

A GENERALIZED STEINBERG SECTION AND BRANCHING RULES FOR QUANTUM GROUPS AT ROOTS OF 1

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Dedicated to the memory of I. M. Gelfand

ABSTRACT. In this paper we construct a generalization of the classical Steinberg section for the quotient map of a semisimple group with respect to the conjugation action. We then give various applications of our result including the construction of a sort of Gelfand–Zeitlin basis for a generic irreducible representation of $U_q(\mathrm{GL}(n))$ when q is a primitive odd root of unity.

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*O cara piota mia che sì t'insusi,
 che come veggion le terreni menti,
 non capère in triangol due ottusi,
 così vedi le cose contingenti
 anzi che sieno in sé, mirando il punto
 a cui tutti li tempi son presenti.*

La Divina Commedia, Paradiso, Canto XVII

1. INTRODUCTION

In his famous paper [12], Steinberg introduces a remarkable section of the quotient map of a simply connected semisimple group modulo its conjugation action. In this paper we are going to extend this construction as follows. Let G be a reductive group with simply connected semisimple factor over an algebraically closed field \mathbb{k} . Let T be a maximal torus, $B \supset T$ a Borel subgroup and B^- the opposite Borel subgroup. We take a sequence $\mathcal{L} = \{L_1 \subset L_2 \subset \cdots \subset L_h = G\}$ of standard Levi subgroups such that for each $i = 1, \dots, h-1$, each simple factor of L_i does not coincide with a simple factor of L_{i+1} . A sequence $\mathcal{M} = \{M_1 \subset \cdots \subset M_h\}$ of connected subgroups of G is said to be admissible if, for all $i = 1, \dots, h$, M_i is a subgroup of L_i containing L_i^{ss} and there exist subtori $S_i \subset T$ such that

- S1: $M_i = S_i \ltimes L_i^{\mathrm{ss}}$;
- S2: $S_{i+1} \cap M_i = \{1\}$.

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Notice that if G is simple and L is any standard Levi factor properly contained in G , then $L \subset G$ is automatically admissible. Also the sequence $\{\mathrm{GL}(1) \subset \mathrm{GL}(2) \subset \cdots \subset \mathrm{GL}(n)\}$ is admissible.

Taking the unipotent radical $U^- \subset B^-$, in Section 2 we define a simultaneous quotient map

$$r_{\mathcal{M}}: B \times U^- \longrightarrow \prod_{i=1}^h M_i // \mathrm{Ad}(M_i)$$

and we show (Theorem 2.19) that $r_{\mathcal{M}}$ has a section, which coincides with the Steinberg section if $\mathcal{M} = \{G\}$ once we identify $B \times U^-$ with the open Bruhat cell $BU^- \subset G$.

In the case of the sequence $\{\mathrm{GL}(1) \subset \mathrm{GL}(2) \subset \cdots \subset \mathrm{GL}(n)\}$ we give, in Theorem 3.3, an alternative elementary proof of this statement which uses a curious explicit parametrization of the Borel subgroup $B \subset \mathrm{GL}(n)$.

As a consequence, the coordinate ring $\bigotimes_{i=1}^h \mathbb{k}[M_i]^{\mathrm{Ad}(M_i)}$ of $\prod_{i=1}^h M_i // \mathrm{Ad}(M_i)$, which is a polynomial ring with some variables inverted, embeds in the coordinate ring of $B \times U^-$. Consider the group

$$H = \{(x, y) \in B \times B^- : \pi_T(x) = \pi_T^-(y)^{-1}\},$$

where π_T, π_T^- are the projections on T of B, B^- respectively, Poisson dual of G . Define $\rho: H \rightarrow G$ by $\rho((x, y)) = xy^{-1}$. The map ρ is a 2^{rkG} to 1 covering map of the open set BU^- . So, we also get an inclusion of $\bigotimes_{i=1}^h \mathbb{k}[M_i]^{\mathrm{Ad}(M_i)}$ into $\mathbb{k}[H]$ and we remark that the image is a Poisson commutative subalgebra of $\mathbb{k}[H]$.

This is particularly interesting in the case of the sequence $\{\mathrm{GL}(1) \subset \mathrm{GL}(2) \subset \cdots \subset \mathrm{GL}(n)\}$. In this case this algebra, which is a multiplicative analogue of the Gelfand–Zeitlin subalgebra of the coordinate ring of the space of $n \times n$ matrices, has transcendence degree $n(n+1)/2$, which is maximal. So, it gives a completely integrable Hamiltonian system. It is worth pointing out that in the case of square matrices and also in the case of the Poisson dual of the unitary groups these systems have been known and studied for quite some time, see for example [1], [9], [10] and reference therein.

Finally in the last two sections, we give an application of our result to the study of the quantum enveloping algebra associated to G when the deformation parameter is a primitive root of one. We show, following ideas in [5], how to decompose a “generic” irreducible representation when it is restricted to the quantum enveloping algebra of an admissible subgroup M satisfying some further assumptions.

In the case in which $G = \mathrm{GL}(n)$ and $M = \mathrm{GL}(n-1)$, this gives a multiplicity one decomposition which can be used to decompose our module into a direct sum of one dimensional subspaces in a way which is quite similar to that appearing in the classical work of Gelfand–Zeitlin [7].

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2. A SIMULTANEOUS VERSION OF STEINBERG RESOLUTION

Let G be a connected reductive group over an algebraically closed field \mathbb{k} whose Lie algebra we denote by \mathfrak{g} . Assume that the semisimple part of G is simply connected. Let $q: G \rightarrow G//\mathrm{Ad}(G)$ be the quotient of G under the adjoint action.

Choose a maximal torus T and a Borel subgroup $B \supset T$ in G . Denote by Φ the corresponding set of roots, by Δ that of simple roots, by Φ^+ that of positive roots and by $W = N_G(T)/T$ the Weyl group. The inclusion $T \subset G$ induces an isomorphism $G//\mathrm{Ad}(G) \simeq T/W$.

If $\alpha \in \Phi$, we denote by $\check{\alpha}(t)$ the corresponding cocharacter. Given α , we can associate to α an $\mathrm{SL}(2)$ embedding γ_α into G and we take the unipotent one parameter subgroups

$$X_\alpha(a) = \gamma_\alpha \left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \right), \quad Y_\alpha(a) = \gamma_\alpha \left(\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \right).$$

Finally set

$$s_\alpha = X_\alpha(1)Y_\alpha(-1)X_\alpha(1) = \gamma_\alpha \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right).$$

2.1. The Steinberg section in the semisimple case. Let us recall the construction of the Steinberg section. Consider the vector space \mathbb{k}^Δ of \mathbb{k} -valued maps on the set of simple roots Δ . If we fix an order $\alpha_1, \dots, \alpha_n$ of the set of simple roots, we identify \mathbb{k}^Δ with \mathbb{k}^n setting for each $a \in \mathbb{k}^\Delta$, $a_i = a(\alpha_i)$. In [12], given such an ordering, Steinberg defined the map $\mathrm{St}: \mathbb{k}^n \rightarrow G$ by

$$\mathrm{St}((a_1, \dots, a_n)) = X_{\alpha_n}(a_n)s_{\alpha_n} \cdots X_{\alpha_1}(a_1)s_{\alpha_1}$$

and proved the following theorem.

Theorem 2.2 [12, Theorem 7.9]. *If G is semisimple and simply connected, the composition $q \circ \mathrm{St}: \mathbb{k}^n \rightarrow G//\mathrm{Ad}(G)$ is an isomorphism.*

Furthermore, the image of St is contained in the regular locus and intersects each regular conjugacy class exactly in one point.

Of course we can use the identification of \mathbb{k}^Δ with \mathbb{k}^n associated to a chosen order, and describe the Steinberg section as a map $\mathrm{St}: \mathbb{k}^\Delta \rightarrow G$ defined by $\mathrm{St}(a) = X_{\alpha_n}(a(\alpha_n))s_{\alpha_n} \cdots X_{\alpha_1}(a(\alpha_1))s_{\alpha_1}$.

We will need to compare two Steinberg sections constructed by considering different orders of the set of simple roots. If $I \subset \Delta$, we identify \mathbb{k}^I with the subspace of \mathbb{k}^Δ of functions vanishing on $\Delta \setminus I$. We set $\mathbb{k}^\alpha := \mathbb{k}^{\{\alpha\}}$.

Lemma 2.3. *Let $\mathrm{St}, \mathrm{St}': \mathbb{k}^\Delta \rightarrow G$ be the Steinberg sections associate to two different orders of the simple roots. Then there exists a morphism $g: \mathbb{k}^\Delta \rightarrow G$, an element w of the Weyl group and an action of T on \mathbb{k}^Δ such that for each $\alpha \in \Delta$ the line \mathbb{k}^α is stable by the action of T and*

$$t \mathrm{St}(a) = g(t \cdot a)w(t) \mathrm{St}'(t \cdot a)g(t \cdot a)^{-1}$$

for all $t \in T$ and $a \in \mathbb{k}^\Delta$, where we denote by $w(t)$ the element ntn^{-1} , n being a representative of w in $N_G(T)$.

Proof. It is known (see for example [8, pp. 74, 75]) that two ordered sequence of simple roots can be obtained from each other by applying recursively the following two operations: commute the order of two adjacent orthogonal simple roots α_i, α_j and move the last simple root, let's say β , to the first position. In the first case $X_{\alpha_i}, s_{\alpha_i}$ and $X_{\alpha_j}, s_{\alpha_j}$ commute, so we can choose $g = 1, w = 1$ and the trivial action of T on \mathbb{k}^Δ . In the second case everything follows, choosing $g(a) = X_\beta(a(\beta)) s_\beta, w = s_\beta$ and

$$(t \cdot a)(\gamma) = \begin{cases} a(\gamma) & \text{if } \gamma \neq \beta, \\ t^\beta a(\gamma) & \text{if } \gamma = \beta. \end{cases} \quad \square$$

Before we proceed, let us write the Steinberg section in a slightly different way. Fix an order of the simple roots. Set $s_i := s_{\alpha_i}$ and define $\beta_i = s_n \cdots s_{i+1}(\alpha_i)$ and $w = s_n \cdots s_1$. The ordered set of roots β_1, \dots, β_n will be called the beta set of roots associated to the given order.

Notice for any i, j , $\beta_i + \beta_j$ is never a root. Indeed if $i = j$ this is clear. Otherwise assume that $i > j$. Then $\beta_i + \beta_j = s_n \cdots s_i(s_{i-1} \cdots s_{j+1}(\alpha_j) - \alpha_i)$. But $s_{i-1} \cdots s_{j+1}(\alpha_j)$ is a positive root whose support does not contain α_i , hence $s_{i-1} \cdots s_{j+1}(\alpha_j) - \alpha_i$ and also $\beta_i + \beta_j$ are not roots.

If $a = (a_1, \dots, a_n) \in \mathbb{k}^n$, we can then rewrite the Steinberg section as

$$\text{St}(a) = \left(\prod_{i=1}^n X_{\beta_i}(a_i) \right) w =: X_G(a)w.$$

2.4. The Steinberg section in the reductive case. We are now going to construct a section also in the reductive case. For this we need a slight extension of Steinberg result which follows in exactly the same way and whose proof we leave to the reader.

Let G^{ss} be the commutator subgroup of G (which by assumption is simply connected) and let $T^{\text{ss}} = G^{\text{ss}} \cap T$, be a maximal torus in G^{ss} . The inclusion $T^{\text{ss}} \subset T$ induces a surjection of character groups $\Lambda(T) \rightarrow \Lambda(T^{\text{ss}})$. Splitting, we obtain a subtorus S of T such that $T = S \times T^{\text{ss}}$. Then $G = S \ltimes G^{\text{ss}}$. Furthermore denote the center of G by $Z(G)$.

Let us recall that a basis for the ring of functions on G^{ss} invariant under conjugation is given by the characters of the irreducible G^{ss} -modules and a set of polynomial generators is given by the characters of the fundamental representations. Notice that, if we take an irreducible module V for G^{ss} , $G^{\text{ss}} \cap Z(G)$ acts on V by a character which can be extended (not uniquely) to a character of $Z(G)$. Thus, in this way, V becomes an irreducible G -module and its character a central function on G . It follows that the restriction map induces a surjective morphism $p: G \rightarrow G^{\text{ss}} // \text{Ad}(G^{\text{ss}})$ whose restriction to G^{ss} is the quotient map. Also, the projection $G \rightarrow S$ commutes with the adjoint action and this gives an identification of $G // \text{Ad}(G)$ with $S \times G^{\text{ss}} // \text{Ad}(G^{\text{ss}})$. Under this identification, the quotient map $q: G = S \ltimes G^{\text{ss}} \rightarrow G // \text{Ad}(G)$ is given by $q((s, g)) = (s, p(sg))$. We have the following generalization of the Steinberg section to the reductive case.

Proposition 2.5. *Let $\text{St}: \mathbb{k}^\Delta \rightarrow G^{\text{ss}}$ be a Steinberg section of the semisimple factor of G . Consider the map $\text{St}_S: S \times \mathbb{k}^\Delta \rightarrow G$ given by*

$$\text{St}_S(s, a) = s \cdot \text{St}(a).$$

Then the composition $q \circ \text{St}_S$ is an isomorphism. Moreover, for each $t \in T$, the map $a \mapsto p(t \cdot \text{St}(a))$ from \mathbb{k}^Δ to $G^{\text{ss}}/\text{Ad}(G^{\text{ss}})$ is an isomorphism.

Proof. In order to show our claim, it clearly suffices to see that our morphism is bijective. Hence, it is enough to prove the second claim. This follows immediately since, p being invariant under the adjoint action, this map is given by $p(X_{\alpha_1}(a_1)s_1 \cdots X_{\alpha_n}(a_n)s_n \cdot t)$ and $s_n \cdot t$ is just another representative of the reflection s_n in the Weyl group for which Steinberg's proof can be repeated verbatim. \square

2.6. Simultaneous Steinberg section for the semisimple factors of a sequence of Levi subgroups. At this point, we have all the ingredients to construct a simultaneous Steinberg section for a sequence of Levi subgroups. We need to introduce some notations. If L is a standard Levi subgroup of G with respect to the given choice of T and B , we denote by Φ_L be its root system and by $\Delta_L = \Delta \cap \Phi_L$ its simple roots. Recall that in Section 2.1 we have introduced the beta set associated to an order of the simple roots. The following Lemma allows us to choose an order of the simple roots such that its beta set has some particularly nice properties with respect to the Levi factor L .

Lemma 2.7. *Let L be a standard Levi subgroup of G which does not contain any simple factor of G . Then there exists an order of the simple roots $\alpha_1, \dots, \alpha_n$ such that*

- (i) $\Delta_L = \{\alpha_1, \dots, \alpha_m\}$.
- (ii) For all $i = 1, \dots, n$ the roots $\beta_i = s_n \cdots s_{i+1}(\alpha_i) \notin \Phi_L$.

Proof. We are going to construct an order $\alpha_1, \dots, \alpha_n$ on Δ with the property that $\alpha_{m+1}, \dots, \alpha_n$ is an arbitrary order on $\Delta \setminus \Delta_L$ and, for every $i \leq m$, there is a $j > m$ such that $\langle s_m \cdots s_{i+1}\alpha_i, \check{\alpha}_j \rangle \neq 0$. If this is the case, this order obviously satisfies our first requirement. The second is clearly satisfied for $i > m$. For $i \leq m$, taking the minimal j for which $\langle s_m \cdots s_{i+1}\alpha_i, \check{\alpha}_j \rangle \neq 0$, we get that the support of the root $s_j \cdots s_{i+1}\alpha_i$ and hence of the root β_i contains α_j .

At this point we can assume $m \geq 1$. By assumption, since L does not contain any simple factor of G , there is a root $\alpha \in \Delta_L$ such that $\langle \alpha, \check{\alpha}_j \rangle \neq 0$ for some $j > m$. We set $\alpha_m = \alpha$ and consider the Levi subgroup L' associated to $\Delta_L \setminus \{\alpha_m\}$. The assumptions of the Lemma are satisfied by L' , hence we may choose an order $\alpha_1, \dots, \alpha_{m-1}$ of $\Delta_L \setminus \{\alpha_m\}$ with the property that, for every $i \leq m-1$, there is a $j \geq m$ such that $\langle s_{m-1} \cdots s_{i+1}\alpha_i, \check{\alpha}_j \rangle \neq 0$. Take $i < m$. If $\langle s_{m-1} \cdots s_{i+1}\alpha_i, \check{\alpha}_m \rangle = 0$, then $\langle s_{m-1} \cdots s_{i+1}\alpha_i, \check{\alpha}_h \rangle \neq 0$ for some $h > m$ and $\langle s_m \cdots s_{i+1}\alpha_i, \check{\alpha}_h \rangle = \langle s_{m-1} \cdots s_{i+1}\alpha_i, \check{\alpha}_h \rangle \neq 0$.

If $\langle s_{m-1} \cdots s_{i+1}\alpha_i, \check{\alpha}_m \rangle \neq 0$, let $\Gamma \subset \Delta_L$ be the support of the root $s_m \cdots s_{i+1}\alpha_i$. Remark that $\alpha_m \in \Gamma$, that $s_{m-1} \cdots s_{i+1}\alpha_i$ is supported on $\Gamma \setminus \{\alpha_m\}$ and that the support of Γ is a connected subset of the Dynkin diagram. By assumption there is a $j > m$ such that $\langle \alpha_m, \check{\alpha}_j \rangle \neq 0$. Then $\langle s_{m-1} \cdots s_{i+1}\alpha_i, \check{\alpha}_j \rangle = 0$. Otherwise there

would be $\alpha_h \in \Gamma \setminus \{\alpha_m\}$ with $\langle \alpha_h, \check{\alpha}_j \rangle \neq 0$ and the Dynkin diagram would contain a cycle. It follows that

$$\begin{aligned} \langle s_m \cdots s_{i+1} \alpha_i, \check{\alpha}_j \rangle &= \langle s_{m-1} \cdots s_{i+1} \alpha_i, s_m \check{\alpha}_j \rangle \\ &= -\langle \alpha_j, \check{\alpha}_m \rangle \langle s_{m-1} \cdots s_{i+1} \alpha_i, \check{\alpha}_m \rangle \neq 0 \end{aligned}$$

as desired. \square

We say that an order of the simple roots which satisfies the condition of the previous lemma is compatible with L .

We explain now what we mean by a simultaneous quotient. Let B^- be the opposite Borel subgroup of B , U and U^- the unipotent radicals of B and B^- . Let $\mathcal{L} = \{L_1 \subset L_2 \subset \cdots \subset L_h = G\}$ be a sequence of standard (with respect to the given choice of T and B) Levi subgroups of G . For each $i = 1, \dots, h$ let $P_i = L_i B_i$ be the standard parabolic subgroup with Levi factor equal to L_i . Let V_i be the unipotent radical of P_i and let $\pi_i: P_i \rightarrow L_i$ be the projection onto the Levi factor. Similarly, considering the opposite parabolic P_i^- , define V_i^- and π_i^- . Let also $q_i^{\text{ss}}: L_i^{\text{ss}} \rightarrow L_i^{\text{ss}} // \text{Ad}(L_i^{\text{ss}})$ be the quotient map. We consider the following “simultaneous” quotient map

$$q_{\mathcal{L}}^{\text{ss}}: U \times U^- \longrightarrow \prod_{i=1}^h L_i^{\text{ss}} // \text{Ad}(L_i^{\text{ss}})$$

defined by

$$q_{\mathcal{L}}^{\text{ss}}(u, v) = (q_1^{\text{ss}}(\pi_1(u)\pi_1^-(v)^{-1}), q_2^{\text{ss}}(\pi_2(u)\pi_2^-(v)^{-1}), \dots, q_h^{\text{ss}}(uv^{-1})). \quad (1)$$

In this generality the map $q_{\mathcal{L}}^{\text{ss}}$ cannot have a section since it is in general not surjective. To avoid this problem we give the following definition.

Definition 2.8. A sequence $\mathcal{L} = \{L_1 \subset L_2 \subset \cdots \subset L_h = G\}$ of standard Levi subgroups is called ss-admissible if for all $i = 1, \dots, h-1$, each simple factor of L_i does not coincide with a simple factor of L_{i+1} .

From now on we fix an ss-admissible sequence of Levi subgroups $\mathcal{L} = \{L_1 \subset L_2 \subset \cdots \subset L_h = G\}$. Let $\Phi_i = \Phi_{L_i}$ and $\Delta_i = \Delta_{L_i}$.

For $i > 1$, fix an order of Δ_i which is compatible with L_{i-1} and an arbitrary order of $\Delta_1 = \Delta_{L_1}$. Let $\text{St}_i: \mathbb{K}^{\Delta_i} \rightarrow L_i^{\text{ss}}$ be the Steinberg section defined by this order.

Remark 2.9. Notice that it is not always possible, even changing the system of simple roots, to choose an order of Δ which is compatible with all the inclusions $L_s \subset L_{s+1}$ at the same time.

Remark that if $i > 1$, whatever compatible order we choose for Δ_i , the roots in Δ_{i-1} form the initial segment in our order. Also notice that, for $i < h$, we get another order on Δ_i , the one obtained by restricting the order of Δ_{i+1} to Δ_i . Let $\text{St}'_i: \mathbb{K}^{\Delta_i} \rightarrow L_i^{\text{ss}}$ be the Steinberg sections defined using this second order.

For $i = 1, \dots, n$ we define “simultaneous” Steinberg sections by

$$\text{St}_{\mathcal{L}_i}: \prod_{i=1}^j \mathbb{K}^{\Delta_i} \longrightarrow L_i^{\text{ss}} \quad \text{by} \quad \text{St}_{\mathcal{L}_i}(a^{(1)}, \dots, a^{(i)}) = \text{St}_i\left(\sum_{j=1}^i a^{(j)}\right)$$

and similarly for $i = 1, \dots, h-1$

$$\mathrm{St}'_{\mathcal{L}_i} : \prod_{i=1}^j \mathbb{K}^{\Delta_i} \rightarrow L_i^{\mathrm{ss}} \quad \text{by} \quad \mathrm{St}'_{\mathcal{L}_i}(a^{(1)}, \dots, a^{(i)}) = \mathrm{St}'_i\left(\sum_{j=1}^i a^{(j)}\right).$$

Let us remark that for $i = h$,

$$\mathrm{St}_{\mathcal{L}_h}(a^{(1)}, \dots, a^{(h)}) = X_G\left(\sum_{j=1}^h a^{(j)}\right) \cdot w$$

We need the following lemma.

Lemma 2.10. *For any ordering $\alpha_1, \dots, \alpha_n$, setting $\beta_i = s_n \cdots s_{i+1}(\alpha_i)$ for each $i = 1, \dots, n$, the element*

$$\prod_{i=1}^n X_{\beta_i}(-1)s_n \cdots s_1$$

lies in U^-U^+ . In particular the element $s_n \cdots s_1$ can be written as a product $u_+u_-v_+$, where $u_+, v_+ \in U_+$ and $u_- \in U_-$.

Proof. Set $\Theta = \prod_{i=1}^n X_{\beta_i}(-1)s_n \cdots s_1$.

Let ω_i denote the fundamental weight such that $\langle \omega_i, \check{\alpha}_j \rangle = \delta_{i,j}$. In the fundamental representation V_{ω_i} of highest weight ω_i , choose a highest weight vector v_i . We denote by $W_i \subset V_{\omega_i}$, the T invariant complement to the one dimensional space spanned by v_i .

In order to prove our claim, we need to see that $\Theta v_i = v_i + w_i$ with $w_i \in W_i$.

Write

$$\prod_{i=1}^n X_{\beta_i}(-1)s_n \cdots s_1 = X_{\alpha_n}(-1)s_n \cdots X_{\alpha_1}(-1)s_1$$

and remark that W_i is stable under $X_{\alpha_j}(-1)s_j$ for each $j \neq i$, and that furthermore $X_{\alpha_j}(-1)s_j v_i = v_i$. On the other hand, a direct computation shows that $X_{\alpha_i}(-1)s_i v_i = v_i + w'_i$ with $w'_i \in W_i$. It follows immediately that $\Theta v_i = v_i + w_i$ with $w_i \in W_i$. This is our claim. \square

Notation 2.11. In what follows we will use the following notation.

If $i_1 < \dots < i_j$ and $f : \prod_{\ell=1}^j \mathbb{K}^{\Delta_{i_\ell}} \rightarrow X$ is a function with values in a set X , then, for all $\underline{a} = (a^{(1)}, \dots, a^{(h)}) \in \prod_{i=1}^h \mathbb{K}^{\Delta_i}$ by $f(\underline{a})$ we will denote $f(a^{(i_1)}, \dots, a^{(i_j)})$.

We will use this notation also if we have an action of a group on $\prod_{\ell=1}^j \mathbb{K}^{\Delta_{i_\ell}}$.

Lemma 2.12. *Given an ss-admissible sequence \mathcal{L} , there exists morphisms*

$$V : \prod_{i=1}^{h-1} \mathbb{K}^{\Delta_i} \rightarrow V_{L_{h-1}}, \quad Z : \mathbb{K}^{\Delta_h} \rightarrow V_{L_{h-1}}, \quad W : \prod_{i=1}^{h-1} \mathbb{K}^{\Delta_i} \rightarrow V_{L_{h-1}}^-$$

such that

$$V(\underline{a}) t \mathrm{St}_{\mathcal{L}_h}(\underline{a}) V^{-1}(\underline{a}) = V(\underline{a}) t Z(\underline{a}) \mathrm{St}'_{\mathcal{L}_{h-1}}(\underline{a}) W(\underline{a}) \quad (2)$$

for all $\underline{a} \in \prod_{i=1}^h \mathbb{K}^{\Delta_i}$ and for all $t \in T$.

Proof. Let $\alpha_1, \dots, \alpha_n$ be the order we have fixed for $\Delta = \Delta_h$ and let $m = |\Delta_{h-1}|$. Set $s_i = s_{\alpha_i}$ and define β_i as in Lemma 2.7. Using Lemma 2.7, we get that the roots β_1, \dots, β_n do not lie in $\Phi_{L_{h-1}}$. Hence the image of X_G is contained in $V_{L_{h-1}}$.

Let us set $w' = s_n \cdots s_{m+1}$. Then we clearly have

$$t \operatorname{St}_{\mathcal{L}}(a^{(1)}, \dots, a^{(h)}) = t X_G(a^{(h)}) w' \operatorname{St}'_{\mathcal{L}_{h-1}}(a^{(1)}, \dots, a^{(h-1)}).$$

Now write, using Lemma 2.10, $w' = u_+ u_- v_+$. Remark that all the elements involved are in the Levi subgroup associated to the simple roots $\Delta_h \setminus \Delta_{h-1}$ and, as a consequence, $u_+, v_+ \in V_{L_{h-1}}$ and $u_- \in V_{L_{h-1}}^-$. Set $V = (\operatorname{St}'_{\mathcal{L}_{h-1}})^{-1} v_+ \operatorname{St}_{\mathcal{L}_{h-1}}$ and $Z := X_G u_+$. Notice that both V and Z take their values in $V_{L_{h-1}}$. Finally set $W = (\operatorname{St}'_{\mathcal{L}_{h-1}})^{-1} u_- \operatorname{St}'_{\mathcal{L}_{h-1}}$ and notice that it takes values in $V_{L_h}^-$. Identity (2) now follows. \square

The previous Lemma can be used inductively.

Lemma 2.13. *For every ss-admissible sequence of Levi factors there exist:*

(a) *morphisms*

$$A: T \times \prod_{i=1}^h \mathbb{k}^{\Delta_i} \rightarrow U, \quad C: T \times \prod_{i=1}^{h-1} \mathbb{k}^{\Delta_i} \rightarrow U^-,$$

(b) *morphisms*

$$U_i: \prod_{j=1}^{i-1} \mathbb{k}^{\Delta_j} \rightarrow L_i,$$

for $i = 1, \dots, h$,

(c) *elements w_i in the Weyl group $N_{L_i}(T)/T$,*

(d) *actions $t \odot_i \underline{a}$ of T on \mathbb{k}^{Δ_i} , for $i = 1, \dots, h$, having the property that \odot_h is trivial and \odot_i leaves every line \mathbb{k}^α stable,*

such that

$$t \pi_i(A(t, \underline{a})) \pi_i^-(C(t, \underline{a})) = U_i(t, \underline{a}) w_i(t) \operatorname{St}_{\mathcal{L}_i}(t \odot_i \underline{a}) U_i(t, \underline{a})^{-1} \quad (3)$$

for every $t \in T$ and $\underline{a} \in \prod_{i=1}^h \mathbb{k}^{\Delta_i}$ (we are using Notation 2.11).

Proof. We proceed by induction on h . If $h = 1$, everything follows from the previous Lemma setting $A(t, \underline{a}) = t^{-1} V(0) t Z(\underline{a})$ and $C(t, \underline{a}) = W(0)$, $U_1(t, \underline{a}) = V(0)$, $w_1 = 1$ and \odot_1 trivial.

If $h > 1$, we can assume that our statement holds for $\mathcal{L}' = \{L_1 \subset \dots \subset L_{h-1}\}$, so there exist morphisms U'_i, A', B' elements w'_i and actions \odot'_i satisfying our requirements. We choose morphisms V, Z, W as in the previous Lemma 2.12.

By Lemma 2.3 there exist a $g_0: \mathbb{k}^{\Delta_{h-1}} \rightarrow L_{h-1}^{\text{ss}}$, an element of the Weyl group w and an action \cdot of T on \mathbb{k}^{Δ_h} which preserves every line \mathbb{k}^α such that

$$t \operatorname{St}'_{h-1}(b) = g_0(t \cdot b) w(t) \operatorname{St}_{h-1}(t \cdot b) g_0^{-1}(t \cdot b)^{-1},$$

for all $t \in T$ and $b \in \mathbb{k}^{\Delta_h}$. Extend g_0 to a map $g: \prod_{i=1}^{h-1} \mathbb{k}^{\Delta_i} \rightarrow L_{h-1}^{\text{ss}}$ by

$$g(a^{(1)}, \dots, a^{(h-1)}) = g_0 \cdot \left(\sum_{i=1}^{h-1} a^{(i)} \right).$$

By definition we have

$$t \operatorname{St}'_{\mathcal{L}_{h-1}}(\underline{a}) = g(t \cdot \underline{a}) w(t) \operatorname{St}_{\mathcal{L}_{h-1}}(\underline{a}) g(t \cdot \underline{a})^{-1}.$$

For all $t \in T$, $\underline{b} \in \prod_{i=1}^{h-1} \mathbb{K}^{\Delta_i}$ and all $\underline{a} \in \prod_{i=1}^h \mathbb{K}^{\Delta_i}$ define

$$\begin{aligned} E_h(t, \underline{b}) &= U'_{h-1}(t, w(t)^{-1} \cdot \underline{b}) g(w(t)^{-1} \cdot \underline{b})^{-1} V(\underline{b}), \\ A_h(t, \underline{a}) &= t^{-1} E_h(t, \underline{a}) V(\underline{a}) t Z(\underline{a}) t^{-1} E_h(t, \underline{a})^{-1} t, \\ C_h(t, \underline{b}) &= E_h(t, \underline{b}) W(\underline{b}) E_h(t, \underline{b})^{-1}, \\ U_h(t, \underline{b}) &= E_h(t, \underline{b}) V(\underline{b}) \end{aligned}$$

and notice that E_h takes values in L_{h-1} , A_h in V_{h-1} , B_h in V_{h-1}^- and U_h in L_h . By (2) we have

$$\begin{aligned} U_h(t, \underline{a}) w(t)^{-1} \operatorname{St}_{\mathcal{L}_h}(\underline{a}) U_h(t, \underline{a})^{-1} \\ = t A_h(t, \underline{a}) A'(t, w(t)^{-1} \cdot \underline{a}) C'(t, w(t)^{-1} \cdot \underline{a}) C_h(t, \underline{a}). \end{aligned}$$

Now, for all $t \in T$, $\underline{b} \in \prod_{i=1}^{h-1} \mathbb{K}^{\Delta_i}$ and all $\underline{a} \in \prod_{i=1}^h \mathbb{K}^{\Delta_i}$, we define

$$\begin{aligned} A(t, \underline{a}) &= A_h(t, \underline{a}) A'(t, w(t)^{-1} \cdot \underline{a}), \\ C(t, \underline{b}) &= C'(t, w(t)^{-1} \cdot \underline{b}) C_h(t, \underline{b}). \end{aligned}$$

Then $\pi_{h-1}(A(t, \underline{a})) = A'(t, w(t)^{-1} \cdot \underline{a})$ and $\pi_{h-1}^-(C(t, \underline{b})) = C'(t, w(t)^{-1} \cdot \underline{b})$. Let $w_h = w^{-1}$, $w_i = w'_i$. Take \odot_h to be the trivial action and, for all $i = 1, \dots, h-1$, $t \in T$, $\underline{c} \in \prod_{j=1}^i \mathbb{K}^{\Delta_j}$, set $t \odot_i \underline{c} := t \odot'_i (w(t)^{-1} \cdot \underline{c})$. Finally for all $i = 1, \dots, h-1$, $t \in T$, $\underline{c} \in \prod_{j=1}^i \mathbb{K}^{\Delta_j}$ and $t \in T$, define

$$U_i(t, \underline{c}) = U'_i(t, w(t)^{-1} \cdot \underline{c}).$$

By the inductive hypothesis and a straightforward computation our claim follows. \square

A special case of the previous lemma gives a section to $q_{\mathcal{L}}^{\text{ss}}$.

Proposition 2.14. *For every ss-admissible sequence of Levi factors \mathcal{L} , there exists a morphism $\chi^{\text{ss}}: \prod_{i=1}^h \mathbb{K}^{\Delta_i} \rightarrow U \times U^-$ such that $q_{\mathcal{L}}^{\text{ss}} \circ \chi$ is an isomorphism.*

Proof. Let A, C be as in the previous lemma and define

$$\chi^{\text{ss}}(\underline{a}) = (A(1, \underline{a}), C(1, \underline{a}))$$

By equation (3) for $(\underline{a}^{(1)}, \dots, \underline{a}^{(h)}) \in \prod_{i=1}^h \mathbb{K}^{\Delta_i}$, we have

$$\begin{aligned} q_{\mathcal{L}}^{\text{ss}} \circ \chi(\underline{a}^{(1)}, \dots, \underline{a}^{(h)}) \\ = \left(q_1^{\text{ss}}(\operatorname{St}_1(\underline{a}^{(1)})), q_2^{\text{ss}}(\operatorname{St}_2(\underline{a}^{(1)} + \underline{a}^{(2)})), \dots, q_h^{\text{ss}}\left(\operatorname{St}_h\left(\sum_{i=1}^h \underline{a}^{(i)}\right)\right) \right), \end{aligned}$$

which, by the properties of the Steinberg map, is clearly an isomorphism. \square

2.15. General simultaneous Steinberg section. We extend now our section taking into account also the fact that the various L_i are not semisimple. Let us start with our ss-admissible sequence $\mathcal{L} = \{L_0 \subset \dots \subset L_h = G\}$ of standard Levi subgroups. Let $q_i: L_i \rightarrow L_i//\text{Ad}(L_i)$ be the quotient map. We can define $q_{\mathcal{L}}: B \times U^- \rightarrow \prod_{i=1}^h L_i//\text{Ad}(L_i)$ as in formula (1). However in general this map is not surjective since the Levi subgroups may have common factors in the center.

Definition 2.16. Fix a ss-admissible sequence of Levi factors $\mathcal{L} = \{L_0 \subset \dots \subset L_h = G\}$. A sequence $\mathcal{M} = \{M_1 \subset \dots \subset M_h\}$ of connected subgroups of G is said to be compatible with \mathcal{L} if for all i , M_i is a subgroup of L_i containing L_i^{ss} .

We say that the compatible sequence \mathcal{M} is admissible, if there exists subtori $S_i \subset T$ such that

- S1: $M_i = S_i \ltimes L_i^{\text{ss}}$;
- S2: $S_{i+1} \cap M_i = \{1\}$.

In this case, notice that $(S_i \times \dots \times S_h) \ltimes L_i^{\text{ss}} \subset G$ is a semi-direct product. If the sequence $\mathcal{M} = \{M \subset G\}$ is admissible, we say that M is admissible.

In what follows we are going to need a number of simple general remarks which we collect in the following:

- Lemma 2.17.**
- (i) *If $S \subset T$ are two tori, there exists a subtorus R of T such that $T = R \times S$. We call R a complement of S in T .*
 - (ii) *Let $M \subset N$ be two reductive connected groups with the same semisimple part. If T is a maximal torus of N , $T \cap M$ is a maximal torus of M .*
 - (iii) *Let D be a lattice. Let $A, B, C \subset D$ be sub-lattices such that A is saturated in D , $A \cap B = C \cap B = \{0\}$ and $A + B$ has finite index in D . Then there exists a sub-lattice B' of D such that $C \cap B' = \{0\}$ and $A \oplus B' = D$.*
 - (iv) *If $M \subset N$ are connected reductive groups with $|Z(N) \cap M| < \infty$ and T a maximal torus of N which intersects M in a maximal torus of M , there exists a subtorus $S \subset T$ such that $N = S \ltimes N^{\text{ss}}$ and $S \cap M = \{1\}$.*

Proof. We have already proved (i) in Section 2.4. To prove (ii) notice that $T \cap M$ contains a maximal torus of M and that it is commutative. Since the centralizer of a maximal torus in a connected group is always connected, $T \cap M$ is a maximal torus of M .

To prove (iii), let a, b, c, d be the ranks of A, B, C, D respectively. By our hypothesis we have $a + b = d$ and $c \leq a$. We can choose a basis e_1, \dots, e_a of A and extend it to a basis e_{a+1}, \dots, e_d of D . Let v_1, \dots, v_c be a basis of C . For $u_1, \dots, u_b \in A$ consider the span $B'(u_1, \dots, u_b)$ of $\{w_i = e_{a+i} + u_i: i = 1, \dots, b\}$. This is a complement of A in D . $B'(u_1, \dots, u_b)$ intersects C if and only if all the maximal minors of the matrix whose columns are given by $w_1, \dots, w_b, v_1, \dots, v_c$ vanish.

Thus, if for each choice of u_1, \dots, u_b the corresponding $B'(u_1, \dots, u_b)$ intersects C , it means that these minors define polynomial functions on A^b which are identically zero. However, if we tensor with the rational numbers, the existence of B guarantees that there exist vectors $u_1, \dots, u_b \in A \otimes_{\mathbb{Z}} \mathbb{Q}$ on which the value of these polynomial functions is non-zero. By the density of the integers in the Zariski topology we get a contradiction.

(iv) is a consequence of (iii). Recall that if R is a torus with lattice of cocharacters $\Lambda_*(R)$, then $R = \mathbb{k}^* \otimes_{\mathbb{Z}} \Lambda_*(R)$. Now, take D to be the lattice of cocharacters of T , A to be the set of cocharacters of $T \cap N^{\text{ss}}$, B the set of cocharacters of the identity component of $Z(N)$ and C the set of cocharacters of $T \cap M$. Choose B' as in (iii) and set $S = \mathbb{k}^* \otimes_{\mathbb{Z}} B'$. \square

A maximal admissible sequence compatible with \mathcal{L} can be constructed in the following way: choose S_h to be a complement torus of L_h^{ss} in L_h and, for every $i = 1, \dots, h-1$, S_i to be a complement of L_i^{ss} in L_{i+1}^{ss} . Then define $M_i = S_i \ltimes L_i^{\text{ss}}$ and notice that \mathcal{M} is an admissible sequence and that L_i is the semi-direct product $S_i \times \dots \times S_h \ltimes L_i^{\text{ss}}$ for $i = 1, \dots, h$.

Another class of admissible sequences is described in the following lemma.

Lemma 2.18. *Let $\mathcal{M} = \{M_1 \subset \dots \subset M_h\}$ be a sequence compatible with a ss-admissible sequence \mathcal{L} . Suppose that for all $i = 1, \dots, h-1$ the intersection $Z(M_{i+1}) \cap M_i$ is finite. Then \mathcal{M} is admissible.*

Proof. By property (iv) above we can construct S_i such that $S_i \cap M_{i-1} = \{1\}$ and $M_i = S_i \ltimes L_i^{\text{ss}}$. \square

We now fix an admissible sequence \mathcal{M} and subgroups S_i with properties S1 and S2. Let $S'_i = S_i \times \dots \times S_h$. For each i let $R_i \subset T$ be such that L_i is the semi-direct product $R_i \times S'_i \ltimes L_i^{\text{ss}}$. Let $p_i: L_i \rightarrow L_i^{\text{ss}} // \text{Ad}(L_i^{\text{ss}})$ be an extension to L_i of the quotient map defined on L_i^{ss} as in Section 2.4. Then we can identify the adjoint quotients $q_i^M: M_i \rightarrow M_i // \text{Ad}(M_i)$ and $q_i: L_i \rightarrow L_i // \text{Ad}(L_i)$ with the maps $S_i \ltimes L_i^{\text{ss}} \rightarrow S_i \times L_i^{\text{ss}} // \text{Ad}(L_i^{\text{ss}})$ given by $(s, g) \mapsto (s, p_i(s \cdot g))$ and $R_i \times S'_i \ltimes L_i^{\text{ss}} \rightarrow R_i \times S'_i \times L_i^{\text{ss}} // \text{Ad}(L_i^{\text{ss}})$ given by $(r, s, g) \mapsto (r, s, p_i(r \cdot s \cdot g))$. Considering the composition of q_i with the projection on the factor S_i of S'_i we get a projection $r_i: L_i \rightarrow S_i \times L_i^{\text{ss}} // \text{Ad}(L_i^{\text{ss}})$ and under the above identifications we obtain $q_i^M(m) = r_i(m)$ for each $m \in M_i$.

With this notations we can define a “simultaneous” quotient map

$$r_{\mathcal{M}}: B \times U^- \longrightarrow \prod_{i=1}^h M_i // \text{Ad}(M_i)$$

defined by

$$r_{\mathcal{M}}(u, v) = (r_1(\pi_1(u)\pi_1^-(v)), r_2(\pi_2(u)\pi_2^-(v)), \dots, r_h(uv)).$$

We can now state our Theorem on simultaneous section.

Theorem 2.19. *Let \mathcal{M} be an admissible sequence as in Definition 2.16 and fix subgroups S_i, R_i as above so that the map $r_{\mathcal{M}}$ is defined. Let $S = S_1 \times \dots \times S_h$. Then there exists $\chi: S \times \prod_{i=1}^h \mathbb{k}^{\Delta_i} \rightarrow B \times U^-$ such that $r_{\mathcal{M}} \circ \chi: S \times \prod_{i=1}^h \mathbb{k}^{\Delta_i} \rightarrow \prod_{i=1}^h M_i // \text{Ad}(M_i)$ is an isomorphism.*

Proof. Let A, C be as in Lemma 2.13 and define

$$\chi(s, \underline{a}) = (sA(s, \underline{a}), C(s, \underline{a})).$$

We prove that $r_{\mathcal{M}} \circ \chi$ is an isomorphism.

Let $s = t_1 \cdots t_h$ with $t_i \in S_i$. By property S1, for each $i = 2, \dots, h$ there exist functions $\lambda_i: S_1 \times \cdots \times S_{i-1} \rightarrow L_i^{\text{ss}}$ and $\mu_i: S_1 \times \cdots \times S_{i-1} \rightarrow S_i$ such that

$$t_1 \cdots t_{i-1} = \lambda_i(t_1, \dots, t_{i-1}) \mu_i(t_1, \dots, t_{i-1}).$$

Then with the notation of Lemma 2.13 we have

$$\begin{aligned} r_i(\chi(s, \underline{a})) &= r_i(s \pi_i(A(s, \underline{a})) \pi_i^-(C(s, \underline{a}))) \\ &= (t_i \mu_i(t_1 \cdots t_{i-1}), p_i(s \pi_i(A(s, \underline{a})) \pi_i^-(C(s, \underline{a})))) \\ &= (t_i \mu_i(t_1 \cdots t_{i-1}), p_i(w_i(s) \text{St}_{\mathcal{L}_i}(s \odot_i \underline{a}))). \end{aligned}$$

At this point everything follows easily from 2.5. \square

We can consider also the following slightly different “simultaneous” quotient map that will be needed later on. Let $S = \prod_{i=1}^h S_i$ be as in Theorem 2.19. Define $q_{\mathcal{M}}^{\text{ss}}: SU \times U^- \rightarrow \prod_{i=1}^h L_i^{\text{ss}} // \text{Ad}(L_i^{\text{ss}})$ by formula (1), and finally define

$$\bar{r}_{\mathcal{M}}: SU \times U^- \longrightarrow S \times \prod_{i=1}^h L_i^{\text{ss}} // \text{Ad}(L_i^{\text{ss}})$$

by $\bar{r}_{\mathcal{M}}(su, v) = (s, q_{\mathcal{M}}^{\text{ss}}(su, v))$ for all $s \in S$, $u \in U$ and $v \in U^-$. A variant of Theorem 2.19 which we are going to use later, is given by

Lemma 2.20. *Let \mathcal{M} be an admissible sequence and let $S, \bar{r}_{\mathcal{M}}$ as above. Then there exists $\chi: S \times \prod_{i=1}^h \mathbb{K}^{\Delta_i} \rightarrow SU \times U^-$ such that $\bar{r}_{\mathcal{M}} \circ \chi: S \times \prod_{i=1}^h \mathbb{K}^{\Delta_i} \rightarrow S \times \prod_{i=1}^h L_i^{\text{ss}} // \text{Ad}(L_i^{\text{ss}})$ is an isomorphism. Moreover the S component of $\bar{r}_{\mathcal{M}}(\chi(s, x))$ is equal to s .*

Proof. The same function χ defined in the proof of Theorem 2.19 satisfies the requirements of the lemma. The proof is completely analogous. \square

3. THE $\text{GL}(n)$ CASE

The proof of Theorem 2.19 is constructive. However it is difficult to write down an explicit general formula. In this section we construct a very explicit section in the case of $\text{GL}(n)$ which seems to us to be particularly nice.

We start by giving a curious parametrization of the Borel subgroup of lower triangular matrices.

Let $A = (a_{i,j})$ be a lower triangular $n \times n$ matrix and let $C = (c_{i,j})$ be the upper $n \times n$ triangular matrix with $c_{i,j} = 1$ for all $i \leq j$.

Take the product matrix $D = AC$ and, for any $1 \leq i_1 < i_2 < \dots < i_r \leq n$, denote by $[i_1, i_2, \dots, i_r]$ the determinant of the principal minor of D consisting of the rows (and columns) of index i_1, i_2, \dots, i_r .

Proposition 3.1. *We have, setting $i_0 = 0$,*

$$[i_1, i_2, \dots, i_r] = \prod_{h=1}^r (a_{i_h, i_h} + \dots + a_{i_h, i_{h-1}+1}). \quad (4)$$

Proof. Remark that, if $d_{h,k}$ is the entry of D , in the h -th row and k -th column,

$$d_{h,k} = \begin{cases} a_{h,1} + \dots + a_{h,k} & \text{if } k \leq h, \\ a_{h,1} + \dots + a_{h,h} & \text{if } k \geq h. \end{cases} \quad (5)$$

From this our result is clear for $r = 1$ and we can proceed by induction.

Set $\underline{i} := (i_1, \dots, i_r)$ and denote by $D(\underline{i})$ the corresponding principal minor. The last two columns of $D(\underline{i})$ are

$$\begin{pmatrix} \sum_{s=1}^{i_1} a_{i_1,s} & \sum_{s=1}^{i_1} a_{i_1,s} \\ \sum_{s=1}^{i_2} a_{i_2,s} & \sum_{s=1}^{i_2} a_{i_2,s} \\ \dots & \dots \\ \sum_{s=1}^{i_{r-1}} a_{i_{r-1},s} & \sum_{s=1}^{i_{r-1}} a_{i_{r-1},s} \\ \sum_{s=1}^{i_{r-1}} a_{i_r,s} & \sum_{s=1}^{i_r} a_{i_r,s} \end{pmatrix}.$$

Substituting the last column with the difference of the last two columns, we deduce

$$[i_1, i_2, \dots, i_r] = [i_1, i_2, \dots, i_{r-1}](a_{i_r, i_r} + \dots + a_{i_r, i_{r-1}+1}).$$

From this everything follows. \square

This proposition has some simple consequences.

Proposition 3.2. *Let P_r be the r -th coefficient of the characteristic polynomial of D . Then*

- (1) P_r does not depend from $a_{h,k}$ if $n - h + k < r$.
- (2) P_r depends linearly from $a_{h,k}$ if $n - h + k = r$. Furthermore, if $n - h + k = r$, the coefficient of $a_{h,k}$ is $\prod_{t=1}^{k-1} a_{t,t} \prod_{t=h+1}^n a_{t,t}$

Proof. Denote by Λ_r the set of subsets of $\{1, \dots, n\}$ of cardinality r . Any such subset has a obvious total order. We have

$$P_r = \sum_{\{i_1, \dots, i_r\} \in \Lambda_r} [i_1, i_2, \dots, i_r].$$

As above, set $i_0 = 0$ and observe that, given $\underline{i} := (i_1, \dots, i_r)$, for each s , $i_s - i_{s-1} \leq n - r + 1$. If $a_{h,k}$ appears in $[i_1, i_2, \dots, i_r]$, then necessarily there is a $1 \leq s \leq r$ with $i_s = h$. Furthermore we must also have $h - k \leq h - i_{s-1} - 1 \leq n - r$ and so $n - h + k \geq r$.

This proves our first claim.

As for our second claim, notice that if $n - h + k = r$ and $i_s = h$ then necessarily, since $i_s - i_{s-1} = n - r + 1$, $i_{s-1} = k$. It follows that $[i_1, i_2, \dots, i_r] = [1, 2, \dots, k-1, h, h+1, \dots, n]$. But, by Proposition 3.2,

$$[1, 2, \dots, k-1, h, h+1, \dots, n] = \sum_k^h a_{h,j} \prod_{t=1}^{k-1} a_{t,t} \prod_{t=h+1}^n a_{t,t}$$

and everything follows. \square

Given an $n \times n$ matrix X , set X_i equal to the $i \times i$ matrix obtained from X erasing the last $n - i$ rows and columns. Notice that $D_i = A_i C_i$.

If we let A vary in the Borel subgroup $B \subset \mathrm{GL}(n)$ of lower triangular matrices, we obtain, for each $i = 1, \dots, n$, a map $\phi_i: B \rightarrow \mathrm{GL}(i)$ defined by $\phi_i(A) = A_i$. Composing with the map associating to each matrix in $\mathrm{GL}(i)$ the coefficients $P_1^{(i)}, \dots, P_i^{(i)}$ of its characteristic polynomial (we assume that $P_h^{(i)}$ has degree h , so that in particular $P_i^{(i)}$, being the determinant, takes non-zero values on $\mathrm{GL}(i)$), we get a morphism

$$c_i: B \rightarrow \mathbb{K}^{i-1} \times \mathbb{K}^*$$

defined by $c_i(A) = (P_1^{(i)}(D_i), \dots, P_i^{(i)}(D_i))$. If we take the map $\Pi = \times_{i=1}^n c_i$, we get a map

$$\Pi: B \rightarrow \mathbb{K}^{\frac{n(n-1)}{2}} \times (\mathbb{K}^*)^n.$$

Theorem 3.3. *The map Π is an isomorphism.*

Proof. We are going to explain how to construct an inverse to Π . This will follow if we show that once we have fixed the values $z_h^{(k)}$ of the functions $P_h^{(k)}$, $1 \leq k \leq h \leq n$, there exists a unique $A = (a_{i,j})$ in B such that $z_h^{(k)} = P_h^{(k)}(D_k)$.

Let us start with the diagonal entries. We must have, for $i = 1, \dots, n$,

$$z_i^{(i)} = \prod_{h=1}^i a_{h,h}.$$

Since $z_h^{(h)} \in \mathbb{K}^*$ we get

$$a_{h,h} = \frac{z_h^{(h)}}{z_{h-1}^{(h-1)}}.$$

Let us now do induction on $h - k = r$. Assume that the entries $a_{p,q}$ with $p - q < r$ can be uniquely determined by $z_s^{(t)}$ with $s - t < r$. By Proposition 3.2 we deduce,

$$\left(\prod_{t=1}^{h-1} a_{t,t} \right)^{-1} z_h^{(h+r)} = \frac{z_h^{(h+r)}}{z_{h-1}^{(h-1)}} = a_{h+r,h} + F_h(r),$$

where $F_h(r)$ is a degree one polynomial in the entries $a_{m+r,m}$ with $m < h$ whose coefficients are polynomials in the entries $a_{s,s}^{\pm}$, $a_{s,t}$ with $0 \leq s - t < r$. Since, by induction all the $a_{s,t}$ with $0 \leq s - t < r$ have been already determined, the entries $a_{h+r,h}$ are solutions of a linear system in triangular form with 1's on the diagonal and hence are uniquely determined. \square

We consider now the Gelfand–Zeitlin sequence

$$\{\mathrm{GL}(1) \subset \dots \subset \mathrm{GL}(n)\},$$

where $\mathrm{GL}(i)$ is the subgroup leaving invariant the last $n - i$ coordinates. Notice that this is an admissible sequence. The map $r = r_{\mathcal{M}}: B \times U^- \rightarrow \prod_{i=1}^n \mathrm{GL}(i) // \mathrm{Ad}(\mathrm{GL}(i))$ of the previous section can be described in the following way

$$r(U, V) = (P_j^{(i)}(U_i V_i)) \quad \text{for } 1 \leq j \leq i \leq n.$$

Hence Theorem 3.3 gives an explicit form of Theorem 2.19:

Corollary 3.4. *Let $\chi: B \rightarrow B \times U^-$ be defined by $\chi(A) = (A, C)$. Then $r \circ \chi$ is an isomorphism.*

Notice that a similar result holds if we take a subsequence of the sequence above:

$$\mathrm{GL}(n_1) \subset \mathrm{GL}(n_2) \subset \cdots \subset \mathrm{GL}(n_h)$$

with $n_1 < \cdots < n_h = n$. In this case define \mathcal{A} as the subset of B of matrices $A = (a_{i,j})$ such that for $m \neq n_h$, $a_{m,j} = \delta_{m,j}$. Then, if we define $\chi: \mathcal{A} \rightarrow B \times U^-$ by $\chi(A) = (A, C)$, the same proof shows that $r \circ \chi$ is an isomorphism.

4. POISSON COMMUTATIVE SUBALGEBRAS OF THE ALGEBRA OF THE POISSON DUAL OF G

We want to apply our result on the existence of a generalized Steinberg section to the study of some Poisson algebras arising from Manin triples. Recall that a Manin triple is a triple $(\mathfrak{g}, \mathfrak{h}, \mathfrak{k})$, where \mathfrak{g} is a Lie algebra equipped with a non-degenerate invariant bilinear form κ , $\mathfrak{h}, \mathfrak{k}$ are Lie subalgebras of \mathfrak{g} which are maximal isotropic subspaces with respect to κ and such that κ induces a perfect pairing between \mathfrak{h} and \mathfrak{k} , so that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{h}$. If H is a connected group with Lie algebra equal to \mathfrak{h} , considering left invariant vector fields, we identify the tangent bundle on H with $H \times \mathfrak{h}$, the cotangent bundle on H with $H \times \mathfrak{k}$ and if f is a function on H , we denote with $\delta_x f \in \mathfrak{k}$ the differential of f at x with respect to this isomorphism. Assume now that H is a subgroup of a group G with Lie algebra equal to \mathfrak{g} so that H acts on \mathfrak{g} preserving the form κ . A Poisson structure on H is then defined in the following way

$$\{f, g\}(x) = \kappa(\delta_x f; \mathrm{Ad}_x(\delta_x g)) - \kappa(\delta_x g; \mathrm{Ad}_x(\delta_x f))$$

for all $x \in H$ and f, g functions on H . If $(\mathfrak{g}_1, \mathfrak{h}_1, \mathfrak{k}_1)$ and $(\mathfrak{g}_2, \mathfrak{h}_2, \mathfrak{k}_2)$ are two Manin triples and $\varphi: \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ is a morphism of Lie algebras such that $\varphi(\mathfrak{h}_1) \subset \mathfrak{h}_2$ and $\varphi(\mathfrak{k}_1) \subset \mathfrak{k}_2$ and $\phi: H_1 \rightarrow H_2$ is a group homomorphism whose differential is equal to φ , then ϕ does not need to be a Poisson map. However we have the following Lemma.

Lemma 4.1. *Let $(\mathfrak{g}_1, \mathfrak{h}_1, \mathfrak{k}_1)$ and $(\mathfrak{g}_2, \mathfrak{h}_2, \mathfrak{k}_2)$ be two Manin triples. Let κ_i be the invariant bilinear form on \mathfrak{g}_i , $i = 1, 2$. Let $\varphi: \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ be such that*

- (i) φ is a morphism of Lie algebras;
- (ii) $\varphi(\mathfrak{k}_1) \subset \mathfrak{k}_2$, and $\varphi(\mathfrak{h}_1) \subset \mathfrak{h}_2$;
- (iii) $\kappa_2(\varphi(u), \varphi(v)) = \kappa_1(u, v)$ for all $u, v \in \mathfrak{g}_{1\mathrm{up}n}$;
- (iv) $\psi = \varphi^*: \mathfrak{h}_2 \rightarrow \mathfrak{h}_1$ is a morphism of Lie algebras.

For $i = 1, 2$, let G_i be a group with Lie algebra equal to \mathfrak{g}_i and let $H_i \subset G_i$ be a connected subgroup with Lie algebra equal to \mathfrak{h}_i and consider on H_i the Poisson structure introduced above. Let $\Phi: H_1 \rightarrow H_2$ and $\Psi: H_2 \rightarrow H_1$ be homomorphisms whose differentials are φ and ψ .

Then the map Ψ is a morphism of Poisson groups.

Proof. Notice first that, since the bilinear form κ_1 is non-degenerate, we have $\psi \circ \varphi = \mathrm{id}$, hence $\Psi \circ \Phi = \mathrm{id}$. So, if $N = \ker \Psi$ we have $H_2 \simeq N \rtimes H_1$, in particular N is

connected. Now we prove that for all $u, v \in \mathfrak{k}_1$ and for all $x \in H_2$

$$\kappa_2(\varphi(u), \text{Ad}_x(\varphi(v))) = \kappa_1(u, \text{Ad}_{\Psi(x)}v).$$

Indeed if $x = \Phi(y)$, since $\Psi \circ \Phi = \text{id}$, this is clear by property (iii). We prove now the statement for $x \in N$. In this case we need to prove that $\kappa_2(\varphi(u), \text{Ad}_x(\varphi(v))) = 0$. Let \mathfrak{n} be the Lie algebra of N . Consider the subspace $V = \mathfrak{n} \oplus \varphi(\mathfrak{k}_1)$. Notice that V is maximal isotropic with respect to κ_2 . Moreover, a simple computation shows that it is stable under the action of \mathfrak{n} , so it is stable also under the action of N . In particular for $x \in N$ and $u, v \in \mathfrak{k}_1$ we have $\kappa_2(\varphi(u), \text{Ad}_x(\varphi(v))) = 0$ as desired.

Let now f, g be two functions on H_1 , $x \in H_2$ and $y = \Psi(x)$. An easy computation shows that $\delta_x(\Psi^*f) = \varphi(\delta_y f)$. Hence we have

$$\begin{aligned} \{\Psi^*f, \Psi^*g\}(x) &= \kappa_2(\varphi(\delta_y f), \text{Ad}_x(\varphi(\delta_y g))) - \kappa_2(\varphi(\delta_y g), \text{Ad}_x(\varphi(\delta_y f))) \\ &= \kappa_1(\delta_y f, \text{Ad}_y(\delta_y g)) - \kappa_1(\delta_y g, \text{Ad}_y(\delta_y f)) = \{f, g\}(y) \end{aligned}$$

proving the claim. \square

We apply this result to our situation. Recall some simple facts about the definition of the Poisson dual of a group G . Fix a maximal torus T of G . Let \mathfrak{g} be the Lie algebra of G and \mathfrak{t} the Lie algebra of T . Fix an invariant non-degenerate bilinear form $(-, -)$ on \mathfrak{g} . On the Lie algebra $\mathfrak{g} \oplus \mathfrak{g}$ define the non-degenerate bilinear form $\kappa((x, y), (u, v)) = (x, u) - (y, v)$. The triple $(\mathfrak{g} \oplus \mathfrak{g}, \Delta, \mathfrak{h})$, where Δ is the diagonal subalgebra and $\mathfrak{h} = \{(x + t, y - t) \in \mathfrak{b} \oplus \mathfrak{b}^- : x \in \mathfrak{u}, y \in \mathfrak{u}^- \text{ and } t \in \mathfrak{t}\}$ is a Manin triple. Indeed it is immediate to see that both subalgebras are isotropic and that they are disjoint. Correspondingly the Poisson dual of G is the group

$$H = \{(x, y) \in B \times B^- : \pi_T(x) = \pi_T^-(y)^{-1}\},$$

where π_T, π_T^- are the projections on T of B, B^- respectively. As we have explained above the Manin triple induces on H the structure of a Poisson Lie group and, if one consider the map

$$\rho = \rho_G : H \longrightarrow G \tag{6}$$

defined by $\rho(x, y) = xy^{-1}$, the symplectic leaves in H are the connected components of the pre-images of the conjugacy classes in G (indeed in the simply connected case in [3] it is shown that such pre-images are always connected unless they are zero dimensional). Furthermore the map ρ is a fiber bundle onto the open Bruhat cell of G with fiber $\{t \in T : t^2 = 1\}$. Let $q : G \rightarrow G//\text{Ad}(G)$ be the quotient under the adjoint action and let $\theta = \theta_G = q \circ \rho : H \rightarrow G//\text{Ad}(G)$. We denote the algebra $\theta^*(\mathbb{K}[G//\text{Ad}(G)])$ by Z^{HC} or Z_G^{HC} . It is not difficult to check that each Hamiltonian vector field kills the elements in Z^{HC} so that this algebra is central with respect to the Poisson structure.

Consider now a standard Levi subgroup L of G and denote by \mathfrak{l} its Lie algebra. Let $B_L = L \cap B$ and $B_L^- = L \cap B^-$ be the standard Borel subgroup and the opposite standard Borel subgroup of L , and denote by $\pi_L : B \rightarrow B_L$, and $\pi_L^- : B^- \rightarrow B_L^-$ the projections onto the Levi factor. Notice that the restriction of the form $(-, -)$ to \mathfrak{l} is non-degenerate so we can define a Manin triple $(\mathfrak{l} \oplus \mathfrak{l}, \Delta_{\mathfrak{l}}, \mathfrak{h}_{\mathfrak{l}})$ taking the intersection of $\mathfrak{l} \oplus \mathfrak{l}$ with Δ and \mathfrak{h} . Define also $H_L = H \cap L \times L$ and $\rho_L : H_L \rightarrow L$

and $Z_L^{\text{HC}} \subset \mathbb{k}[H_L]$ as before. We notice that the transpose ψ of the inclusion, from \mathfrak{h} to \mathfrak{h}_L is a morphism of Lie algebras and can be integrated to a map $\Psi_L: H \rightarrow H_L$ given by $\Psi_L(u, v) = (\pi_L^+(u), \pi_L^-(v))$. We can apply Lemma 4.1 and we get that Ψ_L^* is a morphism of Poisson Lie groups. In particular $A_L^G = \Psi_L^*(Z_L^{\text{HC}})$ is a Poisson commutative subalgebra of $\mathbb{k}[H]$. In the case that $L = T$, we can define in a similar way a Poisson commutative subalgebra larger than A_T^G . In this case we can identify H_T with T and the Manin triple $(\mathfrak{t} \oplus \mathfrak{t}, \Delta_{\mathfrak{t}}, \mathfrak{h}_i)$ is commutative so $H_T = T$ is Poisson commutative. Hence $A_T' = \Psi_T^*(\mathbb{k}[T])$ is a Poisson commutative subalgebra of $\mathbb{k}[H]$. Notice that A_T' is an extension of degree $2^{\dim T}$ of A_T^G .

Let now $\mathcal{L} = \{L_1 \subset \cdots \subset L_h\}$ be a ss-admissible sequence of standard Levi subgroups and let $\mathcal{M} = \{M_1 \subset \cdots \subset M_h\}$ be an admissible sequence compatible with \mathcal{L} . Choose subtori S_i which satisfy properties S1 and S2. Define $S = \prod_{i=1}^h S_i$. Choose a complement of S in T and denote by $p_S: T \rightarrow S$ the associated projection. Let also, as in section 2.6, denote by $p_i: L_i \rightarrow L_i^{\text{ss}} // \text{Ad}(L_i^{\text{ss}})$ the projection on the adjoint quotient of the semisimple factor of L_i . Define $\theta_i = p_i \circ \rho_{L_i} \circ \Psi_{L_i}: H \rightarrow L_i^{\text{ss}} // \text{Ad}(L_i^{\text{ss}})$ and $\theta_{\mathcal{M}}: H \rightarrow S \times \prod_{i=1}^h L_i // \text{Ad}(L_i^{\text{ss}})$ by

$$\theta_{\mathcal{M}}(u, v) = (p_S(\Psi_T(u)), \theta_1(u, v), \dots, \theta_h(u, v)).$$

We can use this map and the simultaneous Steinberg section to construct a big Poisson commutative subalgebra of $\mathbb{k}[H]$.

Theorem 4.2. *The map $\theta_{\mathcal{M}}^*: \mathbb{k}[S] \otimes \bigotimes_{i=1}^h \mathbb{k}[L_i^{\text{ss}} // \text{Ad}(L_i^{\text{ss}})] \rightarrow \mathbb{k}[H]$ is injective and its image is a Poisson commutative subalgebra of $\mathbb{k}[H]$.*

Proof. In order to show the injectivity of the map $\theta_{\mathcal{M}}^*$ we are going to see that the map $\theta_{\mathcal{M}}$ is surjective. We can use the section χ constructed in Lemma 2.20. Using the notation of Section 2, let A, C be as in Lemma 2.13 and define $\chi': S \times \prod_{i=1}^h \mathbb{k}^{\Delta_i} \rightarrow H$ by

$$\chi'(s, \underline{a}) = (s A(s, \underline{a}), s^{-1} C(s, \underline{a})^{-1}).$$

Then $\theta_{\mathcal{M}}(\chi'(s, \underline{a})) = (s, q_{\mathcal{M}}^{\text{ss}}(\chi(s^2, \underline{a})))$. Since by Lemma 2.20 for every fixed s the map $a \mapsto q_{\mathcal{M}}^{\text{ss}}\chi(s^2, \underline{a})$ is bijective we get that also the map $\theta_{\mathcal{M}} \circ \chi'$ is bijective.

To prove that the image is a Poisson commutative subalgebra we notice that it is contained in the product of the subalgebras $A_{L_1}^G, \dots, A_{L_h}^G$ and A_T' . Recall $A_L^G = \Psi_L^*(Z_L^{\text{HC}})$ is Poisson commutative. So it is enough to prove that $A_{L_i}^G$ commutes with $A_{L_j}^G$ when $i > j$ and with A_T' . Notice that if $i > j$, then $A_{L_j}^G = \Psi_{L_i}^*(A_{L_j}^{L_i})$. Hence, since by Lemma 4.1 $\Psi_{L_i}^*$ is a Poisson map and $Z_{L_i}^{\text{HC}}$ is in the center of $\mathbb{k}[H_{L_i}]$, $A_{L_j}^G$ commutes with $A_{L_i}^G$. We can argue in a similar way for the algebra A_T' . \square

In the case of $\text{GL}(n)$ we can produce in this way a commutative subalgebra of maximal dimension. We consider again the Gelfand–Zeitlin admissible sequence

$$\mathcal{M} = \{\text{GL}(1) \subset \cdots \subset \text{GL}(n)\}.$$

In this case the product $S \times \prod_{i=1}^n \text{SL}(i) // \text{Ad}(\text{SL}(i))$ has dimension $\binom{n+1}{2}$ and its coordinate ring A is the localization of a polynomial algebra in $\binom{n+1}{2}$ with respect to some variables.

By what we have recalled above, the generic symplectic leaves of H have dimension equal to the regular orbits in G , hence to $n^2 - n$. So maximal isotropic subspaces in the tangent space of a generic point of H have dimension $n^2 - \binom{n}{2} = \binom{n+1}{2}$.

Generically the quotient map $\theta_{\mathcal{M}}$ is a smooth map and its fibers are maximal dimensional isotropic sub-varieties of H . So the dimension of A is the maximal possible dimension of a commutative Poisson subalgebra of $\mathbb{k}[H]$, which can be stated by saying that it defines a completely integrable Hamiltonian system.

5. THE CENTER OF A QUANTUM GROUP IN THE REDUCTIVE CASE

In this section we recall the description of the center of a quantum group at roots of unity in the simply connected case proved in [4]. We also give an extension of this result to the case of a reductive group, which we could not find in the literature. We assume the characteristic of \mathbb{k} to be equal to zero from now on.

We start by giving the definition of the quantum group associated to a reductive group. If G is a connected reductive group we denote by T_G a maximal torus of G and by Λ_G the lattice of characters of the chosen maximal torus T_G . We choose a set of simple roots $\Delta_G = \{\alpha_1, \dots, \alpha_r\}$ and we set $a_{i,j} = \langle \alpha_i, \check{\alpha}_j \rangle$ so that $C = (a_{i,j})$ is the Cartan matrix. Let (d_1, \dots, d_r) be the usual non-zero entries of the diagonal matrix D such that CD is symmetric. We also assume that we have a nondegenerate symmetric invariant bilinear form $(-, -)$ on the Lie algebra of G . If we restrict $(-, -)$ to the Cartan subalgebra $\mathfrak{t} = \text{Lie } T_G$, we get a non-degenerate form. We assume that the corresponding form on \mathfrak{t}^* takes integer values on $\Lambda_G \times \mathbb{Z}[\Delta_G]$ and furthermore $CD = ((\alpha_i, \alpha_j))$. Notice that $\langle \lambda, \check{\alpha} \rangle = 0$ if and only if $(\lambda, \alpha) = 0$. We set $q_i = q^{d_i}$.

For a non-zero complex number q the algebra $U_q(G)$ is the algebra with generators $E_1, \dots, E_r; F_1, \dots, F_r$ and $K_\lambda, \lambda \in \Lambda_G$ and relations:

$$(R1) \quad K_\lambda K_\mu = K_{\lambda+\mu}, \quad K_0 = 1,$$

$$(R2) \quad K_\lambda E_i K_{-\lambda} = q^{(\lambda, \alpha_i)} E_i,$$

$$(R3) \quad K_\lambda F_i K_{-\lambda} = q^{-(\lambda, \alpha_i)} F_i,$$

$$(R4) \quad [E_i, F_j] = \delta_{i,j} \frac{K_{\alpha_i} - K_{-\alpha_i}}{q_i - q_i^{-1}},$$

$$(R5) \quad \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} E_i^{1-a_{ij}-k} E_j E_i^k = 0 \quad (i \neq j),$$

$$(R6) \quad \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} F_i^k F_j F_i^{1-a_{ij}-k} = 0 \quad (i \neq j),$$

for all $\lambda, \mu \in \Lambda$, $i, j = 1, \dots, r$, where we set for any t , and for $h \leq m$,

$$\begin{bmatrix} m \\ h \end{bmatrix}_t = \frac{[m]_t!}{[m-h]_t! [h]_t!}, \quad [h]_t! = [h]_t \dots [2]_t [1]_t, \quad [h]_t = \frac{t^h - t^{-h}}{t - t^{-1}}.$$

Recall that $U_q(G)$ is a Hopf algebra (see [6]). We denote the center of $U_q(G)$ by $Z_q(G)$.

If we have an isogeny $\phi: G_1 \rightarrow G_2$ then this determines an inclusion of Λ_{G_2} in Λ_{G_1} and of $U_q(G_2)$ in $U_q(G_1)$. Let Γ be the kernel of ϕ and notice that $\Gamma = \text{Spec } \mathbb{k}[\Lambda_{G_1}/\Lambda_{G_2}] = \text{Hom}(\Lambda_{G_1}/\Lambda_{G_2}, \mathbb{k}^*)$ so we can identify it with a group of characters of Λ_{G_1} . We have the following action of Γ on $U_q(G_1)$:

$$\gamma E_i = E_i, \quad \gamma F_i = F_i, \quad \gamma K_\lambda = \gamma(\lambda) K_\lambda$$

for all $\lambda \in \Lambda_{G_1}$ and for all i . Notice that since ϕ is an isogeny all the roots are contained in Λ_{G_2} , hence the action is well defined. Notice also that K_μ is fixed by Γ if and only if μ is an element of Λ_{G_2} . Hence using the Poincaré Birkoff Witt basis [11], we obtain the following Lemma.

Lemma 5.1. *With the notation above, we have*

$$U_q(G_2) = U_q(G_1)^\Gamma$$

and $U_q(G_1)$ is a free module over $U_q(G_2)$ with basis given by K_μ with μ varying in a set of representatives of $\Lambda_{G_1}/\Lambda_{G_2}$.

Let \tilde{S}_G be the connected component of the center of our reductive group G . Then $\tilde{S}_G = M \otimes_{\mathbb{Z}} \mathbb{k}^*$, where M is the lattice of cocharacters which are trivial on Δ_G . In the previous setting choose $G_1 = G^{\text{ss}} \times \tilde{S}_G$, $G_2 = G$ and ϕ the multiplication map. The kernel Γ is the set of pairs (γ, γ^{-1}) with $\gamma \in \tilde{S}_G \cap G^{\text{ss}}$ and we will identify it with a subgroup of G .

In this case we have $\Lambda_{G_1} = \Lambda_{G^{\text{ss}}} \oplus \Lambda'$, where Λ' is the dual of M . We have a surjective projection from Λ_G to $\Lambda_{G^{\text{ss}}}$ given by restriction of characters, whose kernel is equal to $N = \Delta^\perp$ and a surjective projection from Λ_G to Λ' whose kernel is equal to N^\perp .

Since the algebra $U_q(G^{\text{ss}} \times \tilde{S}_G)$ is clearly isomorphic to $U_q(G^{\text{ss}}) \otimes \mathbb{k}[\tilde{S}_G]$, applying Lemma 5.1, we deduce that $U_q(G)$ can be identified with $(U_q(G^{\text{ss}}) \otimes \mathbb{k}[\tilde{S}_G])^\Gamma$.

We now start to describe the center $Z_q(G)$ of $U_q(G)$. We have

Proposition 5.2. *Under the identification of $U_q(G)$ with $(U_q(G^{\text{ss}}) \otimes \mathbb{k}[\tilde{S}_G])^\Gamma$,*

$$Z_q(G) = (Z_q(G^{\text{ss}}) \otimes \mathbb{k}[\tilde{S}_G])^\Gamma = Z_q(G^{\text{ss}} \times \tilde{S}_G)^\Gamma.$$

Moreover $Z_q(G^{\text{ss}} \times \tilde{S}_G)$ is a free module over $Z_q(G)$ with a basis given by the elements K_μ with μ varying in a set of representatives of $(\Lambda_{G^{\text{ss}}} \oplus \Lambda')/\Lambda_G$.

Proof. It is clear that $Z_q(G) \supset Z_q(G^{\text{ss}} \times \tilde{S}_G)^\Gamma$. We prove the other inclusion. Notice that we can choose a set A of representatives of $(\Lambda_{G^{\text{ss}}} \oplus \Lambda')/\Lambda_G$ in the lattice Λ' . Take $x \in U_q(G^{\text{ss}} \times \tilde{S}_G)$. By Lemma 5.1, x can be written as $x = \sum_\mu x_\mu K_\mu$ with $\mu \in A$ and $x_\mu \in U_q(G)$. Since the K_μ are central, if $z \in Z_q(G)$, then it commutes also with x . Hence $z \in Z_q(G^{\text{ss}} \times \tilde{S}_G)$.

Our argument also proves that the elements K_μ with $\mu \in A$ are a basis of $Z_q(G^{\text{ss}} \times \tilde{S}_G)$ over $Z_q(G)$. \square

We now give a geometric description of $Z_q(G)$ in the case G has simply connected semisimple factor and q is a primitive ℓ -th root of 1 with ℓ prime with the entries of the Cartan matrix. This description generalizes the one given in [5]. Set $X_G = \text{Spec } Z_q(G)$. We will use the notations introduced in sections 4 and 2. So $H_G = H$

is the Poisson dual of G , $\rho_G = \rho: H_G \rightarrow G$ is the map $\rho_G(u, v) = uv^{-1}$, W_G is the Weyl group of G with respect to T_G , $q_G^{\text{ss}}: G \rightarrow G^{\text{ss}}/\text{Ad}(G^{\text{ss}}) = T_{G^{\text{ss}}}/W_G$ is an extension of the adjoint quotient of G^{ss} as constructed in section 2.4. If we set $p_{S_G}: G \rightarrow S_G$ to be the quotient of G by its semisimple factor, then $q_G = q_G^{\text{ss}} \times p_{S_G}: G \rightarrow (T_{G^{\text{ss}}}/W_G) \times S_G$ is the quotient map under the adjoint action and $\theta_G = q_G \circ \rho_G$. Notice that p_{S_G} induces an isogeny from \tilde{S}_G and S_G whose kernel is equal to Γ and that, as we did in section 2.4, we can identify S_G with a complement of $T_{G^{\text{ss}}}$ in T . When necessary (mainly for the construction of the Steinberg section as in Section 2.4) we will identify S_G with this torus. Finally, we denote by $\eta_G: T_G/W_G \rightarrow T_G/W_G$ the map induced by $t \mapsto t^\ell$ from $T_G \rightarrow T_G$. This is a finite flat covering of T_G/W_G and it is an unramified Galois covering over the set $(T_G/W_G)^r = T_G^r/W_G$, where T_G^r is the set of regular elements of T_G .

Recall, [2], that $Z_q(G)$ is endowed with a Poisson bracket and contains the elements $E_i^\ell, F_i^\ell, K_\lambda^\ell = K_{\ell\lambda}$, for all $i = 1, \dots, r$, $\lambda \in \Lambda$. Let $Z_0(G)$ be the smallest subalgebra of $U_q(G)$ closed under the Poisson bracket and containing the elements $E_i^\ell, F_i^\ell, K_\lambda^\ell$.

We now recall the description of X_G given in [5] in the simply connected case. In this case the subalgebra $Z_0(G)$ is isomorphic to the coordinate ring of the Poisson dual H_G of G . There exists also another subalgebra $Z_1(G)$ in $Z_q(G)$ which is isomorphic to the ring of functions on T_G invariant under the action of W_G . The Poisson bracket is trivial on $Z_1(G)$.

The map η_G induces a map η_G^* from $Z_1(G)$ to $Z_1(G)$. We denote by $Z_1^{(\ell)}(G)$ its image. Identifying $\mathbb{k}[T_G/W_G]$ with $Z_1^{(\ell)}(G)$ and $\mathbb{k}[H_G]$ with $Z_0(G)$ we get an inclusion $Z_1^{(\ell)}(G) \subset Z_0(G)$. One then shows (see e.g. [4]) that

$$Z_q(G) \simeq Z_0(G) \otimes_{Z_1^{(\ell)}(G)} Z_1(G).$$

Equivalently, setting $X_G = \text{Spec } Z_q(G)$, if G is simply connected, the following diagram is cartesian

$$\begin{array}{ccc} X_G & \xrightarrow{\zeta_G^1} & T_G/W_G \\ \zeta_G \downarrow & & \downarrow \eta_G \\ H_G & \xrightarrow{\theta_G} & T_G/W_G, \end{array} \quad (7)$$

where we denoted by $\zeta_G: X_G \rightarrow H_G$ the finite morphism of degree $\ell^{\dim T_G}$ induced by the inclusion of $Z_0(G)$ in $Z_q(G)$ and by $\zeta_G^1: X_G \rightarrow T_G/W_G$ the one induced by the inclusion of $Z_1(G)$ in $Z_q(G)$.

We now give a similar description in the case in which the semisimple factor of G is simply connected. Before stating our result, we make some remarks on the action of Γ on the objects introduced so far.

We consider the actions of Γ on $H_{G^{\text{ss}}} \times \tilde{S}_G$ and on $(T_{G^{\text{ss}}}/W_G) \times \tilde{S}_G$ given by

$$\gamma \cdot ((u, v), s) = (\gamma u, \gamma^{-1}v, \gamma^{-1}s) \quad \text{and} \quad \gamma \circ (tW_G, s) = (\gamma^2 tW_G, \gamma^{-1}s).$$

With these actions, the map $\theta_{G^{\text{ss}}} \times \text{id}: H_{G^{\text{ss}}} \times \tilde{S}_G \rightarrow (T_{G^{\text{ss}}}/W_G) \times \tilde{S}_G$ is equivariant while the map η_{G_1} satisfies $\eta_{G_1}(\gamma \circ x) = \gamma^\ell \circ \eta_{G_1}(x)$. Moreover, we consider

$\phi_0: H_{G^{\text{ss}}} \times \tilde{S}_G \rightarrow H_G$ given by $\phi_0((u, v), s) = (us, vs^{-1})$ and we notice that this is the quotient map by the action of Γ .

By what we have said about the semisimple case, we can identify the coordinate ring of $H_{G^{\text{ss}}} \times \tilde{S}_G$ with $Z_0(G^{\text{ss}} \times \tilde{S}_G)$, where $K_{\ell\lambda}$, for $\lambda \in \Lambda'$, corresponds to the character λ on \tilde{S}_G . Then the action of Γ corresponds to the following action on $Z_0(G^{\text{ss}} \times \tilde{S}_G)$:

$$\gamma \cdot E_i^\ell = E_i^\ell, \quad \gamma \cdot F_i^\ell = F_i^\ell, \quad \gamma \cdot K_{\ell\lambda} = \gamma(\lambda)K_{\ell\lambda}$$

for all $\lambda \in \Lambda_{G_1}$ and for all i . Notice that with this action we have $Z_0(G) = (Z_0(G^{\text{ss}} \times \tilde{S}_G))^\Gamma$. Hence

$$\text{Spec } Z_0(G) \simeq (H_{G^{\text{ss}}} \times \tilde{S}_G)/\Gamma \simeq H_G,$$

where the last isomorphism is given by ϕ_0 .

Now we want to describe the subalgebra $Z_1(G) := (Z_1(G^{\text{ss}}) \otimes \mathbb{k}[\tilde{S}_G])^\Gamma$, where we are now considering the usual action of Γ on $U_q(G^{\text{ss}} \times \tilde{S}_G)$. Notice that this makes sense since the subring $Z_1(G^{\text{ss}}) \otimes \mathbb{k}[\tilde{S}_G]$ is stable under Γ .

Lemma 5.3. *There exists $\phi_1: (T_{G^{\text{ss}}}/W_G) \times \tilde{S}_G \rightarrow (T_{G^{\text{ss}}}/W_G) \times S_G$ such that the following properties hold:*

- (i) ϕ_1 is Γ invariant with respect to the \circ -action, more precisely it is the quotient of $(T_{G^{\text{ss}}}/W_G) \times \tilde{S}_G$ by the \circ -action;
- (ii) $\phi_1(x, s) = (\psi(x, s), p_{S_G}(s))$ with $\psi(x, s) \in T_{G^{\text{ss}}}/W_G$;
- (iii) the following diagram is commutative and cartesian

$$\begin{array}{ccc} H_{G^{\text{ss}}} \times \tilde{S}_G & \xrightarrow{\theta_{G^{\text{ss}}} \times \text{id}} & (T_{G^{\text{ss}}}/W_G) \times \tilde{S}_G \\ \phi_0 \downarrow & & \downarrow \phi_1 \\ H_G & \xrightarrow{\theta_G} & (T_{G^{\text{ss}}}/W_G) \times S_G; \end{array} \quad (8a)$$

- (iv) the following diagram is commutative

$$\begin{array}{ccc} (T_{G^{\text{ss}}}/W_G) \times \tilde{S}_G & \xrightarrow{\eta_{G^{\text{ss}}} \times \eta_{\tilde{S}_G}} & T_{G^{\text{ss}}}/W_G \times \tilde{S}_G \\ \phi_1 \downarrow & & \downarrow \phi_1 \\ (T_{G^{\text{ss}}}/W_G) \times S_G & \xrightarrow{\eta_G} & (T_{G^{\text{ss}}}/W_G) \times S_G. \end{array} \quad (8b)$$

Proof. Let ξ_1, \dots, ξ_n be the characters of the fundamental representations of G^{ss} . Then the coordinate ring of $T_{G^{\text{ss}}}/W_G$ is the polynomial ring in ξ_1, \dots, ξ_n . As in section 2.4 we can extend these characters to get characters of G that we denote with ξ'_1, \dots, ξ'_n . Notice that by definition the map q_G^{ss} is given by evaluating (ξ'_1, \dots, ξ'_n) . Now define $f_i: G^{\text{ss}} \times \tilde{S}_G \rightarrow \mathbb{k}$ by

$$f_i(x, s) = \xi_i(x)\xi'_i(s^2).$$

Then f_i is Γ invariant and

$$\mathbb{k}[G^{\text{ss}}//\text{Ad}(G^{\text{ss}}) \times \tilde{S}_G] \simeq \mathbb{k}[f_1, \dots, f_n] \otimes \mathbb{k}[\tilde{S}_G]$$

since $\xi'_i(s)$ is a character of \tilde{S}_G . Taking invariants, we obtain

$$\mathbb{k}[G^{\text{ss}}//\text{Ad}(G^{\text{ss}}) \times \tilde{S}_G]^{(\Gamma, \circ)} \simeq \mathbb{k}[f_1, \dots, f_n] \otimes \mathbb{k}[S_G].$$

Now define ϕ_1 by

$$\phi_1(x, s) = ((f_1(x, s), \dots, f_n(x, s), p_{S_G}(s)).$$

By the description of the invariants ϕ_1 satisfies (i).

(ii) is clear by definition.

To prove (iii), notice that for all $((u, v), s) \in H_{G^{\text{ss}}} \times \tilde{S}_G$, we have

$$\theta_G(\phi_0((u, v), s)) = \theta_G(us, vs^{-1}) = (q_G^{\text{ss}}(us^2v^{-1}), p_{S_G}(s))$$

and, since \tilde{S}_G is central,

$$\begin{aligned} q_G^{\text{ss}}(us^2v^{-1}) &= (\xi'_1(uv^{-1}s^2), \dots, \xi'_n(uv^{-1}s^2)) \\ &= (f_1(q_{G^{\text{ss}}}(uv^{-1}), s), \dots, f_n(q_{G^{\text{ss}}}(uv^{-1}), s)), \end{aligned}$$

from which the commutativity of diagram (8a) follows. Since the vertical maps in that diagram are quotients by Γ , the varieties are smooth and the action on the the varieties in the top line is free we get that it is also cartesian.

The commutativity of (8b) is clear. \square

The previous Lemma implies that $\text{Spec } Z_1(G) \simeq (T_{G^{\text{ss}}}/W_G) \times S_G$. As we have done in the simply connected case, we denote by $\zeta_G: X_G \rightarrow H_G$, and $\zeta_G^1: X_G \rightarrow (T_{G^{\text{ss}}}/W_G) \times S_G$ the maps induced by the inclusion $Z_0(G) \subset Z_q(G)$ and $Z_1(G) \subset Z_q(G)$ respectively.

We can now give our second description of the center of $U_q(G)$.

Proposition 5.4. *Let G be reductive with simply connected semisimple factor. Then the following diagram is cartesian*

$$\begin{array}{ccc} X_G & \xrightarrow{\zeta_G^1} & (T_{G^{\text{ss}}}/W_G) \times S_G V \\ \zeta_G \downarrow & & \downarrow \eta_G \\ H_G & \xrightarrow{\theta_G} & (T_{G^{\text{ss}}}/W_G) \times S_G. \end{array} \quad (9)$$

Proof. If $G = G^{\text{ss}} \times \tilde{S}$ with \tilde{S} a torus then \tilde{S} is also the connected component of the center of G and by Proposition 5.2, we immediately have that $Z_q(G) = Z_q(G) \otimes \mathbb{k}[\tilde{S}]$. Hence $X_G = X_{G^{\text{ss}}} \times \tilde{S}$. We can write a diagram similar to (7). Notice that in this case $H_G = H_{G^{\text{ss}}} \times \tilde{S}$, $T_G = T_{G^{\text{ss}}} \times \tilde{S}$ and $T_G/W_G = (T_{G^{\text{ss}}}/W_G) \times \tilde{S}$. Also the ℓ power map η_G is the product of the two power maps $\eta_{G^{\text{ss}}}$ and $\eta_{\tilde{S}}: \tilde{S} \rightarrow \tilde{S}$. Then we have the following cartesian diagram:

$$\begin{array}{ccc} X_G & \longrightarrow & (T_{G^{\text{ss}}}/W_G) \times \tilde{S} \\ \zeta_G \downarrow & & \downarrow \eta_{G^{\text{ss}}} \times \eta_{\tilde{S}} \\ H_{G^{\text{ss}}} \times \tilde{S} & \xrightarrow{\theta_{G^{\text{ss}}} \times \text{id}} & (T_{G^{\text{ss}}}/W_G) \times \tilde{S}. \end{array}$$

So, in this case, everything is an immediate consequence of what we know for G^{ss} .

Now we consider the case of an arbitrary reductive group G with simply connected semisimple factor. Let $\phi : G^{\text{ss}} \times \tilde{S}_G \rightarrow G$ be the multiplication map. By Proposition 5.2 we have that $X_G = X_{G^{\text{ss}} \times \tilde{S}_G} / \Gamma$. We denote by Y the pull back of the maps η_G, θ_G and we prove that it is isomorphic to X_G .

Arguing exactly as in [5], one sees that Y is irreducible, normal, Cohen-Macaulay, and the map $Y \rightarrow H_G$ has degree $\ell^{\dim T_G}$. By the commutativity of the diagrams in (8) we have a natural map $\psi : X_G \rightarrow Y$.

Moreover the morphism ψ is finite and since the morphism $\zeta_G : X_G \rightarrow H_G$ has also degree $\ell^{\dim T_G}$ it is also birational, hence it is an isomorphism. \square

6. BRANCHING RULES FOR QUANTUM GROUPS AT ROOTS OF 1

In this section we are going to show how to obtain some branching rules for quantum groups at roots of 1 following the ideas of [5]. In order to do this, let us recall the notion of a Cayley–Hamilton algebra and some results on the representation theory of quantum groups.

6.1. Cayley–Hamilton algebras. An algebra with trace, over a commutative ring A is an associative algebra R with a 1-ary operation

$$t : R \rightarrow R$$

which is assumed to satisfy the following axioms:

- (1) t is A -linear.
- (2) $t(a)b = bt(a), \quad \forall a, b \in R$.
- (3) $t(ab) = t(ba), \quad \forall a, b \in R$.
- (4) $t(t(a)b) = t(a)t(b), \quad \forall a, b \in R$.

This operation is called a formal trace.

The ring of $n \times n$ matrices with entries in A with the usual trace is an example.

At this point we will restrict the discussion to the case in which A is a field of characteristic 0. We remark that there are universal polynomials $P_i(t_1, \dots, t_i)$ with rational coefficients, such that the characteristic polynomial $\chi_M(t) := \det(t - M)$ of a $n \times n$ matrix M can be written as:

$$\chi_M(t) = t^n + \sum_{i=1}^n P_i(\text{tr}(M), \dots, \text{tr}(M^i)) t^{n-i}.$$

We can thus formally define, in an algebra with trace R , for every element a , a formal n -characteristic polynomial:

$$\chi_a^n(t) := t^n + \sum_{i=1}^n P_i(t(a), \dots, t(a^i)) t^{n-i}.$$

The Cayley–Hamilton Theorem, stating that a matrix M satisfies its characteristic polynomial, suggests the following definition.

Definition 6.2 [5, Definition 2.5]. An algebra with trace R is said to be an n -Cayley–Hamilton algebra, or to satisfy the n^{th} Cayley–Hamilton identity if:

- (1) $t(1) = n$.
- (2) $\chi_a^n(a) = 0, \quad \forall a \in R$.

In [5, Theorem 4.1] it is proved that if a \mathbb{k} algebra R is a domain which is a finite module over its center A , and A is integrally closed in his quotient field F , then R is a n -Cayley–Hamilton algebra with $n^2 = \dim_F R \otimes_A F$.

As usual, let now G be a reductive group with simply connected semisimple factor, ℓ be a natural number prime with the entries of the Cartan matrix and q a primitive ℓ -th root of unity.

The algebra $U_q(G)$ is a domain. Indeed this is shown in [2], [3] for G semisimple and follows for a general reductive G from the fact that $U_q(G) = (U_q(G^{\text{ss}}) \otimes \mathbb{k}[\tilde{S}_G])^\Gamma$. Furthermore the fact that the center $Z_q(G)$ is integrally closed is immediate from Proposition 5.2 since $Z_q(G) = (Z_q(G^{\text{ss}}) \otimes \mathbb{k}[\tilde{S}_G])^\Gamma$ and $Z_q(G^{\text{ss}}) \otimes \mathbb{k}[\tilde{S}_G]$ is integrally closed. Thus $U_q(G)$ is a n -Cayley–Hamilton algebra and by Lemma 5.1 and Proposition 5.2 it follows that $n = \ell^{|\Phi^+|}$.

As a consequence (see [5] Theorem 3.1), the variety X_G parametrizes semisimple representations compatible with the trace (i. e., representation of $U_q(G)$ such that the formal trace coincide with the trace computed by considering the action of the element of $U_q(G)$ on the representation). Furthermore there is a dense open set X_G^{ir} of X_G such that for any $x \in X_G^{\text{ir}}$ the corresponding trace compatible representation is irreducible and it is the unique irreducible representation on which $Z_q(G)$ acts via the evaluation on x . We denote this representation by $V_x(G)$.

We define also $G^{\text{sr}} = q_G^{-1}((T/W)^r)$ to be the open set of semisimple elements of G , $H_G^{\text{sr}} = \rho_G^{-1}(G^{\text{sr}})$ and $X_G^{\text{sr}} = \zeta^{-1}(H^{\text{sr}})$. Hence by Proposition 5.4 $\zeta_G: X_G^{\text{sr}} \rightarrow H_G^{\text{sr}}$ is an unramified covering of degree $\ell^{\dim T_G}$. Finally by [3], it is known that $X_G^{\text{sr}} \subset X_G^{\text{ir}}$.

6.3. Branching rules. Let L be a standard Levi subgroup of G and let $M \subset G$ be an admissible subgroup compatible with L with the property that there exists a homomorphism $\sigma: L \rightarrow M$ which splits the inclusion $M \subset L$.

This is satisfied in the following two examples, which are the main applications we have in mind. The first is when G is semisimple simply connected and $L = M$. The second is when $G = \text{GL}(n)$,

$$L = \left\{ \begin{pmatrix} A & 0 \\ 0 & b \end{pmatrix} : A \in \text{GL}(n-1), b \in \mathbb{C}^* \right\},$$

and

$$M = \left\{ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} : A \in \text{GL}(n-1) \right\}.$$

Let $T_M = T \cap M$ be a maximal torus of M and let $R = \ker \sigma$ be such that $T = R \times T_M$. We can identify the algebra $U_q(M)$ with the subalgebra of $U_q(G)$ generated by the elements K_λ , λ vanishing on R , and by the element E_i, F_i with $\alpha_i \in \Delta_L$. Our goal is to describe how “generic” irreducible representations of $U_q(G)$ decompose when restricted to $U_q(M)$.

The discussion of the previous section about the representations and the center of $U_q(G)$ applies to $U_q(M)$ as well. Hence $U_q(M)$ is itself a n_L -th Cayley–Hamilton algebra, this time with $n_L = \ell^{|\phi_L^+|}$.

We define $\Psi_M: H_G \rightarrow H_L$ as

$$\Psi_M(u, v) = (\sigma(\pi_L^+(u)), \sigma(\pi_L^-(v))).$$

Then, under the identification of $Z_0(G)$ and $Z_0(M)$ with the coordinate rings of H_G and H_M respectively, the inclusion of $Z_0(M)$ in $Z_0(G)$ corresponds to the map Ψ_M^* .

For $x \in X_G$ we define

$$\mathcal{M}_x = \{y \in X_M : \zeta_M(y) = \Psi_M(\zeta_G(x))\}.$$

Theorem 6.4. *Let $x \in X_G^{\text{ir}}$ be such that $\Psi_M(\zeta_G(x)) \in H_M^{\text{sr}}$. Then, taking the associated graded for any Jordan-Hölder filtration of $V_x(G)$ we get:*

$$\text{Gr}(V_x(G))|_{U_q(M)} \simeq \bigoplus_{y \in \mathcal{M}_x} V_y(M) \oplus^b$$

with $b = \ell|\Phi^+| - |\Phi_M^+| - \dim T_M$.

Proof. In view of Lemma 5.7 in [5], our Theorem will follow once we show

Proposition 6.5. *The algebra $Z_q(G) \otimes_{Z_0(M)} Z_q(M)$ is an integral domain.*

Proof. Set $\mathcal{B} = H_G^{\text{sr}} \cap \Psi_M^{-1}(H_M^{\text{sr}})$ and $\mathcal{A} = \zeta_G^{-1}(\mathcal{B})$.

Following almost verbatim the proof of Proposition 7.4 in [5], one is reduced to show that taking the fiber product

$$\begin{array}{ccc} Y & \longrightarrow & T_{M^{\text{ss}}}/W_M \times S_M \\ \downarrow & & \downarrow \eta_M \\ \mathcal{A} & \xrightarrow{\theta_M \circ \Psi_M \circ \zeta_G} & T_{M^{\text{ss}}}/W_M \times S_M \end{array}$$

the variety Y is irreducible.

To show the irreducibility of Y , notice that we have the following cartesian diagram

$$\begin{array}{ccc} Y & \longrightarrow & (T_{M^{\text{ss}}}/W_M \times S_M) \times ((T_{G^{\text{ss}}}/W_G) \times S_G) \\ \downarrow & & \downarrow \eta_M \times \eta_G \\ \mathcal{B} & \xrightarrow{\tilde{\theta}} & (T_{M^{\text{ss}}}/W_M \times S_M) \times ((T_{G^{\text{ss}}}/W_G) \times S_G), \end{array}$$

where $\tilde{\theta}$ is the restriction to \mathcal{B} of the map $\theta = (\theta_M \circ \Psi_M) \times \theta_G$. By the definition of H_G^{sr} and H_M^{sr} , the set $\tilde{\theta}(\mathcal{B})$ is contained in $\mathcal{C} = (T_M/W_M)^r \times (T_G/W_G)^r$, and the map $\eta_M \times \eta_G$ over \mathcal{C} is an unramified covering of degree $\ell^{\dim T_G + \dim T_M}$. So the map $Y \rightarrow \mathcal{B}$ is an unramified covering of smooth varieties. Hence, to prove that Y is irreducible it is enough to prove that it is connected. We prove this by giving a section of $\theta : H_G \rightarrow (T_{M^{\text{ss}}}/W_M \times S_M) \times ((T_{G^{\text{ss}}}/W_G) \times S_G)$.

Consider the admissible sequence $\mathcal{M} = \{M \subset G\}$. We can choose S_M and S_G subtori of T_M and T_G satisfying conditions S1 and S2 of Section 2.15, and we identify these tori with the quotients of G by G^{ss} and of M by M^{ss} respectively, using the maps p_{S_G} and p_{S_M} . Set $S = S_M \times S_G$ and let $\chi : S \times \mathbb{k}^{\Delta_L} \times \mathbb{k}^{\Delta_G} \rightarrow B \times U^-$ be the generalized Steinberg section constructed in Theorem 2.19. Recall that $\chi(s, a, b) = (sA(s, a, b), C(s, a, b))$ and that, as it is shown in the proof of Theorem 2.19 and it is clear by construction, $r_{\mathcal{M}}(\chi(s, a, b)) = (\varphi(s), \psi(s, a, b)) \in S \times (T_{M^{\text{ss}}}/W_M \times$

$T_{G^{\text{ss}}}/W_G$). So, φ is an isomorphism and for all s the map $(a, b) \mapsto \psi(s, a, b)$ is a bijection between $\mathbb{k}^{\Delta_L} \times \mathbb{k}^{\Delta_G}$ and $T_{M^{\text{ss}}}/W_M \times T_{G^{\text{ss}}}/W_G$.

Now define $\chi': S \times \mathbb{k}^{\Delta_L} \times \mathbb{k}^{\Delta_G} \rightarrow H_G$ by

$$\chi'(s, a, b) = (sA(s^2, a, b), s^{-1}C(s^2, a, b)^{-1})$$

and notice that

$$\theta(\chi'(s, a, b)) = (\varphi(s), \psi(s^2, a, b)).$$

So $\theta \circ \chi'$ is bijective and, being the involved varieties smooth, it is an isomorphism.

This finishes the proof of the Proposition and hence of Theorem 6.4. \square

Remark 6.6. (1) Notice that since the algebras $U_q(G)$ and $U_q(G^{\text{ss}})$ have the same degree, equal to $\ell^{|\Phi^+|}$, and we are dealing with generic irreducible representations, one could have assumed right away that G is semisimple.

(2) If G is semisimple with simply connected cover \tilde{G} again the algebras $U_q(G) \subset U_q(\tilde{G})$ have the same degree and our result holds verbatim also for $U_q(G)$.

(3) Assume G semisimple. When $L = T_G$, the algebra $U_q(L)$ is the algebra of functions on T_G , which has the K_λ 's as a basis. Each element in T_G is a character for this algebra. Given a finite dimensional $U_q(L)$ module we can consider its character as a non-negative, integer valued function on Λ_G of finite support. This applies in particular to $U_q(G)$ modules. Consider the map $\Psi_T: H_G \rightarrow T_G = H_{T_G}$ and take an irreducible V lying over an element $h \in H_G^{\text{sr}}$. The set $\mathcal{A} := \eta^{-1}(\Psi(h))$ has $\ell^{\dim T}$ elements. Our result gives that the character of V equals $\ell^{|\Phi^+| - \dim T}$ times the characteristic function of \mathcal{A} .

6.7. The case of $\text{GL}(n)$. We want to apply Theorem 6.4 in the special case in which $G = \text{GL}(n)$ and $M = \text{GL}(n-1)$. We keep the notations of the previous section

Theorem 6.8. *Let $x \in X_{\text{GL}(n)}^{\text{ir}}$. Assume that $\Psi_{\text{GL}(n-1)}(\zeta_{\text{GL}(n)}(x)) \in H_{\text{GL}(n-1)}^{\text{sr}}$. Then the restriction of $V_x(\text{GL}(n))$ to $U_q(\text{GL}(n-1))$ is semisimple and*

$$V_x(\text{GL}(n))|_{U_q(\text{GL}(n-1))} \simeq \bigoplus_{y \in \mathcal{M}_x} V_y(\text{GL}(n-1)).$$

Proof. Both statements follow immediately from Theorem 6.4 once we remark that in this case

$$|\Phi^+| - |\Phi_M^+| - \dim T_M = \binom{n}{2} - \binom{n-1}{2} - (n-1) = 0. \quad \square$$

We can of course iterate this process restricting first to $U_q(\text{GL}(n-1))$ then to $U_q(\text{GL}(n-2))$ and so on. Since $U_q(\text{GL}(1))$ is a polynomial ring in one variable, hence commutative, we deduce that there is a dense open set in $X_0 \subset X_{\text{GL}(n)}$, whose simple definition we leave to the reader, with the property that if $x \in X_0$ then $V_x(\text{GL}(n))$ has a standard decomposition into a direct sum of one dimensional subspaces. This phenomenon is a counter part for quantum groups of what we have seen at the end of section 4 and it is analogous to the Gelfand–Zeitlin phenomenon.

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