A Richardson variety in $\mathcal{G}r$ is the closure of the intersection of a $\mathcal{B}$ orbit with an orbit of the Borel subgroup opposite to $\mathcal{B}$. An SMT for general Richardson varieties has been constructed by Lakshmibai and Littelmann in [16]. In this paper, we will be interested in a particular case of this construction, namely given a Schubert variety $\mathcal{S}$, we will consider the Richardson variety $\mathcal{S}_0 = \{y \in \mathcal{S} : f_0(y) = 0\}$. The SMT described for $\mathcal{S}$ immediately generalizes to $\mathcal{S}_0$ by choosing as set of generators $\mathbb{F}_0(\mathcal{S}_0) = \mathbb{F}(\mathcal{S}) \smallsetminus \{f_0\}$.

In the case of the multicone over a flag variety, some changes to this general setting are needed. We will not need these results in this paper, but we will briefly explain these changes to recall some results for the total ring of $\bar{X}$ proved in [3]. Let $L^+ \subset \mathrm{Pic}(\mathcal{F})$ be a free monoid contained in the set of elements of $\mathrm{Pic}(\mathcal{F})$ generated by global sections, and let $\mathcal{L}_1, \ldots, \mathcal{L}_r$ be the generators of L^+. Define $\Gamma_{L^+}(\mathcal{F}) = \bigoplus_{\mathcal{L} \in L^+} \Gamma(\mathcal{F}, \mathcal{L})$ and $\mathbb{F}_{L^+} = \mathbb{F}_{\mathcal{L}_1} \cup \cdots \cup \mathbb{F}_{\mathcal{L}_r}$. It is still possible to define a t.a.b.r. $\leftarrow$ on $\mathbb{F}_{L^+}$ and a total order $<_t$ on the set of all monomials with the same properties of the total order as in Section 2.2 such that the standard monomials of degree 2 form a basis of $\bigoplus_{i \le j} \Gamma(\mathcal{F}, \mathcal{L}_i \otimes \mathcal{L}_j)$, and such that every product $f f'$, with $f, f' \in \mathbb{F}_{L^+}$ non $\leftarrow$comparable, is a sum $P_{f,f'}$ of standard monomials that are strictly smaller than f, f' with respect to the total order $<_t$. Moreover, the straightening relations $R_{f,f'} = f f' - P_{f,f'}$ generate the ideal of relations in the generators $\mathbb{F}_{L^+}$. However, if one defines in this case the standard monomials as above as monomials of ordered elements, then they are not anymore linearly independent. One therefore has to give a more restrictive definition of a standard monomial (see Appendix A and [2]). For $\mathcal{L} \in L^+$, we denote by $\mathbb{SM}_{L^+}(\mathcal{L})$ the set of all standard monomials with respect to to this new definition belonging to $\Gamma(\mathcal{F}, \mathcal{L})$, and we denote by $\mathbb{SM}_{L^+}$ the set of all standard monomials.

2.4 Standard monomial theory for the total ring of $\bar{X}$

We describe now the connection between $\Bbbk[G/H]$ and the total ring of $\bar{X}$, and we recall some properties of this ring.

For an admissible sublattice $L \subset \Omega_{\mathrm{sc}}$, we introduce an analog of the total ring studied in [3] (in that paper, the ring was called the ring of all sections). Let $\varepsilon_1, \ldots, \varepsilon_\ell \in L$ be the basis $\mathcal{B}$ of the admissible lattice L as in Definition 3. If $\lambda = a_1\varepsilon_1 + \cdots + a_\ell\varepsilon_\ell$, then we define $\mathcal{L}_\lambda = \mathcal{L}_{\varepsilon_1}^{\otimes a_1} \otimes \mathcal{L}_{\varepsilon_2}^{\otimes a_2} \otimes \cdots \otimes \mathcal{L}_{\varepsilon_\ell}^{\otimes a_\ell}$ when we have chosen once and for all the line bundles $\mathcal{L}_{\varepsilon_1}, \ldots, \mathcal{L}_{\varepsilon_\ell}$. Then

$$\Gamma_L(\bar{X}) = \bigoplus_{\lambda \in L} \Gamma(\bar{X}, \mathcal{L}_\lambda)$$

has a natural structure of a G ring.

Set L^+ as the subset of elements λ such that $\mathcal{L}_\lambda$ restricted to the closed orbit Y is generated by global sections. To construct an SMT for $\Gamma_L(\bar{X})$, we use the SMT for the multicone $\Gamma_{L^+}(Y)$ briefly explained above. For all $i = 1, \ldots, \ell$ and for all $f \in \mathbb{F}_{L^+}(\mathcal{L}_{\varepsilon_i}|_Y)$, fix a section $f^X \in \Gamma(\bar{X}, \mathcal{L}_{\varepsilon_i})$ such that $f^X|_Y = f$. For a monomial $m = f_1 \cdots f_r$ in the elements of $\mathbb{F}_{L^+}$, we denote by $m^X = f_1^X \cdots f_r^X$ the corresponding product of the elements f^X.

We define $\mathbb{F}_L^X = \{s_1, \ldots, s_\ell\} \cup \{f^X : f \in \mathbb{F}_{L^+}\}$; we extend the t.a.b.r. on $\mathbb{F}_{L^+}$ to $\mathbb{F}_L^X$ by: $s_1 \leftarrow \cdots \leftarrow s_\ell \leftarrow f^X$ for all $f \in \mathbb{F}_{L^+}$.

Any monomial $s^\nu m^X$, with $\nu \in \langle \widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_\ell \rangle_{\mathbb{N}}$ and $m \in \mathbb{SM}_{L^+}$, is called *standard*; moreover, we denote the set of all standard monomials by $\mathbb{SM}_L^X$. This set is a $\Bbbk$ basis of $\Gamma_L(X)$ and more precisely, we have the following.

Theorem 14 [3]. The set $\{s^{\lambda-\mu}m^X : \mu \in L^+, \quad \mu \leq_\sigma \lambda \text{ and } m \in \mathbb{SM}_{L^+}(\mathcal{L}_\mu|_Y)\}$ is a $\Bbbk$ basis of $\Gamma(\bar{X}, \mathcal{L}_\lambda)$. □

We can also give a rough description of a set of straightening relations in terms of the elements of $\mathbb{F}_L^X$. We define a total order on the set of monomials: let μ, ν be sums with non-negative coefficients of the restricted roots $\widetilde{\alpha}_i$, and let m, n be monomials in the elements in $\mathbb{F}_{L^+}$. We set $s^\mu m^X <_t s^\nu n^X$ if μ is less than ν with respect to the lexicographic order, or if $\mu = \nu$ and $m <_t n$ with respect to the total order of the monomials in the elements in $\mathbb{F}_{L^+}$.

Let now $f \in \mathbb{F}_{L^+}(\mathcal{L}_{\varepsilon_i}\big|_Y)$ and $h \in \mathbb{F}_{L^+}(\mathcal{L}_{\varepsilon_j}\big|_Y)$ be such that they are non $\leftarrow$ comparable as elements of $\mathbb{F}_{L^+}$. Since the standard monomials form a basis of $\Gamma_L(\bar{X})$, we can express the product $f^X h^X$ as a sum of standard monomials

$$f^X h^X = P_{f,h}^X = \sum_{\mu \in L^+ \text{ and } \mu \leq_\sigma \varepsilon_i + \varepsilon_j} s^{\varepsilon_i+\varepsilon_j-\mu} \left(P_{f,h}^\mu\right)^X,$$

where $P_{f,h}^\mu \in \mathbb{SM}_{L^+}(\mathcal{L}_\mu\big|_Y)$.

In the symmetric algebra $\mathsf{S}(\mathbb{F}_L^X)$, set

$$R_{f,h} = fh - P_{f,h}^X.$$

This is a straightening relation. In fact, we have the following theorem.

Theorem 15 [3]. The set of straightening relations $R_{f,h}$ for $f, h \in \mathbb{F}_{L^+}$ non $\leftarrow$ comparable, generates the ideal of relations in the set of generators $\mathbb{F}_L^X$ of $\Gamma_L(\bar{X})$. □

The proof in [3] of the theorem above has been given only for $L = \mathrm{Pic}(\bar{X})$, but extends to the general case without changes.

It is easy to describe the part of highest degree of the relation $R_{f,h}$. Indeed, by restricting this equation to Y, we see that $fh - P_{f,h}^{\varepsilon_i+\varepsilon_j}$ is the usual straightening relation for the multicone L^+ over Y. In a certain sense, the aim of this paper is to give a description of the polynomials $P_{\pi,\pi'}^\mu$ for $\mu \neq \varepsilon_i + \varepsilon_j$.

2.5 A first description of the coordinate ring of X_q

In the construction above, we now choose $L = \Omega_q$ and we describe the relation between $\Gamma_{\Omega_q} = \Gamma_{\Omega_q}(\bar{X})$ and $\Bbbk[X_q]$.

We consider the map $\jmath : X_q \longrightarrow \bar{X}$ given by the composition $X_q \longrightarrow X_{\mathrm{ad}} \hookrightarrow \bar{X}$, and we observe that for all $\lambda \in \Omega_q$ the pullback $\jmath^*(\mathcal{L}_\lambda)$ is the trivial line bundle. Indeed, as a representation of H_q, the fiber of $\jmath^*(\mathcal{L}_\lambda)$ over the point $H_q \in X_q$ is the line $\Bbbk\, h_\lambda$, so the bundle is trivial.

In particular, if $\lambda \in \Omega_q$ and we choose an isomorphism $\varphi_\lambda : \jmath^*(\mathcal{L}_\lambda) \longrightarrow \mathcal{O}$, then we get an inclusion $\varphi_\lambda : \Gamma(\bar{X}, \mathcal{L}_\lambda) \hookrightarrow \Bbbk[X_q]$. If Ω_q is admissible and $\mathcal{B}$ is its admissible basis, then we can choose isomorphisms φ_ε for $\varepsilon \in \mathcal{B}$ and define $\varphi_\lambda = \bigotimes_{\varepsilon\in\mathcal{B}} \varphi_\varepsilon^{\otimes a_\varepsilon} : \mathcal{O} \longrightarrow \jmath^*(\mathcal{L}_\lambda) = \bigotimes_{\varepsilon\in\mathcal{B}} \jmath^*(\mathcal{L}_\varepsilon)^{\otimes a_\varepsilon}$ for $\lambda = \sum_{\varepsilon\in\mathcal{B}} a_\varepsilon \varepsilon$. With this choice of isomorphisms, we get for all $\lambda, \mu \in \Omega_q$ the following commutative diagram:

$$\begin{tikzcd}
\Gamma(\bar{X}, \mathcal{L}_\lambda) \otimes \Gamma(\bar{X}, \mathcal{L}_\mu) \arrow[r, "\text{multipl.}"] \arrow[d, "\varphi_\lambda \otimes \varphi_\mu"] & \Gamma(\bar{X}, \mathcal{L}_{\lambda+\mu}) \arrow[d, "\varphi_{\lambda+\mu}"] \\
\Bbbk[X_q] \otimes \Bbbk[X_q] \arrow[r, "\text{multipl.}"] & \Bbbk[X_q].
\end{tikzcd}$$

Hence we can define a morphism of rings $\jmath^* := \bigoplus_{\lambda\in\Omega_q} \varphi_\lambda : \Gamma_{\Omega_q} \longrightarrow \Bbbk[X_q]$.

Observe also that $\jmath^*_{\alpha_i}(s_i)$ is a nonzero G-invariant function on X_q, and hence we can normalize this function so that $\jmath^*_{\alpha_i}(s_i) = 1$. The relation between the ring Γ_{Ω_q} and the coordinate ring of X_q is given by the following proposition whose proof is easy.

Proposition 16. The map $\jmath^*$ gives an isomorphism

$$\frac{\Gamma_{\Omega_q}}{(s_i - 1 \,:\, i = 1, \ldots, \ell)} \simeq \Bbbk[X_q]. \qquad \square$$

In particular, we have the following corollary.

Corollary 17. If Ω_q is quadratic, then the ring $\Bbbk[X_q]$ has quadratic relations in the generators $\bigcup_{\varepsilon\in\mathcal{B}} V_\varepsilon^*$. $\square$

We believe that the converse is also true.

Conjecture 18. Suppose that Ω_q is admissible. If Ω_q is not quadratic, then also the relations are not quadratic. $\square$

2.6 Surjectivity of multiplication and applications

We now discuss some consequences of the surjectivity of multiplication of sections of line bundles generated by global sections.

If $\lambda, \mu \in \Omega^+_{\mathrm{sc}}$, then the line bundles $\mathcal{L}_\lambda, \mathcal{L}_\mu$ are generated by global sections. By [4], the multiplication map $m_{\lambda,\mu} : \Gamma(\bar{X}, \mathcal{L}_\lambda) \otimes \Gamma(\bar{X}, \mathcal{L}_\mu) \longrightarrow \Gamma(\bar{X}, \mathcal{L}_{\lambda+\mu})$ is surjective. We consider now the restriction $n_{\lambda,\mu}$ of the multiplication map to the submodule

$$V^*_\lambda \otimes V^*_\mu \subset \Gamma(\bar{X}, \mathcal{L}_\lambda) \otimes \Gamma(\bar{X}, \mathcal{L}_\mu),$$

and we define $N(\lambda, \mu) = \{\nu \in \Lambda^+ : \nu \leq_\sigma \lambda + \mu \text{ and } s^{\lambda+\mu-\nu} V^*_\nu \subset \operatorname{Im} n_{\lambda,\mu}\}$.

We now provide a different construction of the set $N(\lambda, \mu)$. For $\lambda, \mu \in \Omega^+_{\mathrm{sc}}$, consider the element $h_\lambda \otimes h_\mu \in V_\lambda \otimes V_\mu$. Let $W^{\lambda,\mu}_\nu$ be the isotypic component of type V_ν of $V_\lambda \otimes V_\mu$. Denote by $\pi^{\lambda,\mu}_\nu$ the G-equivariant projection of $V_\lambda \otimes V_\mu$ onto its isotypic component of type V_ν,

$$\pi^{\lambda,\mu}_\nu : V_\lambda \otimes V_\mu \longrightarrow W^{\lambda,\mu}_\nu.$$

We define $N'(\lambda, \mu) := \{\nu \in \Lambda^+ : \pi^{\lambda,\mu}_\nu(h_\lambda \otimes h_\mu) \neq 0\}$.

Lemma 19. With the same notation as above: for all $\lambda, \mu \in \Omega^+$, we have $N(\lambda, \mu) = N'(\lambda, \mu)$. □

Proof. Consider the Segre embedding $S : \mathbb{P}(V_\lambda) \times \mathbb{P}(V_\mu) \longrightarrow \mathbb{P}(V_\lambda \otimes V_\mu)$ and define the morphism $\Delta_{\bar{X}} : \bar{X} \longrightarrow \mathbb{P}(V_\lambda \otimes V_\mu)$ by $\Delta_{\bar{X}}(x) = S(\psi_\lambda(x), \psi_\mu(x))$. The image of $\Delta_{\bar{X}}$ is the closure of the G orbit of the vector $h_\lambda \otimes h_\mu$, and $\Delta^*_{\bar{X}} : V^*_\lambda \otimes V^*_\mu \longrightarrow \Gamma(\bar{X}, \mathcal{L}_{\lambda+\mu})$ is the multiplication map $n_{\lambda,\mu}$. So $\operatorname{Im} n_{\lambda,\mu} \supset s^{\lambda+\mu-\nu} V^*_\nu$ if and only if $\langle G \cdot (h_\lambda \otimes h_\mu); (W^{\lambda,\mu}_\nu)^* \rangle \not\equiv 0$, where $(W^{\lambda,\mu}_\nu)^*$ is the annihilator of a G-stable complement of $W^{\lambda,\mu}_\nu$. Hence $\operatorname{Im} n_{\lambda,\mu} \supset V^*_\nu$ if and only if $\pi^{\lambda,\mu}_\nu(G \cdot (h_\lambda \otimes h_\mu)) \not\equiv 0$ and this happens if and only if $\pi^{\lambda,\mu}_\nu(h_\lambda \otimes h_\mu) \neq 0$. ■

We shall need the following corollary in Section 4.

Corollary 20. Suppose $\tilde{\Phi}$ is a simple root system of type A_ℓ, B_ℓ, C_ℓ, or BC_ℓ and let $\varepsilon_1, \dots, \varepsilon_\ell$ be the quadratic basis as in Convention 6. Then for $i = 2, \dots, \ell$, we have

$$\pi^{\varepsilon_1, \varepsilon_{i-1}}_{\varepsilon_i}(h_{\varepsilon_1} \otimes h_{\varepsilon_{i-1}}) \neq 0. \qquad \square$$

Proof. The corollary follows from Lemmata 7 and 19, the description of the sections of a line bundle in Theorem 9, and the surjectivity of the multiplication map $m_{\lambda,\mu}$. ■

3 Construction and Properties of the Group ${}^e G$ and Its Lie Algebra

In this section, we describe the Lie algebra ${}^e\mathfrak{g}$ and some of its properties. ${}^e\mathfrak{g}$ is a Kac–Moody algebra endowed with a grading and an involution; it contains the Lie algebra $\mathfrak{g}$ of G as a Levi factor in degree 0 and a spherical representation in degree 1. This construction depends on the choice of a dominant spherical weight ε that we consider to be fixed.

We assume from now on the involution σ to be simple (i.e. $\mathfrak{g}$ is an irreducible $G \rtimes \{\mathrm{id}, \sigma\}$ module) or equivalently, $\tilde{\Phi}$ is an irreducible root system. We keep the notation introduced in the previous sections. In particular, the enumeration of the basis $\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_\ell$ of the irreducible root system $\tilde{\Phi}$ is as in [1]. Let $\widetilde{\omega}_1, \ldots, \widetilde{\omega}_\ell \in \Omega_{\mathrm{sc}}$ be the fundamental weights corresponding to this basis.

3.1 The extended Lie algebra

To define the extended Lie algebra, we define its Dynkin diagram. The new Dynkin diagram is constructed by adding to the old Dynkin diagram a node, indexed by 0. We join the new node 0 with the node corresponding to the simple root $\alpha \in \Delta$ with $\langle \varepsilon, \alpha^\vee \rangle$ lines, and we put an arrow pointing toward the node corresponding to α if this number is bigger than or equal to 2. In general, this is not a Dynkin diagram of finite type. The coefficients of the extended Cartan matrix are given by the following rule: if $\alpha \in \Delta$, then we have

$$\langle \alpha_0, \alpha^\vee \rangle := -\langle \varepsilon, \alpha^\vee \rangle \quad \text{and} \quad \langle \alpha, \alpha_0^\vee \rangle := \begin{cases} 0 & \text{if } \langle \varepsilon, \alpha^\vee \rangle = 0; \\ -1 & \text{if } \langle \varepsilon, \alpha^\vee \rangle \neq 0. \end{cases}$$

Let ${}^e\Delta := \Delta \cup \{\alpha_0\}$ and choose a realization $({}^e\mathfrak{t}, {}^e\Delta, {}^e\Delta^\vee)$ of this Cartan matrix. We define ${}^e\mathfrak{g}$ as the Lie algebra constructed using this realization and we denote by ${}^e\mathfrak{t}$ its standard maximal toral subalgebra. Denote by ${}^e\Phi$ the set of roots of ${}^e\mathfrak{g}$ with respect to ${}^e\mathfrak{t}$ and ${}^e\Phi^+$ (resp. ${}^e\Phi^-$) the positive (resp. negative) roots with respect to the basis ${}^e\Delta$. For all $\alpha \in {}^e\Delta$, let e_α and f_α be the Chevalley generators of ${}^e\mathfrak{g}$ and set $\alpha^\vee = [\mathsf{e}_\alpha, \mathsf{f}_\alpha]$. We can naturally identify the Lie algebra $\mathfrak{t}$ of the maximal torus T, with the subspace of ${}^e\mathfrak{t}$ spanned by the $\alpha^\vee$ for $\alpha \in \Delta$. Moreover, we identify the Lie algebra $\mathfrak{g}$ of G with the semisimple part of the Levi subalgebra of ${}^e\mathfrak{g}$ associated to the simple roots $\neq \alpha_0$.

We have also an inclusion of $\mathfrak{t}^* \subset {}^e\mathfrak{t}^*$ induced by $\Delta \subset {}^e\Delta$. Note that the restriction of the pairing between ${}^e\mathfrak{t}$ and ${}^e\mathfrak{t}^*$ to the subspaces $\mathfrak{t}$ and $\mathfrak{t}^*$ induces the usual pairing between $\mathfrak{t}$ and $\mathfrak{t}^*$. In particular, if $\mathfrak{t}^*_\perp$ is the annihilator of $\mathfrak{t}^*$ in ${}^e\mathfrak{t}$, then we have the natural decompositions ${}^e\mathfrak{t} = \mathfrak{t} \oplus \mathfrak{t}^*_\perp$ and ${}^e\mathfrak{t}^* = \mathfrak{t}^* \oplus \mathfrak{t}_\perp$. Here $\mathfrak{t}_\perp$ denotes the annihilator of $\mathfrak{t}$ in ${}^e\mathfrak{t}^*$.

We denote by ${}^e\mathfrak{g}'$ the derived subalgebra of ${}^e\mathfrak{g}$ and let ${}^e\mathfrak{t}' = {}^e\mathfrak{t} \cap {}^e\mathfrak{g}'$ be the subspace of ${}^e\mathfrak{t}'$ spanned by the elements in ${}^e\Delta^\vee$. Choose an element C generating the intersection ${}^e\mathfrak{t}' \cap \mathfrak{t}^*_\perp$ and an element $D \in \mathfrak{t}^*_\perp$ such that $\langle D, \alpha_0 \rangle = 1$. We normalize C in such a way that $\alpha_0^\vee \in C + \mathfrak{t}$. Observe that C, D generate ${}^e\mathfrak{t}_\perp$, and that they are linearly independent if and only if the new Dynkin diagram is of affine type.

The Lie algebra ${}^e\mathfrak{g}$ is graded according to the action of D,

$$ {}^e\mathfrak{g}_i := \{x \in {}^e\mathfrak{g} \;:\; [D, x] = ix\}. $$

We now define an involution $\boldsymbol{\sigma}$ of ${}^e\mathfrak{g}$ in the following way:

$$ \boldsymbol{\sigma}(x) = \sigma(x) \text{ if } x \in \mathfrak{g}; \quad \boldsymbol{\sigma}(C) = -C; \quad \boldsymbol{\sigma}(D) = -D; \quad \boldsymbol{\sigma}(\mathsf{e}_0) = \mathsf{f}_0 \text{ and } \boldsymbol{\sigma}(\mathsf{f}_0) = \mathsf{e}_0. $$

We denote by $\boldsymbol{\sigma}$ also the induced involution of ${}^e\mathfrak{t}^*$ and we observe that, since $\boldsymbol{\sigma}(\mathfrak{t}^*_\perp) = \mathfrak{t}^*_\perp$, we have $\boldsymbol{\sigma}\big|_{\mathfrak{t}^*} = \sigma$.

We note that for all $\alpha \in \Delta$, we have $\langle \sigma(\alpha), \alpha_0^\vee \rangle = -\langle \alpha, \alpha_0^\vee \rangle$ and $\langle \alpha_0, \sigma(\alpha^\vee) \rangle = -\langle \alpha_0, \alpha^\vee \rangle$, hence $\boldsymbol{\sigma}(\alpha_0^\vee) = -\alpha_0^\vee$ and $\boldsymbol{\sigma}(\alpha_0) = -\alpha_0$. By using this, one can easily show that $\boldsymbol{\sigma}$ is a well-defined Lie algebra involution of ${}^e\mathfrak{g}$.

3.2 Some remarks and conventions concerning the weights of $\mathfrak{g}$ and ${}^e\mathfrak{g}$

For $\alpha \in \Delta$, we denote by $\omega_\alpha \in \mathfrak{t}^*$ the corresponding fundamental weight with respect to the basis Δ. Let Δ_0 be the set of simple roots fixed by σ and let Δ_1 be the complement of Δ_0 in Δ. Recall [7] that σ induces an involution $\bar{\sigma}$ of Δ_1 characterized by the following property: $\sigma(\alpha) + \bar{\sigma}(\alpha)$ is in the vector space spanned by Δ_0. Furthermore, $\bar{\sigma}$ is the restriction to Δ_1 of an automorphism of the Dynkin diagram of Φ.

The following connection between fundamental weights with respect to Δ and fundamental weights with respect to $\tilde{\Delta}$, as explained in [3], is a direct consequence of the Helgason criterion. For a weight $\widetilde{\omega}_i$, we have the following three possibilities:

$$ \widetilde{\omega}_i = \begin{cases} \omega_\alpha & \text{if } \widetilde{\alpha} = \widetilde{\alpha}_i \text{ and } \bar{\sigma}(\alpha) = \alpha \text{ and } \sigma(\alpha) \neq -\alpha; \\ 2\omega_\alpha & \text{if } \widetilde{\alpha} = \widetilde{\alpha}_i \text{ and } \bar{\sigma}(\alpha) = \alpha \text{ and } \sigma(\alpha) = -\alpha; \\ \omega_\alpha + \omega_\beta & \text{if } \widetilde{\alpha} = \widetilde{\alpha}_i \text{ and } \bar{\sigma}(\alpha) = \beta \neq \alpha. \end{cases} $$

We fix some notation for the fundamental weights of ${}^e\mathfrak{g}$. Choose $\gamma, \delta \in \mathfrak{t}_\perp$ univocally determined by $\langle \gamma, C \rangle = \langle \delta, D \rangle = 1$ if the new Dynkin diagram is not affine, and by $\langle \gamma, C \rangle = \langle \delta, D \rangle = 1$ and $\langle \gamma, D \rangle = \langle \delta, C \rangle = 0$ if it is affine. Note that we have

$$\alpha_0^\vee = C - \sum_{\alpha \,:\, \langle \widetilde{\alpha}; \varepsilon \rangle \neq 0} \omega_\alpha^\vee \qquad \text{and} \qquad \alpha_0 = \delta - \varepsilon, \tag{1}$$

where $\omega_\alpha^\vee \in \mathfrak{t}$ are the fundamental weights with respect to $\Delta^\vee$. Note also that for $\alpha \in \Delta$, the weight $\omega_\alpha \in \mathfrak{t}^*$ is not anymore the fundamental weight of α with respect to to ${}^e\Delta$, since in general we do not have $\langle \omega_\alpha, \alpha_0^\vee \rangle = 0$. We denote by ${}^e\omega_\alpha$ the fundamental weight of α with respect to the extended root system. In the affine case, we normalize it in such a way that $\langle {}^e\omega_\alpha, D \rangle = 0$ for all $\alpha \in {}^e\Delta$. So we have

$${}^e\omega_\alpha = \omega_\alpha - \langle \omega_\alpha, \alpha_0^\vee \rangle \, \gamma \qquad \text{and} \qquad {}^e\omega_0 := {}^e\omega_{\alpha_0} = \gamma.$$

It is necessary to pay attention to the fact that in the affine case, with these choices, we do not have $\alpha = \sum_{\beta \in {}^e\Delta} \langle \alpha, \beta^\vee \rangle {}^e\omega_\beta$ for all $\alpha \in {}^e\Delta$. Indeed, this formula holds for $\alpha \neq \alpha_0$, while for α_0 we have $\alpha_0 = \sum_{\beta \in {}^e\Delta} \langle \alpha_0, \beta^\vee \rangle {}^e\omega_\beta + \delta$. In particular, $\alpha\big|_{{}^e\mathfrak{t}'} = \sum_{\beta \in {}^e\Delta} \langle \alpha, \beta^\vee \rangle {}^e\omega_\beta\big|_{{}^e\mathfrak{t}'}$ still holds for every $\alpha \in {}^e\Delta$.

3.3 The restricted root system of the extended Lie algebra

We now want to study some properties of the involution $\boldsymbol{\sigma}$.

As in the case of the root system Φ, if $\alpha \in {}^e\Phi$ and $\boldsymbol{\sigma}(\alpha) \neq \alpha$, then we define $\widetilde{\alpha} := \alpha - \boldsymbol{\sigma}(\alpha)$. In particular, we have $\widetilde{\alpha}_0 := \alpha_0 - \boldsymbol{\sigma}(\alpha_0) = 2\,\alpha_0$. For $i = 1, \ldots, \ell$, we consider the elements $\widetilde{\alpha}_i^\vee \in \mathfrak{t}$ defined in Section 1 as elements of ${}^e\mathfrak{t} \supset \mathfrak{t}$ and we define $\widetilde{\alpha}_0^\vee = \frac{1}{2}\alpha_0^\vee$.

As in the classical case, we define ${}^e\widetilde{\omega}_0, {}^e\widetilde{\omega}_1, \ldots, {}^e\widetilde{\omega}_\ell$ and we note that we have

$${}^e\widetilde{\omega}_0 = 2\,{}^e\omega_0 = 2\,\gamma \qquad \text{and} \qquad {}^e\widetilde{\omega}_i = \widetilde{\omega}_i - \langle \widetilde{\omega}_i, \alpha_0^\vee \rangle \gamma \;\text{ for } i = 1, \ldots, \ell.$$

In general, we do not know if the set of the $\widetilde{\alpha}$ with $\alpha \in {}^e\Phi$ is a root system (see Conjecture 33 below for some comments). But we can always define the Cartan matrix of this hypothetical root system as the $(\ell+1) \times (\ell+1)$ matrix

$$\tilde{\mathbf{A}} := (\langle \widetilde{\alpha}_i ; \widetilde{\alpha}_j^\vee \rangle)_{i,j=0,\ldots,\ell}.$$

The next proposition asserts that the Cartan matrix $\tilde{\mathbf{A}}$ is determined only by the restricted root system and the weight ε. In particular, it is very easy to compute.

Proposition 21. The Cartan matrix $\tilde{\mathbf{A}}$ is given by the coefficients of the Cartan matrix of $\tilde{\Phi}$ and by the following numbers, where $i = 1, \ldots, \ell$,

$$\langle \widetilde{\alpha}_0; \widetilde{\alpha}_0^\vee \rangle = 2; \qquad \langle \widetilde{\alpha}_0; \widetilde{\alpha}_i^\vee \rangle = -2\langle \varepsilon; \widetilde{\alpha}_i^\vee \rangle \quad \text{and} \quad \langle \widetilde{\alpha}_i; \widetilde{\alpha}_0^\vee \rangle = \begin{cases} 0 & \text{if } \langle \varepsilon; \widetilde{\alpha}_i^\vee \rangle = 0; \\ -1 & \text{if } \langle \varepsilon; \widetilde{\alpha}_i^\vee \rangle \neq 0. \end{cases} \qquad \square$$

Proof. Let $\alpha \in \Delta$ be such that $\widetilde{\alpha} = \widetilde{\alpha}_i$. We have

$$\langle \widetilde{\alpha}_i; \widetilde{\alpha}_0^\vee \rangle = \tfrac{1}{2} \langle \alpha - \sigma(\alpha); \alpha_0^\vee \rangle = \langle \alpha; \alpha_0^\vee \rangle,$$

which proves the third equality, while using equation (1) we obtain $\langle \widetilde{\alpha}_0; \widetilde{\alpha}_i^\vee \rangle = \langle 2\,\delta - 2\,\varepsilon; \widetilde{\alpha}_i^\vee \rangle = -2\langle \varepsilon; \widetilde{\alpha}_i^\vee \rangle$. ■

In the same way, the restricted root system controls many properties of $\mathfrak{g}$ related to the involution σ, the Cartan matrix $\tilde{\mathbf{A}}$ controls some of the properties of ${}^e\mathfrak{g}$.

Proposition 22. The Cartan matrix of ${}^e\mathfrak{g}$ is symmetrizable if and only if $\tilde{\mathbf{A}}$ is symmetrizable. Moreover, in this case the standard bilinear form on ${}^e\mathfrak{g}$ defined in [14] is $\boldsymbol{\sigma}$-invariant; we denote this bilinear form by $\boldsymbol{\kappa}$. □

Proof. Recall that we assume that $\mathfrak{g}$ is simple for the action of $G \rtimes \{\mathrm{id}, \sigma\}$ (the proof in the general case is similar). In this case, there are two possibilities: either $\mathfrak{g}$ is simple or $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}$, with $\mathfrak{h}$ a simple Lie algebra and $\sigma(x, y) = (y, x)$.

Assume first that $\mathfrak{g}$ is simple. Let $d_\alpha = \kappa(\alpha, \alpha)$ for $\alpha \in \Delta$ and $\tilde{d}_\alpha = \kappa(\widetilde{\alpha}, \widetilde{\alpha})$ for $\widetilde{\alpha} \in \widetilde{\Delta}$. Then ${}^e\mathfrak{g}$ is symmetrizable if and only if there exists d_0 such that $d_0 \langle \alpha, \alpha_0^\vee \rangle = d_\alpha \langle \alpha_0, \alpha^\vee \rangle$ for all $\alpha \in \Delta$. Similarly, $\tilde{\mathbf{A}}$ is symmetrizable if and only if there exists $\tilde{d}_0$ such that $\tilde{d}_0 \langle \widetilde{\alpha}, \widetilde{\alpha}_0^\vee \rangle = \tilde{d}_\alpha \langle \widetilde{\alpha}_0, \widetilde{\alpha}^\vee \rangle$ for all $\widetilde{\alpha} \in \widetilde{\Delta}$. Now note that $\langle \widetilde{\alpha}, \widetilde{\alpha}_0^\vee \rangle = \frac{1}{2} \langle \alpha - \sigma(\alpha), \alpha_0^\vee \rangle = \langle \alpha, \alpha_0^\vee \rangle$ and that $\langle \widetilde{\alpha}_0, \widetilde{\alpha}^\vee \rangle = 2\langle \alpha_0, \widetilde{\alpha}^\vee \rangle$, and since $\widetilde{\alpha}^\vee \in \mathfrak{t}^*$ which is spanned by the coroots $\alpha^\vee \in \Delta^\vee$, we have $2\langle \alpha_0, \widetilde{\alpha}^\vee \rangle = -2\langle \varepsilon, \widetilde{\alpha}^\vee \rangle = -\frac{4}{\tilde{d}_{\widetilde{\alpha}}} \kappa(\varepsilon, \alpha - \sigma(\alpha)) = \frac{4\, d_\alpha}{\tilde{d}_{\widetilde{\alpha}}} \langle \alpha_0, \alpha^\vee \rangle$. So the two conditions are equivalent and $\tilde{d}_0 = 4 d_0$.

Assume now that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}$ and $\sigma(x, y) = (y, x)$, let $\mathfrak{t}_{\mathfrak{h}}$ be a maximal toral subalgebra of $\mathfrak{h}$ and $\Delta_{\mathfrak{h}}$ a choice of a simple basis for the roots of $\mathfrak{h}$. Then $\mathfrak{t} = \mathfrak{t}_{\mathfrak{h}} \oplus \mathfrak{t}_{\mathfrak{h}}$, $\Delta = \Delta_1 \cup \Delta_2$, where $\Delta_1 = \{(\beta, 0) : \beta \in \Delta_{\mathfrak{h}})\}$ and $\Delta_2 = \{(0, -\beta) : \beta \in \Delta_{\mathfrak{h}})\}$ and $\varepsilon = (\varepsilon_{\mathfrak{h}}, -\varepsilon_{\mathfrak{h}})$. Since $\tilde{\Phi}$ is simple, the condition for $\tilde{\mathbf{A}}$ to be symmetrizable is the same as that given in the discussion of $\mathfrak{g}$ simple, while the condition for ${}^e\mathfrak{g}$ to be symmetrizable becomes equivalent to the existence of d_0 and two nonzero scalars λ_1 and λ_2 such that $d_0\langle\alpha,\alpha_0^\vee\rangle = \lambda_1 d_\alpha\langle\alpha_0\alpha^\vee\rangle$ for all $\alpha \in \Delta_1$, and $d_0\langle\alpha,\alpha_0^\vee\rangle = \lambda_2 d_\alpha\langle\alpha_0\alpha^\vee\rangle$ for all $\alpha \in \Delta_2$. Now if $\varepsilon_1 = 0$, the statement is trivial, while if $\langle\varepsilon_1,\beta^\vee\rangle \neq 0$ and $\alpha_1 = (\beta, 0)$ and $\alpha_2 = (0, -\beta)$, we deduce that we must have $\lambda_1 = \lambda_2$. The proofs can now be completed as above.

From this description, it is also clear that the standard symmetric bilinear form is σ-invariant. ■

Proposition 23.

(i) The Lie algebra ${}^e\mathfrak{g}$ is finite-dimensional if and only if $\tilde{\mathbf{A}}$ is of finite type.

(ii) The Lie algebra ${}^e\mathfrak{g}$ is of affine type if and only if $\tilde{\mathbf{A}}$ is of affine type. □

Proof. In both cases, we can assume that ${}^e\mathfrak{g}$ is symmetrizable.

Consider the bilinear form κ' obtained by the restriction of $\boldsymbol{\kappa}$ to the real span ${}^e\!E$ of ${}^e\Delta$, and the bilinear form $\tilde{\kappa}'$ obtained by the restriction of $\boldsymbol{\kappa}$ to the real span ${}^e\tilde{E}$ of ${}^e\tilde{\Delta}$.

Then we have that ${}^e\mathfrak{g}$ is of finite type if and only if κ' is positive definite, and ${}^e\mathfrak{g}$ is of affine type if and only if κ' is degenerate and positive semidefinite. In the same way, the Cartan matrix $\tilde{\mathbf{A}}$ is of finite (resp. affine) type if and only if $\tilde{\kappa}'$ is positive definite (resp. semidefinite).

Let E be the real span of Δ and let $\tilde{E}$ be the real span of $\widetilde{\Delta}$, then ${}^e\!E = E \oplus \mathbb{R}C$ and ${}^e\tilde{E} = \tilde{E} \oplus \mathbb{R}C$. The restriction of the bilinear form κ to E, respectively, $\tilde{E}$ is positive definite, and recall that C is orthogonal to E.

Hence κ' is positive definite (resp. semidefinite) if and only if $\boldsymbol{\kappa}(C, C) > 0$ (resp. $\boldsymbol{\kappa}(C, C) = 0$). The same condition holds for $\tilde{\kappa}'$, so that ${}^e\mathfrak{g}$ is of finite (resp. affine) type if and only if the Cartan matrix $\tilde{\mathbf{A}}$ is of finite (resp. affine) type. ■

Remark 24. If ${}^e\mathfrak{g}$ is affine, then α_0 does not always correspond to the "affine" root in the new Dynkin diagram. In particular, the grading $\bigoplus {}^e\mathfrak{g}_i$ is not always the "loop graduation." It is clear that ${}^e\mathfrak{g}$ is the (nontwisted) affinization of $\mathfrak{g}$ if and only if $\mathfrak{g}$ is a spherical representation, and the highest root θ is equal to ε.

If $\mathfrak{g}$ is a spherical representation and $z \in \mathfrak{g}$ is a spherical vector, then it is easy to prove that $H = Z_G(z)$ and hence it is a Levi subgroup. On the other hand, if H is a Levi subgroup, then $\mathfrak{g}$ is obviously spherical. So $\sigma(\theta) = -\theta$ and $\tilde{\theta} = \theta - \sigma(\theta)$ is equal to 2θ, and it is the highest root of $\tilde{\Phi}$. So $\tilde{\theta}$ divided by 2 must be in the weight lattice of $\tilde{\Phi}$. This happens if and only if $\tilde{\theta} = 2\widetilde{\omega}_1$ and the restricted root system is of type C or BC or A_1

(but it is not always true that if the restricted root system is of type C or BC, then $\mathfrak{g}$ is a spherical representation).

In particular, $\theta = \varepsilon$ if and only if H is a Levi factor and $\varepsilon = \widetilde{\omega}_1$ (in the reduced case of rank 2, the numbering is given by the fact that here we consider the root system to be of type C_2 and not B_2). □

Remark 25. The cases $\varepsilon = \widetilde{\omega}_1$ or $\varepsilon = 2\widetilde{\omega}_1$ and $\tilde{\Phi}$ of type B_1 (see Convention 6) will be of particular interest to us. We make the results of Proposition 21 explicit in these cases.

Note that with our convention, for $\ell \geqslant 2$ there is no difference between the Cartan matrices of a root system of type B_ℓ and of a root system of type BC_ℓ. By the special choice of ε, there is no difference between the Cartan matrix $\tilde{\mathbf{A}}$ obtained starting from $\tilde{\Phi}$ of type B_ℓ and that obtained starting from $\tilde{\Phi}$ of type BC_ℓ (see Conventions 4 and 6), and in both cases we obtain the Cartan matrix of the affine Dynkin diagram of type $\mathsf{A}_{2\ell}^{(2)}$.

In all other cases, $\tilde{\mathbf{A}}$ is the Cartan matrix associated to the Dynkin diagram obtained by adding a "longer" node and a double arrow from this node to the node associated to $\widetilde{\alpha}_1$, so that it is quite easy to compute.

In particular, note that $\tilde{\mathbf{A}}$ is a Cartan matrix of finite type if and only if $\tilde{\Phi}$ is of type A, and it is of affine type if and only if $\tilde{\Phi}$ is of type $\mathsf{B}, \mathsf{BC}, \mathsf{C}$, or D. Here is a list of what we obtain in these cases.

Type of $\tilde{\Phi}$	A_ℓ, $\ell \geqslant 1$	B_ℓ, $\ell \geqslant 1$	C_ℓ, $\ell \geqslant 2$	D_ℓ, $\ell \geqslant 4$	BC_ℓ, $\ell \geqslant 1$
Type of $\tilde{\mathbf{A}}$ with $\varepsilon = \widetilde{\omega}_1$ or $2\widetilde{\omega}_1$ for B_1	$\mathsf{C}_{\ell+1}$	$\mathsf{A}_{2\ell}^{(2)}$	$\mathsf{C}_\ell^{(1)}$	$\mathsf{A}_{2\ell-1}^{(2)}$	$\mathsf{A}_{2\ell}^{(2)}$

□

3.4 First properties of the extended Lie algebra

From now on, we assume ${}^e\mathfrak{g}$ to be symmetrizable. In general, it is not true that the restriction of $\boldsymbol{\kappa}$ to $\mathfrak{g}$ is a multiple of the Killing form κ. So if we identify ${}^e\mathfrak{t}^*$ with ${}^e\mathfrak{t}$ using $\boldsymbol{\kappa}$ and we define for an element x of nonzero length $x^\vee = \frac{2x}{\boldsymbol{\kappa}(x,x)}$, then for $x \in \mathfrak{t}$ this definition need not to agree with the definition of $x^\vee$ given in Section 1. However, if one has an ideal of $\mathfrak{g}$ which is simple for the action of $G \rtimes \{\mathrm{id}, \sigma\}$, then by the uniqueness of the σ-invariant bilinear form, the restriction of $\boldsymbol{\kappa}$ to such an ideal must be a multiple of the Killing form. So the two possible definitions of $x^\vee$ coincide for elements that belong to such an ideal. In particular, they coincide for all elements in Φ and $\tilde{\Phi}$. For this reason, we keep the same symbol $\alpha^\vee$.

We list now some properties of the Lie algebra ${}^e\mathfrak{g}$.

Proposition 26.

(i) ${}^e\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} {}^e\mathfrak{g}_i$ and $\sigma({}^e\mathfrak{g}_i) = {}^e\mathfrak{g}_{-i}$ for all $i \in \mathbb{Z}$;

(ii) ${}^e\mathfrak{g}_0 = \mathfrak{g} \oplus \mathfrak{t}^*_\perp$, and so $\mathfrak{g}$ is the semisimple part of a Levi factor of ${}^e\mathfrak{g}$ and any ${}^e\mathfrak{g}_i$ is a $\mathfrak{g}$ module;

(iii) ${}^e\mathfrak{g}_{-1} \simeq V_\varepsilon$ as $\mathfrak{g}$ modules;

(iv) the subalgebra ${}^e\mathfrak{g}_- := \bigoplus_{i>0} \mathfrak{g}_{-i}$ is generated by ${}^e\mathfrak{g}_{-1}$;

(v) the subalgebra ${}^e\mathfrak{g}_+ := \bigoplus_{i>0} \mathfrak{g}_i$ is generated by ${}^e\mathfrak{g}_1$;

(vi) for all $i \in \mathbb{Z}$, we have $\dim {}^e\mathfrak{g}_i < \infty$;

(vii) for all $i \in \mathbb{Z}$, we have ${}^e\mathfrak{g}_{-i} \simeq {}^e\mathfrak{g}_i^*$ as a $\mathfrak{g}$ module. □

Proof. Parts (i) and (ii) are direct consequences of the definition.

To prove part (iii), we show that f_{α_0} generates ${}^e\mathfrak{g}_{-1}$ as a $\mathfrak{g}$ module, and that it is the highest weight vector for the action of $\mathfrak{g}$ of weight ε. The second assertion is trivial, since $[\mathsf{e}_\alpha, \mathsf{f}_{\alpha_0}] = 0$ for $\alpha \in \Delta$ and $\langle -\alpha_0, \alpha^\vee\rangle = \langle \varepsilon, \alpha^\vee\rangle$ by definition. Consider the subalgebra ${}^e\mathfrak{g}_- = \bigoplus_{i<0} {}^e\mathfrak{g}_i$. By [14], it is contained in the subalgebra generated by the elements f_i. Hence ${}^e\mathfrak{g}_{-1}$ is generated by the elements of the form $[\mathsf{f}_{\alpha_{i_1}} \dots [\mathsf{f}_{\alpha_{i_m}} [\mathsf{f}_{\alpha_0} [\mathsf{f}_{\alpha_{j_1}} \dots [\mathsf{f}_{\alpha_{j_{n-1}}}, \mathsf{f}_{\alpha_{j_n}}] \dots]$ with $\alpha_{i_1}, \dots, \alpha_{j_n} \in \Delta$. Since $x = [\mathsf{f}_{\alpha_{j_1}} \dots [\mathsf{f}_{\alpha_{j_{n-1}}}, \mathsf{f}_{\alpha_{j_n}}] \dots] \in \mathfrak{g}$, we can rewrite the element above as $-[\mathsf{f}_{\alpha_{i_1}} \dots [\mathsf{f}_{\alpha_{i_m}} [x, \mathsf{f}_{\alpha_0}]] \dots]$ which proves the claim.

Similarly, we observe that ${}^e\mathfrak{g}_{-i-1}$ is the $\mathfrak{g}$ module spanned by $[\mathsf{f}_{\alpha_0}, {}^e\mathfrak{g}_{-i}]$. Now if $x \in \mathfrak{g}$ and $y \in {}^e\mathfrak{g}_{-i}$, we have $[x, [\mathsf{f}_{\alpha_0}, y]] = [\mathsf{f}_{\alpha_0}, [x, y]] + [[x, \mathsf{f}_{\alpha_0}], y] \in [{}^e\mathfrak{g}_{-1}, {}^e\mathfrak{g}_{-i}]$. Hence ${}^e\mathfrak{g}_{-i-1} = [{}^e\mathfrak{g}_{-1}, {}^e\mathfrak{g}_{-i}]$, and this implies part (iv). Assertion (v) is similar and part (vi) follows from parts (iii) and (iv).

Finally, to prove part (vii), note that with respect to the nondegenerate bilinear form κ, the subspace ${}^e\mathfrak{g}_i$ is in duality with ${}^e\mathfrak{g}_{-i}$. ■

We introduce now a triple of elements in ${}^e\mathfrak{g}$. By Lemma 26, we know that ${}^e\mathfrak{g}_1 \simeq V_\varepsilon^*$ and ${}^e\mathfrak{g}_{-1} \simeq V_\varepsilon$, and so we can choose spherical vectors $h_1 \in {}^e\mathfrak{g}_1$ and $h_{-1} \in {}^e\mathfrak{g}_{-1}$ and define $K = [h_1, h_{-1}]$.

Lemma 27.

(i) $\kappa(h_1, h_{-1}) \neq 0$ and $K = [h_1, h_{-1}] \neq 0$;

(ii) if ${}^e\mathfrak{g}$ is not of affine type, then we can choose h_1 and h_{-1} in such a way that h_{-1}, K, h_1 is an sl(2) triple;

(iii) If ${}^e\mathfrak{g}$ is of affine type, then $[K, h_1] = [K, h_{-1}] = 0$. □

Proof. H is reductive, there is only one line of elements fixed by H and κ gives a G-equivariant isomorphism between ${}^e\mathfrak{g}_{-1}$ and ${}^e\mathfrak{g}_1^*$, and hence we must have that $\kappa(h_1, h_{-1}) \neq 0$.

The Lie bracket defines a surjective map ${}^e\mathfrak{g}_{-1} \otimes {}^e\mathfrak{g}_1 \longrightarrow {}^e\mathfrak{g}_0' := {}^e\mathfrak{g}_0 \cap {}^e\mathfrak{g}'$ and ${}^e\mathfrak{g}_0' = \mathfrak{g} \oplus \Bbbk C$ as $\mathfrak{g}$ modules. The composition with the projection on the trivial factor is the only G-equivariant map from ${}^e\mathfrak{g}_{-1} \otimes {}^e\mathfrak{g}_1$ to a trivial representation, so that it must be a nonzero multiple of the map given by $x_1 \otimes x_{-1} \mapsto \kappa(x_1, x_{-1})C$. In particular, $K \neq 0$ and, up to a nonzero scalar, we have $K = C + x$ with $x \in \mathfrak{g}$. Since h_1 and h_{-1} are fixed by H, so is x, and hence either x is a nonzero spherical vector or $x = 0$. In the first case (see Remark 24), it is easy to prove that $\mathfrak{h} = Z_{\mathfrak{g}}(x)$. In particular, $x \in \mathfrak{h}$, $[x, h_1] = 0$, and $[x, h_{-1}] = 0$, since h_1 and h_{-1} are spherical. So in either of the cases ($x = 0$ or not), we have $[K, h_1] = [C, h_1]$ (and the same for h_{-1}).

Now parts (ii) and (iii) follow by the fact that if the diagram is affine, then C is central, and if it is not affine, then C is a nonzero scalar multiple of D, and hence K acts nontrivially on ${}^e\mathfrak{g}_1$ and ${}^e\mathfrak{g}_{-1}$. ■

3.5 The Weyl group of the extended Lie algebra

Note that if $\alpha \in {}^e\Phi^+$ and $\sigma(\alpha) \neq \alpha$, then $\sigma(\alpha) \in {}^e\Phi^-$. If ${}^e\mathfrak{g}$ is finite-dimensional, then this implies that the maximal toral subalgebra ${}^e\mathfrak{t}$ is maximally split. In the infinite-dimensional case, we would like to consider this property as the analog for the toral subalgebra ${}^e\mathfrak{t}$ to be maximally split, and we would like to prove for this situation the basic structural properties analogous to those in the finite-dimensional case.

In [24], the relation between the Weyl group W of the root system Φ and the Weyl group $\widetilde{W}$ of the root system $\tilde{\Phi}$ is described. Let $\mathfrak{s} \subset \mathfrak{t}$ be the (-1)-eigenspace of the action of σ on $\mathfrak{t}$ and set $W_1 = \{w \in W : w(\mathfrak{s}) \subset \mathfrak{s}\}$ the subgroup of W preserving the span of spherical weights, and $W_2 = \{w \in W : w|_{\mathfrak{s}} = \mathrm{id}_{\mathfrak{s}}\}$ the subgroup of W_1 acting trivially on spherical weights. The restriction to $\mathfrak{s}$ gives an injective map $r : W_1/W_2 \longrightarrow \mathrm{Aut}(\mathfrak{s})$. The relation between W and $\widetilde{W}$ is given by the following proposition.

Proposition 28 [24, Proposition 4.7]**.** r defines an isomorphism between W_1/W_2 and $\widetilde{W}$. □

We now generalize this result to the extended situation. We first prove a weak form. Let ${}^e\mathfrak{s}$ be the (-1)-eigenspace of the action of σ on ${}^e\mathfrak{t}$ and for $i = 0, \ldots, \ell$, let $\tilde{s}_i$ the reflection of ${}^e\mathfrak{s}$ be defined by the simple root $\tilde{\alpha}_i$. Let ${}^eW \subset \mathrm{Aut}({}^e\mathfrak{t})$ be the Weyl group of the root system ${}^e\Phi$ and ${}^e\widetilde{W} \subset \mathrm{Aut}({}^e\mathfrak{s})$ the group generated by the reflections $\tilde{s}_i$ for $i = 0, \ldots, \ell$. As in the finite-dimensional case, define ${}^eW_1 = \{w \in {}^eW : w({}^e\mathfrak{s}) \subset {}^e\mathfrak{s}$ and $w|_{{}^e\mathfrak{s}} \in {}^e\widetilde{W}\}$ and

${}^eW_2 = \{w \in {}^eW : w|_{{}^e\mathfrak{s}} = \mathrm{id}_{{}^e\mathfrak{s}}\}$. The restriction to ${}^e\mathfrak{s}$ gives an injective map $\boldsymbol{r} : {}^eW_1/{}^eW_2 \longrightarrow \mathrm{Aut}({}^e\mathfrak{s})$, and the analog of Proposition 28 holds.

Lemma 29. $\boldsymbol{r}$ defines an isomorphism between ${}^eW_1/{}^eW_2$ and ${}^e\widetilde{W}$. □

Proof. Note that $\{s_\alpha : \alpha \in \Delta\}$ (resp. $\{\tilde{s}_i : i = 1, \ldots, \ell\}$) generates a subgroup of ${}^eW$ (resp. ${}^e\widetilde{W}$) isomorphic to W (resp. $\widetilde{W}$), which acts trivially on ${}^e\mathfrak{t}' \cap \mathfrak{t}^*_\perp$ (resp. ${}^e\mathfrak{s} \cap \mathfrak{t}^*_\perp$). So we have the following commutative diagram:

$$\begin{array}{ccc} W_1 & \xrightarrow{r} & \widetilde{W} \\ \cap & & \cap \\ {}^eW_1 & \xrightarrow{r} & {}^e\widetilde{W}. \end{array}$$

Hence it is clear that $\tilde{s}_1, \ldots, \tilde{s}_\ell \in \mathrm{Im}\, \boldsymbol{r}$ by the result of Proposition 28 in the finite case and it remains to prove that $\tilde{s}_0 \in \mathrm{Im}\, \boldsymbol{r}$. But $\widetilde{\alpha}_0 = 2\alpha_0$, and $\tilde{s}_0 = \boldsymbol{r}(s_{\alpha_0})$. ■

It is possible to describe explicit covers of the generators of the Weyl group ${}^e\widetilde{W}$; i.e. explicit elements $w_i \in {}^eW_1$ such that $\boldsymbol{r}(w_i) = \tilde{s}_i$.

For $i = 0, 1, \ldots, \ell$, let $\Sigma_i = \{\alpha \in {}^e\Delta : \widetilde{\alpha} = \widetilde{\alpha}_i\} \cup \Delta_0$.

Proposition 30.

(i) Let w_Δ be the longest element in W (with respect to the basis Δ), then $w_\Delta(\Delta_0) = -\Delta_0$;

(ii) $w_\Delta \circ \sigma = \sigma \circ w_\Delta$;

(iii) denote by w_i the longest element of the Weyl group of Σ_i. Then $w_i \in {}^eW_1$ and $\boldsymbol{r}(w_i) = \tilde{s}_i$. □

Proof. First of all, we note that part (i) is a special case of Lemma 15.5.8 in [26].

Now we show that part (i) implies part (ii). Note first that to prove part (ii), it is enough to examine the case of a simple involution. Note also that in the case of the flip: $\sigma(x, y) = (y, x)$, the claim is trivial. So we can assume that $\mathfrak{g}$ is simple. If $\alpha \in W$, then $\sigma \circ s_\alpha \circ \sigma = s_{\sigma(\alpha)}$, and hence σ acts on W by conjugation. If $w_\Delta(\Delta_0) = -\Delta_0$, then w_Δ preserves Φ_0. Hence, if we consider $w' = \sigma \circ w_\Delta \circ \sigma$, then we have that it is an element of the Weyl group which takes positive roots into negative roots, so $w' = w_\Delta$ and $w_\Delta \circ \sigma = \sigma \circ w_\Delta$.

Finally, we show that part (ii) implies part (iii). For $i = 0$, it is trivial: ${}^e\mathfrak{s}$ is orthogonal to Δ_0, and hence if w_{Δ_0} is the longest element of the Weyl group associated to Δ_0, then $w_{\Delta_0} \in {}^eW_2$. We also note that α_0 is not joined to Δ_0, so we have that $w_0 = s_{\alpha_0} \circ w_{\Delta_0}$ and $\boldsymbol{r}(w_0) = \tilde{s}_0$ follow from $\boldsymbol{r}(s_{\alpha_0}) = \tilde{s}_0$.

Thus we can reduce the proof to the finite-dimensional case. Let $\widetilde{\omega}_h$ be a fundamental weight of $\tilde{\Phi}$ orthogonal to $\widetilde{\alpha}_i$. Then $\widetilde{\omega}_h$ is the sum of fundamental weights ω_α orthogonal to any root in Σ_i. This shows that $w_i(\widetilde{\omega}_h) = \widetilde{\omega}_h$. So it suffices to show that $w_i(\widetilde{\alpha}_i) = -\widetilde{\alpha}_i$.

Let $\mathfrak{t}_i$ be the vector space spanned by Σ_i and Φ_i the root system generated by Σ_i. σ preserves Φ_i, and so by considering $\sigma|_{\mathfrak{g}_i}$, we can assume that the rank of the involution σ is 1. In particular, $w_\Delta = w_i$ in this case and it commutes with σ, hence w_i preserves $\mathfrak{s}_i := \mathfrak{s} \cap \mathfrak{t}_i$. We have already seen that the orthogonal complement to $\widetilde{\alpha}_i$ in $\mathfrak{s}_i$ is fixed by w_i and hence, since w_i is a real isometry, $w_i(\widetilde{\alpha}_i) = \pm\widetilde{\alpha}_i$. Moreover, observe that $\widetilde{\alpha}_i \in \mathbb{N}[\Phi_i^+]$ so that $w_i(\widetilde{\alpha}_i) \in -\mathbb{N}[\Phi_i^+]$, and hence $w_i(\widetilde{\alpha}_i) = -\widetilde{\alpha}_i$. ■

Now let ${}^e\!E = \Lambda \otimes_{\mathbb{Z}} \mathbb{R} + \mathbb{R}\gamma + \mathbb{R}\delta \subset {}^e\mathfrak{t}^*$. Let ${}^e\Phi_{re}$ be the ${}^eW$ orbit of ${}^e\Delta$ and define the subsets A and U of E as

$$A = \{x \in {}^e\!E \;:\; \langle x; \alpha^\vee\rangle \geqslant 0 \text{ for all } \alpha \in {}^e\Delta\},$$
$$U = \{x \in {}^e\!E \;:\; \langle x; \alpha^\vee\rangle \geqslant 0 \text{ for all but a finite number of } \alpha \in {}^e\Phi_{re}\}.$$

Then ${}^eW A = U$ and A is a fundamental domain for the action of ${}^eW$ on U. Define also ${}^e\tilde{E} = \Lambda^s \otimes_{\mathbb{Z}} \mathbb{R} + \mathbb{R}\gamma + \mathbb{R}\delta$, let ${}^e\tilde{\Phi}_{\rm re}$ be the ${}^e\widetilde{W}$ orbit of ${}^e\tilde{\Delta}$. Let $\tilde{A}$, $\tilde{U}$ be defined in the same way as A and U, then $\tilde{U}$ is stable under the action of ${}^e\widetilde{W}$ and $\tilde{A}$ is a fundamental domain for the action of ${}^e\widetilde{W}$ on $\tilde{U}$.

Lemma 31.

(i) For all $x \in {}^e\tilde{\Phi}_{\rm re}$, there exists an $\alpha \in {}^e\Phi_{\rm re}$ such that $x = \tilde{\alpha} = \alpha - \boldsymbol{\sigma}(\alpha)$;

(ii) $A \cap {}^e\tilde{E} = \tilde{A}$ and $U \cap {}^e\tilde{E} = \tilde{U}$; □

Proof. (i) If $w \in {}^eW_1$, then w commutes with $\boldsymbol{\sigma}$. Indeed, ${}^e\mathfrak{t} = {}^e\mathfrak{t}_+ \oplus {}^e\mathfrak{s}$ where ${}^e\mathfrak{t}_+$ is the subspace fixed by $\boldsymbol{\sigma}$. By σ invariance of $\boldsymbol{\kappa}$, this is an orthogonal decomposition of ${}^e\mathfrak{t}$. So if $w \in {}^eW$ preserves ${}^e\mathfrak{s}$, it also preserves ${}^e\mathfrak{t}_+$ and by consequence commutes with $\boldsymbol{\sigma}$.

If $x \in {}^e\tilde{\Phi}_{\rm re}$, then $x = w(\widetilde{\beta})$ with $\beta \in \Delta$ and, as in Lemma 29, $w \in W_1$. So $x = w(\beta) - w(\boldsymbol{\sigma}(\beta)) = w(\beta) - \boldsymbol{\sigma}(w(\beta)) = \widetilde{\alpha}$ with $\alpha = w(\beta) \in {}^e\Phi_{\rm re}$.

(ii) The statement about A is obvious, since $\langle x, \widetilde{\alpha}\rangle = 2\langle x, \alpha\rangle$ for all $x \in {}^e\mathfrak{s}$ and for all $\alpha \in {}^e\Delta$. Moreover, $U \cap {}^e\tilde{E} \supset {}^eW_1(A) \cap {}^e\tilde{E} = {}^eW_1(A \cap {}^e\tilde{E}) = {}^e\widetilde{W}(\tilde{A}) = \tilde{U}$. Finally, if $x \in U \cap {}^e\tilde{E}$ by part (i) and $\langle x, \widetilde{\alpha}\rangle = 2\langle x, \alpha\rangle$, we also have $x \in \tilde{U}$. ■

In the next section, we will need the following integral form of ${}^e\tilde{E}$: ${}^e\Omega = \Omega + \mathbb{Z}\gamma + \mathbb{Z}\delta$.

Corollary 32.

(i) ${}^eW_1 = \{w \in {}^eW : w({}^e\mathfrak{s}) = {}^e\mathfrak{s}\}$;

(ii) if $\lambda \in U \cap {}^e\Omega$ and $w \in {}^eW$ is such that $w(\lambda) \in A$, then $w(\lambda) \in {}^e\Omega$ and there exists a $\tilde{w} \in {}^e\widetilde{W}$ such that $\tilde{w}(\lambda) = w(\lambda)$. $\square$

Proof. We shall prove part (ii), the proof of part (i) is similar.

Choose $\tilde{w} \in {}^e\widetilde{W}$ such that $\tilde{w}(\lambda) \in \tilde{A} \subset A$. By Corollary 29, $\tilde{w}$ is the restriction to ${}^e\mathfrak{s}$ of an element of ${}^eW$, so since A is a fundamental domain, we have $\tilde{w}(\lambda) = w(\lambda)$.

Hence it is enough to prove that, if $\lambda \in {}^e\Omega$ and $\tilde{w} \in {}^e\widetilde{W}$, then $\tilde{w}(\lambda) \in {}^e\Omega$. This is clear if $\tilde{w} \in \widetilde{W}$, since $\widetilde{W}$ preserves ${}^e\Omega$ and fixes γ and δ. So it is enough to consider the case $\tilde{w} = \tilde{s}_0 = s_0$. In this case, the claim follows from $\alpha_0 = \delta - \varepsilon$ and $\langle \lambda ; \alpha_0^\vee \rangle \in \mathbb{Z}$ if $\lambda \in {}^e\Omega$. ■

If one tries to develop an analog of the classical finite-dimensional theory for this situation, one of the first questions that one needs to address is to clarify the relationship between ${}^e\tilde{\Phi}$ and the Cartan matrix $\tilde{\mathbf{A}}$. More precisely, we have the following conjecture.

Conjecture 33. Suppose that $\tilde{\Phi}$ is not of type BC. Consider the realization of the Cartan matrix $\tilde{\mathbf{A}}$ given by $({}^e\mathfrak{s}, {}^e\tilde{\Phi}, {}^e\tilde{\Phi}^\vee)$ and the root system Ψ of its associated Kac–Moody algebra. Then $\Psi = {}^e\tilde{\Phi}$. $\square$

When $\tilde{\Phi}$ is of type BC, we could modify the conjecture to give it a reliable appearance, but there seems to be no general theory of nonreduced Kac–Moody root systems.

Note that the conjecture is true in the case ${}^e\mathfrak{g}$ is finite-dimensional or in the case $\sigma\big|_{{}^e\mathfrak{t}} = -\mathrm{id}_{{}^e\mathfrak{t}}$. It is also easy to verify the conjecture in the case ${}^e\mathfrak{g}$ is the affinization of $\mathfrak{g}$, since we have an explicit description of the root system.

4 The Representation Z and the Richardson Variety $\mathcal{R}$

In this section, we introduce a representation Z of ${}^e\mathfrak{g}$ and a Richardson variety $\mathcal{R}$, and we prove the main technical results of the paper.

We keep the notation introduced in the previous section. Moreover, from now on we fix a simple involution of G and a subgroup of the form H_q (see Section 1) such that Ω_q is quadratic. We denote Ω_q by Ω and H_q by H. We denote by $\varepsilon_1, \ldots, \varepsilon_\ell$ the admissible basis of Ω as in Convention 6 and we choose $\varepsilon = \varepsilon_1$ in the construction of ${}^e\mathfrak{g}$ given in the previous section. We set for convenience $\varepsilon_0 = 0$ and we define ${}^e\varepsilon_i = \varepsilon_i - \langle \varepsilon_i, \alpha_0^\vee \rangle \gamma \in {}^e\mathfrak{t}^*$, and for convenience ${}^e\varepsilon_0 = {}^e\tilde{\omega}_0$. In particular, the restricted root system $\tilde{\Phi}$ is of type A, B, C, or BC, and the Lie algebra ${}^e\mathfrak{g}$ is of finite or affine type (see Remark 25).

4.1 The representation Z

Let Z be the integrable highest-weight module of ${}^e\mathfrak{g}$ with highest weight ${}^e\omega_0$, and let z_0 be a highest-weight vector in Z. We define a grading of Z using the action of D in the following way: let $n_0 = \langle {}^e\omega_0, D\rangle$ and set

$$Z_n = \{z \in Z \,:\, D \cdot z = (n + n_0)z\}.$$

This grading is compatible with the grading of ${}^e\mathfrak{g}$ introduced in the previous section and, by Proposition 26(vi), each Z_n is a $\mathfrak{g}$ module, is finite-dimensional, and is zero for $n > 0$. We define the restricted dual Z^* of Z as $Z^* = \bigoplus_{n\geqslant 0}(Z_{-n})^*$. Z^* is the integrable lowest-weight module with lowest weight $-{}^e\omega_0$, and is graded by the action of D with $(Z^*)_n = (Z_{-n})^*$. We choose z_0^* as a lowest-weight vector such that $\langle z_0, z_0^*\rangle = 1$.

Remark 34. We have $n_0 = 0$ if ${}^e\mathfrak{g}$ is of affine type and $n_0 = \frac{\ell+1}{2}$ if it is of finite type. This can easily be computed by observing that in the finite type case, ${}^e\tilde{\Phi}$ is of type $\mathsf{C}_{\ell+1}$ and $D = {}^e\omega_0^\vee$. □

We need some information on the decomposition of Z_n into $\mathfrak{g}$ modules. We denote by $\leq$ the dominant order on ${}^e\mathfrak{t}^*$ and we extend the order $\leq_\sigma$ to ${}^e\mathfrak{t}^*$ by requiring that $\mu \leq_\sigma \lambda$ if $\lambda - \mu \in \mathbb{N}[{}^e\tilde{\Delta}]$. Furthermore, if $\lambda \in {}^e\mathfrak{t}^*$ is such that $\boldsymbol{\sigma}(\lambda) = -\lambda$, then λ can be written in the form $\lambda = \sum_{i=0}^{\ell} a_i\,\varepsilon_i + a\,\gamma + b\,\delta$ and we define

$${}^e\mathrm{gr}(\lambda) := \sum_{i=0}^{\ell} i\,a_i - \langle D, \lambda\rangle.$$

Note that we have

$${}^e\mathrm{gr}({}^e\omega_0) = n_0, \qquad {}^e\mathrm{gr}(\widetilde{\alpha}_0) = \cdots = {}^e\mathrm{gr}(\widetilde{\alpha}_{\ell-1}) = 0, \quad \text{and} \quad {}^e\mathrm{gr}(\widetilde{\alpha}_\ell) > 0.$$

More generally, if $\lambda \in {}^e\mathfrak{t}^*$, then we define ${}^e\mathrm{gr}(\lambda) = \frac{1}{2}{}^e\mathrm{gr}(\lambda - \boldsymbol{\sigma}(\lambda))$. Recall that ${}^e\Omega = \Omega + \mathbb{Z}\,\gamma + \mathbb{Z}\,\delta$ and that $\Delta_0 = \{\alpha \in {}^e\Delta \,:\, \boldsymbol{\sigma}(\alpha) = \alpha\}$.

Proposition 35. Let $\lambda \in {}^e\mathfrak{t}^*$ be a weight of the ${}^e\mathfrak{g}$ module Z. Then ${}^e\mathrm{gr}(\lambda) \leq {}^e\mathrm{gr}({}^e\omega_0)$ and moreover, if $\lambda \in {}^e\Omega$ and ${}^e\mathrm{gr}(\lambda) = {}^e\mathrm{gr}({}^e\omega_0)$, then $\lambda \in {}^e\widetilde{W}_\ell({}^e\omega_0)$, where ${}^e\widetilde{W}_\ell$ is the subgroup generated by $\tilde{s}_0, \tilde{s}_1, \ldots, \tilde{s}_{\ell-1}$. □

Proof. The fact that ${}^e\mathrm{gr}(\lambda) \leq {}^e\mathrm{gr}({}^e\omega_0)$ follows from $\lambda \leq {}^e\omega_0$ and ${}^e\mathrm{gr}(\alpha) = {}^e\mathrm{gr}(\widetilde{\alpha}) \geqslant 0$ for $\widetilde{\alpha} \in {}^e\Delta \smallsetminus \Delta_0$, and ${}^e\mathrm{gr}(\alpha) = 0$ if $\alpha \in \Delta_0$.

Assume now that $\lambda \in {}^e\Omega$ and that ${}^e\mathrm{gr}(\lambda) = {}^e\mathrm{gr}({}^e\omega_0)$. Let $\tilde{w} \in {}^e\widetilde{W}$ be such that $\mu = \tilde{w}(\lambda) \in \tilde{A}$ (see Corollary 32). By the description of the weights of the integrable module Z and Lemma 32, we have $\mu \leq {}^e\omega_0$ and $\mu \in {}^e\Omega$, and hence ${}^e\mathrm{gr}(\lambda) = {}^e\mathrm{gr}(\mu) = {}^e\mathrm{gr}({}^e\omega_0)$. Moreover, since ${}^e\mathrm{gr}(\widetilde{\alpha}_\ell) < 0$, we can choose $\tilde{w}$ in ${}^e\widetilde{W}_\ell$.

Hence it is enough to prove that ${}^e\omega_0$ is the only element ν of ${}^e\Omega \cap A$ such that $\nu \leq {}^e\omega_0$ and ${}^e\mathrm{gr}(\nu) = {}^e\mathrm{gr}({}^e\omega_0)$. Take ν with these properties and consider $\tilde{\nu} = 2\,\nu$. Then $\tilde{\nu} \leq_\sigma \widetilde{\omega}_0$ and let $\widetilde{\omega}_0 - \tilde{\nu} = \sum_{i=0}^{\ell} b_i\,\widetilde{\alpha}_i$. Then from ${}^e\mathrm{gr}(\tilde{\nu}) = {}^e\mathrm{gr}({}^e\widetilde{\omega}_0)$ and ${}^e\mathrm{gr}(\widetilde{\alpha}_\ell) > 0$, we deduce that $b_\ell = 0$. Now by Proposition 21 and Remark 25, the root system generated by $\widetilde{\alpha}_0, \ldots, \widetilde{\alpha}_{\ell-1}$ is of type C_ℓ (numbered from $\ell - 1$ to 0). Further, $\tilde{\nu}$ is a weight with respect to this root system, and $\tilde{\nu}$ is less than or equal to ${}^e\widetilde{\omega}_0$ with respect to the dominant order of this root system (since $b_\ell = 0$). A simple computation for a root system of type C_ℓ then shows that the elements with these properties are given by the following list:

$$ {}^e\widetilde{\omega}_0 > {}^e\varepsilon_2 + \delta_0 > {}^e\varepsilon_4 + 2\delta_0 > \cdots, $$

where $\delta_0 = 0$ if ${}^e\mathfrak{g}$ is of finite type and $\delta_0 = \delta$ if ${}^e\mathfrak{g}$ is of affine type. In particular, $\tilde{\nu}$ must be one of these weights and $\nu = \frac{1}{2}\tilde{\nu}$ belongs to ${}^e\Omega$ only if $\tilde{\nu} = {}^e\widetilde{\omega}_0$ and $\nu = {}^e\omega_0$. ■

As we have already noticed in the proof of Proposition 35, the root system generated by $\widetilde{\alpha}_0, \ldots, \widetilde{\alpha}_{\ell-1}$ is always of type C_ℓ, so that we can easily compute the orbit ${}^e\widetilde{W}_\ell {}^e\omega_0$. In particular, we are interested in the weights in this orbit that are dominant with respect to Δ (or equivalently, $\widetilde{\Delta}$). We now describe these weights. Recall that a root system of type C_ℓ can be realized in $\mathbb{R}^\ell$, with standard basis $e_1, \ldots, e_\ell$, as the set $\{\pm e_i \pm e_j \,:\, i, j = 1, \ldots, \ell\} \smallsetminus \{0\}$ and $\alpha_1^{\mathsf{C}} = e_1 - e_2, \ldots, \alpha_{\ell-1}^{\mathsf{C}} = e_{\ell-1} - e_\ell, \alpha_\ell^{\mathsf{C}} = 2\,e_\ell$ is a basis. Then an element $x = \sum x_i\,e_i$ is an integral weight if the coefficients x_i are integers, is a dominant weight with respect to $\alpha_1^{\mathsf{C}}, \ldots, \alpha_{\ell-1}^{\mathsf{C}}$ if and only if $x_1 \geqslant \ldots \geqslant x_\ell$, and the fundamental weight ω_i^{C} is the element $\sum_{j \leq i} e_i$. The Weyl group is isomorphic to $\mathsf{S}_\ell \ltimes (\mathbb{Z}/2)^\ell$ where S_ℓ acts by permutations and $(\mathbb{Z}/2)^\ell$ acts by changing the sign of the elements e_i. Hence the elements of the Weyl group orbit of ω_ℓ^{C} that are dominant with respect to the first $\ell - 1$ roots are the elements $e_1 + \cdots + e_i - e_{i+1} \cdots - e_\ell$ for $i = 0, \ldots, \ell$. In particular, there are $\ell + 1$ of these elements.

4.2 Some special elements of the Weyl group

We describe the elements $\hat{\tau}_0, \ldots, \hat{\tau}_\ell$ of ${}^e\widetilde{W}_\ell$ whose action on ${}^e\omega_0$ gives all the weights in ${}^e\widetilde{W}_\ell({}^e\omega_0)$ that are dominant with respect to Δ. Let

$$ \hat{\tau}_0 = \mathrm{id} \quad \text{ and for } m = 0, \ldots, \ell - 1 \qquad \hat{\tau}_{m+1} = \tilde{s}_0 \tilde{s}_1 \tilde{s}_2 \cdots \tilde{s}_m \hat{\tau}_m, $$

and define $\tau_m = w_\Delta \hat{\tau}_i$. Also set $\hat{\tau} = \hat{\tau}_\ell$ and $\tau = \tau_\ell$. Then

Lemma 36.

(i) For $i = 0, \dots, \ell$, we have

$$\hat{\tau}_i({}^e\omega_0) = \begin{cases} {}^e\varepsilon_i - {}^e\omega_0 & \text{if } {}^e\mathfrak{g} \text{ is of finite type;} \\ {}^e\varepsilon_i - {}^e\omega_0 - i\delta & \text{if } {}^e\mathfrak{g} \text{ is of affine type;} \end{cases}$$

in particular, $\hat{\tau}_i({}^e\omega_0)|_{{}^e\mathfrak{t}'} = ({}^e\varepsilon_i - {}^e\omega_0)|_{{}^e\mathfrak{t}'}$;

(ii) For $i = 0, \dots, \ell$, we have $\langle \hat{\tau}_i({}^e\omega_0), D \rangle = n_0 - i$;

(iii) $\{\hat{\tau}_m({}^e\omega_0)) : m = 0, \dots, \ell\} = \{\lambda \in {}^e\widetilde{W}_\ell({}^e\omega_0) : \lambda \text{ dominant with respect to } \Delta\}$. □

Proof. To prove part (i), observe that this is a computation which involves only objects related to the Weyl group ${}^e\widetilde{W}$. So it is enough to note that ${}^e\varepsilon_0 = {}^e\widetilde{\omega}_0 = 2\,{}^e\omega_0$, and that by Remark 25 (and also see Convention 4 for the BC_1 case),

$$\widetilde{\alpha}_0 = 2\,{}^e\varepsilon_0 - 2{}^e\varepsilon_1 + 2\,\delta_0;$$
$$\widetilde{\alpha}_i = 2\,{}^e\varepsilon_i - {}^e\varepsilon_{i-1} - {}^e\varepsilon_{i+1} \ \text{ for } 1 \le i \le \ell - 1,$$

where $\delta_0 = 0$ if ${}^e\mathfrak{g}$ is of finite type and $\delta_0 = \delta$ if ${}^e\mathfrak{g}$ is of affine type.

Parts (ii) and (iii) now follow from part (i) and the fact that, by the above discussion the set on the right-hand side in part (iii) has $\ell + 1$ elements. ■

We now restate the results of this discussion in the form we shall use in Section 5. For $\lambda \in \Omega$ and $\lambda = \sum_i a_i\,\varepsilon_i$, we define $\mathrm{gr}(\lambda) = \sum_i i\,a_i$.

Corollary 37. Let $\lambda \in \Omega$ be such that V_λ^* appears as a G module in $(Z^*)_n$. Then $gr(\lambda) \le n$ and if $gr(\lambda) = n$, then $n \le \ell$ and $\lambda = \varepsilon_n$. Moreover, the multiplicity of $V_{\varepsilon_n}^*$ in $(Z^*)_n$ is 1. □

Proof. The first part of the corollary is just a restatement of Proposition 35. The last statement follows from the fact that each weight in the orbit ${}^eW({}^e\omega_0)$ appears with multiplicity 1. ■

4.3 The Schubert Variety $\mathcal{S}$ and the Richardson Variety $\mathcal{R}$

We denote by ${}^eG$ the (minimal) Kac–Moody group (see [15, p. 228]) associated to the Lie algebra ${}^e\mathfrak{g}$ and by ${}^eP$ the stabilizer of the line $\Bbbk z_0$, so that $\mathcal{G}r = {}^eG/{}^eP$ is its associated Grassmannian. On this Grassmannian, we consider the line bundle $\mathcal{L}$ whose space of sections is the ${}^e\mathfrak{g}$ module Z^*.

Let ${}^e\!B$ be the Borel subgroup of ${}^e\!G$ corresponding to the positive roots. Recall that the ${}^e\!B$ orbits in $\mathcal{G}r$ are parametrized by ${}^eW/{}^eW_{{}^eP}$ and that ${}^eW_{{}^eP}$, the Weyl group associated to ${}^eP$, is equal to W. For $w \in {}^eW$, we denote by $[w]$ its class in ${}^eW/W$; we recall that the set ${}^eW/W$ is partially ordered by the inclusion relations corresponding to the orbit closures of the ${}^e\!B$-orbits. In particular, the closure of ${}^e\!Bw{}^eP/{}^eP$ is given by all the orbits ${}^e\!Bw'{}^eP/{}^eP$ with $[w'] \leq [w]$.

Consider the Schubert variety $\mathcal{S}_{\tau_m} := \overline{{}^e\!B\tau_m{}^eP/{}^eP}$ and the ${}^e\!B$ module of sections $\Gamma(\mathcal{S}_{\tau_m}) = \Gamma(\mathcal{S}_{\tau_m}, \mathcal{L})$. This module is a graded quotient of Z^* and we denote by $\Gamma_n(\mathcal{S}_{\tau_m})$ its graded components.

Lemma 38. For $m = 1, \dots, \ell$, we have $\mathcal{S}_{\tau_m} = \overline{{}^eP\tau_m{}^eP/{}^eP}$; in particular, the Schubert varieties $\mathcal{S}_{\tau_m}$ are G-stable. □

Proof. Recall that the ${}^eP$ orbits in $\mathcal{G}r$ are parametrized by $W\backslash{}^eW/W$, and if $w \in {}^eW$, we denote by $[w]_{{}^eP}$ its class in $W\backslash{}^eW/W$. Since ${}^ePw{}^eP$ is the union of all classes ${}^e\!Bw'{}^eP$ with $[w']_{{}^eP} = [w]_{{}^eP}$, our claim follows from the fact that $[w\tau_i] \leq [\tau_i]$ for all $w \in W$, or equivalently, from $[w\hat{\tau}_i] \geqslant [\hat{\tau}_i]$ for all $w \in W$. ■

Note that $\Gamma(\mathcal{S}_{\tau_m})$ is a ${}^eP$ module, and so it is also a G module. The following two theorems collect the essential properties of $\Gamma(\mathcal{S}_{\tau_m})$ which we will need for the constructions in the next section. We first describe the structure of $\Gamma_n(\mathcal{S}_{\tau_m})$ as a G module.

Theorem 39. Let $m \in \{1, \dots, \ell\}$, then

(i) for any $0 \leq i \leq m$, we have $\Gamma_i(\mathcal{S}_{\tau_m}) \simeq V^*_{\varepsilon_i}$ as G modules and $\Gamma_i(\mathcal{S}_{\tau_m}) = 0$ for any $i > m$;

(ii) $\Gamma(\mathcal{S}_{\tau_m}, \mathcal{L}^{\otimes n}) \simeq \bigoplus_{0 \leq i_1 \leq \cdots \leq i_n \leq m} V^*_{\varepsilon_{i_1} + \cdots + \varepsilon_{i_n}}$ as G modules. □

Proof. For $\rho \in {}^eW$, define $S(\rho) = \{\eta \in {}^eW/W : \eta \leq [\rho]\}$ and set

$$S^+(\rho) = \{\eta \in S(\rho) : \eta({}^e\omega_0)|_{\mathfrak{t}} \text{ is dominant for the Lie algebra } \mathfrak{g}\}.$$

For $\eta \in {}^eW/W$, denote by $\hat{\eta}$ the minimal element in $W\eta$. Observe that if $\eta \in S(\rho)$, then $\eta \in S^+(\rho)$ if and only if $\hat{\eta} = \eta$, in particular, $\widehat{[\tau_h]} = [\hat{\tau}_h]$. The first step in the proof is to show that $S^+(\tau_m) = \{[\hat{\tau}_0], [\hat{\tau}_1], \dots, [\hat{\tau}_m]\}$.

Let $\eta \in S^+(\tau_m)$ and suppose $\eta \leq [\tau_h]$ for some $0 \leq h \leq m$. We want to show that either $\eta = [\hat{\tau}_h]$ or $\eta \leq [\tau_{h-1}]$. Once this is established, our claim follows by induction on h, since by hypothesis $\eta \leq [\tau_m]$.

First recall that by Proposition 30, $s_0 = s_{\alpha_0}$ does not appear in any reduced expression for $\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_\ell$. Hence there exists a reduced expression $s_0 s_{\beta_1} s_{\beta_2} \cdots s_{\beta_q}$ with $\beta_i \in {}^e\!\Delta$ for $\hat{\tau}_h$, and in turn there exists a reduced expression $s_{\gamma_1} s_{\gamma_2} \cdots s_{\gamma_p} s_0 s_{\beta_1} s_{\beta_2} \cdots s_{\beta_q}$ for τ_h with $\gamma_i \in \Delta$ for all $1 \leq i \leq p$.

Recall also that η is the minimal element in $W\eta$, so if $\eta \neq [e]$ and if w is the minimal element in ${}^eW$ such that $[w] = \eta$ is any reduced expression for w must start with s_0, suppose that $s_0 s_{\delta_1} s_{\delta_2} \cdots s_{\delta_r}$ is such an expression. By the characterization of the Bruhat order in terms of subwords and since $\eta \leq [\tau_h]$, we can choose the decomposition of w such that $s_0 s_{\delta_1} s_{\delta_2} \cdots s_{\delta_r}$ is a subword of $s_{\gamma_1} s_{\gamma_2} \cdots s_{\gamma_p} s_0 s_{\beta_1} s_{\beta_2} \cdots s_{\beta_q}$. But $\gamma_i \in \Delta$ for all $1 \leq i \leq p$, and hence $s_0 s_{\delta_1} \cdots s_{\delta_r}$ is a subword of $s_0 s_{\beta_1} \cdots s_{\beta_q} = \hat{\tau}_h$; this shows that $\eta \leq [\hat{\tau}_h]$ as elements of ${}^eW/W$.

Next we show that $[s_0 \hat{\tau}_h]$ is the unique element in ${}^eW/W$ covered by $[\hat{\tau}_h]$ with respect to the Bruhat order. (If a, b are elements of a partially ordered set, we say that a *covers* b if $a > b$ and $a > c \geqslant b$ implies $c = b$.) Recall that $\kappa' < \kappa$ for $\kappa, \kappa' \in {}^eW/W$ if and only if for the corresponding Demazure modules in Z we have $Y_{\kappa'} \subseteq Y_\kappa$. For $\hat{\tau}_h$, the Demazure module is generated by an extremal weight vector $v_{\hat{\tau}}$ of weight $\hat{\tau}_h({}^e\omega_0)\big|_{{}^e\mathfrak{t}'} = (\varepsilon_h - {}^e\omega_0)|_{{}^e\mathfrak{t}'}$. It follows that $e_\alpha v_{\hat{\tau}} = 0$ for any root operator corresponding to a simple root $\alpha \neq \alpha_0$, and $e_{\alpha_0} v_{\hat{\tau}} = v_{s_{\alpha_0}\hat{\tau}}$ is a generator for the Demazure module $Y_{s_{\alpha_0}\hat{\tau}}$. This shows that a Demazure module properly contained in $Y_{\hat{\tau}}$ is also contained in $Y_{s_{\alpha_0}\hat{\tau}}$, which proves the claim. So we can now conclude that $\eta = [\hat{\tau}_h]$ or $\eta \leq [s_0 \hat{\tau}_h] \leq [\tau_{h-1}]$ and the claimed description of $S^+(\tau_m)$ is thus proved.

Now we prove part (i) using the LS-path branching rule [21]. Let $\mathbb{B}$ be the LS-path model for the ${}^e\mathfrak{g}$ module Z and let $\mathbb{B}(\tau_m)$ be the path submodel for the ${}^eP$ module $\Gamma(\mathcal{S}_{\tau_m})$, and recall that

$$\mathrm{Res}_G^{{}^eP}\,\Gamma(\mathcal{S}_{\tau_m}, \mathcal{L}^{\otimes n}) \simeq \oplus_\pi V^*_{\pi(1)\big|_{\mathfrak{t}}},$$

where the sum runs over all LS-paths $\pi \in \mathbb{B}(\tau_m)$ of degree n such that $\pi(x)\big|_{\mathfrak{t}}$ belongs to the dominant Weyl chamber of $\mathfrak{g}$ for all $0 \leq x \leq 1$. Suppose such a path is written as $\pi = \pi_1 * \cdots * \pi_r$ with $\pi_i = \pi_{a_i \eta_h({}^e\omega_0)}$ for some elements $\eta_1 < \cdots < \eta_r$ in $S(\tau_m)$ and some rational numbers $0 < a_1, \ldots, a_r$ such that $a_1 + \cdots + a_r = n$. The requirement $\pi(x)\big|_{\mathfrak{t}}$ is dominant for all x implies that $\eta_r({}^e\omega_0)\big|_{\mathfrak{t}}$ is dominant or equivalently, $\eta_r \in S^+(\tau_m)$, so $\eta_r = [\hat{\tau}_h]$ for some $0 \leq h \leq m$.

Now the requirement for π to be an LS-path implies that $a_{r-1}\langle\hat{\tau}_h({}^e\omega_0), \alpha_0^\vee\rangle \in \mathbb{Z}$. But $\langle\hat{\tau}_h({}^e\omega_0), \alpha_0^\vee\rangle$ is equal to -1 if $h > 0$ and to 1 if $h = 0$, so if $n = 1$, this implies that $a_r = 1$ and $r = 1$, $\pi = \pi_{\eta_h({}^e\omega_0)}$ and $\pi(1)\big|_{\mathfrak{t}} = \varepsilon_h$, which proves our claim since $\langle\hat{\tau}_h({}^e\omega_0), D\rangle = -h$ so $V^*_{\varepsilon_h}$ is in degree h.

To simplify the presentation, we prove part (ii) only in the case $n = 2$, the proof for the general case is completely analogous. In this case, we can have $a_r = 2$ and $r = 1$, $\pi = \pi_{2\eta_h({}^e\omega_0)}$ and $\pi(1)\big|_{\mathfrak{t}} = 2\varepsilon_h$, or $a_r = 1$ and $r > 1$. In the second case, the requirement $\pi(x)\big|_{\mathfrak{t}}$ is dominant for all x implies that $(a_{r-1}\eta_{r-1}({}^e\omega_0) + \hat{\tau}_h({}^e\omega_0))\big|_{\mathfrak{t}}$ is dominant. Now note that if $a, b > 0$ and $\eta < [\hat{\tau}_h]$ are such that $(a\eta({}^e\omega_0) + b\hat{\tau}_h({}^e\omega_0))\big|_{\mathfrak{t}}$ is dominant, then $\eta \in S^+(\tau_m)$. Indeed, if $\alpha \in \Delta$, then $\langle\hat{\tau}_h({}^e\omega_0), \alpha^\vee\rangle \neq 0$ implies $\widetilde{\alpha} = \widetilde{\alpha}_h$. So it is enough to prove that $\langle\eta({}^e\omega_0), \alpha^\vee\rangle \geqslant 0$ for all $\alpha \in \Delta$ such that $\widetilde{\alpha} = \widetilde{\alpha}_h$. By construction, we have $\hat{\tau}_h({}^e\omega_0) = {}^e\omega_0 - \sum_{i \leq h-1} a_i\widetilde{\alpha}_i$ with $a_i \in \mathbb{N}$, and $\hat{\tau}_h({}^e\omega_0) = {}^e\omega_0 - \sum_{\alpha\in{}^e\Delta:\widetilde{\alpha}\neq\widetilde{\alpha}_h} b_\alpha\alpha$ with $b_\alpha \in \mathbb{N}$. So if $\eta < [\hat{\tau}_h]$, then we must have $\eta({}^e\omega_0) = {}^e\omega_0 - \sum_{\alpha\in{}^e\Delta:\widetilde{\alpha}\neq\widetilde{\alpha}_h} c_\alpha\alpha$, where $c_\alpha \in \mathbb{N}$. In particular, $\langle\eta({}^e\omega_0), \alpha^\vee\rangle \geqslant 0$ for all $\alpha \in \Delta$ such that $\widetilde{\alpha} = \widetilde{\alpha}_h$. ∎

The above theorem will be more convenient for us in the following form. For $m = 1, \ldots, \ell$, define $\mathcal{R}_m$ as the Richardson subvariety of $\mathcal{S}_{\tau_m}$ defined by $z_0^* = 0$ and also set $\mathcal{R} = \mathcal{R}_\ell$.

Corollary 40. For $m = 1, \ldots, \ell$, we have the following isomorphism of G modules:

$$\Gamma(\mathcal{R}_m, \mathcal{L}^{\otimes n}) \simeq \bigoplus_{1\leq i_1\leq\cdots\leq i_n\leq m} V^*_{\varepsilon_{i_1}+\cdots+\varepsilon_{i_n}}.$$

In particular, $\Gamma_{\mathcal{R}} = \bigoplus_{n\geqslant 0}\Gamma(\mathcal{R}, \mathcal{L}^n)$ is isomorphic to $\Bbbk[G/H]$ as a G module. □

We need the following simple result in the proof of the next theorem.

Lemma 41. The module $V_{\varepsilon_{i+1}}$ appears with multiplicity 1 in the tensor product $V_{\varepsilon_1} \otimes V_{\varepsilon_i}$ for $i = 1, 2, \ldots, \ell - 1$. □

Proof. Let us denote by $\mathbb{B}$ a path model for the G module V_{ε_k} and denote by π_{ε_1} the path $\mathbb{Q} \ni t \mapsto t\varepsilon_1 \in \Lambda \otimes \mathbb{Q}$. We have the path tensor product formula (see [21])

$$V_{\varepsilon_k} \otimes V_{\varepsilon_1} \simeq \oplus V_{\eta(1)+\varepsilon_1},$$

where the sum runs over all paths $\eta \in \mathbb{B}$ such that the concatenation $\eta * \pi_{\varepsilon_1}$ is completely contained in the dominant Weyl chamber. So in order to obtain the module $V_{\varepsilon_{k+1}}$, we have to look for paths in $\mathbb{B}$ ending in $\varepsilon_{k+1} - \varepsilon_1$.

Using the same description of restricted roots we used in the proof of Lemma 36, we have $s_{\widetilde{\alpha}_1} s_{\widetilde{\alpha}_2} \cdots s_{\widetilde{\alpha}_k}(\varepsilon_k) = \varepsilon_{k+1} - \varepsilon_1$.

Since the restricted Weyl group is a quotient of a subgroup of the Weyl group of G, we have proved that the weight $\varepsilon_{k+1} - \varepsilon_1$ is an extremal weight for the G module V_{ε_k}. This shows that exactly one path in $\mathbb{B}$ ends in $\varepsilon_{k+1} - \varepsilon_1$ and this finishes the proof. ■

The nonvanishing of the following specific vector will be important for us in the next section. Recall that, by Proposition 26, we have ${}^e\mathfrak{g}_1 \simeq V^*_{\varepsilon_1}$ and so we can choose a spherical vector $h_1 \in {}^e\mathfrak{g}_1$.

Theorem 42. For any $0 \le i \le \ell$, the element $h_1^i \cdot z_0^*\big|_{\mathcal{S}_\tau}$ is a nonzero section in $\Gamma_i(\mathcal{S}_\tau)$. □

Proof. Consider the enveloping algebra of ${}^e\mathfrak{g}_+$: $\mathbf{U}_+ = \mathbf{U}({}^e\mathfrak{g}_+)$. Note that it is generated by ${}^e\mathfrak{g}_1 \subset {}^e\mathfrak{g}_+ \subset \mathbf{U}_+$, and that the map from $\mathbf{U}_+$ to Z^* given by $x \mapsto x \cdot z_0^*$ is surjective. Moreover, $\mathbf{U}_+$ and Z^* are compatibly graded, hence for all $n > 0$ we have a surjective morphism

$$ {}^e\mathfrak{g}_1^{\otimes n} \longrightarrow Z_n^* \quad \text{given by} \quad x_1 \otimes \cdots \otimes x_n \longmapsto x_1 \cdot (x_2 \cdot (\ldots x_n \cdot z_0^*)). $$

Similarly, we have a surjective map from ${}^e\mathfrak{g}_1^{\otimes n}$ onto $\Gamma_n(\mathcal{S}_\tau)$ and by induction a surjective map

$$ a : {}^e\mathfrak{g}_1 \otimes \Gamma_i(\mathcal{S}_\tau) \longrightarrow \Gamma_{i+1}(\mathcal{S}_\tau) \quad \text{given by} \quad x \otimes v \longmapsto x \cdot v. $$

Now we have $\Gamma_i(\mathcal{S}_\tau) \simeq V^*_{\varepsilon_i}$ and ${}^e\mathfrak{g}_1 \simeq V^*_{\varepsilon_1}$. By the above lemma, the multiplicity of $V^*_{\varepsilon_{i+1}}$ in $V^*_{\varepsilon_1} \otimes V^*_{\varepsilon_i}$ is 1. Since a is G-equivariant, the morphism a must be equal (up to a nonzero scalar) to the projection $\pi^{\varepsilon_1, \varepsilon_i}_{\varepsilon_{i+1}}$. In particular, $a(h_1 \otimes h_{\varepsilon_i}) \neq 0$ by Corollary 20; such a nonzero element is H-invariant and must hence be a nonzero multiple of $h_{\varepsilon_{i+1}}$, which proves the claim by induction. ■

5 The Equations of the Symmetric Variety

In this section, we describe the relation between the symmetric space G/H and the Grassmannian $\mathcal{G}r$. The naive approach is the following: let $h_{-1} \in {}^e\mathfrak{g}_{-1}$ be fixed by H as in

Section 3.4 and define $x = e^{h_{-1}}(\Bbbk z_0) \in \mathbb{P}(Z)$.The point x is certainly fixed by H (since both h_{-1} and z_0 are fixed by H). So we can define an immersion $G/H \longrightarrow \mathcal{G}r$ by $gH \mapsto gx$, and we deduce the defining equations for G/H from the defining equations for $\mathcal{G}r$.

Of course, this naive approach has a difficulty, since the exponential map is not defined for all elements in the Lie algebra in the affine case. Nevertheless, the reader should keep this simple idea as a travel guide in mind. To make the idea work despite the obvious mistake, we have to go through a sometimes rather technical-looking detour.

5.1 The completion of $\mathcal{U}^-$ and some notation for Schubert varieties

In order to define $e^{h_{-1}}$, we introduce a completion of the negative unipotent subgroup of ${}^e G$. Let ${}^e\mathfrak{b}^-$ be the Borel algebra defined by the negative roots and let ${}^e B^-$ be the associated Borel subgroup. We define ${}^e\mathfrak{n}^-$ as the nilpotent radical of ${}^e\mathfrak{b}^-$ and ${}^e\mathcal{U}^-$ as the unipotent radical of ${}^e B^-$. Also we denote by ${}^e\hat{\mathfrak{n}}^-$ the pro-Lie algebra $\prod_{\alpha \in {}^e\Phi^-} {}^e\mathfrak{g}_\alpha$ and we define ${}^e\hat{\mathcal{U}}^- := \exp({}^e\hat{\mathfrak{n}}^-)$ (see [15, p. 221]) and we have an inclusion ${}^e\mathcal{U}^- \hookrightarrow {}^e\hat{\mathcal{U}}^-$. In particular, $e^{h_{-1}}$ is an element of ${}^e\hat{\mathcal{U}}^-$.

The group ${}^e\hat{\mathcal{U}}^-$ does not act on Z but for all finite codimensional ${}^e\mathcal{U}^-$ submodules J of Z, the action of ${}^e\mathcal{U}^-$ on Z/J extends uniquely to an action of ${}^e\hat{\mathcal{U}}^-$. Moreover, if J is G-stable, then the orbit ${}^e\hat{\mathcal{U}}^- z_0$ in Z/J is also stable by the action of G.

Let ${}^e P^-$ be the parabolic subgroup opposite to ${}^e P$. The ${}^e G$-orbit of the line $\Bbbk z_0^*$ in $\mathbb{P}(Z^*)$ is isomorphic to ${}^e G/{}^e P^-$. For an element η of the Weyl group ${}^e W$, let $\mathcal{S}_\eta$ be the Schubert variety $\overline{{}^e B\, \eta\, {}^e P/{}^e P}$ and denote by $\mathcal{S}_\eta^\vee$ the Schubert variety $\overline{{}^e B^-\, \eta\, {}^e P^-/{}^e P^-}$. Let $Y_\eta \subset Z$ be the associated Demazure module, i.e. Y_η is the vector subspace of Z generated by the cone over $\mathcal{S}_\eta$. Similarly, let $Y_\eta^\vee \subset Z^*$ be the associated Demazure module. Denote by $J_\eta \subset Z$ (resp. $J_\eta^\vee \subset Z^*$) the annihilator of $Y_\eta^\vee$ (resp. Y_η). Then J_η is a ${}^e\mathcal{U}^-$-stable complement of Y_η, and if $\mathcal{S}_\eta$ is G-stable, then J_η is also G-stable.

For an element η of the Weyl group ${}^e W$, we denote by $\hat{A}_\eta$ the orbit ${}^e\hat{\mathcal{U}}^-(\Bbbk z_0) \subset Z/J_\eta$. If $\mathcal{S}_{\eta'} \subset \mathcal{S}_\eta$, then we have an inclusion $J_{\eta'} \supset J_\eta$ of the annihilators. Denote by $p^\eta_{\eta'}$ the projection

$$p^\eta_{\eta'} : \mathbb{P}(Z/J_\eta) \smallsetminus \mathbb{P}(J_{\eta'}/J_\eta) \longrightarrow \mathbb{P}(Z/J_{\eta'}).$$

Note that $\hat{A}_\eta \subset \mathbb{P}(Z/J_\eta) \smallsetminus \mathbb{P}(J_{\eta'}/J_\eta)$, and so $p^\eta_{\eta'}$ is well defined on $\hat{A}_\eta$. Let $A := {}^e\mathcal{U}^-(\Bbbk z_0) \subset \mathcal{G}r$ be the open cell and set $A_\eta = A \cap \mathcal{S}_\eta$. The projection from $\mathbb{P}(Z) \smallsetminus \mathbb{P}(J_\eta)$ to $\mathbb{P}(Z/J_\eta)$ becomes an isomorphism when restricted to A_η, and its image is contained in $\hat{A}_\eta$.

5.2 The immersion ι_η

Let $\mathcal{S}_\eta$ be the closure of a ${}^eP$-orbit and set $x_\eta := e^{h_{-1}}(\Bbbk z_0) \in \mathbb{P}(Z/J_\eta)$. Consider the G-equivariant map

$$\iota_\eta : G/H \longrightarrow \hat{A}_\eta \subset \mathbb{P}(Z/J_\eta); \quad \iota_\eta(gH) = gx_\eta. \tag{2}$$

Since $e^{h_{-1}}z_0$ is fixed by H, the pullback $\iota_\eta^*(\mathcal{O}_{\mathbb{P}(Z/J_\eta)})$ on G/H is trivial and we have an induced map $\iota_\eta^* : (Z/J_\eta)^* \simeq Y_\eta^\vee \longrightarrow \Bbbk[G/H]$. We can normalize this map in such a way that $\iota_\eta^*(z_0^*)$ is the constant function, with value 1 on G/H. Note that if $\mathcal{S}_{\eta'} \subset \mathcal{S}_\eta$, then we have the following commutative diagram:

$$\begin{tikzcd} G/H \arrow[r, "\iota_\eta"] \arrow[dr, "\iota_{\eta'}"'] & \hat{A}_\eta \arrow[d, "p^\eta_{\eta'}"] & \subset \mathbb{P}(Z/J_\eta) \\ & \hat{A}_{\eta'} & \subset \mathbb{P}(Z/J_{\eta'}). \end{tikzcd}$$

If we normalize the pullback $(p^\eta_{\eta'})^*$ of the projections $p^\eta_{\eta'}$ to map z_0^* into z_0^*, then $(p^\eta_{\eta'})^*$ restricted to $Y_{\eta'}^\vee$ is just given by the inclusion $Y_{\eta'}^\vee \subset Y_\eta^\vee$. So if $f \in Y_\eta^\vee$, then $\iota_\eta^*(f) = \iota_{\eta'}^*(f)$ and we can define $\iota^* : Z^* \longrightarrow \mathbb{C}[G/H]$ as the limit of the maps ι_η^*. Consider the morphism of rings $S\iota^* : S(Z^*) \longrightarrow \Bbbk[G/H]$ given by the symmetric product of the map ι. Recall that the ring $\Gamma_{\mathcal{G}r} := \Gamma_{\mathcal{L}}(\mathcal{G}r) = \bigoplus_{n\geqslant 0} \Gamma(\mathcal{G}r, \mathcal{L}^n)$ is a quotient of $S(Z^*)$, and let $I \subset S(Z^*)$ be the ideal defining $\Gamma_{\mathcal{G}r}$.

Lemma 43. $S\iota^*(I) = 0$, so $S\iota^*$ determines a morphism of rings $\varphi : \Gamma_{\mathcal{G}r} \longrightarrow \Bbbk[G/H]$. □

Proof. Let $f \in I$. We can assume that f is a homogeneous element contained in a suitable symmetric power of $Y_\eta^\vee$ for an appropriate η, so that $S\iota^*(f) = S\iota_\eta^*(f)$. We wish to prove that $f(\iota_\eta(x)) = 0$ for all $x \in G/H$. By [15, §VII.3], there exists a Schubert variety $\mathcal{S}_\vartheta$ such that we have $p^\vartheta_\eta(A_\vartheta) = \hat{A}_\eta$. So let $y \in A_\vartheta$ be such that $p^\vartheta_\eta(y) = \iota_\eta(x)$. Then $f(y) = 0$, since $y \in \mathcal{G}r$. But note that $f(y) = f(p^\vartheta_\eta(y))$, since f is in a symmetric power of $Y_\eta^\vee$ and so it is zero on J_η. ■

5.3 Standard monomial theory for G/H

Now we use the morphism φ and the SMT for the ring $\Gamma_{\mathcal{G}r}$ (see Section 2.3) to construct an SMT for the ring $\Bbbk[G/H]$. Let $\mathbb{F} = \mathbb{F}_{\mathcal{L}}$ be the basis of $\Gamma(\mathcal{G}r, \mathcal{L}) = Z^*$ constructed in [22] and denote by $\leftarrow$ the t.a.b.r. on $\mathbb{F}$. The construction can be arranged in such a way that $f_0 = z_0^*$ is the minimal element in $\mathbb{F}$. Denote by $\mathbb{SM}$ (resp. by $\mathbb{M}$) the set of standard monomials (resp. the set of monomials) in the elements of $\mathbb{F}$. For $f \in \mathbb{F}$, set $g_f = \varphi(f)$, we define similarly g_m for $m \in \mathbb{M}$.

For an element $\eta \in {}^eW$, we define $\mathbb{F}(\eta) = \{f \in \mathbb{F} : f|_{\mathcal{S}_\eta} \neq 0\}$ and $\mathbb{F}_0(\eta) = \mathbb{F}(\eta) \setminus \{f_0\}$. If η is the special element τ (see Section 4.2), then we denote $\mathbb{F}_0(\eta)$ just by $\mathbb{F}_0$. Recall that $\mathbb{F}_0 = \{f \in \mathbb{F} : f|_{\mathcal{R}} \neq 0\}$. Let $\mathbb{SM}_0$ (resp. $\mathbb{M}_0$) be the set of all standard monomials (resp. all monomials) in the elements of $\mathbb{F}_0$. Recall that, as in Section 2.3, the set $\{m|_{\mathcal{R}} : m \in \mathbb{SM}_0\}$ is a $\Bbbk$ basis of $\Gamma_{\mathcal{R}} = \bigoplus_{n \geqslant 0} \Gamma(\mathcal{R}, \mathcal{L}^n)$.

We are now in a position to apply all the various technical results of this and the previous sections to conclude with our main theorem.

By Theorem 42, for all $f \in \mathbb{F}_0$, the functions g_f do not vanish identically. Hence by Corollary 40, the set

$$\mathbb{G}_0 = \{g_f : f \in \mathbb{F}_0\}$$

is a $\Bbbk$ basis of $\mathbb{V} := V_{\varepsilon_1}^* \oplus \cdots \oplus V_{\varepsilon_\ell}^* \subset \Bbbk[G/H]$. We introduce the following t.a.b.r. on $\mathbb{G}_0$ induced by the t.a.b.r. on $\mathbb{F}_0$: $g_f \leftarrow g_{f'}$ if and only if $f \leftarrow f'$.

Theorem 44. The set $\{g_m : m \in \mathbb{SM}_0\}$ is a basis of $\Bbbk[G/H]$, hence $(\mathbb{G}_0, \leftarrow)$ is an SMT for the ring $\Bbbk[G/H]$. □

Proof. Let E be the span in $\Gamma_{\mathcal{G}r}$ of the monomials g_m with $m \in \mathbb{SM}_0$, this set is G-stable. By Corollary 40, E is isomorphic to $\bigoplus_{\lambda \in \Omega} V_\lambda^*$ as a G module. Let E_λ be the G-submodule of E isomorphic to V_λ^*. By Theorem 42, we know that $\varphi(E_{\varepsilon_i}) \neq 0$, and hence also $\varphi(E_\lambda) \neq 0$ because $\Bbbk[G/H]$ is a domain (the product of the two highest-weight vectors in E_μ and E_ν is the highest-weight vector in $E_{\mu+\nu}$). So $\varphi|_E$ is injective, and by the descriptions of $\Bbbk[G/H]$ and $\Gamma_{\mathcal{R}}$ as G modules (Corollary 40), it follows that the map is surjective. ■

Remark 45. In Section 2.4, we gave a description for an SMT for the ring $\Gamma_{\bar{X}}$. In particular, by the description of $\Bbbk[G/H]$ as the quotient $\Gamma_{\bar{X}}/(s_i = 1)$, we obtain a set of generators of $\Bbbk[G/H]$. These generators coincide with the functions $g_f \in \mathbb{G}_0$. This follows from the fact that the G modules that we are considering (the submodules V_{ε_i} of Z_i^*) are generated by extremal weight vectors of the modules Z^*, by the construction of the SMT in [22], and standard arguments.

It is also not difficult to prove that the SMT of Theorem 44 is compatible with G modules in the following sense: there exists a filtration of $\mathbb{k}[G/H]$ by G modules F_i with simple quotients such that, for all i, the set $\{g_m : m \in \mathbb{SM}_0\} \cap F_i$ is a $\mathbb{k}$ basis of F_i. $\square$

5.4 Straightening relations for $\mathbb{k}[G/H]$

We now describe straightening relations for the standard monomial theory using the Plücker relations for the Grassmannian. We denote by $<_t$ the total order on $\mathbb{M}$ and for $f, f' \in \mathbb{F}$ non $\leftarrow$comparable, let $R_{f,f'} = f f' - P_{f,f'} \in I \cap S^2(Z^*)$ be the Plücker relation as in Section 2.3.

Let $\mathbb{k}[u] = \mathbb{k}[u_f \mid f \in \mathbb{F}_0]$ be the polynomial ring with generators indexed by the elements of $\mathbb{F}_0$. For a monomial $m = f_1 \cdots f_s \in \mathbb{M}_0$, let $u_m = u_{f_1} \cdots u_{f_s}$ be the corresponding monomial in $\mathbb{k}[u]$. Denote by ψ the morphism of rings from the polynomial algebra $\mathbb{k}[u]$ to $\mathbb{k}[G/H]$ defined by $\psi(u_f) = g_f$, and let Rel be the kernel of this morphism.

We introduce on $\mathbb{k}[u]$ a degree: for $f \in \mathbb{F}_0(\tau_i) \smallsetminus \mathbb{F}_0(\tau_{i-1})$, let u_f be of degree i and we indicate by $\mathrm{gr}(r)$ the degree of an element r in $\mathbb{k}[u]$. If $m, m' \in \mathbb{M}_0$, then we define $u_m \prec_t u_{m'}$ if $\mathrm{gr}(u_m) < gr(u_{m'})$ or if $\mathrm{gr}(u_m) = gr(u_{m'})$ and $m <_t m'$.

This order has the properties explained in Section 2.2. The compatibility of this order with the t.a.b.r. $\leftarrow$ on $\mathbb{G}_0$ follows from the compatibility of the t.a.b.r. $\leftarrow$ between elements of $\mathbb{F}$, with the dominant order of the associated weights recalled in Section 2.3.

Fix an element $\eta \geqslant \tau$ such that for all $f, f' \in \mathbb{F}_0$ that are not comparable, the relation $R_{f,f'}$ is in $S^2(Y_\eta^\vee)$. Equivalently, $P_{f,f'}$ is a polynomial in the functions in $\mathbb{F}(\eta)$. We define $\mathbb{F}_1 = \mathbb{F}(\eta) \smallsetminus \mathbb{F}_0$.

For each $f \in \mathbb{F}$, let $n_f = -\langle D, \mathrm{weight}(f)\rangle + n_0$. Recall that weight$(f)$ is the weight of f with respect to $\mathfrak{t}$ and so $f \in (Z^*)_{n_f}$. Note that the set $\{f : f \notin \mathbb{F}_0$, and $n_f = n\}$ is a G-stable complement for $V^*_{\varepsilon_n}$ in $(Z^*)_n$ for $n = 1, \ldots, \ell$ and is equal to $(Z^*)_n$ otherwise. Hence if $f \in \mathbb{F}_1$, then by Corollary 37, we have

$$g_f \in \bigoplus_{\lambda \in \Omega \text{ and } gr(\lambda) < n_f} V^*_\lambda.$$

In particular, for each $f \in \mathbb{F}_1$, we can choose an element $F_f(u) \in \mathbb{k}[u]$ such that $\mathrm{gr}(F_f) < n_f$ and such that $\psi(F_f) = F_f((g_{f'})_{f' \in \mathbb{F}_0}) = g_f$. We also set $F(u) = (F_f(u))_{f \in \mathbb{F}_1}$.

Remark 46. The computation of the polynomials F depends only on the expansion of $e^{h_{-1}} z_0^*$ and on the representation theory of G and not anymore on the geometry of G/H.

Indeed, once $e^{h_{-1}} z_0^*$ is computed, we can determine the map φ, hence the decomposition of the functions g_f in the irreducible factors in $\Bbbk[G/H]$ (we have explicit bases of the irreducible modules given, for example, by the basis in [22]). Now given an element in $V_\lambda^* \subset \Bbbk[G/H]$, we have $\lambda = \sum n_i \varepsilon_i$ and V_λ^* appears with multiplicity 1 in the tensor product $TP = V_{\varepsilon_1}^{\otimes n_1} \otimes \cdots \otimes V_{\varepsilon_1}^{\otimes n_1}$. In particular, the G-equivariant projection π from TP to V_λ^* is unique up to scalar. Now consider the product map from TP to the ring $\Bbbk[G/H]$ followed by the projection onto V_λ^*. This is also a G-equivariant nonzero map, so it should coincide with π up to a nonzero scalar. By fixing highest-weight vectors, this scalar can be normalized to be 1. So the functions F_f are determined by the decomposition of the tensor product TP. $\square$

We will use the set $\mathbb{G}_1 = \{g_f : f \in \mathbb{F}_1\}$ as a set of auxiliary variables, so for each $f \in \mathbb{F}_1$ we introduce a new variable v_f and we set $v = (v_f)_{f \in \mathbb{F}_1}$.

For non $\leftarrow$comparable elements $f, f' \in \mathbb{F}_0$, we have the polynomials $R_{f,f'}$ and $P_{f,f'}$ in the symmetric algebra $\mathsf{S}(\mathbb{F}_0 \cup \mathbb{F}_1)$. Let $R_{f,f'}(u,v)$ and $P_{f,f'}(u,v)$ be the polynomials obtained by substituting an element $h \in \mathbb{F}_0 \cup \mathbb{F}_1$ by u_h if $h \in \mathbb{F}_0$ and by v_h if $h \in \mathbb{F}_1$, so that $R_{f,f'}(u,v) = u_f u_{f'} - P_{f,f'}(u,v)$. Note that $P_{f,f'}(u,v)$ is a homogeneous polynomial of degree 2 which is the sum of monomials of the form $u_{f_1} u_{f_2}$ or $u_{f_1} v_{f_2}$ or $v_{f_1} v_{f_2}$, where $f_1\, f_2 <_t f\, f'$ and $n_{f_1} + n_{f_2} = n_f + n_{f'}$ (by the fact that the relations are $\mathfrak{t}$-homogeneous).

Now let ψ_1 be the morphism of rings from the polynomial ring $\Bbbk[u,v]$ to $\Bbbk[G/H]$, defined by $\psi_1(u_f) = g_f$ if $f \in \mathbb{F}_0$ and $\psi_1(v_f) = g_f$ if $f \in \mathbb{F}_1$, and let Rel_1 be the kernel of this map. By Lemma 43 and by the definition above, we have the following equations in $\Bbbk[u,v]$:

$$v_f = F_f(u) \qquad (\mathrm{mod}\ Rel_1) \qquad \text{for all } f \in \mathbb{F}_1; \tag{3}$$

$$R_{f,f'}(u,v) = 0 \qquad (\mathrm{mod}\ Rel_1) \qquad \text{for all } f, f' \in \mathbb{F}_0 \text{ which are non} \leftarrow\text{comparable.} \tag{4}$$

Now we can substitute equations (3) in equations (4) and define

$$\hat{P}_{f,f'}(u) = P_{f,f'}(u, F(u))$$
$$\hat{R}_{f,f'}(u) = R_{f,f'}(u, F(u)) = u_f u_{f'} - \hat{P}_{f,f'}(u)$$

for all $f, f' \in \mathbb{F}_0$ which are non $\leftarrow$comparable. The new polynomials $\hat{R}_{f,f'}(u)$ obtained in this way are obviously elements of $\mathrm{Rel} \subset \Bbbk[u]$. More precisely, the following theorem states that these polynomials form a set of straightening relations.

Theorem 47. The relations $\hat{R}_{f,f'}(u)$ for $f, f' \in \mathbb{F}_0$ that are non $\leftarrow$comparable are a set of straightening relations for the order $\prec_t$ introduced above. In particular, they generate the ideal $\mathrm{Rel} = \ker\psi$ in $\Bbbk[u]$. □

Proof. We have to prove for all $f, f' \in \mathbb{F}_0$ which are non $\leftarrow$comparable: the polynomial $\hat{P}_{f,f'}(u)$ is a sum of monomials $u_m \prec_t u_f u_{f'}$.

Let $u_{f_1} v_{f_2}$ be a monomial that appears in $P_{f,f'}(u, v)$. Then $\mathrm{gr}(u_{f_1} F_{f_2}(u)) = gr(u_{f_1}) + gr(F_{f_2}(u)) < n_{f_1} + n_{f_2} = n_f + n_{f'}$, by the discussion above and so all the monomials that appear in $u_{f_1} F_{f_2}(u)$ are $\prec_t$ of $u_f u_{f'}$. Similarly, we can treat the monomials $v_{f_1} v_{f_2}$. Finally, the monomials $u_{f_1} u_{f_2}$ that appear in $P_{f,f'}(u, v)$ are such that $f_1\ f_2 <_t f\ f'$, and so $u_{f_1} u_{f_2} \prec_t u_f u_{f'}$. This proves that the relations $\hat{R}_{f,f'}(u)$ for $f, f' \in \mathbb{F}_0$ are a set of straightening relations. The second part of the statement follows now by Theorem 44 and Lemma 10. ■

Despite the fact that the computation of the polynomials F_f depends only on the expansion of $e^{h_{-1}}$ and the representation theory of G, it seems to be complicated to get explicit formulas and check basic properties for these polynomials. For example, by Corollary 17 we know that the relations in the generators $\mathbb{G}_0$ are quadratic. However, a priori the relations $\hat{R}_{f,f'}$ can be of higher degree. From this point of view, it is natural to ask whether it is possible to fix $\eta \geq \tau$ such that the functions F_f can be chosen to be linear in the generators, in this case it would be clear that the relations $\hat{R}_{f,f'}$ are quadratic.

A more precise way to state this is the following: let η be minimal such that $p^{\eta}_{\tau}(\mathcal{S}_\eta)$ contains x_τ. Is it true that any spherical module in $\Gamma(\mathcal{S}_\eta, \mathcal{L})$ is one of the modules $V^*_{\varepsilon_i}$? In the last section, we show that in the case ${}^e\mathfrak{g}$ is of finite type this question has an affirmative answer.

Remark 48. We have seen above that the coordinate rings of G/H and $\Gamma_{\mathcal{R}}$ have similar properties. Indeed, we can perform a two-step flat and G-equivariant deformation of $\Bbbk[G/H]$ to $\Gamma_{\mathcal{R}}$. Let $\Gamma_\tau = \bigoplus_{n\geqslant 0} \Gamma(\mathcal{S}_\tau, \mathcal{L}^n)$ and define A to be the quotient of Γ_τ modulo the ideal generated by $(f_0 - 1)$. It is clear that the ring A can be deformed to $\Gamma_{\mathcal{R}}$ in a flat and a G-equivariant way. We exhibit now a deformation of $\Bbbk[G/H]$ to A. For this purpose, we first need to change the choice of our generators $\mathbb{F}_1$.

Let $\mathbb{F}'_1$ be a set of elements such that

(i) $f_0 \in \mathbb{F}'_1$;
(ii) $\mathbb{F}'_1$ is a basis of the vector space generated by $\mathbb{F}_1$;

(iii) the elements of $\mathbb{F}_1'$ are $\mathfrak{t}$-homogeneous and compatible with G modules; in particular, for each f in $\mathbb{F}_1'$ there exists an irreducible submodule M of $\Gamma(\mathcal{Gr}, \mathcal{L})$ such that $f \in M$, and let $M \simeq V^*_{\lambda_f}$. If $\lambda_f = \sum a_i \varepsilon_i$ and $f \in Z_n$ then we define $\tilde{n}_f = n - \sum i\, a_i$ and note that this number is greater than 0 if $f \neq f_0$.

Note that conditions (i) and (ii) are compatible, since the vector space spanned by $\mathbb{F}_1$ is G-stable and $\mathfrak{t}$-homogeneous. With this choice of generators for each $f \in \mathbb{F}_1'$, we have that $\varphi(f)$ is then in the image of the product

$$m : S^{a_1}(V^*_{\varepsilon_1}) \otimes \cdots \otimes S^{a_\ell}(V^*_{\varepsilon_\ell}) \longrightarrow \mathbb{k}[G/H],$$

where $\lambda_f = \sum a_i \varepsilon_i$. In particular, there exists an element $F_f' \in S^{a_1}(V^*_{\varepsilon_1}) \otimes \cdots \otimes S^{a_\ell}(V^*_{\varepsilon_\ell})$ such that $m(F_f') = \varphi(f)$. We consider F_f' as a multihomogeneous polynomial in the variables $f \in \mathbb{F}_0$.

Finally, note that the old basis $\mathbb{F}_1$ can be written in terms of the basis $\mathbb{F}_1'$. So we can write the relations $R_{f,f'}$ with respect to this new basis by expressing the elements in $\mathbb{F}_1$ as linear combinations of elements of $\mathbb{F}_1'$. We call these relations $R'_{f,f'}$.

Now consider a set of variables $u = (u_f)_{f \in \mathbb{F}_0}$ as in the previous discussion and a set of new variables $v' = (v'_{f'})_{f' \in \mathbb{F}_1'}$. Consider now in the polynomial ring $\mathbb{k}[u, v, t]$ the ideal generated by $R'_{f,f'}(u, v'))$ for $f, f' \in \mathbb{F}_0$ not comparable and by the elements $v_f - t^{\tilde{n}_f} F_f'(u)$; let B be the quotient of $\mathbb{k}[u, v, t]$ by this ideal and finally for $a \in \mathbb{C}^*$, let $B_a = B/(t - a)$.

Now note that there is a $\mathbb{k}^*$-action on B defined for all $z \in \mathbb{k}^*$ by $z \cdot u_f = z^n u_f$ if $f \in \mathbb{F}_0 \cap Z_n$, and by $z \cdot v_f = z^n v_f$ if $f \in \mathbb{F}_1' \cap Z_n$. Finally, note that $B_0 \simeq A$ and that $B_1 \simeq \mathbb{k}[G/H]$. In particular, B gives the claimed flat deformation from $\mathbb{k}[G/H]$ to A. □

6 The Finite-Type Case

In the case ${}^e\mathfrak{g}$ is of finite type (or equivalently, by Proposition 23: when $\tilde{\Phi}$ is of type A_ℓ), part of the proof and construction described in the previous paragraphs can be simplified and also some other additional properties hold. In this section, we describe some of these special properties.

Proposition 49.

(i) $\mathcal{S}_\tau$ is a codimension 1 Schubert variety in $\mathcal{Gr}$;

(ii) $\Gamma_i(\mathcal{Gr}) = \Gamma_i(\mathcal{S}_\tau)$ for $i = 0, \ldots, \ell - 1$;

(iii) $\Gamma_\ell(\mathcal{Gr}) = \mathbb{k}$ and $\Gamma_i(\mathcal{Gr}) = 0$ for $i > \ell$;

(iv) $h_1^i z_0^* \neq 0$ for all $i = 0, \ldots, \ell$. □

Proof. To prove part (i), it is enough to show that $[s_0\tau] = [w_{{}^e\Delta}]$ in ${}^eW/W$ or equivalently, since $s_0 = \tilde{s}_0$, that $\tau(\omega_0) = \omega_0 - \varepsilon_1$. This is a computation essentially in the restricted root system which in this case we know to be of type A_ℓ. We have $\tau = w_\Delta \hat{\tau}$, and so by Lemma 36 we have $\tau(\omega_0) = w_\Delta(\varepsilon_\ell - \omega_0) = w_\Delta(\varepsilon_\ell) - \omega_0$. Now $w_\Delta(\varepsilon_\ell) = -\varepsilon_1 = \varepsilon_\ell - (\widetilde{\alpha}_1 + \cdots + \widetilde{\alpha}_\ell)$, so $\tau(\omega_0) = \omega_0 - \varepsilon_1$.

Parts (ii) and (iii) follow immediately and part (iv) is a consequence of Lemma 27. ■

Remark 50. In the case where the restricted root system is of type B, C, or BC, a similar computation gives $\tau(\omega_0) = -\varepsilon_\ell + 3\omega_0$. □

The theory developed in the previous section becomes particularly simple for $\tilde{\Phi}$ of type A and we restate parts of Theorems 44 and 47 in the following more explicit way.

Theorem 51.

(i) $\mathbb{F} = \mathbb{F}_0 \cup \{f_0, f_1\}$, where f_1 is the highest-weight vector in Z^*;

(ii) we can normalize f_0 and f_1 in such a way that $F_{f_0} = F_{f_1} = 1$;

(iii) the map φ induces the following isomorphism:

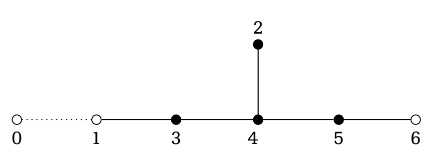
$$\Bbbk[G/H] \simeq \frac{\Gamma_{\mathcal{G}r}}{(f_0 = f_1 = 1)}. \qquad \square$$

It should also be pointed out that for almost all of the involutions in which the restricted root system is of type A, the described results were already obtained as special cases by other authors as explained in the introductory section.

There are two more families of involutions in which the restricted root system is of type A which are not in the literature: one for the group $SO(n)$ in which the restricted root system is of type A_1 and for this reason, it is particularly simple, and the involution of E_6 with fixed-point subalgebra of type F_4.

In the last part of this section, we would like to make as explicit as possible the case of the involution of E_6 with fixed-point subalgebra of type F_4.

In this case, we have that ${}^eG$ is of type E_7. In the picture below, we have numbered the nodes of E_7 following the notations of the previous sections and we have colored the nodes according to the Satake diagram of the corresponding involution.



The module Z is of dimension 56 and it is a minuscule module, so that we can identify an element of the basis $\mathbb{F}$ by giving its weight. Also $S^2 Z \simeq Z_{2\omega_0} \oplus {}^e\mathfrak{g}$, so that the Plücker relations are generated as ${}^eG$ modules by the following single relation:

$$x_0\, y_0 - x_1\, y_1 + x_2\, y_2 - x_3\, y_3 + x_4\, y_4 - x_5\, y_5 = 0,$$

where $x_0 = z_0^*$ and $x_1 = \mathsf{f}_0(x_0)$, $x_2 = \mathsf{f}_1(x_1)$, $x_3 = \mathsf{f}_3(x_2)$, $x_4 = \mathsf{f}_4(x_3)$, $x_5 = \mathsf{f}_5(x_4)$, $y_5 = \mathsf{f}_2(x_4)$, $y_4 = \mathsf{f}_5(y_5)$, $y_3 = \mathsf{f}_4(y_4)$, $y_2 = \mathsf{f}_3(y_3)$, $y_1 = \mathsf{f}_1(y_2)$, and $y_0 = \mathsf{f}_0(y_1)$ where $\mathsf{f}_i = \mathsf{f}_{\alpha_i}$ are the Chevalley generators.

Appendix A. SMT from Above and from Below

We have actually proved the existence of two bases for the coordinate ring of the symmetric space. One basis is given by the standard monomials $\{g_m : m \in \mathbb{SM}_0\}$ obtained by restricting the standard monomial theory on the (affine) Grassmannian to the symmetric space (Theorem 44). The other basis comes from below in the following sense: it is obtained via lifting and pullback from the SMT for the multicone over the closed orbit in the wonderful compactification (Proposition 16). To be more precise, in the last case we have a description of $\mathbb{k}[X_q]$ as the quotient

$$\frac{\Gamma_{\Omega_q}}{(s_i - 1 \,:\, i = 1, \ldots, \ell)} \simeq \mathbb{k}[X_q].$$

By Theorem 14, Γ_{Ω_q} has as basis monomials of the form $s^{\mu} m^X$, where s^{μ} is a product of the s_i and the m^X are appropriate lifts of the standard monomials on the closed orbit Y in the wonderful compactification (see Sections 2.4 and 2.5). So the images $\overline{m^X}$ of the m^X also form a basis for the coordinate ring. We would like to compare these two bases and the two different indexing systems.

Theorem 52. The possible choices relevant for the construction of the two bases can be arranged such that the two bases coincide. $\square$

Before coming to the proof, note that this comparison is also interesting from the combinatorial point of view. The definition of a standard monomial on a Grassmannian is rather straightforward; see also Section 2.3. The set of generators of the ring is indexed by certain LS-paths of shape ${}^e\omega_0$. For details see [21], we recall here only the properties needed in the following. An LS-path of shape ${}^e\omega_0$ is a pair of sequences

$\pi = (\underline{x}, \underline{a})$, where $\underline{x} = (x_1, \ldots, x_r)$ is a strictly increasing sequence (in the Bruhat order) of elements in ${}^eW/{}^eW_{{}^e\omega_0}$ (here ${}^eW$ is the Weyl group of ${}^e\mathfrak{g}$ and ${}^eW_{{}^e\omega_0}$ is the stabilizer of ${}^e\omega_0$), and $\underline{a} = (1 > a_1 > \cdots > a_{r-1} > 0)$ is a strictly decreasing sequence of rational numbers (satisfying certain properties; see [21]).

Let $\eta = (\underline{\kappa}, \underline{b})$ be a second LS-path of shape where $\underline{\kappa} = (\kappa_1, \ldots, \kappa_s)$. We say that

$$\pi \leftarrow \eta \quad \text{if and only if} \quad x_r \leq \kappa_1. \tag{5}$$

Note that $\pi \geq \kappa$ and $\kappa \geq \pi$ implies $r = s = 1$, and hence $\pi = \eta = (x)$; moreover, the relation $\leftarrow$ is clearly transitive, hence it is a t.a.b.r. in the sense of Section 2.2. By definition, a product

$$f_{\pi_1} \cdots f_{\pi_s} \quad \text{is standard if and only if} \quad \pi_1 \leftarrow \pi_2 \leftarrow \ldots \leftarrow \pi_s. \tag{6}$$

As mentioned in Remark 45, in the multicone picture the definition of a standard monomial is much more involved. The generators are again indexed by certain LS-paths, but of a different type. Let $\epsilon_1, \ldots, \epsilon_n$ be the generators of the admissible lattice. The generators of type ϵ_i (see Section 2.3) are indexed by LS-paths of type ϵ_i, i.e. pairs of sequences $\pi = (\underline{x}, \underline{a})$, where $\underline{x} = (x_1, \ldots, x_r)$ is a strictly increasing sequence (in the Bruhat order) of elements in the quotient W/W_{ϵ_i}, and $\underline{a}$ is a strictly decreasing sequence of rational numbers (satisfying certain conditions; see [21]). By a *defining sequence* for π, we mean a weakly increasing sequence $\underline{\tilde{x}} = (\tilde{x}_1, \ldots, \tilde{x}_r)$ of elements in W such that $\tilde{x}_j \equiv x_j \bmod W_{\epsilon_i}$. Given LS-paths $\pi_{1,1}, \ldots, \pi_{1,a_1}, \ldots, \pi_{n,1}, \ldots, \pi_{n,a_n}$, where $\pi_{i,j} = (\underline{x}^{i,j}, \underline{a}^{i,j})$ is an LS-path of type ϵ_i, the monomial

$$\underbrace{f_{\pi_{1,1}} \cdots f_{\pi_{1,a_1}}}_{\text{type } \epsilon_1} \cdot \underbrace{f_{\pi_{2,1}} \cdots f_{\pi_{2,a_2}}}_{\text{type } \epsilon_2} \cdots \underbrace{f_{\pi_{\ell,1}} \cdots f_{\pi_{\ell,a_\ell}}}_{\text{type } \epsilon_\ell}$$

is called standard if there exist defining sequences $\underline{\tilde{x}}^{i,j}$ for the $\pi_{i,j} = (\underline{x}^{i,j}, \underline{a}^{i,j})$ such that the defining sequences give rise to a weakly increasing sequence of Weyl group elements

$$\underbrace{\tilde{x}_1^{1,1} \leq \tilde{x}_2^{1,1} \leq \ldots \leq \tilde{x}_r^{1,1}}_{\underline{\tilde{x}}^{1,1}} \leq \underbrace{\tilde{x}_1^{1,2} \leq \cdots \leq \tilde{x}_p^{1,2}}_{\underline{\tilde{x}}^{1,2}} \leq \cdots \leq \underbrace{\tilde{x}_1^{n,a_n} \leq \ldots \leq \tilde{x}_s^{n,a_n}}_{\underline{\tilde{x}}^{n,a_n}}. \tag{7}$$

This definition of a standard monomial is far away from the definition given in Section 2.2. It depends on the choice of the enumeration $\epsilon_1, \ldots, \epsilon_n$ of the basis of the admissible lattice, and there is no obvious canonical choice.

In the case where the admissible lattice is the weight lattice, there exist special "nice enumerations" for certain groups (see [18], for a Young-diagram-like version, see [20]). In these cases, the definition above simplifies dramatically and becomes similar to the one given above for the Grassmannian. The bijection below, together with the comparison theorem above gives a beautiful geometric interpretation of this combinatorial fact and provides yet another connection between Young tableaux like indexing systems and combinatorics of the affine Weyl group.

Proof. We have already pointed out in Remark 45 that the possible choices for the set of generators can be arranged for both constructions such that the generators actually coincide. It remains to prove that the notion of a *standard monomial* coincides for both constructions.

Let us recall a few facts and definitions related to LS-paths. By Lemma 36, we can enumerate the basis of the lattice $\epsilon_1, \ldots, \epsilon_\ell$ such that there exist elements in ${}^eW$ (the enumeration is different from the one in the lemma above)

$$\hat{\tau}_1 > \hat{\tau}_2 > \cdots > \hat{\tau}_\ell \quad \text{and} \quad \hat{\tau}_h({}^e\omega_0)|_{\mathfrak{t}} = \epsilon_h, \; h = 1, \ldots, \ell; \tag{8}$$

and the $\hat{\tau}_j$ are of minimal length with this property. Consider first an LS-path $\pi = (\underline{x}, \underline{a})$ of type ϵ_i, where $\underline{x} = (x_1, \ldots, x_r)$. By abuse of notation, we also write $x_j \in W$ for a minimal representative. By the definition of an LS-path and by equation (8), it follows that

$${}^e\pi = ({}^e\mathrm{x}, \underline{a}), \quad \text{where} \quad {}^e\mathrm{x} = (x_1\hat{\tau}_i, \ldots, x_r\hat{\tau}_i)$$

is an LS-path of type ${}^e\omega_0$. So the map $\pi \mapsto {}^e\pi$ defines an injective (and also a surjective) map between the union $\bigcup_{i=1}^{\ell}\{\text{LS-paths of type } \epsilon_i\}$ and the set of LS-paths standard on the Richardson variety $\mathcal{R}$, i.e. the associated sections do not vanish identically on $\mathcal{R}$.

It remains to check that the notion of a standard monomial in both pictures is the same. In order not to get drowned in indices, we consider only a product of two elements. Let $\pi = (\underline{x}, \underline{a})$ be of type ϵ_i and $\eta = (\underline{y}, \underline{b})$ of type ϵ_j such that $i > j$ and $f_\pi f_\eta$ be standard. By definition, this implies that we can find defining sequences $(\tilde{x}_1, \ldots, \tilde{x}_r)$ for

$\underline{x} = (x_1, \ldots, x_r)$ and $(\tilde{y}_1, \ldots, \tilde{y}_s)$ for $\underline{y} = (y_1, \ldots, y_s)$ such that in W, we have

$$\tilde{x}_1 \le \cdots \le \tilde{x}_r \le \tilde{y}_1 \le \cdots \le \tilde{y}_s, \quad \text{and hence in } {}^eW\text{: } \tilde{x}_1\hat{\tau}_i \le \cdots \le \tilde{x}_r\hat{\tau}_i \le \tilde{y}_1\hat{\tau}_j \le \cdots \le \tilde{y}_s\hat{\tau}_j.$$

Recall that an element $\tilde{x}_k$ is of the form $x_k w_k$, where w_k is an element in the stabilizer W_{ϵ_i}. Similarly, $\tilde{y}_m$ is of the form $y_m w_m$, where $w_m \in W_{\epsilon_j}$. So the linearly ordered sequence in ${}^eW$ above gives rise to a linearly ordered sequence

$$x_1\hat{\tau}_i \le \cdots \le x_r\hat{\tau}_i \le y_1\hat{\tau}_j \le \cdots \le y_s\hat{\tau}_j \quad \text{in} \quad {}^eW/{}^eW_{e_{\omega_0}}.$$

Now by equation (6) this implies that $f_{e_\pi} f_{e_\eta}$ is a standard monomial.

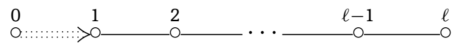
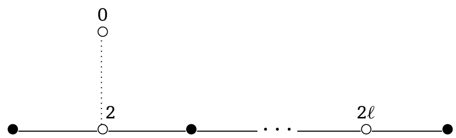
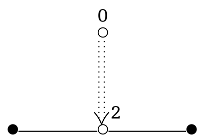
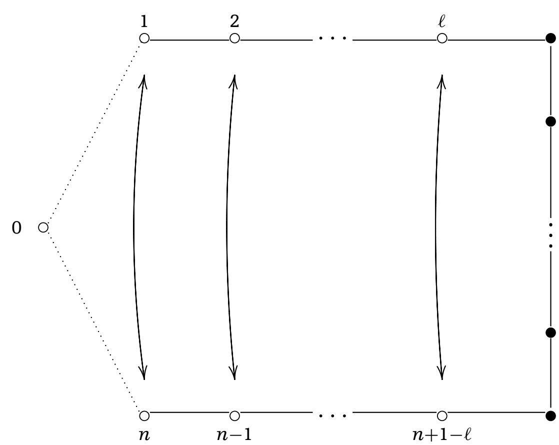
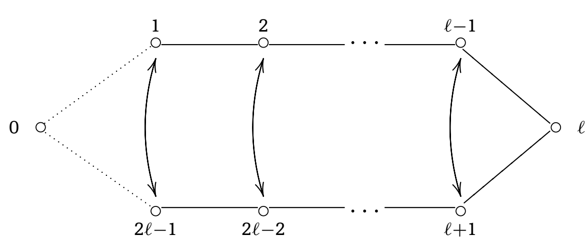
This argument extends to arbitrary standard monomials on the multicone. Summarizing, $\Bbbk[X_q]$ has as a basis of the standard monomials from below, i.e. the classes $\overline{m^X}$, where the m^X are appropriate lifts of the standard monomials (with respect to the enumeration of the basis of the admissible lattice chosen above) on the closed orbit Y in the wonderful compactification. The map defined on the set of standard monomials

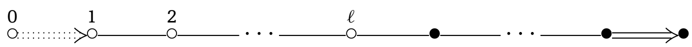
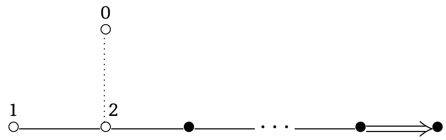
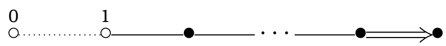
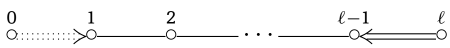
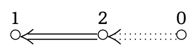
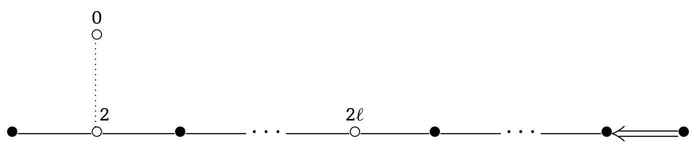
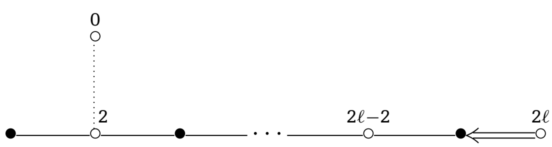
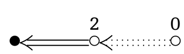
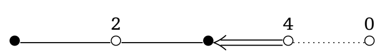
$$f_{\pi_1} \cdots f_{\pi_s} \mapsto f_{e_{\pi_1}} \cdots f_{e_{\pi_s}}$$

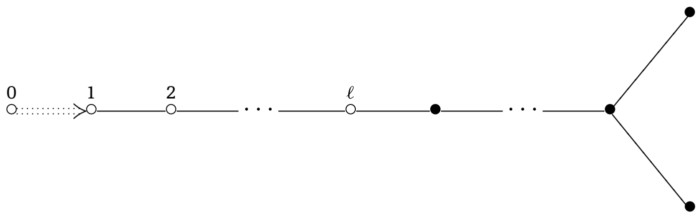
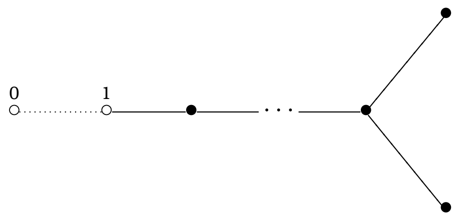
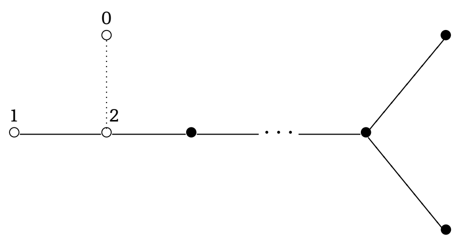
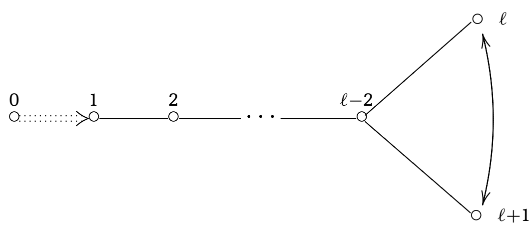
induces a bijection between the standard monomials from below and the standard monomials from above, i.e. the set $\{g_m \,:\, m \in \mathbb{SM}_0\}$ (Theorem 44). ■

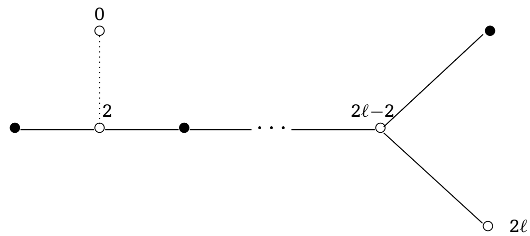
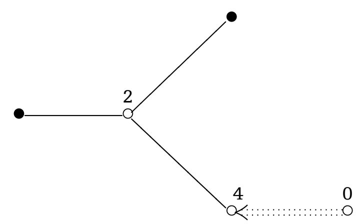
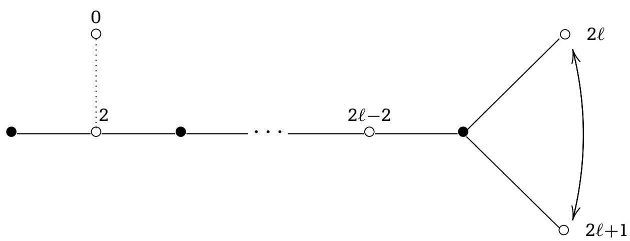
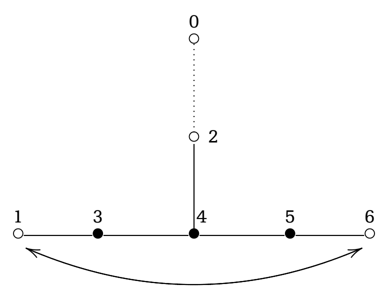
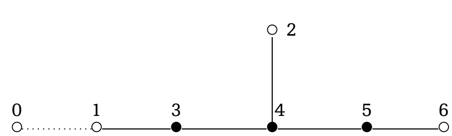
Appendix B. The Satake Diagrams

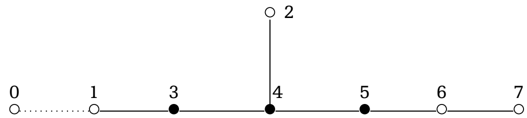
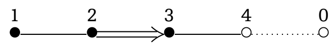
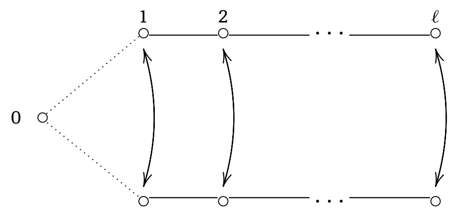
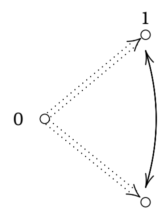
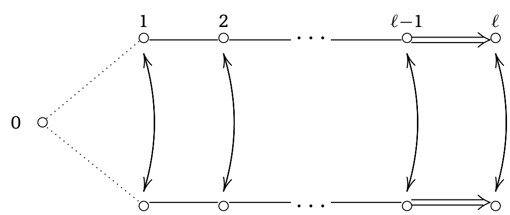
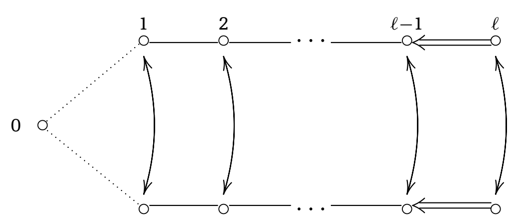
In this appendix, we list the Satake diagrams of all involutions. We add a node to a Satake diagram as described in the previous sections and we obtain in this way the extended Dynkin diagram; this special node is joined to the rest of the diagram with a dotted line (or lines). Beside each diagram, we indicate the Lie algebra $\mathfrak{h}$ of the set of fixed points, the type of the restricted root system, and the isogeny type of the group, in particular, "SC" means "simply connected" and "ADJ" means "adjoint."

$\mathfrak{h}$	Extended Satake diagram	Roots and isogeny type
$\mathfrak{so}(\ell+1)$		$\ell \geqslant 1$ A_ℓ SC
		$\ell=1$ B_1 ADJ
$\mathfrak{sp}(2\ell+2)$		$\ell \geqslant 1$ A_ℓ SC
		$\ell=1$ B_1 ADJ
$\mathfrak{sl}(\ell)\oplus\mathfrak{sl}(n+1-\ell)$		BC_ℓ SC=ADJ
$\mathfrak{sl}(\ell)\oplus\mathfrak{sl}(\ell)\oplus\mathbb{C}$		C_ℓ SC

$\mathfrak{h}$	Extended Satake diagram	Roots and isogeny type
$\mathfrak{so}(\ell)\oplus\mathfrak{so}(2n+1-\ell)$		$\ell\geqslant 1$ B_ℓ ADJ
		$\ell=2$ C_2 SC
		$\ell=1$ A_1 SC
$\mathfrak{gl}(\ell)$		$\ell\geqslant 2$ C_ℓ SC
		$\ell=2$ B_2 ADJ
$\mathfrak{sp}(2\ell)\oplus\mathfrak{sp}(2n-2\ell)$		BC_ℓ SC=ADJ
$\mathfrak{sp}(2\ell)\oplus\mathfrak{sp}(2\ell)$		$\ell\geqslant 1$ C_ℓ SC
		$\ell=1$ B_1 ADJ
		$\ell=2$ B_2 ADJ

$\mathfrak{h}$	Extended Satake diagram	Roots and isogeny type
$\mathfrak{so}(\ell)\oplus\mathfrak{so}(2n-\ell)$		$\ell\geqslant 1$ B_ℓ ADJ
		$\ell=1$ A_1 SC
		$\ell=2$ C_2 SC
$\mathfrak{so}(\ell)\oplus\mathfrak{so}(\ell+2)$		B_ℓ ADJ

$\mathfrak{h}$	Extended Satake diagram	Roots and isogeny type
$\mathfrak{gl}(2\ell)$		$\ell \geqslant 2$ C_ℓ SC
		$\ell = 2$ B_2 ADJ
$\mathfrak{gl}(2\ell+1)$		BC_ℓ SC=ADJ
$\mathfrak{so}(10) \oplus \mathbb{C}$		BC_2 SC=ADJ
F_4		A_2 SC

$\mathfrak{h}$	Extended Satake diagram	Roots and isogeny type
$E_6 \oplus \mathbb{C}$		C_3 SC
$\mathfrak{so}(9)$		BC_1 SC=ADJ
$\mathfrak{sl}(\ell+1)$		$\ell \geqslant 1$ A_ℓ SC
		$\ell = 1$ B_1 ADJ
$\mathfrak{so}(2\ell+1)$		B_ℓ ADJ
$\mathfrak{sp}(2\ell)$		C_ℓ SC

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References

[1] Bourbaki, N. "Groupes et algébres de Lie." *Éléments de mathématique*, Chap. 4, 5, 6. Section 34. Actualités Scientifiques et Industrielles 1337. Paris: Hermann, 1968.

[2] Chirivì, R. "Deformation and Cohen–Macaulayness of the multicone over the flag variety." *Commentarii Mathematici Helvetici* 76, no. 3 (2001): 436–66.

[3] Chirivì, R., and A. Maffei. "The ring of sections of a complete symmetric variety." *Journal of Algebra* 261, no. 2 (2003): 310–26.

[4] Chirivì, R., and A. Maffei. "Projective normality of complete symmetric varieties." *Duke Mathematical Journal* 122, no. 1 (2004): 93–123.

[5] De Concini, C. "Symplectic standard tableaux." *Advances in Mathematics* 34, no. 1 (1979): 1–27.

[6] De Concini, C., and C. Procesi. "A characteristic free approach to invariant theory." *Advances in Math.* 21, no. 3 (1976): 330–54.

[7] De Concini, C., and C. Procesi, "Complete symmetric varieties." In *Invariant Theory* (Montecatini, 1982), 1–44. Berlin: Springer, 1983.

[8] Doubilet, P., G. Rota, and J. Stein. "On the foundations of combinatorial theory 9: Combinatorial methods in invariant theory." *Studies in Applied Mathematics* 53 (1974): 185–216.

[9] Eisenbud, D. *Commutative Algebra with a View Toward Algebraic Geometry*. Graduate Text in Mathematics 150. New York: Springer, 1995.

[10] Helgason, S. "A duality for symmetric spaces with applications to group representations." *Advances in Mathematics* 5 (1970): 1–154.

[11] Hodge, W. V. "Some enumerative results in the theory of forms." *Proceedings of the Cambridge Philosophical Society* 39 (1943): 22–30.

[12] Hodge, W. V., and D. Pedoe. *Methods of Algebraic Geometry*, vol. 2. Cambridge: Cambridge University Press, 1952.

[13] Humphreys, J. E. *Introduction to Lie algebras and Representation Theory*. Graduate Text in Mathematics 9. New York: Springer, 1972.

[14] Kac, V. *Infinite Dimensional Lie Algebras*, 3rd ed. Cambridge: Cambridge University Press, 1990.

[15] Kumar, S. *Kac–Moody Groups, Their Flag Varieties and Representation Theory*. Progress in Mathematics 204. Boston, MA: Birkhäuser Boston, 2002.

[16] Lakshmibai, V., and P. Littelmann. "Richardson varieties and equivariant K-theory." Special issue celebrating the 80th birthday of Robert Steinberg. *Journal of Algebra* 260, no. 1 (2003): 230–60.

[17] Lakshmibai, V., P. Littelmann, and P. Magyar. “Standard monomial theory and applications.” In *Representation Theories and Algebraic Geometry*, edited by A. Broer. Dordrecht, the Netherlands: Kluwer Academic, 1998.

[18] Lakshmibai, V., C. Musili, and C. S. Seshadri. “Geometry of G/P-IV. A standard monomial theory for classical types.” *Indian Academy of Sciences: Proceedings Part A* 88 (1979): 279–362.

[19] Lakshmibai, V., and C. S. Seshadri. “Geometry of G/P-II. The work of De Concini and Procesi and the basic conjectures.” *Indian Academy of Sciences: Proceedings Part A* 87 (1978): 1–54.

[20] Littelmann, P. “A generalization of the Littlewood–Richardson rule.” *Journal of Algebra* 130, no. 2 (1990): 328–68.

[21] Littelmann, P. “A Littlewood–Richardson rule for symmetrizable Kac–Moody algebras.” *Inventiones Mathematicae* 116, no. 1–3 (1994): 329–46.

[22] Littelmann, P. “Contracting modules and standard monomial theory for symmetrizable Kac–Moody algebras.” *Journal of the American Mathematical Society* 11, no. 3 (1998): 551–67.

[23] Onishchik, A., and È. Vinberg. *Lie Groups and Algebraic Groups*. Springer Series in Soviet Mathematics. Berlin: Springer, 1990.

[24] Richardson, R. “Orbits, invariants, and representations associated to involutions of reductive groups.” *Inventiones Mathematicae* 66, no. 2 (1982): 287–312.

[25] Seshadri, C. S. “Geometry of G/P. I. Theory of standard monomials for minuscule representations.” In *C. P. Ramanujam—A Tribute*, 207–39. Tata Institute of Fundamental Research Studies in Mathematics 8. Berlin: Springer, 1978.

[26] Springer, T. *Linear Algebraic Groups*, 2nd ed. Progress in Mathematics 9. Boston, MA: Birkhäuser Boston, 1998.

[27] Vust, T. “Opération de groupes réductifs dans un type de cônes presque homogènes.” *Bulletin de la Societe Mathematique de France* 102 (1974): 317–33.