These extensions were studied by Helton and Howe [73]. They correspond to elements of $H_1(X;\mathbb{Z})$, which is a canonical subgroup of $K_{-1}(X)$. The corresponding extensions are the 'uninteresting' ones which we found in Chapter 4.

The best way to think of the situation is probably in terms of pseudo-differential operators [144]. In practice J will be given by a pseudo-differential operator of order zero. The commutator $[J, M_f]$, when f is a smooth function, will then be an operator of order -1. On a manifold of dimension d such an operator belongs to the ideal $\mathcal{I}_r$ if $r > d$. It will thus not normally be Hilbert–Schmidt if $d > 1$.

Example. Let us consider the polarization corresponding to the Dirac operator on a torus X of odd dimension $d = 2m-1$, i.e. $X = \mathbb{R}^d/2\pi\mathbb{Z}^d$. The spin bundle on X is a trivial bundle whose fibre $\Delta \cong \mathbb{C}^N$ (where $N = 2^{m-1}$) is an irreducible module for the Clifford algebra C_d generated by elements $e_1, \ldots, e_d$ such that $e_i^2 = 1$ and $e_i e_j = -e_j e_i$ when $i \neq j$. The Dirac operator on the space H of maps $X \to \Delta$ is

$$D = -i \sum e_j \frac{\partial}{\partial \theta_j}.$$

If we expand the functions in Fourier series, so that H is identified with $\ell^2(\mathbb{Z}^d; \Delta)$ then D becomes the multiplication operator

$$\{f_p\} \mapsto \{pf_p\}.$$

(Here $p \in \mathbb{Z}^d$, and $pf_p \in \Delta$ is got by acting with $p \in \mathbb{R}^d \subset C_d$ on $f_p \in \Delta$.) The corresponding polarization operator J is multiplication by $p/\|p\|$. The commutator $[J, M_f]$, where M_f is multiplication by the scalar-valued function $f = \sum f_p e^{i\langle p, \theta\rangle}$, is represented by the kernel

$$(p, q) \mapsto f_{p-q} \cdot \{p/\|p\| - q/\|q\|\} \tag{6.10.3}$$

on $\mathbb{Z}^d \times \mathbb{Z}^d$. Now $p/\|p\| - q/\|q\|$ is a self-adjoint operator on Δ whose square is

$$2\left(1 - \frac{\langle p, q\rangle}{\|p\|\,\|q\|}\right) = 4\sin^2\frac{\phi}{2},$$

where ϕ is the angle between p and q. If $p - q$ is held fixed then $4\sin^2(\phi/2)$ decays like $\|p\|^{-2}$ as $p \to \infty$. The kernel (6.10.3) is therefore square-summable only if $\dim(X) = 1$. In general it belongs to the Schatten class $\mathcal{I}_r$ when $r > \dim(X)$.

7

THE GRASSMANNIAN OF HILBERT SPACE AND THE DETERMINANT LINE BUNDLE

Because we are studying loop groups by regarding them as groups of operators in Hilbert space we shall need to have a rather detailed knowledge of the structure of the Grassmannian of Hilbert space. This chapter is devoted to that subject. The most important part is the construction of the determinant line bundle in Section 7.7, and the reader interested in that can omit everything between Sections 7.1 and 7.7 except for the definition of an 'admissible basis' in Section 7.5.

7.1 The definition of Gr(H)

Suppose that H is a separable Hilbert space with a given polarization $H = H_+ \oplus H_-$: we assume that H_+ and H_- are infinite dimensional orthogonal closed subspaces. We shall study the Grassmannian of closed subspaces of H which are 'comparable' in size with H_+. Before giving the formal definition of this class of subspaces, let us explain that they are a completion of the class of subspaces W which are *commensurable* with H_+, i.e. those such that $W \cap H_+$ has finite codimension in both W and H_+. They may, however, have zero intersection with H_+: for example the graph W_T of every Hilbert–Schmidt operator $T: H_+ \to H_-$ is included, but W_T is commensurable with H_+ only if T is of finite rank.

Definition (7.1.1). Gr(H) *is the set of all closed subspaces W of H such that*

(i) *the orthogonal projection* $\mathrm{pr}_+ : W \to H_+$ *is a Fredholm operator, and*

(ii) *the orthogonal projection* $\mathrm{pr}_- : W \to H_-$ *is a Hilbert–Schmidt operator.*

Fredholm and Hilbert–Schmidt operators have been discussed already in Section 6.2. We recall that a bounded operator is Fredholm if its kernel and cokernel are finite dimensional.

Another way of stating the definition (7.1.1) is: W belongs to Gr(H) if it is the image of an operator $w: H_+ \to H$ such that $\mathrm{pr}_+ \circ w$ is Fredholm and $\mathrm{pr}_- \circ w$ is Hilbert–Schmidt. As the sum of a Fredholm operator and a Hilbert–Schmidt operator is Fredholm, we see that if W belongs to Gr(H) then so does the graph of every Hilbert-Schmidt operator $W \to W^\perp$. These graphs form the subset U_W of Gr(H) consisting of all W' for which the orthogonal projection $W' \to W$ is an isomorphism: it is in one-to-one correspondence with the Hilbert space $\mathcal{I}_2(W; W^\perp)$ of Hilbert–Schmidt operators $W \to W^\perp$. In fact

Proposition (7.1.2). *$\mathrm{Gr}(H)$ is a Hilbert manifold modelled on $\mathcal{I}_2(H_+; H_-)$.*

Before proving this we need one further observation. The group $GL_{\mathrm{res}}(H)$ introduced in Section 6.2 acts on the set $\mathrm{Gr}(H)$. We have

Proposition (7.1.3). *The subgroup $U_{\mathrm{res}}(H)$ of $GL_{\mathrm{res}}(H)$ acts transitively on $\mathrm{Gr}(H)$, and the stabilizer of H_+ is $U(H_+) \times U(H_-)$.*

Proof of (7.1.3). Suppose $W \in \mathrm{Gr}(H)$; we shall find $A \in U_{\mathrm{res}}(H)$ such that $A(H_+) = W$. Let $w: H_+ \to H$ be an isometry with image W, and $w^\perp: H_- \to H$ an isometry with image $W^\perp$. Then

$$w \oplus w^\perp : H_+ \oplus H_- \to H_+ \oplus H_-$$

is a unitary transformation A such that $A(H_+) = W$. We write it

$$A = \begin{pmatrix} w_+ & w_+^\perp \\ w_- & w_-^\perp \end{pmatrix}.$$

Because W belongs to $\mathrm{Gr}(H)$ we know that w_+ is Fredholm and w_- is Hilbert–Schmidt. But because A is unitary it follows that $w_+^\perp$ is Hilbert–Schmidt also (for $w_+^* w_+^\perp + w_-^* w_-^\perp = 0$), and so A belongs to $U_{\mathrm{res}}(H)$.

The assertion about the stabilizer of H_+ is obvious.

Proof of (7.1.2). Suppose that U_{W_0} and U_{W_1} are the subsets of $\mathrm{Gr}(H)$ described above corresponding to the Hilbert spaces $I_0 = \mathcal{I}_2(W_0; W_0^\perp)$ and $I_1 = \mathcal{I}_2(W_1; W_1^\perp)$. Let $U_{W_0} \cap U_{W_1}$ correspond to I_{01} in I_0 and I_{10} in I_1. We must show that I_{01} and I_{10} are open sets, and that the 'change of coordinates' $I_{01} \to I_{10}$ is smooth.

Let the matrix of the identity transformation

$$W_0 \oplus W_0^\perp \to W_1 \oplus W_1^\perp$$

be

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{7.1.4}$$

(i.e. a is a map $W_0 \to W_1$, etc.) From the proof of (7.1.3) we know that a and d are Fredholm, and b and c are Hilbert–Schmidt. Suppose that $W \in \mathrm{Gr}(H)$ is simultaneously the graph of $T_0: W_0 \to W_0^\perp$ and $T_1: W_1 \to W_1^\perp$. Then the operators

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ T_0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ T_1 \end{pmatrix} q$$

from W_0 to $W_1 \oplus W_1^\perp$ must coincide, where q is some isomorphism $W_0 \to W_1$. We conclude that

$$T_1 = (c + dT_0)(a + bT_0)^{-1}. \tag{7.1.5}$$

Thus T_1 is a holomorphic function of T_0 in the open set

$$I_{01} = \{T_0 \in I_0 : a + bT_0 \text{ is invertible}\}.$$

For a subspace W of H which is commensurable with H_+ it is natural to define the *virtual dimension* of W relative to H_+ as

$$\dim(W/W \cap H_+) - \dim(H_+/W \cap H_+).$$

The generalization of this for an arbitrary $W \in \mathrm{Gr}(H)$ is the *index* of the perpendicular projection $\mathrm{pr}_+ : W \to H_+$, i.e.

$$\mathrm{virt.dim}\, W = \dim(\ker \mathrm{pr}_+) - \dim(\mathrm{coker}\, \mathrm{pr}_+).$$

Equivalently,

$$\mathrm{virt.dim}\, W = \dim(W \cap H_-) - \dim(W^\perp \cap H_+).$$

The virtual dimension separates $\mathrm{Gr}(H)$ into disconnected pieces. In fact the subspaces with a given virtual dimension form a connected set; we shall see presently, for example, that the spaces of virtual dimension zero are the closure of the coordinate patch consisting of the graphs of all Hilbert-Schmidt operators $H_+ \to H_-$. Notice also that if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

belongs to $GL_{\mathrm{res}}(H)$, then

$$\mathrm{virt.dim}\, A(W) = \mathrm{virt.dim}\, W + \chi(a),$$

where $\chi(a)$ is the index of the Fredholm operator a.

To proceed further we shall introduce an orthonormal basis in H. That amounts to identifying H with the space $L^2(S^1; \mathbb{C})$ with its natural basis $\{z^k\}_{k \in \mathbb{Z}}$. (As usual $z = e^{i\theta}$.) We then have a collection of special points $\{H_S\}$ in $\mathrm{Gr}(H)$: H_S is just the closed subspace spanned by z^s for $s \in S$, where S is a subset of $\mathbb{Z}$ which has finite difference from the positive integers $\mathbb{N}$ (i.e. S is bounded below, and contains all sufficiently large integers). We shall write $\mathcal{S}$ for the collection of such sets S. Notice that

$$\mathrm{virt.dim}\, H = \mathrm{card}(S - \mathbb{N}) - \mathrm{card}(\mathbb{N} - S).$$

We shall call this number the *virtual cardinal* of S.

Proposition (7.1.6). *For any $W \in \mathrm{Gr}(H)$ there is a set $S \in \mathcal{S}$ such that the orthogonal projection $W \to H_S$ is an isomorphism. In other words the sets $\{U_S\}_{S \in \mathcal{S}}$, where $U_S = U_{H_S}$, form an open covering of $\mathrm{Gr}(H)$.*

Proof. Because the projection $W \to H_+$ has finite dimensional kernel one can find $S_0 \in \mathcal{S}$ such that the projection $W \to H_{S_0}$ is injective. If it is not also surjective there is some $s \in S_0$ such that z^s is not in its range. Then the projection $W \to H_{S_1}$, where $S_1 = S_0 - \{s\}$, is still injective. Repeating this finitely many times gives us the desired S.

We now have quite explicit coordinate charts on Gr(H), indexed by $\mathcal{S}$. A point of U_S is the graph of a Hilbert–Schmidt operator $H_S \to H_S^\perp$, and is represented by an $\bar{S} \times S$ matrix, where $\bar{S} = \mathbb{Z} - S$. The transitions between the charts are given by (7.1.5) where the matrix (7.1.4) is a permutation matrix; in particular, the components b and c have only finitely many non-zero entries.

7.2 Some dense submanifolds of Gr(H)

We shall describe in terms of the coordinate charts just introduced four important dense submanifolds of Gr(H). The reason for being interested in them will appear later.

(i) $\mathrm{Gr}_0(H)$ consists of all subspaces W such that $z^k H_+ \subset W \subset z^{-k} H_+$ for some k. Such subspaces can be identified with subspaces of $H_{-k,k} = z^{-k}H_+ / z^k H_+$, and so $\mathrm{Gr}_0(H)$ is the union of the finite dimensional classical Grassmannians $\mathrm{Gr}(H_{-k,k})$. In terms of the coordinate charts, $\mathrm{Gr}_0(H)$ consists of the graphs of operators $H_S \to H_S^\perp$ with only finitely many non-zero matrix entries: these are dense in $\mathcal{I}_2(H_S; H_S^\perp)$.

(ii) $\mathrm{Gr}_1(H)$ consists of all subspaces W which are commensurable with H_+. These are the graphs of all operators $H_S \to H_S^\perp$ of *finite rank*.

(iii) $\mathrm{Gr}_\omega(H)$ consists of the graphs of all operators $T: H_S \to H_S^\perp$ whose matrix entries T_{pq} (for $p \in \bar{S}$, $q \in S$) are such that $r^{p-q} T_{pq}$ is bounded for some r with $0 < r < 1$.

(iv) $\mathrm{Gr}_\infty(H)$ consists of graphs of all operators $T: H_S \to H_S^\perp$ whose entries T_{pq} are *rapidly decreasing*, i.e. such that $|p-q|^m T_{pq}$ is bounded for each m.

Without entering fully into the motivation for introducing these subspaces, let us notice that if W belongs to $\mathrm{Gr}_\infty(H)$ then it has a dense subspace consisting of smooth functions: that is so because the finite linear combinations of the smooth functions

$$w_q = z^q + \sum_p T_{pq} z^p$$

are dense in W. Similarly, if W belongs to $\mathrm{Gr}_\omega(H)$ then real-analytic functions are dense in W, and if W belongs to $\mathrm{Gr}_0(H)$ then trigonometric

polynomials are dense in it. These conditions do not, however, characterize Gr_0, Gr_ω and Gr_∞. Thus the graph W_T of $T: H_+ \to H_-$, where

$$Tz^k = \frac{1}{k} z^{-k},$$

does not belong to any of them, though the trigonometric polynomials are obviously dense in it. (At the other extreme, it is not hard to show that a generic $W \in \mathrm{Gr}(H)$ contains no non-zero smooth function at all.)

The subspace $\mathrm{Gr}_\infty(H)$ can be described in the following way.

Proposition (7.2.1). *$\mathrm{Gr}_\infty(H)$ consists precisely of the subspaces $W \in \mathrm{Gr}(H)$ for which the images of both orthogonal projections*

$$\mathrm{pr}_-: W \to H_- \quad \text{and} \quad \mathrm{pr}_+: W^\perp \to H_+$$

consist of smooth functions.

Proof. That the images do consist of smooth functions if W belongs to Gr_∞ is immediate from the definition. (Notice that if W is the graph of $T: H_S \to H_S^\perp$ then $W^\perp$ is the graph of $-T^*: H_S^\perp \to H_S$.) Conversely, if W is the graph of $T: H_S \to H_S^\perp$ and the image of pr_- consists of smooth functions then so does the image of T. Thus T defines a map from H_S to the space of smooth functions on the circle. By the closed graph theorem this must be continuous, and it can be thought of as a smooth map from the circle into the dual of H_S. Its smoothness is equivalent to the condition that

$$|p|^m \Big\{\sum_q |T_{pq}|^2\Big\}^{\frac{1}{2}}$$

is bounded as $p \to \infty$ for each m. Similarly the smoothness of the image of $\mathrm{pr}_+: W^\perp \to H_+$ is equivalent to the boundedness of

$$|q|^m \Big\{\sum_p |T_{pq}|^2\Big\}^{\frac{1}{2}}$$

as $q \to \infty$ for each m; and the two conditions together imply that W belongs to Gr_∞.

An exactly analogous description can be given of $\mathrm{Gr}_0(H)$ and $\mathrm{Gr}_\omega(H)$: for the former the proof is trivial.

The four subspaces can be considered as manifolds in their own right. The most important for us will be $\mathrm{Gr}_\infty(H)$, which we shall refer to as the *smooth* Grassmannian. The description above shows that it is a manifold modelled on the metrizable nuclear space (cf. [60]) of matrices $T = \{T_{pq} : p < 0,\ q \geq 0\}$ whose topology is defined by the sequence of seminorms ρ_m, where

$$\rho_m(T) = \sup_{p,q} |p-q|^m \, |T_{pq}|.$$

One can also describe $\mathrm{Gr}_\infty(H)$ directly in terms of the space C^∞ of smooth functions on the circle (with its usual topology): it consists of all closed subspaces W of C^∞ such that the projection $W \to C^\infty_+$ is Fredholm and the projection $W \to C^\infty_-$ is compact. (Here $C^\infty_\pm = C^\infty \cap H_\pm$.) We shall, however, omit the justification of this.

We conclude this section with a very simple application of the existence of the dense submanifold $\mathrm{Gr}_0 = \mathrm{Gr}_0(H)$.

Proposition (7.2.2). *Every holomorphic function $f: \mathrm{Gr}(H) \to \mathbb{C}$ is constant on each connected component.*

Proof. It is enough to show that f is locally constant on Gr_0. But Gr_0 is the union of the finite dimensional Grassmannians $\mathrm{Gr}(H_{-n,n})$. As these are compact algebraic varieties every holomorphic function on them is locally constant.

7.3 The stratification of Gr(H)

A generic element W of $\mathrm{Gr}(H)$, if it has virtual dimension zero, is transversal to H_-, i.e. $W \cap H_- = 0$ and $W + H_- = H$. These generic elements form a dense open subset in their connected component. The other elements W in the same component meet H_- non trivially: it follows from the discussion below that those such that $\dim(W \cap H_-) \geq k$ form a closed subset of codimension k^2. The most obvious stratification of $\mathrm{Gr}(H)$ would be by the dimension of the intersection $W \cap H_-$, which is necessarily finite. We shall need, however, a finer stratification, which records the dimension of $W \cap z^m H_-$ for every m.

Let us say that an element f of $H = L^2(S^1; \mathbb{C})$ is of *finite order s* if it is of the form

$$\sum_{k=-\infty}^{s} f_k z^k \tag{7.3.1}$$

with $f_s \neq 0$. In other words, f is the boundary value of a function f which is holomorphic in the hemisphere $|z| > 1$ except for a pole of order s at $z = \infty$. For any $W \in \mathrm{Gr}(H)$, let W^{fin} denote the set of elements of finite order in W. Because elements of finite order are dense in any H_S, and because the projection $W \to H_S$ is an isomorphism for suitable S, we have

Proposition (7.3.2). *W^{fin} is dense in W.*

The elements of W of order $\leq m$ form the finite dimensional space

$W_m = W \cap z^{m+1} H_-$. For given W we define

$$S_W = \{s \in \mathbb{Z} : W \text{ contains an element of order } s\}.$$

The set S_W belongs to $\mathscr{S}$, and its virtual cardinal is the virtual dimension d of W, for the number of elements of S_W which are $\leq m$ is $\dim W_m$, which is $m + 1 + d$ providing m is large enough for the projection $W \to z^{m+1} H_+$ to be surjective.

For each $s \in S_W$ let w_s be an element of W of the form (7.3.1) with $f_s = 1$. Evidently $\{w_s\}$ is a basis of W^{fin} in the algebraic sense, and the projection $W \to H_{S_W}$ is an isomorphism. We can choose w_s uniquely so that it projects to z^s; we shall call this the *canonical basis* of W. (This choice is precisely the process of choosing a basis for a subspace in 'reduced echelon form', familiar in elementary linear algebra.)

For given $S \in \mathscr{S}$ we shall call the set

$$\Sigma_S = \{W \in \mathrm{Gr}(H) : S_W = S\}$$

the *stratum* of $\mathrm{Gr}(H)$ corresponding to S. In other words, Σ_S consists of all W such that $\dim(W_m) = d_m(S)$ for all m, where $d_m(S)$ is the number of elements of S which are $\leq m$.

An indexing set S of virtual cardinal d can be written canonically

$$S = \{s_{-d}, s_{-d+1}, s_{-d+2}, \ldots\},$$

with $s_{-d} < s_{-d+1} < s_{-d+2} < \ldots$ and $s_k = k$ for large k. We shall order the sets of the same virtual cardinal by defining

$$\begin{aligned} S \leq S' \quad &\Leftrightarrow \quad s_k \geq s'_k \text{ for all } k \\ &\Leftrightarrow \quad d_m(S) \leq d_m(S') \text{ for all } m. \end{aligned}$$

We shall also define the *length* $\ell(S)$ of S by

$$\ell(S) = \sum_{k \geq 0} (k - s_k).$$

Then $S < S'$ implies $\ell(S) < \ell(S')$.

Finally, it will be convenient to introduce the 'strictly lower triangular' subgroup $\mathcal{N}_-$ of GL_{res}, consisting of all elements A such that $A(z^k H_-) = z^k H_-$ and $(A - 1)(z^k H_-) \subset z^{k-1} H_-$ for all k.

The stratification is described by

Proposition (7.3.3).

(i) *The stratum Σ_S is a contractible closed submanifold of the open set U_S, of codimension $\ell(S)$.*

(ii) *Σ_S is the orbit of H_S under $\mathcal{N}_-$.*

(iii) *If $W \in U_S$ then $S \geq S_W$.*

(iv) *The closure of Σ_S is the union of the strata $\Sigma_{S'}$ with $S' \geq S$.*

Proof.

(i) We have already shown that Σ_S is contained in U_S. Now if $W \in U_S$ then $W \to H_S$ is an isomorphism, and so W has a unique basis $\{w_s\}$ which projects to $\{z^s\}$. Because of the uniqueness, W belongs to Σ_S if and only if w_s has order s for each s. If W is described as the graph of $T: H_S \to H_S^\perp$, so that $w_s = z^s + Tz^s$, then W belongs to Σ_S precisely when the matrix elements T_{pq} vanish when $p > q$. The number of pairs (p, q) in $\tilde{S} \times S$ such that $p > q$ is the length $\ell(S)$. Thus Σ_S corresponds to a sub-Hilbert-space of codimension $\ell(S)$.

(ii) Suppose that $W \in \Sigma_S$ is the graph of $T: H_S \to H_S^\perp$. Let $\mathrm{pr}_S: H \to H_S$ be the projection. Then $A = 1 + T \circ \mathrm{pr}_S$ belongs to $\mathcal{N}_-$, and $A(H_S) = W$.

(iii) Perpendicular projection on to H_S can only lower the order of an element, and so if $W \to H_S$ is an isomorphism then H_S must have at least as many linearly independent elements of order $\leq m$ as W does, i.e.

$$\mathrm{card}\{s \in S : s \leq m\} \geq \mathrm{card}\{s \in S_W : s \leq m\}$$

for each m. This is equivalent to the assertion $S \geq S_W$.

(iv) It follows from (iii) that the closure of Σ_S is contained in the union of the $\Sigma_{S'}$ with $S' \geq S$. But if $S' > S$ let W_t be the subspace spanned by

$$(1-t)z^{s_k} + tz^{s'_k}$$

for $k \geq -d$. If $0 \leq t < 1$ then W_t belongs to Σ_S; if $t = 1$ then $W_t = H_{S'} \in \Sigma_{S'}$. This proves that the closure of Σ_S meets $\Sigma_{S'}$. The closure must then contain $\Sigma_{S'}$ because $\Sigma_{S'}$ is a single orbit of the group $\mathcal{N}_-$.

7.4 The cellular decomposition of $\mathrm{Gr}_0(H)$

The Grassmannian of a finite dimensional vector space has a classical decomposition into Schubert cells. (Cf. [68] or [116].) Our Grassmannian $\mathrm{Gr}_0(H)$ is the union of the finite dimensional Grassmannians $\mathrm{Gr}(H_{-n,n})$, and it too can be decomposed into Schubert cells. This decomposition is dual to the stratification of $\mathrm{Gr}(H)$ described in the last section in the following sense:

(i) the same set $\mathcal{S}$ indexes the cells $\{C_S\}$ and the strata $\{\Sigma_S\}$;
(ii) the dimension of C_S is the codimension of Σ_S;
(iii) C_S meets Σ_S transversally in a single point, and meets no other stratum of the same codimension.

To describe C_S we begin by defining the *co-order* of a polynomial element

$$f = \sum_{k=-N}^{N} f_k z^k$$

of H as the smallest k such that $f_k \neq 0$. Then for $W \in \mathrm{Gr}_0(H)$ the set

$$S^W = \{s \in \mathbb{Z} : W \text{ contains an element of co-order } s\}$$

belongs to $\mathcal{S}$, and for $S \in \mathcal{S}$ we define

$$C_S = \{W \in \mathrm{Gr}_0(H) : S^W = S\}.$$

Proposition (7.4.1).

(i) *C_S is a closed submanifold of the open set U_S of* $\mathrm{Gr}(H)$ *and is diffeomorphic to* $\mathbb{C}^{\ell(S)}$.

(ii) *C_S is the orbit of H_S under the 'strictly upper triangular' subgroup $\mathcal{N}_+$ of GL_{res}.*

(iii) *If $W \in \mathrm{Gr}_0(H)$ belongs to U_S then $S \leq S^W$.*

(iv) *The closure of C_S is the union of the $C_{S'}$ with $S' \leq S$.*

(v) *C_S intersects $\Sigma_{S'}$ if and only if $S \geq S'$, and C_S intersects Σ_S transversally in the single point H_S.*

The strictly upper triangular subgroup $\mathcal{N}_+$ consists of all A such that $A(z^k H_+) = z^k H_+$ and $(A-1)(z^k H_+) \subset z^{k+1} H_+$ for all k.

Proof. This is precisely analogous to (7.3.3). The essential observation is that C_S consists of the graphs of all operators $T: H_S \to H_S^\perp$ whose matrix elements T_{pq} vanish unless $p > q$.

7.5 The Plücker embedding

Points of a finite dimensional Grassmannian are traditionally described by Plücker coordinates. We can do exactly the same with $\mathrm{Gr}(H)$.

We have pointed out in Section 7.3 that any $W \in \mathrm{Gr}(H)$ has a canonical basis. We shall find it useful, however, to introduce a class of 'admissible bases' for W. Suppose that W has virtual dimension d.

Definition (7.5.1). *A sequence $\{w_k\}_{k \geq -d}$ in W is called an* admissible *basis for W if*

(i) *the linear map $w: z^{-d}H_+ \to W$ which takes z^k to w_k is a continuous isomorphism, and*

(ii) *the composite $\mathrm{pr} \circ w$, where $\mathrm{pr}: W \to z^{-d}H_+$ is the orthogonal projection, is an operator with a determinant.*

Remarks.

(i) We recall that an operator with a determinant is one which differs from the identity by an operator of trace class. (Cf. Section 6.6.)

(ii) We shall usually not distinguish between the basis $\{w_k\}$ and the corresponding linear map w.

(iii) The canonical basis for W is admissible: for it, the composite $\mathrm{pr} \circ w$ differs from the identity by an operator of finite rank.

It is clear from the definitions that any two admissible bases for the same space W are related to each other by a matrix which has a determinant. Furthermore, if w is an admissible basis for W, and $S \in \mathcal{S}$ is a set of virtual cardinal d, and $\mathrm{pr}_S : W \to H_S$ is the projection, then $\mathrm{pr}_S \circ w$ is also an operator with a determinant. We define the *Plücker coordinate* $\pi_S(w)$ of the basis w as the determinant $\det(\mathrm{pr}_S \circ w)$. If $S \in \mathcal{S}$ does not have virtual cardinal d we define $\pi_S(w) = 0$. If w' is another admissible basis for w then

$$\pi_S(w') = \Delta_{ww'} \pi_S(w),$$

where $\Delta_{ww'}$ is the determinant of the matrix relating w' and w; so if one thinks of $\{\pi_S\}_{S \in \mathcal{S}}$ as *projective* coordinates then they depend only on W.

Proposition (7.5.2). *The Plücker coordinates $\{\pi_S\}_{S \in \mathcal{S}}$ define a holomorphic embedding*

$$\pi : \mathrm{Gr}(H) \to P(\mathcal{H})$$

into the projective space of the Hilbert space $\mathcal{H} \doteq \ell^2(\mathcal{S})$.

Remark. In Section 7.7 we shall give a more invariant description of $\mathcal{H}$.

Proof. We must first show that for an admissible basis w we have

$$\sum_{S \in \mathcal{S}} |\pi_S(w)|^2 < \infty.$$

In fact we shall prove

$$\sum_S |\pi_S(w)|^2 = \det(w^* w). \tag{7.5.3}$$

(The right-hand-side is defined, for if we write $w : z^{-d} H_+ \to H$ as $w_+ \oplus w_-$ with respect to $H = z^{-d} H_+ \oplus z^{-d} H_-$ then $w^* w = w_+^* w_+ + w_-^* w_-$, which has a determinant because w_+ has a determinant and w_- is Hilbert–Schmidt.)

It is enough to prove (7.5.3) for any one admissible basis w for each subspace W. And by continuity it is enough to prove it when W belongs to $\mathrm{Gr}_0(H)$. So we may assume that w_+ differs from the identity matrix in only finitely many entries, and that w_- has only finitely many non-zero entries. In that case (7.5.3) reduces to the following assertion:

If P and Q are $n \times m$ and $m \times n$ matrices, with $n \leq m$, then

$$\det(PQ) = \sum_S \det(P_S)\det(Q_S),$$

where S runs through the n element subsets of $\{1, 2, \ldots, m\}$, and P_S, Q_S are the corresponding $n \times n$ submatrices of P and Q. (This assertion simply expresses the functoriality of the n^{th} exterior power: $\Lambda^n(P \circ Q) = (\Lambda^n P) \circ (\Lambda^n Q)$.)

To prove that π is an embedding, let us consider first the case of a subspace W which is the graph of an operator $T : H_+ \to H_-$. The canonical basis $\{w_k\}_{k \geq 0}$ for W is given by

$$w_q = z^q + \sum_{p<0} T_{pq} z^p.$$

Suppose that $S \in \mathcal{S}$ has virtual cardinal 0, and write $A = S - \mathbb{N}$, $B = \mathbb{N} - S$. These are two finite sets of the same size. A moment's reflection reveals that $\pi_S(w)$ is the determinant of the finite submatrix of (T_{pq}) formed from the rows A and columns B. In particular, each entry T_{pq} occurs among the Plücker coordinates. So π is certainly an embedding in the coordinate patch $U_{\mathbb{N}}$.

The other coordinate patches can be treated in exactly the same way. We remark finally that W belongs to the patch U_S if and only if $\pi_S(w) \neq 0$.

The stratification of $\mathrm{Gr}(H)$, and also the three dense subspaces Gr_0, Gr_ω and Gr_∞, can be described very simply in terms of Plücker coordinates, as follows.

Proposition (7.5.4).

(i) $W \in U_S \Leftrightarrow \pi_S(W) \neq 0$,
(ii) $W \in \Sigma_S \Leftrightarrow \pi_S(W) \neq 0$ *and* $\pi_{S'}(W) = 0$ *when* $S' < S$,
(iii) $W \in C_S \Leftrightarrow \pi_S(W) \neq 0$ *and* $\pi_{S'}(W) = 0$ *unless* $S' \leq S$,
(iv) $W \in \mathrm{Gr}_0 \Leftrightarrow \pi_S(W) = 0$ *except for finitely many* S,
(v) $W \in \mathrm{Gr}_\omega \Leftrightarrow r^{-\ell(S)}\pi_S(W)$ *is bounded for* $S \in \mathcal{S}$, *for some* $r < 1$,
(vi) $W \in \mathrm{Gr}_\infty \Leftrightarrow \ell(S)^m \pi_S(W)$ *is bounded for* $S \in \mathcal{S}$, *for each* m.

All of these assertions are obvious except perhaps for the last two, whose validity will become clear in the next section.

7.6 The $\mathbb{C}^\times_{\leq 1}$-action

The circle $\mathbb{T}$ acts unitarily on $H = L^2(S^1; \mathbb{C})$ by rotating S^1, and the action preserves the polarization $H = H_+ \oplus H_-$. This means that $\mathbb{T}$ acts on $\mathrm{Gr}(H)$. It is easy to see that the fixed points are precisely the subspaces H_S for $S \in \mathcal{S}$. We shall write $R_u : \mathrm{Gr}(H) \to \mathrm{Gr}(H)$ for the action of $u \in \mathbb{T}$.

The map $\mathbb{T} \times \mathrm{Gr}(H) \to \mathrm{Gr}(H)$ describing the action is continuous, but not differentiable. In the coordinate chart $U_S \cong \mathcal{I}_2(H_S; H_S^\perp)$ the action of R_u on $T : H_S \to H_S^\perp$ multiplies the matrix element T_{pq} by u^{q-p}. From this we find

Proposition (7.6.1). *The $\mathbb{T}$-orbit of a point $W \in \mathrm{Gr}(H)$ is smooth (i.e. the map $u \mapsto R_u W$ is smooth) if and only if W belongs to $\mathrm{Gr}_\infty(H)$. The orbit is* *real-analytic if and only if W belongs to $\mathrm{Gr}_\omega(H)$. Furthermore, $\mathbb{T}$ acts smoothly on the manifold $\mathrm{Gr}_\infty(H)$ with its own C^∞ topology.*

The description in terms of coordinate charts shows that the action of $\mathbb{T}$ extends to an action

$$\mathbb{C}^\times_{\leq 1} \times \mathrm{Gr}(H) \to \mathrm{Gr}(H) \tag{7.6.2}$$

of the semigroup $\mathbb{C}^\times_{\leq 1}$ of non-zero complex numbers of modulus ≤ 1. The map (7.6.2) is holomorphic on the open set $\mathbb{C}^\times_{<1} \times \mathrm{Gr}(H)$. If $|u| < 1$ then R_u maps $\mathrm{Gr}(H)$ into $\mathrm{Gr}_\omega(H)$. On the submanifold $\mathrm{Gr}_0(H)$ the action of $\mathbb{T}$ extends to a holomorphic action of the whole group $\mathbb{C}^\times$.

The action of the semigroup $\mathbb{C}^\times_{\leq 1}$ is very closely connected with the stratification of $\mathrm{Gr}(H)$.

Proposition (7.6.3).

(i) *Σ_S consists precisely of the points $W \in \mathrm{Gr}(H)$ such that $R_u W$ tends to H_S as $u \to 0$.*

(ii) *C_S consists precisely of the points $W \in \mathrm{Gr}_0(H)$ such that $R_u W$ tends to H_S as $u \to \infty$.*

If we restrict u to real values then the situation described in Proposition (7.6.3) is very reminiscent of Morse theory. If the trajectories $u \mapsto R_u W$ were the gradient flow of a function F on $\mathrm{Gr}(H)$ then the H_S would be the critical points of F, and Σ_S and C_S would be the *stable* and *unstable* manifolds of H_S in the sense of Morse theory [142]. This picture is essentially valid; the only qualification is that the function F is defined only on the smooth Grassmannian Gr_∞, where the trajectories are smooth. We shall find the function F in Section 7.8.

Proposition (7.6.3) follows at once from the behaviour of the Plücker coordinates with respect to the $\mathbb{T}$-action.

Proposition (7.6.4). *We have*

$$\pi_S(R_u W) = \lambda u^{\ell(S)} \pi_S(W),$$

where λ is non-zero and independent of S.

In other words the Plücker embedding

$$\pi : \mathrm{Gr}(H) \to P(\mathcal{H})$$

is equivariant with respect to $\mathbb{C}^\times_{\leq 1}$ when R_u acts on $\mathcal{H} = \ell^2(\mathcal{S})$ by

$$(R_u \xi)_S = u^{\ell(S)} \xi_S.$$

Remark. This proposition will be superseded in the next section by the more precise result (7.7.5).

Proof. If $w : z^{-d} H_+ \to W$ is an admissible basis for W then we can take $R_u \circ w \circ R_u^{-1}$ as a basis for $R_u W$. By continuity it is enough to prove the result for a dense set of ws. So for the component of virtual dimension zero we can suppose

$$w_q = z^q + \sum_p T_{pq} z^p.$$

If $S - \mathbb{N} = A = \{a_1, \ldots, a_k\}$, and $\mathbb{N} - S = B = \{b_1, \ldots, b_k\}$, then $\ell(S) = \Sigma (b_i - a_i)$. The Plücker coordinate $\pi_S(w)$ is the determinant of the submatrix of T formed from the rows A and columns B. Conjugation by R_u multiplies this determinant by $z^{-\Sigma a_i + \Sigma b_i} = z^{\ell(S)}$. The other connected components can be treated similarly.

Remark. It is worth pointing out that if $\mathcal{H}_\infty$ and $\mathcal{H}_\omega$ are the smooth and real-analytic vectors in $\mathcal{H}$ in the sense of representation theory [153], i.e. the vectors whose $\mathbb{T}$-orbits are smooth or real-analytic, then

$$\mathrm{Gr}_\infty(H) = \pi^{-1} P(\mathcal{H}_\infty), \text{ and}$$
$$\mathrm{Gr}_\omega(H) = \pi^{-1} P(\mathcal{H}_\omega).$$

7.7 The determinant bundle

In this section we shall construct a holomorphic line bundle Det on the Grassmannian $\mathrm{Gr}(H)$. Its fibre $\mathrm{Det}(W)$ at $W \in \mathrm{Gr}(H)$ is to be thought of as the 'top exterior power' of W. We can make sense of this by using the concept of an 'admissible basis', introduced in Section 7.5. An element of $\mathrm{Det}(W)$ is represented by definition by a formal expression

$$\lambda w_{-d} \wedge w_{-d+1} \wedge w_{-d+2} \wedge \cdots, \tag{7.7.1}$$

where $\lambda \in \mathbb{C}$ and $w = \{w_k\}$ is an admissible basis of W. We shall denote the expression (7.7.1) simply by $[\lambda, w]$. If w' is another admissible basis of W, then $[\lambda, w]$ is identified with $[\lambda \det(t), w']$, where $t = (t_{ij})$ is the matrix relating w and w':

$$w_i = \sum_j t_{ij} w'_j.$$

$\mathrm{Det}(W)$ is clearly a one-dimensional complex vector space, and the union of the $\mathrm{Det}(W)$ for $W \in \mathrm{Gr}(H)$ is the line bundle Det. We must, however, explain how Det is a complex manifold, and why the bundle is locally trivial.

For each indexing set $S \in \mathcal{S}$ we have the open set U_S of $\mathrm{Gr}(H)$, identified with the graphs of Hilbert–Schmidt operators $T : H_S \to H_S^\perp$. The graph W_T of T has the admissible basis $\{w_i\}$, where

$$w_i = z^q + \sum_{p \notin S} T_{pq} z^p \tag{7.7.2}$$

with $q = s_i$ and $S = \{s_{-d}, s_{-d+1}, \ldots\}$. We identify the part of Det above

U_S with $\mathbb{C} \times U_S$ by

$$(\lambda, W_T) \in \mathbb{C} \times U_S \leftrightarrow [\lambda, w] \in \text{Det},$$

where w is given by (7.7.2). The transitions between these local trivializations are as follows. Suppose that W_T belongs to $U_S \cap U_{S'}$, and $W_T = W_{T'}$, where $T' : H_{S'} \to H_{S'}^{\perp}$. We know from (7.1.5) that

$$T' = (c + dT)(a + bT)^{-1},$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is the matrix of the permutation relating S to S'. Then

$$(\lambda, W_T) \in \mathbb{C} \times U_S \leftrightarrow (\lambda', W_{T'}) \in \mathbb{C} \times U_{S'},$$

where

$$\lambda' = \lambda \det(a + bT).$$

This is a holomorphic function of (λ, T), as we require. (To be quite concrete, $\det(a + bT)$ is simply the finite dimensional determinant formed from the rows A and columns B of T, where $A = S' - S$ and $B = S - S'$.)

The Grassmannian $\text{Gr}(H)$ is a homogeneous space under the action of the restricted general linear group $GL_{\text{res}}(H)$. It would be natural to expect the action of GL_{res} to lift to an action on the line bundle Det. This, however, is not quite the case, for if w is an admissible basis for $W \in \text{Gr}(H)$, and $A \in GL_{\text{res}}$, then Aw is not in general an admissible basis of $A(W)$. The extension $GL_{\text{res}}^{\sim}$ of GL_{res} by $\mathbb{C}^{\times}$ described in Chapter 6 was constructed precisely to deal with this situation.

Theorem (7.7.3). *The action of* GL_{res} on $\text{Gr}(H)$ *is covered by an action of* $GL_{\text{res}}^{\sim}$ *on the line bundle* Det.

Proof. Let us first consider the connected component Gr^0, consisting of spaces W of virtual dimension 0. An admissible basis for such a W is an isomorphism $w : H_+ \to W$, which we can write as a $\mathbb{Z} \times \mathbb{N}$ matrix

$$w = \begin{pmatrix} w_+ \\ w_- \end{pmatrix}$$

such that $w_+ : H_+ \to H_+$ has a determinant. Recall that the subgroup $\mathscr{E}$ of $GL_{\text{res},0} \times GL(H_+)$ is defined as the set of pairs (A, q) such that aq^{-1} has a determinant, where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We define an action of $\mathscr{E}$ on the set of admissible bases by

$$(A, q) \,.\, w = Awq^{-1}.$$

This is well-defined because $(Awq^{-1})_+ = aw_+q^{-1} + bw_-q^{-1}$ has a determinant. Then $\mathscr{E}$ acts on Det by

$$(A, q) \,.\, [\lambda, w] = [\lambda, (A, q)w].$$

The subgroup $\mathscr{T}_1$ of $\mathscr{E}$, which consists of pairs $(1, q)$ with $\det(q) = 1$, acts trivially on Det, and so we have defined an action of $\mathscr{E}/\mathscr{T}_1$. This is the identity component $GL_{\text{res},0}^{\sim}$ of $GL_{\text{res}}^{\sim}$.

To make $GL_{\text{res},0}^{\sim}$ act on the part of Det over Gr^d, the set of subspaces W of virtual dimension d, recall that we defined an automorphism $\tilde{\sigma}$ of $GL_{\text{res},0}^{\sim}$ which covered the automorphism $A \mapsto \sigma A \sigma^{-1}$ of $GL_{\text{res},0}$. Here $\sigma : H \to H$ is the shift map, given by multiplication by z. We define the action of $\tilde{A} \in GL_{\text{res},0}^{\sim}$ on $\text{Det} \,|\, \text{Gr}^d$ as the action of $\sigma^{-d} \circ \tilde{\sigma}^d(\tilde{A}) \circ \sigma^d$, where

$$\sigma : \text{Det} \to \text{Det}$$

is defined by $\sigma \,.\, [\lambda, w] = [\lambda, \sigma w]$. As $GL_{\text{res}}^{\sim}$ is the semidirect product of $GL_{\text{res},0}^{\sim}$ and the cyclic subgroup generated by σ, we now have an action of $GL_{\text{res}}^{\sim}$ on Det.

Remarks.

(i) The group extension $GL_{\text{res},0}^{\sim}$ can be constructed directly from the line bundle Det. For $GL_{\text{res},0}^{\sim}$ is the group of all holomorphic automorphisms of $\text{Det} \,|\, \text{Gr}^0$ which cover the actions of elements of $GL_{\text{res},0}$ on Gr^0. (If $\tilde{A}_0$ and $\tilde{A}_1$ are automorphisms of $\text{Det} \,|\, \text{Gr}^0$ which cover the same map on Gr^0, then $\tilde{A}_0^{-1}\tilde{A}_1$ must be the operation of multiplication by a non-vanishing holomorphic function on Gr^0. But any such function is constant (see Proposition (7.2.2)).

(ii) The line bundle Det has a natural hermitian metric for which

$$\|[\lambda, w]\|^2 = |\lambda|^2 \det(w^* w).$$

This is preserved by the action of $U_{\text{res}}^{\sim}$. The unit circle bundle in Det can therefore be identified with $U_{\text{res}}^{\sim}/U(H_+) \times U(H_-)$, and its Chern class is represented by the invariant form defined by (6.6.5).

Let us return to the Plücker embedding defined in Section 7.5. Each Plücker coordinate π_S can be regarded as a holomorphic section of the line bundle Det^* dual to Det. For a holomorphic section of Det^* is a holomorphic function $\text{Det} \to \mathbb{C}$ which is linear on each fibre. The coordinate π_S defines such a function by

$$[\lambda, w] \mapsto \lambda \pi_S(w).$$

The Hilbert space $\mathscr{H}$ of Proposition (7.5.2) is therefore contained in the dual of the space of all holomorphic sections of Det^*. We shall see in

Chapter 10 that it is a dense subspace of the dual. Meanwhile, let us notice simply that the embedding $\pi:\mathrm{Gr}(H)\to P(\mathscr{H})$ arises from a holomorphic map

$$\pi:\mathrm{Det}\to\mathscr{H} \tag{7.7.4}$$

which is linear on each fibre. The line bundle Det is thus the pull-back of the tautological line bundle on $P(\mathscr{H})$. (Cf. Section 2.9.)

The map $\pi:\mathrm{Det}\to\mathscr{H}$ is norm-preserving, as we see from the formula (7.5.3).

Proposition (7.7.5). *The map $\pi:\mathrm{Det}\to\mathscr{H}$ is equivariant with respect to $\mathbb{C}^\times_{\leq 1}$ when $R_u\in\mathbb{C}^\times_{\leq 1}$ acts on $\{\xi_S\}\in\mathscr{H}$ by*

$$(R_u\xi)_S=u^{\ell^*(S)}\xi_S.$$

Here $\ell^*(S)=\ell(S)+\frac{1}{2}d(d+1)$, where $d=\mathrm{card}(S)$.

Proof. We combine the proof of (7.6.4) with the fact that the action of R_u on $[\lambda, w]\in\mathrm{Det}$, where $w:z^{-d}H_+\to H$ is an admissible basis, is given by

$$R_u[\lambda, w]=[\lambda u^{\frac{1}{2}d(d+1)}, R_uwR_u^{-1}].$$

More general determinant bundles

The determinant bundle can actually be defined on a larger space than $\mathrm{Gr}(H)$. Let $\mathrm{Gr}_{\mathrm{cpt}}(H)$ denote the set of closed subspaces W of H such that the projection $W\to H_+$ is Fredholm and the projection $W\to H_-$ is *compact.* Then our construction applies without change to define a holomorphic line bundle Det on $\mathrm{Gr}_{\mathrm{cpt}}(H)$. The crucial difference, however, is that the bundle on $\mathrm{Gr}_{\mathrm{cpt}}(H)$ is not homogeneous: it is acted on only by the subgroup of $GL^{\sim}_{\mathrm{res}}(H)$ consisting of elements whose off-diagonal blocks are of trace class.

The line bundle Det on $\mathrm{Gr}_{\mathrm{cpt}}(H)$ is essentially the same thing as the determinant bundle defined by Quillen [124] on the space $\mathrm{Fred}(H_+)$ of Fredholm operators in H_+. The fibre of Quillen's bundle at $T:H_+\to H_+$ is

$$\det(\ker T)^*\otimes\det(\mathrm{coker}\ T).$$

The relation between the two bundles is the following. Let $\mathscr{B}$ denote the space of injective maps $w:H_+\to H$ such that $w(H_+)$ belongs to $\mathrm{Gr}_{\mathrm{cpt}}(H)$. Then we have holomorphic maps

$$\mathrm{Gr}_{\mathrm{cpt}}(H)\ \leftarrow\ \mathscr{B}\ \rightarrow\ \mathrm{Fred}(H_+).$$

Both of these maps have contractible fibres. The determinant bundles on $\mathrm{Gr}_{\mathrm{cpt}}(H)$ and $\mathrm{Fred}(H_+)$ pull back to the same bundle on $\mathscr{B}$, and the bundle on $\mathrm{Fred}(H_+)$ is the quotient of the one on $\mathscr{B}$ by the obvious free action of $GL(H_+)$.

7.8 Gr(*H*) as a Kähler manifold and a symplectic manifold

Because the group U_{res} acts transitively on $\mathrm{Gr}(H)$ we can define a hermitian metric on $\mathrm{Gr}(H)$ by giving a hermitian form on its tangent space at the base-point H_+ which is invariant under the action of the isotropy group $U(H_+)\times U(H_-)$. The tangent space at H_+ is the space $\mathscr{I}_2(H_+;H_-)$ of Hilbert–Schmidt operators $H_+\to H_-$ (on which the isotropy group acts by left- and right-composition), and the unique invariant inner product is

$$(X, Y)\mapsto 2\,\mathrm{trace}(X^*Y),$$

up to a scalar multiple. This inner product defines a *Kähler* structure on $\mathrm{Gr}(H)$. Indeed its imaginary part

$$\omega(X, Y)=-i\,\mathrm{trace}(X^*Y-Y^*X) \tag{7.8.1}$$

is the closed 2-form which we have already encountered in Proposition (6.6.5) as the form on the Lie algebra $\mathfrak{u}_{\mathrm{res}}$ which defines the central extension $\mathfrak{u}^{\sim}_{\mathrm{res}}$. (We recall that an invariant differential form on the homogeneous space $U_{\mathrm{res}}/(U_+\times U_-)$ is the same thing as a skew form ω on $\mathfrak{u}_{\mathrm{res}}$ which is invariant under the adjoint action of $U_+\times U_-$ and in addition satisfies $\omega(\xi,\eta)=0$ when ξ or η belongs to $\mathfrak{u}_+\oplus\mathfrak{u}_-$.) To see that the form (7.8.1) coincides with that of (6.6.5) we map $\mathscr{I}_2(H_+;H_-)$ into $\mathfrak{u}_{\mathrm{res}}$ by

$$X\mapsto\begin{pmatrix}0 & -X^*\\ X & 0\end{pmatrix}.$$

We saw at the end of the last section (see Remark (ii), p. 115) that the form ω represents the Chern class of the line bundle Det on $\mathrm{Gr}(H)$. An equivalent statement is that the Kähler structure of $\mathrm{Gr}(H)$ is induced from the standard structure on the projective space $P(\mathscr{H})$ by the Plücker embedding.

On a simply connected symplectic manifold X—even if it is infinite dimensional—any vector field ξ which preserves the 2-form ω arises from a so-called *Hamiltonian function* $F:X\to\mathbb{R}$, in the sense that the gradient $\mathrm{d}F$ is the 1-form $\omega(\xi,\ )$ on X. On $\mathrm{Gr}(H)$ the vector field defined by any element of the Lie algebra $\mathfrak{u}_{\mathrm{res}}$ preserves the form ω, and we can ask for the corresponding function.

Proposition (7.8.2). *The Hamiltonian function $F:\mathrm{Gr}(H)\to\mathbb{R}$ which defines the flow on $\mathrm{Gr}(H)$ corresponding to $\xi\in\mathfrak{u}_{\mathrm{res}}$ is given by*

$$F(W)=-i\ \mathrm{trace}\ \xi(J_W-J).$$

Here J and J_W are the operators of square 1 which define the decompositions $H=H_+\oplus H_-$ and $H=W\oplus W^\perp$. We leave it to the reader to check that the operator $\xi(J_W-J)$ is necessarily of trace class.

Proof. The gradient of F at W along the tangent vector corresponding to $\eta \in \mathfrak{u}_{\text{res}}$ is

$$dF(W; \eta) = -i \text{ trace } \xi[\eta, J_W].$$

Suppose that $W = gH_+$, with $g \in U_{\text{res}}$. Then $J_W = gJg^{-1}$, and the value of the invariant form ω at W on the tangent vectors defined by ξ, η is

$$\begin{aligned}\omega(W; \xi, \eta) &= \omega(g^{-1}\xi g, g^{-1}\eta g)\\ &= -i \text{ trace } g^{-1}\xi g[g^{-1}\eta g, J]\\ &= -i \text{ trace } \xi[\eta, J_W]\\ &= dF(W; \eta). \qquad (7.8.3)\end{aligned}$$

(Here (7.8.3) is obtained from (6.6.5) by noticing that trace $(c_1b_2 - b_1c_2) = \text{trace } A_1[A_2, J]$.)

We cannot apply Proposition (7.8.2) directly to the rotation action of $\mathbb{T}$ on $\text{Gr}(H)$, for we saw in Section 7.6 that the action was smooth only on the submanifold $\text{Gr}_\infty(H)$. This corresponds to the fact that the infinitesimal generator $-d/d\theta$ is an unbounded operator on H and does not belong to the Lie algebra $\mathfrak{u}_{\text{res}}$. Nevertheless (7.8.2) does hold for the rotation flow on $\text{Gr}_\infty(H)$. We shall call the corresponding Hamiltonian function the *energy* $\mathscr{E}: \text{Gr}_\infty(H) \to \mathbb{R}$. Thus

$$\mathscr{E}(W) = \text{trace}\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\theta}\right)(J_W - J). \qquad (7.8.4)$$

The critical points of $\mathscr{E}$ are the stationary points of the rotation action, i.e. the points H_S for $S \in \mathscr{S}$. Let us notice that

$$\mathscr{E}(H_S) = \ell^*(S) = \ell(S) + \tfrac{1}{2}d(d+1),$$

where $d = \text{card}(S)$ (Cf. (7.7.5).) More generally, we have

Proposition (7.8.5).

$$\begin{aligned}\mathscr{E}(W) &= \sum_S \ell^*(S)\, |\pi_S(W)|^2\\ &= \left\langle \Omega_W, \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\theta}\cdot \Omega_W\right\rangle,\end{aligned}$$

where $\{\pi_S(W)\}$ are the Plücker coordinates of W, normalized so that $\Sigma\, |\pi_S(W)|^2 = 1$, and Ω_W is the corresponding unit vector in $\mathscr{H}$.

Thus $\mathscr{E}$ takes only positive values.

In the language of quantum mechanics we can regard $\text{Gr}(H)$ as the space of states of a classical system, and $P(\mathscr{H})$ as the corresponding quantum state space. Then Ω_W represents the quantum state corresponding to W, and (7.8.5) asserts that the classical energy $\mathscr{E}(W)$ is the expected value of the quantum energy operator $\mathrm{i}(\mathrm{d}/\mathrm{d}\theta)$ in the state Ω_W. The result follows from the fact that $\text{Gr}(H)$ has the Kähler structure induced from $P(\mathscr{H})$. For in general if T is any skew-adjoint operator in $\mathscr{H}$ then the Hamiltonian function corresponding to the flow on $P(\mathscr{H})$ induced by T is

$$\xi \mapsto \langle \xi, \mathrm{i}T\xi \rangle.$$

It can be shown fairly easily that the Morse decomposition of $\text{Gr}(H)$ into the ascending and descending stable manifolds of the stationary points of the gradient flow of $\mathscr{E}$ is precisely the stratification and cell decomposition which we found in Sections 7.3 and 7.4. We shall not pursue this discussion any further—but see Section 8.9.

8

THE FUNDAMENTAL HOMOGENEOUS SPACE

8.1 Introduction: the factorization theorems

The most important results proved in this chapter are three factorization theorems. We shall state them here for the loop group of the general linear group $GL_n(\mathbb{C})$, but this can be replaced by $LG_{\mathbb{C}}$ for any compact G. The first involves the subgroup $L^+GL_n(\mathbb{C})$ of $LGL_n(\mathbb{C})$ consisting of loops γ which are the boundary values of holomorphic maps

$$\gamma:\{z \in \mathbb{C}:|z|<1\} \to GL_n(\mathbb{C}).$$

Theorem (8.1.1). *Any loop $\gamma \in LGL_n(\mathbb{C})$ can be factorized uniquely*

$$\gamma = \gamma_u \cdot \gamma_+,$$

with $\gamma_u \in \Omega U_n$ and $\gamma_+ \in L^+GL_n(\mathbb{C})$. In fact the product map

$$\Omega U_n \times L^+GL_n(\mathbb{C}) \to LGL_n(\mathbb{C})$$

is a diffeomorphism.

Here ΩU_n denotes the base-point-preserving loops in LU_n, i.e. those such that $\gamma(1)=1$. Proposition (8.1.1) will be proved in Section 8.3.

The second theorem, which is due to Birkhoff [11, 12], involves also the subgroup $L^-GL_n(\mathbb{C})$ consisting of loops $\gamma \in LGL_n(\mathbb{C})$ which are the boundary values of holomorphic maps

$$\gamma:\{z \in \mathbb{C}\cup\infty:|z|>1\} \to GL_n(\mathbb{C}).$$

Theorem (8.1.2). *Any loop $\gamma \in LGL_n(\mathbb{C})$ can be factorized*

$$\gamma = \gamma_- \,.\, \lambda \,.\, \gamma_+,$$

where $\gamma_- \in L^-GL_n(\mathbb{C})$, $\gamma_+ \in L^+GL_n(\mathbb{C})$, and $\lambda \in \check{T}$ is a loop which is a homomorphism *from S^1 into the diagonal matrices in $GL_n(\mathbb{C})$, i.e. λ is of the form*

$$z^{\mathbf{a}} = \begin{pmatrix} z^{a_1} & & & \\ & z^{a_2} & & \\ & & \ddots & \\ & & & z^{a_n} \end{pmatrix}.$$

The factor λ is uniquely determined by γ up to conjugation in $GL_n(\mathbb{C})$, i.e. up to the order of $\{a_1, \ldots, a_n\}$. Loops for which $\lambda=1$ form a dense open subset of the identity component of $LGL_n(\mathbb{C})$, and the multiplication map

$$L_1^- \times L^+ \to LGL_n(\mathbb{C}),$$

where $L_1^- = \{\gamma_- \in L^-:\gamma_-(\infty)=1\}$, is a diffeomorphism on to this subset.

We shall describe two important applications of Birkhoff's theorem in the next section. The theorem will be proved in Section 8.4.

Both theorems (8.1.1) and (8.1.2) have exact analogues for the groups of real-analytic, rational, and polynomial loops, but are false for continuous loops. The third theorem, however, applies only to the group of polynomial loops. We shall refer to it as the *Bruhat factorization.* (Cf. [79].)

Theorem (8.1.3). *Any polynomial loop $\gamma \in L_{\mathrm{pol}}GL_n(\mathbb{C})$ can be factorized*

$$\gamma = \gamma_+^{(1)} \,.\, \lambda \,.\, \gamma_+^{(2)},$$

where $\gamma_+^{(1)}$ and $\gamma_+^{(2)}$ both belong to L_{pol}^+, and λ is a homomorphism from S^1 into the diagonal matrices.

The three theorems are precise analogues of the following three well-known facts about $GL_n(\mathbb{C})$.

(i) Any $A \in GL_n(\mathbb{C})$ is the product of a unitary matrix and an upper triangular matrix.

(ii) Any $A \in GL_n(\mathbb{C})$ can be factorized

$$A = P\pi Q,$$

where P is lower triangular, Q is upper triangular, and π is a permutation matrix. Furthermore π is determined uniquely by A, and $\pi=1$ for a dense open subset of $GL_n(\mathbb{C})$—in fact for all A whose leading principal minors do not vanish.

(iii) The same statement as (ii), but with P and Q both upper triangular, and π anti-diagonal for a dense open subset.

Of course (ii) and (iii) are trivially equivalent; they are called the 'Bruhat decomposition' of $GL_n(\mathbb{C})$.

The theorems for loop groups are proved in exactly the same way as the finite dimensional results. The unitary–upper–triangular factorization (i) is simply the 'Gram–Schmidt process' for replacing an arbitrary basis of $\mathbb{C}^n$—the columns of A—by an orthonormal basis. More geometrically, it is the assertion that any flag in $\mathbb{C}^n$ (see Section 2.8) contains an orthonormal basis, i.e. that U_n acts transitively on the flag manifold $GL_n(\mathbb{C})/B$. (B denotes the subgroup of upper-triangular matrices.) The

Bruhat decomposition, likewise, expresses the decomposition of the flag manifold into its Schubert cells.

For a loop group LG the space that plays the role of the flag manifold is the complex homogeneous space $X = LG_{\mathbb{C}}/L^+G_{\mathbb{C}}$. Theorem (8.1.1) is the assertion that LG acts transitively on X, so that $X \cong LG/G$. We shall call X the *fundamental homogeneous space* of LG. The analogy between it and the flag manifold is far-reaching. In particular:

(i) it is a complex projective algebraic variety;
(ii) it has a canonical stratification and cell decomposition;
(iii) irreducible representations of LG can be constructed as spaces of holomorphic sections of line bundles on it.

In Section 8.3 we shall show that when G is U_n there is a beautiful description of the space LG/G as a kind of Grassmannian; from this we shall derive the factorization theorems. A similar description is possible for the other classical groups, and a slightly more complicated one for a general compact Lie group.

The idea of the Grassmannian model comes from 'scattering theory' in the sense of Lax and Phillips [99]. We shall say a little about that point of view in Section 8.12, as an appendix to this chapter. From a completely different point of view the Grassmannian model is an expression of the Bott periodicity theorem; Bott's theorem has been mentioned in Section 6.4, but we shall return to it in Section 8.8.

The Grassmannian model reduces the study of LG/G to linear algebra. It is also interesting, however, to think of LG/G as a Kähler manifold, and to study the Morse theory of the energy function on it. We have discussed that approach in Section 8.9. Another completely different point of view, described in Section 8.10, is to regard LG/G directly as a space of holomorphic vector bundles on the Riemann sphere.

The space LG/G is not the only complex homogeneous space of LG. Indeed $L^+G_{\mathbb{C}}$ should be thought of as a maximal parabolic subgroup of $LG_{\mathbb{C}}$ in the sense of algebraic groups, so $LG_{\mathbb{C}}/L^+G_{\mathbb{C}}$ is more accurately to be compared with a Grassmannian $GL_n(\mathbb{C})/P$, where P is a group of echelon matrices

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix},$$

than with the flag manifold $GL_n(\mathbb{C})/B$. The difference is not, however, very important: the subgroup B^+ of $L^+GL_n(\mathbb{C})$ consisting of loops γ such that $\gamma(0)$ is upper triangular is a minimal parabolic subgroup, and we shall see in Section 8.7 that $LGL_n(\mathbb{C})/B^+$ can be regarded as a space of 'periodic flags' in Hilbert space. Much more interesting, however, is the existence of a quite different complex homogeneous space for $LGL_n(\mathbb{C})$ which is associated to a Riemann surface. This is described in Section 8.11.

We end this section with a technical remark. The Lie group LG is the semidirect product of the subgroup G of constant loops and the normal subgroup ΩG of loops γ such that $\gamma(1) = 1$. (G acts on ΩG by conjugation.) In particular $LG = G \times \Omega G$ as a manifold; and the homogeneous space LG/G can be identified with ΩG. We shall often make this identification without comment, and shall think of ΩG as a homogeneous space of LG. The action of $\gamma \in LG$ on ΩG is therefore $\omega \mapsto \tilde{\omega}$, where

$$\tilde{\omega}(z) = \gamma(z)\omega(z)\gamma(1)^{-1},$$

and the rotation R_α of S^1 through the angle α acts on ΩG by

$$(R_\alpha \omega)(\theta) = \omega(\theta - \alpha)\omega(-\alpha)^{-1}. \quad (8.1.4)$$

8.2 Two applications of the Birkhoff factorization

Singularities of ordinary differential equations

What we have called the Birkhoff factorization was discovered by Birkhoff [11] in 1909 when he was investigating the singularities of differential equations of the form

$$\frac{dv}{dz} = A(z)v(z) \quad (8.2.1)$$

for a $\mathbb{C}^n$-valued function v, where A is a given $(n \times n)$-matrix-valued function which is defined and holomorphic in a neighbourhood U of the origin in the complex plane except for a simple pole at the origin. The problem is to 'change coordinates' by multiplying v by a $GL_n(\mathbb{C})$-valued function T, holomorphic in U, so that the new function $\tilde{v} = Tv$ satisfies a simpler equation. We shall use the Birkhoff factorization to prove

Proposition (8.2.2). *A generic equation* (8.2.1) *can be reduced to the form*

$$\frac{d\tilde{v}}{dz} = z^{-1}K\tilde{v}, \quad (8.2.3)$$

where K is a constant matrix.

The precise meaning of 'generic' is that the residue of A at $z = 0$ is a matrix of which no two distinct eigenvalues differ by an integer. It is not true that every equation of the form (8.2.1) can be reduced to (8.2.3).

Proof. An equation of the form (8.2.1) has a 'solution matrix' X, a multivalued holomorphic function defined in $U - \{0\}$ with values in $GL_n(\mathbb{C})$ which satisfies $dX/dz = AX$. The unique solution of (8.2.1) such that $v(z_0) = v_0$ is expressed in terms of X by

$$v(z) = X(z)X(z_0)^{-1}v_0.$$

The solution matrix is unique up to multiplication on the right by a constant invertible matrix. Its many-valuedness can be described by saying that when z travels once anti-clockwise around the origin then $X(z)$ is multiplied on the right by a matrix $M \in GL_n(\mathbb{C})$ called the *monodromy matrix*. A more precise statement is that there is a genuine holomorphic function $\hat{X}$ defined in the set

$$\{\zeta \in \mathbb{C} : e^{\zeta} \in U\}$$

such that $X(z) = \hat{X}(\log z)$, and

$$\hat{X}(\zeta + 2\pi i) = \hat{X}(\zeta)M.$$

Let us notice that a solution matrix of the equation (8.2.3) is given by $X(z) = z^K = e^{K \log z}$.

Beginning with (8.2.1), let us choose X, and then choose K so that $e^{2\pi i K} = M$. If we define

$$Y(z) = X(z)z^{-K},$$

then Y is a single-valued holomorphic function $U - \{0\} \to GL_n(\mathbb{C})$. The restriction of Y to a small circle $|z| = \varepsilon$ contained in U is a loop in $GL_n(\mathbb{C})$. Birkhoff argued, not quite correctly, that in the generic case, providing K is suitably chosen, this loop has a factorization $Y = Y_+Y_-$, where Y_+ is holomorphic in U and Y_- is holomorphic in the whole Riemann sphere except for the origin. (One assumes initially that Y_+ and Y_- are holomorphic for $|z| < \varepsilon$ and $|z| > \varepsilon$; but they are then automatically holomorphic wherever Y is.) We can assume also that $Y_-(\infty) = 1$.

Now define $\tilde{v} = Y_+^{-1}v$. The equation satisfied by $\tilde{v}$ is $d\tilde{v}/dz = \tilde{A}\tilde{v}$, where

$$\tilde{A} = Y_+^{-1}AY_+ - Y_+^{-1} . dY_+/dz.$$

From this we see that $\tilde{A}$ is holomorphic in U except for a simple pole at the origin. But $Y_+^{-1} = Y_- z^K X^{-1}$, and hence

$$\tilde{A} = z^{-1}Y_-KY_-^{-1} + dY_-/dz . Y_-^{-1}.$$

This shows that $\tilde{A}$ is holomorphic everywhere in the Riemann sphere except for the origin. As $Y_-(z)$ is of the form $1 + y_1 z^{-1} + \ldots$ we know that $Y_-'(z)$ is $O(z^{-2})$ as $z \to \infty$, and so $z\tilde{A}(z) \to K$ as $z \to \infty$. By Liouville's theorem the only possibility is that $\tilde{A}(z) = z^{-1}K$. (In fact it follows also that $Y_-(z) = 1$ for all z.)

It is not sensible to try to repair Birkhoff's argument, as its importance was historical. The simplest way to prove (8.2.2) does not use the factorization theorem: it is better to show directly that when K is suitably chosen the map $Y : U - \{0\} \to GL_n(\mathbb{C})$ extends holomorphically to U. For a full discussion of the subject we refer to Turrettin [148].

The classification of holomorphic vector bundles on the Riemann sphere

The best known application—it would really be better to call it a reformulation—of Birkhoff's theorem is to the classification of holomorphic vector bundles on the Riemann sphere S^2. This was first pointed out by Grothendieck [69].

Let us write $S^2 = U_0 \cup U_\infty$, where $U_0 = S^2 - \{\infty\}$ and $U_\infty = S^2 - \{0\}$. The most obvious bundle on S^2 is the line bundle L which is constructed by attaching $U_0 \times \mathbb{C}$ to $U_\infty \times \mathbb{C}$ by the map

$$(z, \lambda) \mapsto (z, z\lambda).$$

There is also the tensor power L^k for any $k \in \mathbb{Z}$: its attaching function is

$$(z, \lambda) \mapsto (z, z^k\lambda).$$

(If S^2 is regarded as the complex projective line $P(\mathbb{C}^2)$ whose points are the rays in $\mathbb{C}^2$ then L is the 'Hopf bundle' whose fibre at $\xi \in P(\mathbb{C}^2)$ is the line $\xi \subset \mathbb{C}^2$.) Birkhoff's theorem is equivalent to

Proposition (8.2.4). *Any holomorphic vector bundle E on S^2 is isomorphic to a sum $L^{a_1} \oplus \ldots \oplus L^{a_n}$, where the integers $\{a_1, \ldots, a_n\}$ are uniquely determined (apart from their order).*

Proof. The restrictions of E to U_0 and to U_∞ are necessarily trivial, as U_0 and U_∞ are Stein manifolds [66]. So E is obtained by attaching $U_0 \times \mathbb{C}^n$ to $U_\infty \times \mathbb{C}^n$ by means of a holomorphic function

$$\gamma : U_0 \cap U_\infty \to GL_n(\mathbb{C}).$$

By Birkhoff's theorem (8.1.2) we can factorize γ as $\gamma_- . \lambda . \gamma_+$, where γ_+ and γ_- are holomorphic in U_0 and U_∞ respectively, and $\lambda = z^a$ is a homomorphism. If we change coordinates in $U_0 \times \mathbb{C}^n$ by γ_+ and in $U_\infty \times \mathbb{C}^n$ by γ_-^{-1} then we find that E can also be constructed by taking λ as the attaching function. But the bundle defined by λ is $L^{a_1} \oplus \ldots \oplus L^{a_n}$.

8.3 The Grassmannian model of ΩU_n

The group $LGL_n(\mathbb{C})$ acts on the Hilbert space $H^{(n)} = L^2(S^1; \mathbb{C}^n)$, and hence, by Proposition (6.3.1), on the Grassmannian $\mathrm{Gr}(H^{(n)})$. The subspaces W of the form γH_+ for $\gamma \in LGL_n(\mathbb{C})$ have the property that $zW \subset W$, as the action of γ commutes with multiplication by the scalar-valued function z. It turns out that the orbit of H_+ under $LGL_n(\mathbb{C})$ is essentially characterized by this property. (We shall usually write γH_+ for $M_\gamma(H_+)$ and zW for $M_z(W)$ when we do not need to emphasize that γ and z are operators.)

Definition (8.3.1). $\mathrm{Gr}^{(n)}$ *denotes the closed subset of* $\mathrm{Gr}(H^{(n)})$ *consisting of subspaces W such that $zW \subset W$.*

This is the Grassmannian model of the loop space. Its crucial property is

Theorem (8.3.2). *The group $L_{\frac{1}{2}}U_n$ acts transitively on $\mathrm{Gr}^{(n)}$, and the isotropy group of H_+ is the group U_n of constant loops.*

We recall (see Proposition (6.3.3)) that $L_{\frac{1}{2}}U_n$ is the commutant of the multiplication operator M_z in $U_{\mathrm{res}}(H^{(n)})$.

It is obvious that $\gamma H_+ = H_+$ if and only if $\gamma \in L^+GL_n(\mathbb{C})$, so the assertion in Theorem (8.3.2) about the isotropy group is a kind of 'maximum modulus' principle. When $n=1$ it is the statement that a map $S^1 \to \mathbb{T}$ which extends to a non-vanishing holomorphic function in the disc is constant.

Proof of (8.3.2). The first step is to see that if W belongs to $\mathrm{Gr}^{(n)}$ then zW has codimension n in W. Consider the commutative diagram

$$\begin{array}{ccc} zW & \rightarrow & W \\ \downarrow & & \downarrow \\ zH_+ & \rightarrow & H_+ \end{array}$$

where the horizontal maps are inclusions and the vertical ones are orthogonal projections. The two vertical maps are Fredholm, and clearly have the same index (equal to the virtual dimension of W). But $zH_+ \to H_+$ is also Fredholm, with index $-n$. It follows that $zW \to W$ is Fredholm, and its index must also be $-n$ in view of the formula $\chi(AB) = \chi(A) + \chi(B)$ for the index of a composite [34]. Thus $\dim(W/zW) = n$.

Now let $\{w_1, \ldots, w_n\}$ be an orthonormal basis for $W \ominus zW$, the orthogonal complement of zW in W. As in the proof of (6.1.1) we put the vector-valued functions w_i side by side to form an $(n \times n)$-matrix-valued function γ on S^1. We then find that $\gamma(\theta)$ is a unitary matrix for almost all $\theta \in S^1$, i.e. γ belongs to $L_{\mathrm{meas}}U_n$. To see this, let us write

$$w_k(\theta) = \sum_m w_{km} e^{im\theta},$$

with $w_{km} \in \mathbb{C}^n$. Then

$$\begin{aligned} \langle w_k(\theta), w_\ell(\theta) \rangle &= \sum_{m,r} \langle w_{km}, w_{\ell r} \rangle e^{i(r-m)\theta} \\ &= \sum_p \langle w_k, z^p w_\ell \rangle_H e^{ip\theta} \\ &= \delta_{k\ell}, \end{aligned}$$

where for the moment we have written $\langle\ ,\ \rangle_H$ for the inner product in $H^{(n)}$ to distinguish it from the inner product in $\mathbb{C}^n$.

The multiplication operator M_γ is therefore a unitary operator in $H^{(n)}$, and by its construction it satisfies

$$M_\gamma(H_+ \ominus z^k H_+) = W \ominus z^k W$$

for all k. To deduce that $M_\gamma(H_+) = W$ we must prove that $\bigcap z^k W = 0$. Suppose that w belongs to $\bigcap z^k W$, and $\|w\| = 1$. Then $z^{-k}w$ belongs to W for all k. Because the projection $\mathrm{pr}_- : W \to H_-$ is a compact operator, one can find a convergent subsequence of $\{\mathrm{pr}_-(z^{-k}w)\}$, converging, say, to $v \in H$. Clearly $\|v\| = 1$, as $z^k\,\mathrm{pr}_-(z^{-k}w) \to w$. But $\|v\|^2 = \lim\langle v, z^{-k}w\rangle$, and for any $v, w \in H^{(n)}$ we have $\langle v, z^{-k}w\rangle \to 0$ as $k \to \infty$; a contradiction, proving that $\bigcap z^k W = 0$.

To see that M_γ belongs to U_{res}, we observe that its $(H_+ \to H_-)$ component factorizes

$$H_+ \to W \xrightarrow{\mathrm{pr}_-} H_-,$$

and is therefore Hilbert–Schmidt. The $(H_- \to H_+)$ component is also Hilbert–Schmidt because M_γ is unitary.

We have now proved that $L_{\frac{1}{2}}U_n$ acts transitively on $\mathrm{Gr}^{(n)}$. But any $\gamma \in L_{\frac{1}{2}}U_n$ such that $\gamma H_+ = H_+$ must preserve the n-dimensional subspace $H_+ \ominus zH_+$, and is completely determined by its action there. It therefore belongs to U_n.

Theorem (8.3.2) shows that $\Omega_{\frac{1}{2}}U_n = L_{\frac{1}{2}}U_n/U_n$ can be identified with $\mathrm{Gr}^{(n)}$ as a set, and justifies the name 'Grassmannian model'. We shall return to the topological aspect of the correspondence later. Meanwhile we shall determine which subsets of $\Omega_{\frac{1}{2}}U_n$ correspond to the four subspaces $\mathrm{Gr}_0^{(n)}$, $\mathrm{Gr}_1^{(n)}$, $\mathrm{Gr}_\omega^{(n)}$, $\mathrm{Gr}_\infty^{(n)}$—where $\mathrm{Gr}_\alpha^{(n)}$ denotes $\mathrm{Gr}^{(n)} \cap \mathrm{Gr}_\alpha(H^{(n)})$—and we shall derive the factorization theorems of Section 8.1.

Proposition (8.3.3). *In the correspondence $\mathrm{Gr}^{(n)} \leftrightarrow \Omega_{\frac{1}{2}}U_n$*

(i) $\mathrm{Gr}_0^{(n)}$ *corresponds to* $\Omega_{\mathrm{pol}}U_n$,
(ii) $\mathrm{Gr}_1^{(n)}$ *corresponds to* $\Omega_{\mathrm{rat}}U_n$,
(iii) $\mathrm{Gr}_\omega^{(n)}$ *corresponds to* $\Omega_{an}U_n$,
(iv) $\mathrm{Gr}_\infty^{(n)}$ *corresponds to* ΩU_n.

We recall that the groups of polynomial, rational, real-analytic and smooth loops have been mentioned in Section 3.5, and the corresponding subspaces of the Grassmannian were defined in Section 7.2.

Proof. In one direction, if W belongs to $\mathrm{Gr}_\alpha^{(n)}$, where $\alpha = 0, 1, \omega$, or ∞, then we must show that $W \ominus zW$ consists of functions of the corresponding kind. Let us consider the smooth case. If W belongs to Gr_∞ then by (7.2.1) the images of both projections $W \to H_-$ and $(zW)^\perp \to H_+$ consist of smooth functions. So a function in $W \ominus zW$ has smooth projections on to both H_+ and H_-, and is therefore smooth. The argument for Gr_0 and Gr_ω is identical.

In the case of Gr_1, to say that the image of the projection $W \to H_-$ consists of rational functions is the same as to say that there exists a polynomial $p(z)$ such that $p(z)W \subset H_+$. This is not automatically true when $W \in \mathrm{Gr}_1$, but it is true under the additional hypothesis that $zW \subset W$. For one can take for p the minimal polynomial of the transformation induced by M_z on the finite dimensional space $W/W \cap H_+$. Similarly, for the image of $(zW)^\perp \to H_+$ to consist of rational functions we need a polynomial $q(z^{-1})$ such that $q(z^{-1})W^\perp \subset H_-$. This is equivalent to $\bar{q}(z)H_+ \subset W$, and one can take for $\bar{q}$ the minimum polynomial of M_z on $H_+/W \cap H_+$.

In the converse direction, it is obvious that the action of $L_{\mathrm{pol}}U_n$ preserves Gr_0. And $L_{\mathrm{rat}}U_n$ preserves Gr_1 because the existence of polynomials $p(z)$ and $\bar{q}(z)$ such that

$$p(z)W \subset H_+ \quad \text{and} \quad \bar{q}(z)H_+ \subset W$$

is obviously sufficient as well as necessary for W to belong to $\mathrm{Gr}_1^{(n)}$. The smooth loop group LU_n preserves Gr_∞ by Proposition (7.2.1), and $L_{\mathrm{an}}U_n$ preserves Gr_ω by the corresponding characterization of Gr_ω. This completes the proof of (8.3.3).

Remarks.

(i) The reason there is no simple model for the *continuous* loops $\Omega_{\mathrm{cts}}U_n$ is that the positive and negative frequency parts of a continuous function are not necessarily continuous. In other words, in contrast with the behaviour of the four classes of function just discussed, if C is the space of continuous functions on S^1 it is not true that $C = C_+ \oplus C_-$, where $C_\pm = C \cap H_\pm$. Suppose, for example, that f is the function already mentioned in Section 6.3, defined by

$$f(\theta) = \sum_{k>1} \frac{\sin k\theta}{k \log k}. \tag{8.3.4}$$

This function is continuous. But $f = f_+ + f_-$, where

$$f_+(\theta) = \frac{1}{2\mathrm{i}} \sum_{k>1} \frac{\mathrm{e}^{\mathrm{i}k\theta}}{k \log k},$$

and f_+ is unbounded in the neighbourhood of $\theta = 0$.

(ii) $\mathrm{Gr}_0^{(n)}$ is not dense in $\mathrm{Gr}^{(n)}$, despite the fact that $\mathrm{Gr}_0(H)$ is dense in $\mathrm{Gr}(H)$. For we saw in Section 3.5 that a loop in U_n cannot be polynomial unless its determinant is of the form z^k. We shall see in Section 8.10, however, that $\mathrm{Gr}_0^{(n)}$ has the same homotopy type as $\mathrm{Gr}^{(n)}$.

The Grassmannian model for ΩU_n gives us at once the first of the three basic factorization theorems for loops. For the complex group $L_{\frac{1}{2}}GL_n(\mathbb{C})$ acts on $\mathrm{Gr}^{(n)}$ as well as $L_{\frac{1}{2}}U_n$, and the stabilizer of H_+ in $L_{\frac{1}{2}}GL_n(\mathbb{C})$ is clearly the closed subgroup $L_{\frac{1}{2}}^+GL_n(\mathbb{C})$ of loops γ which are the boundary values of holomorphic maps

$$\gamma : \{z \in \mathbb{C} : |z| < 1\} \to GL_n(\mathbb{C}).$$

Because $L_{\frac{1}{2}}U_n$ acts transitively on $\mathrm{Gr}^{(n)}$ we have

Proposition (8.3.5). *The group $L_{\frac{1}{2}}GL_n(\mathbb{C})$ is the product*

$$L_{\frac{1}{2}}U_n \,.\, L_{\frac{1}{2}}^+GL_n(\mathbb{C}).$$

Furthermore, exactly the same factorization property holds for smooth, real-analytic, rational, and polynomial loops.

Remark. The proposition is false for the continuous loop group, as we see from the unique factorization $\mathrm{e}^f = \mathrm{e}^{f_- - f_+}\mathrm{e}^{2f_+}$, where f is the function of (8.3.4).

We have still not quite proved Theorem (8.1.1). It remains to show that the multiplication map $\Omega U_n \times L^+ \to LGL_n(\mathbb{C})$ is a diffeomorphism. For this it is enough to prove the smoothness of the map

$$u : LGL_n(\mathbb{C}) \to \Omega U_n$$

which assigns to a loop its unitary component. The map u factorizes as $\gamma \mapsto \tilde{\gamma} \mapsto u(\gamma)$, where

(i) $\gamma \mapsto \tilde{\gamma}$ is defined by projecting the columns $(\gamma_1, \ldots, \gamma_n)$ of $\gamma \in LGL_n(\mathbb{C})$ on to $(zW)^\perp$, where $W = \gamma H_+$; and

(ii) $\tilde{\gamma} \mapsto u(\gamma)$ is defined by orthonormalizing the basis $(\tilde{\gamma}_1, \ldots, \tilde{\gamma}_n)$ of $W \ominus zW$.

The second of these maps is obviously smooth. The first is smooth because $LGL_n(\mathbb{C})$ acts smoothly on the smooth Grassmannian, whose topology, in turn, is designed to ensure the smoothness of the map

$$C^\infty \times \mathrm{Gr}_\infty(H) \to C^\infty$$
$$(f, W) \mapsto f_W$$

which assigns to a smooth function f on the circle and a subspace $W \in \mathrm{Gr}_\infty(H)$ the projection f_W of f on to W.

Notice that the preceding argument does *not* apply to the group $L_{\frac{1}{2}}GL_n(\mathbb{C})$. Using once again the function f of (8.3.4) we observe that for any $t \in \mathbb{R}$ the loop $\mathrm{e}^{tf} \in L_{\frac{1}{2}}$ factorizes as $\mathrm{e}^{t(f_- - f_+)}\mathrm{e}^{2tf_+}$. But the map $t \mapsto \mathrm{e}^{tf}$ is smooth, while the set $\{\mathrm{e}^{t(f_- - f_+)}\}_{t \in \mathbb{R}}$ has the discrete topology on $L_{\frac{1}{2}}U_n$ because $f_- - f_+$ is unbounded.

8.4 The stratification of $\mathrm{Gr}^{(n)}$: the Birkhoff and Bruhat decompositions.

For the rest of this chapter, except for the Appendix, we shall be concerned only with the smooth Grassmannian $\mathrm{Gr}_\infty(H^{(n)})$ and its subspace $\mathrm{Gr}_\infty^{(n)}$, and shall have no use for the Hilbert manifold $\mathrm{Gr}(H^{(n)})$. We shall therefore change notation by dropping the subscript, and shall write $\mathrm{Gr}(H^{(n)})$ and $\mathrm{Gr}^{(n)}$ for the smooth spaces.

Because $\mathrm{Gr}^{(n)}$ can be identified with $LGL_n(\mathbb{C})/L^+GL_n(\mathbb{C})$, Birkhoff's theorem (8.1.2) amounts to the description of the orbits of the action of $L^-GL_n(\mathbb{C})$ on $\mathrm{Gr}^{(n)}$: it asserts that each L^--orbit contains a point of the form $z^{\mathbf{a}}H_+^{(n)}$, unique up to the order of $\{a_1, \ldots, a_n\}$. We shall prove a slightly more precise result. If $N^- = N^-GL_n(\mathbb{C})$ is the subgroup of L^- consisting of loops γ such that $\gamma(\infty)$ is upper triangular with 1s on the diagonal then we shall show that each orbit of N^- on $\mathrm{Gr}^{(n)}$ contains a unique point of the form $z^{\mathbf{a}}H_+^{(n)}$. In fact we shall show that the orbits of N^- are precisely the intersections of $\mathrm{Gr}^{(n)}$ with the strata of $\mathrm{Gr}(H)$ defined in Chapter 7.

The proof we are about to give is very elementary and explicit, but it is nevertheless rather tedious. For that reason we shall mention in advance the following geometrical description of what will ultimately be proved. The fixed points of the rotation action of $\mathbb{T}$ on ΩG (see (8.1.4)) are easily seen to be the homomorphisms $\lambda: S^1 \to U_n$, corresponding to the subspaces $\lambda . H_+ \in \mathrm{Gr}^{(n)}$. But the action of $\mathbb{T}$ extends to an action of the semigroup $\mathbb{C}^{\times}_{\leq 1}$ (see Section 7.6), and for any $W \in \mathrm{Gr}^{(n)}$ the point $R_u W$ tends to a fixed-point of the $\mathbb{T}$-action as $u \to 0$ in $\mathbb{C}^{\times}_{\leq 1}$. It turns out that if $R_u W \to \lambda . H_+$ then W belongs to the L^--orbit of $\lambda . H_+$. (Notice that if $\gamma \in L^-$ then $R_u\gamma$ tends to the constant loop $\gamma(\infty)$ as $u \to 0$.)

The stratification of $\mathrm{Gr}(H)$ was defined by regarding H as $L^2(S^1; \mathbb{C})$, whereas in this chapter we are concerned with $H^{(n)} = L^2(S^1; \mathbb{C}^n)$. In fact all we need is a Hilbert space with an orthonormal basis indexed by the integers. It is surprisingly convenient, however, to identify $H^{(n)}$ with H in the way explained in Section 6.5, and to think of its elements sometimes as vector-valued functions of z and sometimes as scalar-valued functions of ζ, the two being related by the formulae (6.5.1). Thus $H = H^{(n)}$ has the orthonormal basis $\{\zeta^k\}_{k\in\mathbb{Z}}$, and the definition of $\mathrm{Gr}^{(n)}$ can be rewritten

$$\mathrm{Gr}^{(n)} = \{W \in \mathrm{Gr}(H) : \zeta^n W \subset W\}.$$

Recall that $\mathrm{Gr}(H)$ is the union of disjoint strata Σ_S, where W belongs to Σ_S if S is the set of integers s such that W contains an element of order s. If $W \in \mathrm{Gr}^{(n)}$ belongs to Σ_S then obviously

$$S + n \subset S.$$

Sets $S \in \mathcal{S}$ satisfying this condition are completely determined by giving the complement S^* of $S+n$ in S, which must consist of n elements, one in each congruence class modulo n. They correspond precisely to the homomorphisms from $\mathbb{T}$ into the maximal torus of U_n: to the homomorphism $z^{\mathbf{a}}$ there corresponds the set $S_{\mathbf{a}}$ such that $S_{\mathbf{a}}^*$ is

$$\{na_1, na_2 + 1, na_3 + 2, \ldots, na_n + n - 1\}.$$

The subspace $H_{S_{\mathbf{a}}}$ spanned by $\{\zeta^k\}_{k\in S_{\mathbf{a}}}$ is $z^{\mathbf{a}}H_+^{(n)}$. Thus the strata of $\mathrm{Gr}(H)$ which meet $\mathrm{Gr}^{(n)}$ can be indexed by the homomorphisms $z^{\mathbf{a}}$. We shall write $\Sigma_{\mathbf{a}}$ for $\Sigma_{S_{\mathbf{a}}} \cap \mathrm{Gr}^{(n)}$, and $H_{\mathbf{a}}$ for $H_{S_{\mathbf{a}}}$.

Notice that as a group of operators the subgroup N^- of $LGL_n(\mathbb{C})$ is the intersection of $LGL_n(\mathbb{C})$ with the lower triangular subgroup $\mathcal{N}^-$ of (7.3.3).†

Proposition (8.4.1). *The orbit of $H_{\mathbf{a}}$ under N^- is $\Sigma_{\mathbf{a}}$. It can be identified with the subgroup $L_{\mathbf{a}}^-$ of N^-, where $L_{\mathbf{a}}^- = N^- \cap z^{\mathbf{a}}L_1^- z^{-\mathbf{a}}$.*

Proof. The strata of $\mathrm{Gr}(H)$ are the orbits of $\mathcal{N}^-$ by Proposition (7.3.3). As the group N^- is contained in $\mathcal{N}^-$ each orbit of N^- is certainly contained in some $\Sigma_{\mathbf{a}}$.

If $W \in \Sigma_{\mathbf{a}}$ then the projection $W \to H_{\mathbf{a}}$ is an isomorphism. Let w_i be the inverse image of $z^{a_i}\varepsilon_i$, where $\{\varepsilon_1, \ldots, \varepsilon_n\}$ is the standard basis of $\mathbb{C}^n$. The functions w_i are smooth; furthermore $\{z^k w_i : 1 \leq i \leq n \text{ and } k \geq 0\}$ is a basis for W^{fin} because its projection is a basis for $H_{\mathbf{a}}^{\mathrm{fin}}$; and W^{fin} is a dense subspace of the space W^{sm} of smooth functions in W, because $H_{\mathbf{a}}^{\mathrm{fin}}$ is dense in $H_{\mathbf{a}}^{\mathrm{sm}}$. We know from (8.3.3) that the evaluation map $W^{\mathrm{sm}} \to \mathbb{C}^n$ at any point z of S^1 is surjective, for W^{sm} contains n functions which form the columns of a smooth map $S^1 \to U_n$. It follows that $w_1(z), \ldots, w_n(z)$ are independent in $\mathbb{C}^n$ for any $z \in S^1$, and so $w = (w_1, \ldots, w_n)$ is an element of $LGL_n(\mathbb{C})$, and $w(H_+) = W$.

Now

$$w(\varepsilon_i) = z^{a_i}\varepsilon_i + (\text{lower terms}), \tag{8.4.2}$$

where 'lower terms' refers to the lexicographic ordering of the basis elements $\{z^k\varepsilon_j\} = \{\zeta^m\}$ of $H^{(n)} = H$. So

$$wz^{-\mathbf{a}}(\varepsilon_i) = \varepsilon_i + (\text{lower terms}).$$

In other words, $\gamma = wz^{-\mathbf{a}}$ is the boundary value of a holomorphic map from the hemisphere $|z| > 1$ to the $n \times n$ matrices, and $\gamma(\infty)$ is upper triangular. Furthermore $\gamma(H_{\mathbf{a}}) = W$. But the determinant of γ cannot vanish when $|z| > 1$, for if it did then $\det(\gamma)$ would have non-zero winding number on S^1, contradicting the fact that $H_{\mathbf{a}}$ and W have the same virtual dimension.

Finally, the loop γ belongs to $z^{\mathbf{a}}L_1^- z^{-\mathbf{a}}$ as well as to N^-. For the basis elements occurring in the 'lower terms' in (8.4.2) are not only lower than $z^{a_i}\varepsilon_i$ but also belong to $H_{\mathbf{a}}^{\perp} = z^{\mathbf{a}}H_-$. This means that when the operator

† Our terminology concerning upper and lower triangular matrices is a little muddled, for when discussing infinite matrices indexed by $\mathbb{Z} \times \mathbb{Z}$ we regard the i^{th} row as above the j^{th} if $i > j$, while it is customary to do the opposite with finite matrices. We have decided to tolerate the anomaly that the positive Borel subgroup of $GL_n(\mathbb{C})$ is taken to be the lower triangular matrices in this chapter, but the upper triangular matrices everywhere else in the book.

$z^{-\mathbf{a}}$ is applied to equation (8.4.2) we have

$$z^{-\mathbf{a}}\gamma z^{\mathbf{a}}(\varepsilon_i) = \varepsilon_i + (\text{element of } H_-).$$

We have now proved Proposition (8.4.1), but to complete the proof of the Birkhoff factorization theorem (8.1.2) we must show that the multiplication map $L_1^- \times L^+ \to LGL_n(\mathbb{C})$ is a diffeomorphism on to a dense open subset of the identity component. This is, however, very easy. The map $\gamma \mapsto \gamma H_+$ from $LGL_n(\mathbb{C})$ to the smooth Grassmannian is smooth. On the open subset of $LGL_n(\mathbb{C})$ where γH_+ belongs to the coordinate chart $U_{\mathbb{N}}$, i.e. where γH_+ is the graph of an operator $T_\gamma : H_+ \to H_-$, we have $\gamma = \gamma_- \gamma_+$, where the columns of γ_- are $\{\varepsilon_i + T_\gamma \varepsilon_i\}$. Thus γ_- (and hence also γ_+) depends smoothly on γ.

The preceding argument shows also that $\mathrm{Gr}^{(n)}$, with its subspace topology in the smooth Grassmannian $\mathrm{Gr}(H)$, is locally homeomorphic to L_1^- and hence is a smooth manifold. It is also a smooth submanifold of $\mathrm{Gr}(H)$: it is easy to see that in the coordinate patch $U_{\mathbb{N}}$ of $\mathrm{Gr}(H)$, identified with a space $\mathscr{D}(H_+; H_-)$ of linear maps $H_+ \to H_-$, there is an open subset which is the product of $U_{\mathbb{N}} \cap \mathrm{Gr}^{(n)} \cong L_1^-$ with the linear subspace $\mathscr{D}(zH_+; H_-)$.

The proof of (8.4.1) shows that the group $L_{\mathbf{a}}^-$ is diffeomorphic to the homogeneous space $N^-/{}_{\mathbf{a}}N^-$, where ${}_{\mathbf{a}}N^-$ is $N^- \cap z^{\mathbf{a}}L^+z^{-\mathbf{a}}$, a finite dimensional group which is the stabilizer of $H_{\mathbf{a}}$ in N^-. This means that the multiplication map

$$L_{\mathbf{a}}^- \times {}_{\mathbf{a}}N^- \to N^- \tag{8.4.3}$$

is a diffeomorphism.

The splitting (8.4.3) evidently arises from a splitting of the basis elements of the Lie algebra of N^- into two subsets which span the two subgroups on the left. The fact that such a splitting of the Lie algebra induces a splitting of the group is well-known and elementary for a finite dimensional simply connected nilpotent group ([20] Chapter 3 Section 9.5); and (8.4.3) is really a finite dimensional result, because $L_{\mathbf{a}}^-$ contains, for large q, the normal subgroup N_q^- of N^- consisting of loops γ such that $\gamma(z) - 1$ vanishes to order q at $z = \infty$, and N^-/N_q^- is finite dimensional. (N^-/N_q^- is nilpotent because N_r^-/N_{r+1}^- is abelian for $r > 0$.) At present the splitting which is of most interest to us is a slight variant of (8.4.3), namely

$$L_{\mathbf{a}}^- \times L_{\mathbf{a}}^+ \xrightarrow{\cong} z^{\mathbf{a}}L_1^- z^{-\mathbf{a}}, \tag{8.4.4}$$

where $L_{\mathbf{a}}^+ = N^+ \cap z^{\mathbf{a}}L_1^- z^{-\mathbf{a}}$, and N^+ is the subgroup of L^+ consisting of loops γ such that $\gamma(0)$ is lower triangular with ones on the diagonal.

Theorem (8.4.5).

(i) *The map $\gamma \mapsto \gamma H_{\mathbf{a}}$ defines a diffeomorphism between $z^{\mathbf{a}}L_1^- z^{-\mathbf{a}}$ and a contractible open neighbourhood $U_{\mathbf{a}}$ of $H_{\mathbf{a}}$ in $\mathrm{Gr}^{(n)}$.*

(ii) *The stratum $\Sigma_{\mathbf{a}}$ is a contractible closed submanifold of $U_{\mathbf{a}}$, of complex codimension*

$$d(\mathbf{a}) = \sum_{i<j} |a_i - a_j| - \nu(\mathbf{a}),$$

where $\nu(\mathbf{a})$ is the number of pairs i, j with $i < j$ but $a_i > a_j$.

(iii) *The orbit of $H_{\mathbf{a}}$ under N^+ is a complex cell $C_{\mathbf{a}}$, of complex dimension $d(\mathbf{a})$, which meets $\Sigma_{\mathbf{a}}$ transversally in the single point $H_{\mathbf{a}}$. The splitting (8.4.4) defines a diffeomorphism*

$$\Sigma_{\mathbf{a}} \times C_{\mathbf{a}} \to U_{\mathbf{a}}.$$

(iv) *The union of the cells $C_{\mathbf{a}}$ is $\mathrm{Gr}_0^{(n)}$; in fact $C_{\mathbf{a}}$ is the intersection of $\mathrm{Gr}^{(n)}$ with the cell $C_{S_{\mathbf{a}}}$ of Gr_0.*

Remarks.

(i) We shall call the cells $C_{\mathbf{a}}$ the *Bruhat cells* of $\mathrm{Gr}^{(n)}$. Notice that part (iv) of (8.4.5) implies the Bruhat factorization theorem (8.1.3).

(ii) The stratum $\Sigma_{\mathbf{a}}$ is contractible, and diffeomorphic to $L_{\mathbf{a}}^-$. But we cannot assert, as in the analogous finite dimensional theorem, that $L_{\mathbf{a}}^-$ is diffeomorphic to its Lie algebra by the exponential map, for the exponential map of $L_{\mathbf{a}}^-$ is not surjective. Goodman and Wallach have given the following example of an element γ of $N^-SL_2(\mathbb{C})$ which does not belong to the image of exp:

$$\gamma = \begin{pmatrix} 1 + 2z^{-2} & 4z^{-1} \\ z^{-1} + z^{-3} & 1 + 2z^{-2} \end{pmatrix}.$$

This cannot be of the form $\exp(\xi)$, for

$$\gamma(\mathrm{i}) = \begin{pmatrix} -1 & -4\mathrm{i} \\ 0 & -1 \end{pmatrix}$$

does not belong to the image of exp in $SL_2(\mathbb{C})$.

Proof of (8.4.5). Little more needs to be said. The contractibility of $U_{\mathbf{a}}$ and $\Sigma_{\mathbf{a}}$ follows from the contractibility of the groups L_1^- and $N_{\mathbf{a}}^-$; these consist of holomorphic functions in the disc $|z| > 1$, and can be contracted by the homomorphisms $\gamma \mapsto \gamma_t$ (for $0 \leq t \leq 1$), where $\gamma_t(z) = \gamma(t^{-1}z)$.

The orbit $C_{\mathbf{a}}$ is contractible by the same argument (applied to the disc $|z| < 1$). It is a cell because the exponential map of the nilpotent group $L_{\mathbf{a}}^+$ is a diffeomorphism.

The proof that $C_{\mathbf{a}}$ is the intersection of $\mathrm{Gr}^{(n)}$ with $C_{S_{\mathbf{a}}}$ is exactly like that of (8.4.1)—cf. (7.4.1).

It remains to calculate the dimension $d(\mathbf{a})$ of the group $L^+_{\mathbf{a}}$. Because conjugation by $z^{\mathbf{a}}$ multiplies the $(i, j)^{\text{th}}$ entry of a matrix by $z^{a_i - a_j}$, we find that $L^+_{\mathbf{a}}$ is an open subset of the matrix-valued functions (f_{ij}) such that

$$f_{ii} = 1,$$
$$f_{ij} \text{ belongs to } zH_+ \cap z^{a_i - a_j} H_- \text{ if } i < j, \text{ and}$$
$$f_{ij} \text{ belongs to } H_+ \cap z^{a_i - a_j} H_- \text{ if } i > j.$$

This leads at once to the above formula for $d(\mathbf{a})$.

One thing lacking from the preceding Proposition, in comparison with (7.3.3) and (7.4.1), is a description of the closures of the strata $\Sigma_{\mathbf{a}}$ and cells $C_{\mathbf{a}}$. We shall content ourselves with the slightly weaker statement in our next proposition.

At the beginning of our discussion we asked about the orbits of L^- and L^+ on $\mathrm{Gr}^{(n)}$, rather than of N^- and N^+. It is clear that the orbit of $H_{\mathbf{a}}$ under L^- contains $H_{\sigma\mathbf{a}}$ for every permutation σ of $\{1, \ldots, n\}$, where $\sigma\mathbf{a} = (a_{\sigma(1)}, \ldots, a_{\sigma(n)})$. The orbit does not contain $H_{\mathbf{b}}$ for any other $\mathbf{b}$, as one sees from the fact that the action of L^- on W does not change the dimension of $W \cap z^k H_-$. The orbit is therefore the union of the strata $\Sigma_{\sigma\mathbf{a}}$; we shall denote it by $\Sigma_{|\mathbf{a}|}$. The orbit of $H_{\mathbf{a}}$ under L^+, similarly, is the union of the cells $C_{\sigma\mathbf{a}}$, and will be denoted by $C_{|\mathbf{a}|}$. Notice that if $a_1 \geq a_2 \geq \ldots \geq a_n$ then $\Sigma_{\mathbf{a}}$ is a dense open subset of $\Sigma_{|\mathbf{a}|}$, and if $a_1 \leq a_2 \leq \ldots \leq a_n$ then $C_{\mathbf{a}}$ is a dense open subset of $C_{|\mathbf{a}|}$.

The set of multi-indices $\mathbf{a}$ such that $a_1 \geq a_2 \geq \ldots \geq a_n$ can be identified with the set of conjugacy classes of homomorphisms from S^1 into U_n. We shall order such multi-indices by prescribing $\mathbf{a} \leq \mathbf{b}$ if

$$a_1 + a_2 + \ldots + a_k \leq b_1 + b_2 + \ldots + b_k \quad \text{for} \quad 1 \leq k < n,$$

and

$$a_1 + a_2 + \ldots + a_n = b_1 + b_2 + \ldots + b_n.$$

The disposition of the sets $\Sigma_{|\mathbf{a}|}$ and $C_{|\mathbf{a}|}$ is summarized in the following Proposition, where all indices $\mathbf{a}$ are assumed to be written in decreasing order $a_1 \geq a_2 \geq \ldots \geq a_n$.

Proposition (8.4.6).

(i) *The orbits $\{\Sigma_{|\mathbf{a}|}\}$ of L^- on $\mathrm{Gr}^{(n)}$ are indexed by the conjugacy classes of homomorphisms $S^1 \to U_n$. The set $\Sigma_{|\mathbf{a}|}$ is a locally closed submanifold of $\mathrm{Gr}^{(n)}$ of codimension $d(\mathbf{a})$. Furthermore $\Sigma_{|\mathbf{a}|}$ lies in the closure of $\Sigma_{|\mathbf{b}|}$ if and only if $\mathbf{a} \geq \mathbf{b}$.*

(ii) *The orbits $\{C_{|\mathbf{a}|}\}$ of L^+ on $\mathrm{Gr}_0^{(n)}$ are indexed in the same way, and $C_{|\mathbf{a}|}$ is a locally closed submanifold of $\mathrm{Gr}^{(n)}$ of dimension $d(\tilde{\mathbf{a}})$, where $\tilde{\mathbf{a}} = (a_n, a_{n-1}, \ldots, a_1)$. The closure of $C_{|\mathbf{b}|}$ contains $C_{|\mathbf{a}|}$ if and only if $\mathbf{a} \leq \mathbf{b}$.*

(iii) *$C_{|\mathbf{a}|}$ meets $\Sigma_{|\mathbf{b}|}$ if and only if $\mathbf{a} \geq \mathbf{b}$, and $C_{|\mathbf{a}|}$ meets $\Sigma_{|\mathbf{a}|}$ transversally in the set $\Lambda_{\mathbf{a}}$ of homomorphisms $S^1 \to U_n$ which are conjugate to $z^{\mathbf{a}}$.*

Remark. $\Lambda_{\mathbf{a}}$ is a generalized flag manifold of the form $U_n/U_{n_1} \times \ldots \times U_{n_r}$; its dimension is $\nu(\mathbf{a})$ in the notation of (8.4.5).

Proof. What needs to be proved is the assertions about the ordering, and about the intersection of $C_{|\mathbf{a}|}$ with $\Sigma_{|\mathbf{a}|}$.

For any decreasing multi-index $\mathbf{a}$ and any integer p let us define

$$\delta_p(\mathbf{a}) = \sum_{k=1}^{n} (p - a_k)_+,$$

where a_+ means a if $a \geq 0$ and 0 otherwise. It is easy to check that, providing $\Sigma a_i = \Sigma b_i$,

$$\mathbf{a} \leq \mathbf{b} \Leftrightarrow \delta_p(\mathbf{a}) \leq \delta_p(\mathbf{b}) \text{ for all } p,$$

and that

$$W \in \Sigma_{|\mathbf{a}|} \Leftrightarrow \dim(W \cap z^p H_-) = \delta_p(\mathbf{a}) \text{ for all } p,$$

and

$$W \in C_{|\mathbf{a}|} \Leftrightarrow \dim(W / W \cap z^p H_+) = \delta_p(\mathbf{a}) \text{ for all } p.$$

From this it follows that $\mathbf{a} \leq \mathbf{b}$ if $\Sigma_{|\mathbf{b}|}$ is in the closure of $\Sigma_{|\mathbf{a}|}$, or $C_{|\mathbf{a}|}$ is in the closure of $C_{|\mathbf{b}|}$, or $C_{|\mathbf{b}|}$ meets $\Sigma_{|\mathbf{a}|}$.

It also follows that if $W \in C_{|\mathbf{a}|} \cap \Sigma_{|\mathbf{a}|}$ then

$$W = (W \cap z^p H_-) \oplus (W \cap z^p H_+)$$

for all p, and hence that

$$W \ominus zW = \bigoplus_{i=1}^{r} A_i z^{k_i},$$

where $\mathbb{C}^n = A_1 \oplus \ldots \oplus A_r$ is some orthogonal decomposition of $\mathbb{C}^n$. This implies that $W = \lambda H_+$, where $\lambda : S^1 \to U_n$ is the homomorphism defined by

$$\lambda(z) = z^{k_1} \oplus \ldots \oplus z^{k_r}$$

with respect to the decomposition $A_1 \oplus \ldots \oplus A_r$. Thus $C_{|\mathbf{a}|} \cap \Sigma_{|\mathbf{a}|} = \Lambda_{\mathbf{a}}$. The intersection is transversal because the tangent space to $\mathrm{Gr}^{(n)}$ at $H_{\mathbf{a}}$ can be identified with the Lie algebra $\Omega_{\mathbf{a}}$ of $z^{\mathbf{a}} L_1^- z^{-\mathbf{a}}$, and then the tangent spaces to $\Sigma_{|\mathbf{a}|}$, $C_{|\mathbf{a}|}$ and $\Lambda_{\mathbf{a}}$ correspond to the intersection of $\Omega_{\mathbf{a}}$ with the factors of the decomposition $L\mathfrak{g}_{\mathbb{C}} = L_1^- \mathfrak{g}_{\mathbb{C}} \oplus L_1^+ \mathfrak{g}_{\mathbb{C}} \oplus \mathfrak{g}_{\mathbb{C}}$. (Here $\mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}_n(\mathbb{C})$.)

We have proved the 'only if' part of the three assertions about the ordering. For the converse let us first consider a pair $\mathbf{a} < \mathbf{b}$ where

$\mathbf{b}=\mathbf{a}+\mathbf{e}_{pq}$ for some $p<q$, where

$$\mathbf{e}_{pq}=(0,\ldots,0,1,0,\ldots,0,-1,0,\ldots,0)$$

with 1 and -1 in the p^{th} and q^{th} places. We shall show that there is a complex projective line $\mathbb{C}\cup\infty$ in $\mathrm{Gr}^{(n)}$ which joins $H_{\mathbf{a}}$ to $H_{\mathbf{b}'}$, where $\mathbf{b}'$ is $\mathbf{b}$ with its p^{th} and q^{th} elements interchanged, and which lies, except for its end-points $\{0,\infty\}$ in $\Sigma_{|\mathbf{a}|}\cap C_{|\mathbf{b}|}$.

There is an embedding i_{pq} of $SL_2(\mathbb{C})$ in $LGL_n(\mathbb{C})$ which takes

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}$$

to (f_{ij}), where (f_{ij}) differs from the identity matrix only in the p^{th} and q^{th} rows and columns, and

$$\begin{pmatrix} f_{pp} & f_{pq}\\ f_{qp} & f_{qq}\end{pmatrix}=\begin{pmatrix} a & bz\\ cz^{-1} & d\end{pmatrix}.$$

Consider the orbit of $H_{\mathbf{a}}$ under the action of $SL_2(\mathbb{C})$ induced by i_{pq}. The stabilizer consists of the lower triangular matrices, so the orbit is a standard projective line $S^2\cong\mathbb{C}\cup\infty$. All of its points except the point at infinity form the orbit of the strictly upper triangular matrices in $SL_2(\mathbb{C})$. These matrices map into L_1^-, so $S^2-\{\infty\}$ is contained in $\Sigma_{|\mathbf{a}|}$. But the point at infinity is $i_{pq}(A).H_{\mathbf{a}}$, where

$$A=\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix};$$

and this is $H_{\mathbf{b}'}$, which belongs to $\Sigma_{|\mathbf{b}|}$. Thus the closure of $\Sigma_{|\mathbf{a}|}$ contains $\Sigma_{|\mathbf{b}|}$. On the other hand $S^2-\{0\}$ is the orbit of $H_{\mathbf{b}'}$ under the strictly lower triangular matrices, and belongs to $C_{|\mathbf{b}|}$. So the closure of $C_{|\mathbf{b}|}$ contains $C_{|\mathbf{a}|}$, and $C_{|\mathbf{b}|}$ meets $\Sigma_{|\mathbf{a}|}$.

That completes the proof in the case $\mathbf{b}=\mathbf{a}+\mathbf{e}_{pq}$. In the general case the first two assertions about the ordering follow because whenever $\mathbf{a}<\mathbf{b}$ one can get from $\mathbf{a}$ to $\mathbf{b}$ by successively adding multi-indices of the form $\mathbf{e}_{pq}$. (See [108] (1.15).) We shall leave the third assertion to the reader.

8.5 The Grassmannian model for the other classical groups

The Grassmannian description we have given of the loop space of U_n can be modified very easily to treat the orthogonal and symplectic groups.

The orthogonal group

The group O_n consists of the real matrices in U_n, so ΩO_n is a submanifold of ΩU_n.

Proposition (8.5.1). *A subspace $W\in\mathrm{Gr}^{(n)}$ corresponds to a loop in O_n if and only if it belongs to*

$$\mathrm{Gr}^{(n)}_{\mathbb{R}}=\{W\in\mathrm{Gr}^{(n)}:\bar W^{\perp}=zW\}.$$

All the spaces $W\in\mathrm{Gr}^{(n)}_{\mathbb{R}}$ have virtual dimension 0, but it follows from (8.5.1) that $\mathrm{Gr}^{(n)}_{\mathbb{R}}$ has two connected components, which in fact are distinguished by the parity of the dimension of the kernel of the projection $W\to H_+$. (That will appear in Chapter 12: the space $\mathscr{J}(H)$ considered in Section 12.4 is closely related to $\mathrm{Gr}^{(n)}_{\mathbb{R}}$.)

Before proving (8.5.1) let us notice that $\mathrm{Gr}^{(n)}_{\mathbb{R}}$ is a *complex* submanifold of $\mathrm{Gr}^{(n)}$. For the map $W\mapsto z^{-1}\bar W^{\perp}$ is a holomorphic involution on $\mathrm{Gr}(H^{(n)})$: in the coordinate patch consisting of graphs of operators $T:H_+\to H_-$ it is represented by the complex linear map $T\to -z^{-1}\bar T^*$. The condition of (8.5.1) asserts that W is very nearly an *isotropic* subspace of $H^{(n)}$ for the complex bilinear form B on $H^{(n)}$ defined by $B(\xi,\eta)=\langle\bar\xi,\eta\rangle$: more precisely, the radical of W with respect to B is zW.

Notice that we must now avoid identifying $H^{(n)}$ with $H^{(1)}$, for that does not respect the real subspaces.

Proof of (8.5.1). First suppose that γ is a loop in the complex orthogonal group $O_n(\mathbb{C})$. Then the multiplication operator M_γ preserves the complex bilinear form B on $H^{(n)}$, so it commutes with the operation $W\mapsto\bar W^{\perp}$ of forming the orthogonal complement with respect to B. As H_+ satisfies $\bar H_+^{\perp}=zH_+$, so does γH_+.

Conversely, if $\bar W^{\perp}=zW$, then $W\ominus zW=W\cap\bar W$, and so $W\ominus zW$ is the complexification of a real n-dimensional subspace of $L^2(S^1;\mathbb{R}^n)$, and we can find an orthonormal basis for it consisting of real functions. Thus $W=\gamma H_+$ for some loop γ in O_n.

Proposition (8.5.1) gives us two factorization theorems immediately. Because $\mathrm{Gr}^{(n)}_{\mathbb{R}}$ is a homogeneous space of $LO_n(\mathbb{C})$ on which ΩO_n acts transitively, we have

Proposition (8.5.2). *The multiplication map*

$$\Omega O_n\times L^+O_n(\mathbb{C})\to LO_n(\mathbb{C})$$

is a diffeomorphism, where $L^+O_n(\mathbb{C})$ denotes the loops which are boundary values of holomorphic maps

$$\{z\in\mathbb{C}:|z|<1\}\to O_n(\mathbb{C}).$$

Secondly, any element W of $\mathrm{Gr}^{(n)}_{\mathbb{R}}$ in a suitable neighbourhood of H_+ is transversal to H_-, i.e. $W\cap H_-=0$ and $W+H_-=H^{(n)}$. For such spaces W the intersection $W\cap zH_-$ is n-dimensional, and $W=(W\cap zH_-)\oplus zW$. We know then (see the proof of (8.4.1)) that any basis $\{w_1,\ldots,w_n\}$ for $W\cap zH_-$ forms the columns of a loop $\gamma_-\in L^-GL_n(\mathbb{C})$ such that

$\gamma = \gamma_- \gamma_+$, with $\gamma_+ \in L^+GL_n(\mathbb{C})$. If the basis is chosen orthogonal with respect to B—which is possible because the restriction of B to $W \cap zH_-$ is necessarily nondegenerate—then γ_- belongs to $L^-O_n(\mathbb{C})$, because (see the proof of (8.3.2))

$$\begin{aligned}\langle \overline{w_k(e^{i\theta})}, w_\ell(e^{i\theta}) \rangle &= \sum_{p,q \leq 0} \langle \bar{w}_{kp}, w_{\ell q} \rangle e^{+ip\theta + iq\theta} \\ &= \sum_{r \geq 0} \langle \bar{w}_k, z^r w_\ell \rangle_H e^{-ir\theta} \\ &= B(w_k, w_\ell).\end{aligned}$$

This gives us

Proposition (8.5.3). *The multiplication map*

$$L_1^- O_n(\mathbb{C}) \times L^+ O_n(\mathbb{C}) \to LO_n(\mathbb{C})$$

is a diffeomorphism on to an open subset of $LO_n(\mathbb{C})$.†

We could go on to derive Birkhoff and Bruhat decomposition theorems, but we shall postpone that until the next section.

The symplectic group

The group Sp_n is the subgroup of all elements u in U_{2n} which preserve a nondegenerate skew form on $\mathbb{C}^{2n}$. Equivalently, it consists of the unitary transformations u which are quaternionic-linear when $\mathbb{C}^{2n}$ is identified with $\mathbb{H}^n$. If $J: \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ is the antilinear map representing multiplication by the quaternion j, then u belongs to Sp_n if and only if $uJ = Ju$. The complexification of Sp_n is the subgroup $Sp_n(\mathbb{C})$ of all elements of $GL_{2n}(\mathbb{C})$ which preserve the skew complex-bilinear form S defined by $S(\xi, \eta) = \langle J\xi, \eta \rangle$.

Corresponding to (8.5.1) we have

Proposition (8.5.4). *A subspace* $W \in \mathrm{Gr}^{(2n)}$ *corresponds to a loop in* Sp_n *if and only if it belongs to*

$$\mathrm{Gr}_{\mathbb{H}}^{(2n)} = \{W \in \mathrm{Gr}^{(2n)} : (JW)^\perp = zW\}.$$

The proof is identical to that of (8.5.1); and the result implies two factorization theorems precisely analogous to (8.5.2) and (8.5.3).

8.6 The Grassmannian model for a general compact Lie group

In studying ΩG for a compact semisimple group G one may as well assume that the centre of G is trivial. For if $\tilde{G}$ is a covering group of G then the manifold $\Omega \tilde{G}$ is just the union of some of the connected

† The open subset is dense in the group of null-homotopic loops in $O_n(\mathbb{C})$. These form two of the four connected components of $LO_n(\mathbb{C})$.

components of ΩG. If the centre is trivial then G is the identity component of the group of automorphisms of its Lie algebra $\mathfrak{g}$.

The most obvious Hilbert space on which LG acts is $H^{\mathfrak{g}} = L^2(S^1; \mathfrak{g}_{\mathbb{C}})$. This is, essentially, its adjoint representation. We shall identify $H^{\mathfrak{g}}$ with $H^{(n)}$, where n is the dimension of G. Thus we are regarding LG as a subgroup of LU_n by the adjoint representation of G on $\mathbb{C}^n$. The loop space ΩG is a submanifold of ΩU_n, which can be identified with a submanifold of $\mathrm{Gr}(H^{\mathfrak{g}})$.

Definition (8.6.1). $\mathrm{Gr}^{\mathfrak{g}}$ *is the subset of* $\mathrm{Gr}(H^{\mathfrak{g}})$ *consisting of subspaces* W *such that*

(i) $zW \subset W$,
(ii) $\bar{W}^\perp = zW$, *and*
(iii) W^{sm} *is a Lie algebra.*

Here W^{sm} denotes the subspace of smooth functions in W, which we know is dense. To say that it is a Lie algebra means simply that it is closed under the bracket operation defined pointwise for $\mathfrak{g}_{\mathbb{C}}$-valued functions.

Theorem (8.6.2). *The action of* $LG_{\mathbb{C}}$ *on* $\mathrm{Gr}(H^{\mathfrak{g}})$ *preserves* $\mathrm{Gr}^{\mathfrak{g}}$, *and if the centre of* G *is trivial then* $\gamma \mapsto \gamma H_+$ *defines a diffeomorphism* $\Omega G \to \mathrm{Gr}^{\mathfrak{g}}$.

Proof. The first statement is obvious, as any group acts on its own Lie algebra by Lie algebra automorphisms. (The condition (ii) arises as in (8.5.1) because the adjoint action preserves the Killing form, so that $G_{\mathbb{C}}$ is contained in the orthogonal group $O(\mathfrak{g}_{\mathbb{C}})$.)

Conversely, suppose that W satisfies the conditions of (8.6.1). From condition (ii) we have $W \ominus zW = W \cap \bar{W}$. We know that $W \ominus zW$ consists of smooth functions. For any point z of the circle the evaluation map $e_z : W \cap \bar{W} \to \mathfrak{g}_{\mathbb{C}}$ at z is an isomorphism, and must be an isomorphism of Lie algebras. It also commutes with complex conjugation. If γ is defined by $\gamma(z) = e_z e_1^{-1}$ then γ is a loop in the group of automorphisms of $\mathfrak{g}$. For a group with trivial centre this means that γ belongs to ΩG. By our usual argument we have $\gamma H_+ = W$.

As in the preceding section we can deduce from (8.6.2) that the multiplication $\Omega G \times L^+G_{\mathbb{C}} \to LG_{\mathbb{C}}$ is a diffeomorphism, and that the multiplication $L_1^- G_{\mathbb{C}} \times L^+ G_{\mathbb{C}} \to LG_{\mathbb{C}}$ is a diffeomorphism on to a dense open subset of the identity component. We shall now go further and determine the stratification and cell decomposition of ΩG corresponding to the theorems proved earlier for U_n.

Let us choose a maximal torus T of G, and a system of positive roots. Then we can define the nilpotent subgroups $N_0^{\pm}$ of $G_{\mathbb{C}}$ whose Lie algebras are spanned by the root vectors of $\mathfrak{g}_{\mathbb{C}}$ corresponding to the positive (resp.

negative) roots. We can also define subgroups $N^{\pm}$ of $L^{\pm}=L^{\pm}G_{\mathbb{C}}$: $N^{\pm}$ consists of the loops $\gamma\in L^{\pm}$ such that $\gamma(0)\in N_0^{\pm}$ (resp. $\gamma(\infty)\in N_0^{\pm}$). Thus $L_1^{\pm}\subset N^{\pm}\subset L^{\pm}$.

The result we wish to prove is

Theorem (8.6.3).

(i) $\mathrm{Gr}^{\mathfrak{g}}\cong\Omega G$ *is the union of strata* Σ_λ *indexed by the lattice* $\check{T}$ *of homomorphisms* $\lambda:\mathbb{T}\to T$.

(ii) Σ_λ *is the orbit of* $\lambda.H_+$ *under* N^-. *It is a locally closed contractible complex submanifold of finite codimension* d_λ *in* $\mathrm{Gr}^{\mathfrak{g}}$, *and it is diffeomorphic to* $L_\lambda^-=N^-\cap\lambda.L_1^-.\lambda^{-1}$.

(iii) *The orbit of* $\lambda.H_+$ *under* N^+ *is a complex cell* C_λ *of dimension* d_λ. *It is diffeomorphic to* $L_\lambda^+=N^+\cap\lambda.L_1^-.\lambda^{-1}$, *and meets* Σ_λ *transversally in the single point* $\lambda.H_+$.

(iv) *The orbit of* $\lambda.H_+$ *under* $\lambda.L_1^-.\lambda^{-1}$ *is an open subset* U_λ *of* $\mathrm{Gr}^{\mathfrak{g}}$. *The multiplication* $L_\lambda^+\times L_\lambda^-\to\lambda.L_1^-.\lambda^{-1}$ *defines a diffeomorphism* $C_\lambda\times\Sigma_\lambda\to U_\lambda$.

(v) *The union of the cells* C_λ *is* $\mathrm{Gr}_0^{\mathfrak{g}}\cong\Omega_{\mathrm{pol}}G$.

Once again the stratification of $\mathrm{Gr}^{\mathfrak{g}}$ will be induced by that of $\mathrm{Gr}(H^{\mathfrak{g}})$. To define the latter we must choose an orthonormal basis $\{\zeta_k\}$ of $H^{\mathfrak{g}}$ indexed by $\mathbb{Z}$. We shall do this so that H_+ is spanned by $\{\zeta_k\}$ for $k\geq 0$, and $z\zeta_k=\zeta_{k+n}$. Thus $\{\zeta_0,\ldots,\zeta_{n-1}\}$ is a basis for $\mathfrak{g}_{\mathbb{C}}$. We shall choose it to consist of eigenvectors of the action of T. Each vector ζ_i (for $0\leq i<n$) then has a weight with respect to the action of T: either zero or a root of G. We choose the order of the ζ_i so that ζ_i precedes ζ_j whenever the difference between the weight of ζ_j and the weight of ζ_i is a sum of positive roots. In particular, $\{\zeta_0,\ldots,\zeta_{m-1}\}$, where $m=\frac{1}{2}(n-\ell)$ and ℓ is the rank of G, are the negative root vectors, and span the Lie algebra of N_0^-, while $\{\zeta_{m+l},\ldots,\zeta_{n-1}\}$ span the Lie algebra of N_0^+.

With respect to the basis $\{\zeta_k\}$ the group N^- acts on $H^{\mathfrak{g}}$ by elements of the lower triangular subgroup $\mathcal{N}^-$ of (7.3.3).

The strata and cells of $\mathrm{Gr}(H^{\mathfrak{g}})$ are indexed by subsets S of $\mathbb{Z}$. Among them are the sets S_λ corresponding to the lattice $\check{T}$ of homomorphisms $\mathbb{T}\to T$: these are defined by

$$H_{S_\lambda}=\lambda.H_+.$$

We shall write H_λ for H_{S_λ}. Our proof of (8.6.3) depends on the following lemma.

Lemma (8.6.4). *The strata* Σ_S *and cells* C_S *of* $\mathrm{Gr}(H^{\mathfrak{g}})$ *which meet* $\mathrm{Gr}^{\mathfrak{g}}$ *are precisely the* Σ_{S_λ} *and* C_{S_λ} *for* $\lambda\in\check{T}$.

We shall postpone the proof of the lemma to the end of this section, and shall proceed with the proof of (8.6.3).

Proof of (8.6.3). Let us define

$$\Sigma_\lambda=\Sigma_{S_\lambda}\cap\mathrm{Gr}^{\mathfrak{g}}.$$

The lemma tells us that $\mathrm{Gr}^{\mathfrak{g}}$ is the union of the Σ_λ. Because N^- is contained in $\mathcal{N}^-$ it is clear that the orbit of H_λ is contained in Σ_λ. Thus the main point is to show that L_λ^- acts transitively on Σ_λ. Now from (8.4.1) we know that any $W\in\Sigma_\lambda$ can be expressed *uniquely* as γH_λ, where γ belongs to $L_\lambda^-GL_n(\mathbb{C})$. As L_λ^- is the intersection of $L_\lambda^-GL_n(\mathbb{C})$ with $LG_{\mathbb{C}}$, it is enough to show that γ belongs to $LG_{\mathbb{C}}$. The construction of γ in (8.4.1) can be formulated as follows. If W belongs to Σ_λ then $W\in zH_\lambda^\perp$ has dimension n, and the evaluation map $e_z:W\cap zH_\lambda^\perp\to\mathfrak{g}_{\mathbb{C}}$ at each point $z\in S^1$ is an isomorphism. The orthogonal projection $\mathrm{pr}:W\cap zH_\lambda^\perp\to H_\lambda\ominus zH_\lambda$ is also an isomorphism. Now $\gamma(z)$ is the composite

$$\mathfrak{g}_{\mathbb{C}}\xrightarrow{M_\lambda}H_\lambda\ominus zH_\lambda\xrightarrow{\mathrm{pr}^{-1}}W\cap zH_\lambda^\perp\xrightarrow{e_z}\mathfrak{g}_{\mathbb{C}}.$$

Each of the three spaces here is a Lie algebra—in the case of $W\cap zH_\lambda^\perp$, notice that $(zH_\lambda^\perp)^{\mathrm{sm}}=\lambda\bar{H}_+^{\mathrm{sm}}$. Furthermore each map is a homomorphism of Lie algebras—for pr is induced by the projection of $(zH_\lambda^\perp)^{\mathrm{sm}}$ on to $zH_\lambda^\perp\ominus H_\lambda^\perp$, and $(H_\lambda^\perp)^{\mathrm{sm}}$ is an ideal in $(zH_\lambda^\perp)^{\mathrm{sm}}$. So $\gamma(z)$ is a homomorphism of Lie algebras, and hence belongs to $G_{\mathbb{C}}$, as we want.

The rest of the proof is exactly the same as for $GL_n(\mathbb{C})$, and we shall say no more about it.

The orbits of L^- and L^+ on $\mathrm{Gr}^{\mathfrak{g}}$ and $\mathrm{Gr}_0^{\mathfrak{g}}$ are obtained, again just as before, by grouping the Σ_λ and C_λ together into pieces $\Sigma_{|\lambda|}$ and $C_{|\lambda|}$ indexed by the *conjugacy classes* of homomorphisms $\mathbb{T}\to G$, or equivalently by the set of orbits $\check{T}/W$ of the Weyl group W of G acting on the lattice $\check{T}$. This set $\check{T}/W$ can be ordered by prescribing $|\lambda|\leq|\mu|$ if the convex hull of the orbit $W.\lambda$ is contained in the convex hull of $W.\mu$. (Here $\check{T}$ is regarded as a lattice in the vector space $\mathfrak{t}$.) Without any further discussion we record

Proposition (8.6.5).

(i) $\Sigma_{|\lambda|}$ *intersects* $C_{|\lambda|}$ *transversally in the set* Λ_λ *of homomorphisms* $\mathbb{T}\to G$ *which are conjugate to* λ.

(ii) $\left.\begin{array}{l}\Sigma_{|\lambda|} \textit{ is contained in the closure of } \Sigma_{|\mu|}\\ C_{|\mu|} \textit{ is contained in the closure of } C_{|\lambda|}\\ C_{|\lambda|} \textit{ meets } \Sigma_{|\mu|}\end{array}\right\}$ *if and only if* $|\lambda|\geq|\mu|$.

Because we have a cell decomposition of $\mathrm{Gr}_0^{\mathfrak{g}}$ whose cells are all of even dimension the fundamental group $\pi_1(\mathrm{Gr}_0^{\mathfrak{g}})$ must be trivial. Now $\mathrm{Gr}_0^{\mathfrak{g}}$ is the polynomial loop space $\Omega_{\mathrm{pol}}G$. The fundamental group of ΩG is the second homotopy group $\pi_2(G)$. If we show that $\Omega_{\text{pol}}G$ is homotopy equivalent to ΩG then we shall have a proof of the well-known but important fact that $\pi_2(G)$ is zero for any compact Lie group G. This proof is in essence the same as Bott's Morse theory proof [14].

Proposition (8.6.6). *The inclusion* $\mathrm{Gr}_0^{\mathfrak{g}} \to \mathrm{Gr}^{\mathfrak{g}}$, *or equivalently* $\Omega_{\text{pol}}G \to \Omega G$, *is a homotopy equivalence.*

Corollary (8.6.7). *The homotopy group* $\pi_2(G)$ *is zero.*

Proof of (8.6.6). The idea is that $\mathrm{Gr}_0^{\mathfrak{g}}$ and $\mathrm{Gr}^{\mathfrak{g}}$ have corresponding stratifications by homotopy equivalent subsets.

Let us arrange the elements of the lattice $\check{T}$ in a sequence beginning with 0 so that if Σ_μ is contained in the closure of Σ_λ then λ precedes μ (which we shall denote by $\lambda \leqslant \mu$). Let $\mathrm{Gr}^{\leqslant\lambda}$ denote the union of all the open sets U_μ of $\mathrm{Gr}^{\mathfrak{g}}$ such that $\mu \leqslant \lambda$, and $\mathrm{Gr}^{<\lambda}$ the union of the U_μ such that $\mu < \lambda$. We shall also write $\mathrm{Gr}_0^{\leqslant\lambda}$ and $\mathrm{Gr}_0^{<\lambda}$ for the corresponding parts of $\mathrm{Gr}_0^{\mathfrak{g}}$. It is enough for us to prove that $\mathrm{Gr}_0^{\leqslant\lambda} \to \mathrm{Gr}^{\leqslant\lambda}$ is a homotopy equivalence for all λ. (Cf. [114] Appendix.)

Now $\mathrm{Gr}^{\leqslant\lambda}$ is the union of U_λ and $\mathrm{Gr}^{<\lambda}$, and the intersection of the two sets is $U_\lambda - \Sigma_\lambda$. (The fact that all points of U_λ belong to strata $\leqslant\lambda$ follows from the corresponding fact for $\mathrm{Gr}(H)$ proved in (7.3.3).) The set U_λ is contractible, while $U_\lambda - \Sigma_\lambda$ is diffeomorphic to $\Sigma_\lambda \times (C_\lambda - \{H_\lambda\})$ and so homotopy equivalent to $C_\lambda - \{H_\lambda\}$.

The space $\mathrm{Gr}_0^{\leqslant\lambda}$ is likewise the union of $U_{\lambda,0}$ and $\mathrm{Gr}_0^{<\lambda}$, whose intersection is $U_{\lambda,0} - \Sigma_{\lambda,0}$. Whereas U_λ was diffeomorphic to $\lambda \,.\, L_1^- \,.\, \lambda^{-1}$, the set $U_{\lambda,0}$ is homeomorphic to $\lambda \,.\, L_{1,\text{pol}}^- \,.\, \lambda^{-1}$— we do not know that $\mathrm{Gr}_0^{\mathfrak{g}}$ is a manifold—and hence to $\Sigma_{\lambda,0} \times C_\lambda$. Furthermore $U_{\lambda,0}$ and $\Sigma_{\lambda,0}$ are contractible for the same reason as U_λ, and $U_{\lambda,0} - \Sigma_{\lambda,0}$ is homotopy equivalent to $U_\lambda - \Sigma_\lambda$. We can conclude by induction that $\mathrm{Gr}_0^{\leqslant\lambda}$ is homotopy equivalent to $\mathrm{Gr}^{\leqslant\lambda}$. (We are using the fact that if $X = U \cup V$ and $X' = U' \cup V'$, where U, V, U', V' are open subsets, then a map $f: X \to X'$ is a homotopy equivalence if it induces homotopy equivalences $U \to U'$, $V \to V'$, and $U \cap V \to U' \cap V'$. Cf. [67] (16.24).)

Proof of (8.6.4). We end this section with the postponed proof of Lemma (8.6.4).

Let us define an action of the circle $\mathbb{T}$ on $L\mathfrak{g}$ as follows. Choose a homomorphism $\rho = \exp(\dot{\rho}): \mathbb{T} \to T$ such that $\dot{\rho}$ belongs to the positive Weyl chamber in $\mathfrak{t}$, i.e. $\langle \alpha, \dot{\rho} \rangle > 0$ for every root α. For any sufficiently large integer q the centralizer of $\rho(e^{2\pi i/q})$ in G will be T. The action of $\mathbb{T}$ which we want is got by simultaneously rotating with speed q and conjugating by ρ:

$$\mathbb{T} \times L\mathfrak{g} \to L\mathfrak{g}$$

takes (u, ξ) to $S_u\xi$, where

$$S_u\xi(z) = \rho(u)\xi(u^{-q}z)\rho(u)^{-1}.$$

This action extends to the Hilbert space $H^{\mathfrak{g}}$, and is diagonal with respect to the basis $\{\zeta_k\}$. In fact $S_u\zeta_k = u^{m_k}\zeta_k$, where m_k increases monotonically with k providing q is large enough. The action induces an action on $\mathrm{Gr}(H^{\mathfrak{g}})$ which extends to an action of $\mathbb{C}^\times_{\leqslant 1}$ (see Section 7.6). Furthermore the action preserves $\mathrm{Gr}^{\mathfrak{g}}$, and also the strata Σ_S. For any $W \in \mathrm{Gr}(H^{\mathfrak{g}})$ the point S_uW tends to a limit as $u \to 0$, the limit being necessarily fixed under the $\mathbb{T}$-action and contained in the same stratum as W. (For the strata are characterized by (7.5.4), and S_u simply multiplies the Plücker coordinate $\pi_S(W)$ by $u^{m(S)}$, where $m(S)$ increases monotonically with S and tends to ∞ as $\ell(S) \to \infty$.) But the only fixed points of the action on $\mathrm{Gr}^{\mathfrak{g}}$ are the spaces λH_+ with $\lambda \in \check{T}$. Indeed if γH_+ is fixed for some $\gamma \in \Omega G$ then

$$\rho(u)\gamma(u^{-q}z)\gamma(u^{-q})^{-1}\rho(u)^{-1} = \gamma(z) \tag{8.6.8}$$

for all $u, z \in \mathbb{T}$. Putting $u = e^{2\pi i/q}$ we find that $\gamma(z)$ commutes with $\rho(e^{2\pi i/q})$, and so $\gamma(z)$ belongs to T. Equation (8.6.8) then reduces to the condition that $\gamma: \mathbb{T} \to T$ is a homomorphism. Thus every stratum contains a point λH_+, and is therefore of the form Σ_{S_λ}.

The argument for the cells C_S is essentially the same: one considers S_uW as $u \to \infty$.

8.7 The homogeneous space LG/T and the periodic flag manifold

We saw in Section 2.8 that the most important homogeneous space of a compact group G is G/T, where T is a maximal torus of G. We have already mentioned that the analogue of G/T for a loop group is LG/T rather than the more natural space $\Omega G = LG/G$ which we have been studying so far in this chapter. We shall now give a rapid account of LG/T.

We know already that LG/T is a complex manifold, for it can be identified with $LG_{\mathbb{C}}/B^+$, where B^+ consists of the elements $\gamma_0 + \gamma_1 z + \ldots$ of $L^+G_{\mathbb{C}}$ such that γ_0 belongs to the positive Borel subgroup B_0^+ of $G_{\mathbb{C}}$. The main property of LG/T is that it is stratified by the orbits of N^-, and the strata are indexed by the affine Weyl group W_{aff}. This group was defined in Section 5.1. It is the semidirect product $W \tilde{\times} \check{T}$, where W is the Weyl group of G and $\check{T}$ is the lattice of homomorphisms $\mathbb{T} \to T$. We can regard W_{aff} as a subset of LG/T, for $W_{\text{aff}} = (N_T \,.\, \check{T})/T$, where N_T is the normalizer of T in G. The following Proposition is essentially a restatement of the proof of (5.1.2).

Proposition (8.7.1). *The set of fixed points of the rotation action of* $\mathbb{T}$ *on* LG/T *is* W_{aff}.

The properties of the stratification of LG/T are listed in

Theorem (8.7.2).

(i) *The complex manifold $Y = LG/T = LG_{\mathbb{C}}/B^+$ is the union of strata Σ_w indexed by $w \in W_{\text{aff}}$.*

(ii) *The stratum Σ_w is the orbit of w under N^-, and is a locally closed contractible complex submanifold of Y whose codimension is the length $\ell(w)$ of w. It is diffeomorphic to $N_w^- = N^- \cap wN^-w^{-1}$.*

(iii) *Σ_w is a closed subset of the open subset U_w of Y, where $U_w = w \,.\, \Sigma_1$. The action of $A_w = N^+ \cap wN^-w^{-1}$ defines a diffeomorphism*

$$A_w \times \Sigma_w \to U_w.$$

(iv) *The orbit of w under A_w is a complex cell C_w of dimension $\ell(w)$ which intersects Σ_w transversally at w. The union of the cells C_w is $Y_{\text{pol}} = L_{\text{pol}}G/T$.*

(v) *If $\ell(w') = \ell(w) + 1$ then $\Sigma_{w'}$ is contained in the closure of Σ_w if and only if $w' = ws$, where $s \in W_{\text{aff}}$ is the reflection corresponding to a simple affine root.*

Here the length $\ell(w)$ is defined as the dimension of A_w, i.e. as the number of positive affine roots $\boldsymbol{\alpha}$ such that $w \,.\, \boldsymbol{\alpha}$ is negative.

The most important part of the content of Proposition (8.7.2) is the pair of factorization theorems

$$LG_{\mathbb{C}} = \bigcup_{w \in W_{\text{aff}}} N^- wB^+ \tag{8.7.3a}$$

and

$$L_{\text{pol}}G_{\mathbb{C}} = \bigcup_{w \in W_{\text{aff}}} N^+_{\text{pol}} wB^+_{\text{pol}}. \tag{8.7.3b}$$

These follow from (8.6.3) together with the Bruhat decomposition of the finite dimensional group

$$G_{\mathbb{C}} = \bigcup_{w \in W} N_0^- wB_0^+ = \bigcup_{w \in W} N_0^+ wB_0^+. \tag{8.7.4}$$

Granting (8.7.4) the proof of (8.7.2) presents nothing new. It does not seem worth giving the details. For a proof of (8.7.4) we refer to Bourbaki [20] Chapter 6 Section 2. We shall, however, mention the crucial point in the proof of part (v) of (8.7.2), although it is identical with the finite dimensional case.

If $\boldsymbol{\alpha}$ is a simple affine root of LG then there is (see (5.2.4)) an associated homomorphism $i_\alpha : SL_2(\mathbb{C}) \to LG_{\mathbb{C}}$ which maps the torus $\mathbb{T}$ of $SL_2(\mathbb{C})$ into T. This gives us a map $i_\alpha : S^2 \to Y$, where $S^2 = \mathbb{C} \cup \infty$ is $SL_2(\mathbb{C})/\mathbb{T}$. If $w' = ws_\alpha$ in W_{aff}, where s_α is the reflection corresponding to $\boldsymbol{\alpha}$, and $\ell(w') = \ell(w) + 1$, then the map

$$z \mapsto i_\alpha(z) \,.\, w$$

from S^2 to Y defines a holomorphic curve in Y linking w to w'. This curve lies in Σ_w except for the point $i_\alpha(\infty) \,.\, w = w'$, and so the closure of Σ_w contains $\Sigma_{w'}$.

In the case of U_n there is a geometrical model for LU_n/T which we shall now describe: it is analogous to the Grassmannian model of ΩU_n.

Definition (8.7.5). $\mathrm{Fl}^{(n)}$ *consists of all sequences $\{W_k\}_{k \in \mathbb{Z}}$ of subspaces of $H^{(n)}$ such that*

(i) *each W_k belongs to $\mathrm{Gr}(H^{(n)})$,*

(ii) *$W_{k+1} \subset W_k$ for each k, and $\dim(W_k/W_{k+1}) = 1$,*

(iii) *$zW_k = W_{k+n}$.*

It is natural to refer to the points of $\mathrm{Fl}^{(n)}$ as *periodic flags*. We shall also consider the subspace $\mathrm{Fl}_0^{(n)}$ of $\mathrm{Fl}^{(n)}$ consisting of flags such that each W_k belongs to Gr_0.

In this section it is now once again convenient to identify $H^{(n)}$ with $H = L^2(S^1; \mathbb{C})$. Then the flag $\{\zeta^k H_+\}$ is a canonical base-point in $\mathrm{Fl}^{(n)}$; its stabilizer in $LGL_n(\mathbb{C})$ is $B^+GL_n(\mathbb{C})$. The following proposition is proved in the same way as (8.3.2), and even follows from it.

Proposition (8.7.6). *The group LU_n acts transitively on $\mathrm{Fl}^{(n)}$, and the stabilizer of $\{\zeta^k H_+\}$ is the maximal torus T of U_n.*

This means that $\mathrm{Fl}^{(n)} \cong LU_n/T \cong LGL_n(\mathbb{C})/B^+$. Evidently $\mathrm{Fl}^{(n)}$ is fibred over $\mathrm{Gr}^{(n)}$ by the map $\{W_k\} \mapsto W_0$, the fibre being the finite dimensional flag manifold $U_n/T = \mathrm{Fl}(\mathbb{C}^n)$.

The orbits of N^- and N^+ provide a stratification of $\mathrm{Fl}^{(n)}$ and a cellular subdivision of $\mathrm{Fl}_0^{(n)}$. We shall not pursue this any further, however, beyond explaining how the strata and cells are indexed by the affine Weyl group W_{aff} of LU_n. In the present case W_{aff} is the semidirect product of the symmetric group S_n with the lattice $\check{T} \cong \mathbb{Z}^n$, on which S_n acts by permuting the factors. When W_{aff} is identified with a subgroup of LU_n it acts on $H^{(n)}$ by permuting the basis elements $\{\zeta^k\}$. In fact it can be identified with the group of all permutations π of $\mathbb{Z}$ with the property

$$\pi(k+n) = \pi(k) + n \tag{8.7.7}$$

for all k.

Proposition (8.7.8). *The strata and the cells of $\mathrm{Fl}^{(n)}$ are indexed by the affine Weyl group W_{aff} of LU_n.*

Proof. We must show that any flag $\{W_k\}$ belongs to the orbit of $\pi\{H_k\}$, where $H_k = \zeta^k H_+$, for some permutation π satisfying (8.7.7). Now each W_k belongs to some stratum Σ_{S_k} of the Grassmannian, and $S_k - S_{k+1}$ has exactly one element, say s_k. The desired permutation is given by $\pi(k) = s_k$. For $k = 0, 1, 2, \ldots, n-1$ we choose a vector $w_k \in H^{(n)}$

spanning W_k/W_{k+1} which is of order s_k. Just as in the proof of (8.3.2) we find that $\{w_0, w_1, \ldots, w_{n-1}\}$ are the columns of a loop γ such that $\gamma\{H_k\} = \{W_k\}$; and $\gamma\pi^{-1}$ belongs to N^-.

The argument for the cells is precisely analogous.

8.8 Bott periodicity

We have already mentioned in Section 6.4 the Bott periodicity theorem, which asserts that the infinite unitary group $U = \cup_n U_n$ is homotopy equivalent to its second loop space $\Omega^2 U$. Another formulation is that ΩU is homotopy equivalent to

$$\mathrm{Gr}_0(H) = \bigcup_{p \leq q} \mathrm{Gr}(\zeta^p H_+ / \zeta^q H_+).$$

(This formulation is well-known [17]: $\mathrm{Gr}_0(H)$ is the standard model for the space which algebraic topologists call $\mathbb{Z} \times BU$.) The theory which we have built up incorporates a proof of the theorem, for the identification of ΩU_n with a subspace of $\mathrm{Gr}(H)$ is precisely the Bott map.

Before explaining the proof, let us recall that we showed in (8.6.6) that the polynomial loop group $\Omega_{\mathrm{pol}} U_n$ is homotopy equivalent to the smooth loop group ΩU_n (and hence, by very standard arguments, to the continuous loop group $\Omega_{\mathrm{cts}} U_n$). It is a much more elementary fact, but can be proved by the same argument (8.6.6), that $\mathrm{Gr}_0(H)$ is homotopy equivalent to $\mathrm{Gr}(H)$. So Bott's theorem is a consequence of the following.

Proposition (8.8.1). *The inclusion*

$$\Omega_{\mathrm{pol}} U_n = \mathrm{Gr}_0^{(n)} \to \mathrm{Gr}_0(H)$$

induces an isomorphism of homotopy groups up to dimension $2n-2$.

Proof. It is enough to consider the identity components of the two spaces.

$\mathrm{Gr}_0(H)$ is the union of cells C_S indexed by subsets S of $\mathbb{Z}$. If S satisfies

$$n + k + \mathbb{N} \subset S \subset k + \mathbb{N} \qquad (8.8.2)$$

for some k then any $W \in C_S$ is sandwiched between $\zeta^{k+n} H_+$ and $\zeta^k H_+$. This implies that $\zeta^n W \subset W$, and hence that W belongs to $\mathrm{Gr}_0^{(n)}$. The cell C_S is therefore completely contained in $\mathrm{Gr}_0^{(n)}$. (Although we shall not need the fact, it may be worth remarking that in the notation of (8.4.5) the cells $C_{\mathbf{a}}$ of $\mathrm{Gr}_0^{(n)}$ obtained in this way are those such that

$$a_i = \pm 1 \text{ or } 0 \text{ for each } i,$$

and

$$\text{if } a_i = 1 \text{ and } a_j = -1 \text{ then } i < j.)$$

To prove (8.8.1) we must show that every cell C_S of Gr_0 such that S does not satisfy (8.8.2) has complex dimension $\geq n$, and that the same applies to every cell $C_S^{(n)} = C_S \cap \mathrm{Gr}_0^{(n)}$ of $\mathrm{Gr}_0^{(n)}$ when $n + S \subset S$ but S does not satisfy (8.8.2).

The dimension of C_S is given (if S has virtual cardinal zero) by

$$\ell(S) = \sum_{k \geq 0} (k - s_k) \qquad (8.8.3)$$

by (7.4.1). The condition (8.8.2) is easily seen to be equivalent to

$$s_m = m \quad \text{when} \quad m = s_0 + n.$$

So if (8.8.2) does not hold then $k - s_k \geq 1$ when $k \leq s_0 + n$, and

$$\ell(S) \geq -s_0 + (s_0 + n) = n,$$

as we want.

The argument is essentially the same for the cells $C_S^{(n)}$. We leave it to the reader to check that the formula of (8.4.5) for the dimension can be rewritten in the same form as (8.8.3), except that for $C_S^{(n)}$ the sum must be taken only over the n values of k for which s_k does not belong to $n + S$. There are two cases. If $s_0 \leq -n$ the result is clear. If not, then s_k does not belong to $n + S$ when $k \leq s_0 + n$, because $s_k < k$; the preceding argument then applies.

Remark. The possibility of proving the Bott periodicity theorem by the above method was first pointed out in the announcement [57] of Garland and Raghunathan.

8.9 ΩG as a Kähler manifold: the energy flow

In this section we shall look at the homogeneous space ΩG afresh, thinking of it somewhat more geometrically.

Let us choose a positive definite invariant inner product $\langle\ ,\ \rangle$ on the Lie algebra $\mathfrak{g}$. In Chapter 4 we introduced a skew form ω on $L\mathfrak{g}$, defined by

$$\omega(\xi, \eta) = \frac{1}{2\pi} \int_0^{2\pi} \langle \xi(\theta), \eta'(\theta) \rangle \, d\theta.$$

This defines a left-invariant closed 2-form ω on LG; because it is invariant under conjugation by constant loops, and because it vanishes when ξ or η is constant, it defines an invariant closed 2-form ω on the homogeneous space $\Omega G = LG/G$. If ξ is a non-zero element of the tangent space $\Omega\mathfrak{g} = L\mathfrak{g}/\mathfrak{g}$ to ΩG at its base-point then there is always some $\eta \in \Omega\mathfrak{g}$ such that $\omega(\xi, \eta) \neq 0$. (One can take $\eta = \xi'$.) We shall therefore think of ω as defining a *symplectic* structure on ΩG.

If X is a finite dimensional symplectic manifold with symplectic form ω then to each smooth function $F: X \to \mathbb{R}$ there corresponds a so-called *Hamiltonian vector field* ξ_F on X characterized by

$$\omega_x(\xi_F(x), \eta) = \mathrm{d}F(x; \eta), \tag{8.9.1}$$

where $x \in X$ and η is a tangent vector to X at x. In the infinite-dimensional case the existence of ξ_F is not automatic, but *at most* one vector field ξ_F can satisfy (8.9.1). On the other hand it is easy to see that if X is simply connected and ξ is a vector field on X which leaves ω invariant then $\xi = \xi_F$ for some smooth function F. (For de Rham's theorem applies, and $(x; \eta) \mapsto \omega_x(\xi(x), \eta)$ is a closed 1-form on X.) We shall make use of two examples of this construction.

We consider first the *energy* function $\mathscr{E}: \Omega G \to \mathbb{R}$ defined by

$$\mathscr{E}(\gamma) = \frac{1}{4\pi} \int_0^{2\pi} \|\gamma(\theta)^{-1}\gamma'(\theta)\|^2 \, \mathrm{d}\theta. \tag{8.9.2}$$

(Here and elsewhere we always use notation as if a loop γ were a matrix-valued function. Thus $\gamma(\theta)^{-1}\gamma'(\theta)$ denotes the element of $\mathfrak{g}$ got by left-translating to the origin the tangent vector $\gamma'(\theta)$ to G at $\gamma(\theta)$. The formula (8.9.2) is written for $\gamma \in LG$, but is invariant under both left and right multiplication by elements of G.)

Proposition (8.9.3). *The Hamiltonian vector field on ΩG which corresponds to $\mathscr{E}$ is the generator of the flow defined by rotating the loops.*

Note. In this proposition ΩG is regarded as a homogeneous space LG/G rather than as a subset of LG. As a subset it would not be preserved by rotation. If one wants to regard ΩG as a subset of LG then the action on $\gamma \in \Omega G$ of the rotation R_α through the angle α must be defined by

$$(R_\alpha\gamma)(\theta) = \gamma(\theta - \alpha)\gamma(-\alpha)^{-1}. \tag{8.9.4}$$

Proof of (8.9.3). For an infinitesimal change $\delta\gamma$ in γ the change in $\mathscr{E}$ is

$$\mathrm{d}\mathscr{E}(\gamma; \delta\gamma) = \frac{1}{2\pi} \int \langle \gamma^{-1}\gamma', \delta(\gamma^{-1}\gamma') \rangle.$$

On the other hand the value at γ of the vector field corresponding to rotation is γ' (modulo the action of a constant element of $\mathfrak{g}$). So what we have to show is that

$$\frac{1}{2\pi} \int \langle \gamma^{-1}\gamma', \delta(\gamma^{-1}\gamma') \rangle = \omega_\gamma(\gamma', \delta\gamma)$$
$$= \frac{1}{2\pi} \int \langle \gamma^{-1}\gamma', (\gamma^{-1}\delta\gamma)' \rangle.$$

This is true because

$$(\gamma^{-1}\delta\gamma)' = \gamma^{-1}\delta\gamma' - \gamma^{-1}\gamma'\gamma^{-1}\delta\gamma$$
$$= \delta(\gamma^{-1}\gamma') + [\gamma^{-1}\gamma', \gamma^{-1}\delta\gamma],$$

while

$$\langle \gamma^{-1}\gamma', [\gamma^{-1}\gamma', \gamma^{-1}\delta\gamma] \rangle = \langle [\gamma^{-1}\gamma', \gamma^{-1}\gamma'], \gamma^{-1}\delta\gamma \rangle = 0.$$

Corollary (8.9.5). *The critical points of the energy function $\mathscr{E}$ on ΩG are the loops γ which are homomorphisms $\gamma: S^1 \to G$.*

Proof. The critical points are precisely the stationary points of the corresponding Hamiltonian flow. From the formula (8.9.4) we find that $R_\alpha\gamma = \gamma$ for all α if and only if γ is a homomorphism.

Our other example of a Hamiltonian flow is even simpler.

Proposition (8.9.6). *The flow on ΩG generated by $\xi \in \mathfrak{g}$ corresponds to the Hamiltonian function $F_\xi: \Omega G \to \mathbb{R}$ given by*

$$F_\xi(\gamma) = \frac{1}{2\pi} \int_0^{2\pi} \langle \xi, \gamma'(\theta)\gamma(\theta)^{-1} \rangle \, \mathrm{d}\theta.$$

The proof of this is completely straightforward. Cf. [8].

Remark (8.9.7). Combining (8.9.3) and (8.9.6) we see that the Hamiltonian function corresponding to the twisted rotational flow on ΩG which was considered in the proof of (8.6.4) is the *tilted energy* given by

$$\mathscr{E}(\gamma) = \frac{q}{4\pi} \int_0^{2\pi} \left\| \gamma(\theta)^{-1}\gamma'(\theta) - \frac{1}{q}\rho \right\|^2 \mathrm{d}\theta.$$

This function has isolated critical points: they are the homomorphisms $\gamma: S^1 \to T$.

It is our object to investigate the Morse theory of the energy function on ΩG, i.e. the trajectories of the gradient flow of $\mathscr{E}$. For this to make sense, the manifold ΩG must be given a Riemannian structure. There are very many invariant Riemannian metrics on ΩG, but the choice is fixed for us by the fact that ΩG is a *complex* manifold: in Section 8.6 we proved that ΩG can be regarded as $LG_{\mathbb{C}}/L^+G_{\mathbb{C}}$, and is locally diffeomorphic to $L_1^-G_{\mathbb{C}}$.

Proposition (8.9.8). *The complex structure and the symplectic structure of ΩG are compatible, and combine to make ΩG into a Kähler manifold.*

This means that if T_γ is the real tangent space to ΩG at γ, and J_γ is the automorphism of T_γ which corresponds to multiplication by i in terms of the complex structure of ΩG, then

(i) $\omega_\gamma(J_\gamma\xi, J_\gamma\eta) = \omega_\gamma(\xi, \eta)$ for all $\xi, \eta \in T_\gamma$,

(ii) $(\xi, \eta) \mapsto g_\gamma(\xi, \eta) = \omega_\gamma(\xi, J_\gamma \eta)$ is a *positive definite inner product* on T_γ.

The Kähler form on T_γ is then given by

$$(\xi, \eta) \mapsto g_\gamma(\xi, \eta) + i\omega_\gamma(\xi, \eta).$$

Proof. Because both the complex structure and the form ω are invariant, it suffices to prove (i) and (ii) when $\gamma = 1$. If ξ is expanded as $\xi = \sum \xi_k z^k$, with $\xi_k \in \mathfrak{g}_\mathbb{C}$, then the action of J_1 on ξ multiplies ξ_k by i when $k < 0$ and by $-$i when $k > 0$. (The constant term ξ_0 is to be disregarded, as we are really working in $L\mathfrak{g}/\mathfrak{g}$.) Now

$$\omega(\xi, \eta) = i \sum_{k \in \mathbb{Z}} k\langle \xi_{-k}, \eta_k \rangle,$$

where the inner product has been extended to a complex bilinear form on $\mathfrak{g}_\mathbb{C}$. This gives us both (i) and (ii), for

$$g(\xi, \xi) = \omega(\xi, J_1\xi) = 2 \sum_{k>0} k\langle \xi_{-k}, \xi_k \rangle \geq 0.$$

For a function F on a Kähler manifold the gradient flow of F is related to its Hamiltonian flow simply by applying the operators J_γ in the tangent bundle: the two flows are in fact the real and imaginary parts of a flow parametrized by $\mathbb{C}$. Now we have already seen in Section 7.6 and Section 8.6 that the rotation-action of $\mathbb{T}$ on the Grassmannian $\mathrm{Gr}(H)$ and on ΩG extends to a holomorphic action of the semigroup $\mathbb{C}^\times_{\leq 1}$. (Notice that we have by now considered three different actions of $\mathbb{T}$ on H. The action studied in Section 7.6 came from the identification of H with $L^2(S^1; \mathbb{C})$. The one we are concerned with now comes from $H \cong H^{\mathfrak{g}} = L^2(S^1; \mathfrak{g}_\mathbb{C})$. There is also the twisted version of this which was used in Section 8.6, where the rotation was combined with conjugation by the elements of a one-parameter subgroup of G. All three are diagonal with respect to the standard orthonormal basis of H, and in each case $R_u \in \mathbb{T}$ multiplies the k^{th} basis vector by a power of u which increases with k and tends to ∞ as $k \to \infty$.) We can now reinterpret the discussion in Sections 7.6 and 8.6, and in particular the proof of (8.6.4).

First let us recall [115] that a vector field on an infinite dimensional manifold does not always possess trajectories, and even when it does they need not be unique. The theory which we have developed can be summarized as follows.

Theorem (8.9.9).

(i) *There is a downwards trajectory of the energy function emanating from every point* γ *of* ΩG. *It is given by*

$$t \mapsto \gamma_t = R_{e^{-t}}\gamma$$

for $t \geq 0$.

(ii) *The loop* γ_t *is real-analytic when* $t > 0$, *and it converges to a homomorphism* $\gamma_\infty : S^1 \to G$ *as* $t \to \infty$.

(iii) *There is an upwards trajectory* $t \mapsto \gamma_t$ *defined for non-zero time* $\varepsilon < t \leq 0$ *if and only if* γ *is real-analytic. It is defined for all* $t \leq 0$ *if and only if* γ *is a polynomial loop, and in that case* γ_t *converges to a homomorphism* $\gamma_{-\infty}$ *as* $t \to -\infty$.

(iv) *The sets* $\Sigma_{|\lambda|}$ *and* $C_{|\lambda|}$ *of Proposition* (8.6.5) *are the ascending and descending manifolds of the critical level* Λ_λ *of the energy*; *i.e.*

$$\gamma_t \in \Sigma_{|\lambda|} \Leftrightarrow \gamma_t \to \gamma_\infty \in \Lambda_\lambda \quad \text{as} \quad t \to \infty.$$

$$\gamma_t \in C_{|\lambda|} \Leftrightarrow \gamma_t \to \gamma_{-\infty} \in \Lambda_\lambda \quad \text{as} \quad t \to -\infty.$$

It is interesting to specialize this result to the case $G = \mathbb{T}$. The identity component of $\Omega\mathbb{T}$ can be identified with the vector space of smooth functions $f: S^1 \to \mathbb{R}$. We have

$$\mathscr{E}(f) = \frac{1}{4\pi} \int_0^{2\pi} |f'(\theta)|^2 \, d\theta.$$

The downwards gradient flow of $\mathscr{E}$ is $\{f_t\}$, where $\{f_t\}$ is obtained by solving the parabolic pseudo-differential equation

$$\frac{\partial f_t}{\partial t} = (-\Delta)^{\frac{1}{2}} f_t, \qquad (8.9.10)$$

where $\Delta = (\partial/\partial\theta)^2$. If f is expanded as $\sum a_k e^{ik\theta}$, then

$$f_t(\theta) = \sum a_k e^{-|k|t + ik\theta}.$$

The assertions (i), (ii), (iii) of (8.9.9) are quite clear in this case. But for a general group the equation corresponding to (8.9.10) is non-linear; and although our results are very plausible it would probably be rather difficult to prove them by direct methods.

Classical and quantum-mechanical energy

Combining the isomorphism $\Omega G \cong \mathrm{Gr}^{\mathfrak{g}}$ with the Plücker embedding of the Grassmannian (see Sections 7.5 and 7.7) we obtain a holomorphic embedding

$$\pi: \Omega G \to P(\mathscr{H}).$$

The rotation action of the circle on $H^{\mathfrak{g}}$ induces an action on $\mathscr{H}$ generated by an unbounded hermitian operator $i(d/d\theta)$. Let us think of ΩG as a classical state-space, and $P(\mathscr{H})$ as the corresponding quantum state-space. For any loop γ let us choose a unit vector Ω_γ belonging to the ray $\pi(\gamma)$ in $\mathscr{H}$.

Proposition (8.9.11). *We have*

$$\mathscr{E}(\gamma)=\left\langle \Omega_\gamma, \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\theta}\Omega_\gamma \right\rangle,$$

where the classical energy $\mathscr{E}(\gamma)$ is defined using the Killing form on $\mathfrak{g}$.

Notice that the right-hand-side is the expected value of the quantum energy operator $\mathrm{i}\,(\mathrm{d}/\mathrm{d}\theta)$ in the state Ω_γ.

Proof. The result follows from the fact that the canonical Kähler structure of $P(\mathscr{H})$ induces the Kähler structure on ΩG corresponding to the Killing form on $\mathfrak{g}$. This, in turn, is true because the Kähler form of $P(\mathscr{H})$ restricts to the standard $U_{\mathrm{res}}(H^{\mathfrak{g}})$-invariant form on $\mathrm{Gr}(H^{\mathfrak{g}})$, which corresponds to the basic Lie algebra 2-cocycle of $U_{\mathrm{res}}(H^{\mathfrak{g}})$. The latter restricts to the basic 2-cocycle of $\Omega\mathfrak{u}_n$, and so—via the embedding of G in U_n by the adjoint action—to the 2-cocycle of ΩG associated to the Killing form.

Remark. It is instructive to derive (8.9.11) more directly from the formula (7.8.4), i.e. to prove that

$$\mathscr{E}(\gamma)=\operatorname{trace}\left\{\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\theta}\,.\,(\gamma J\gamma^{-1}-J)\right\},$$

where J is the Hilbert transform of (6.3.2). It is easy to check that the operator on the right has the kernel

$$-\frac{1}{2\pi}\frac{\partial}{\partial\theta}\{\cot\tfrac{1}{2}(\theta-\phi)\,.\,\gamma(\phi)^{-1}(\gamma(\theta)-\gamma(\phi))\}$$

at $(\theta,\phi)\in S^1\times S^1$. When $\theta=\phi$ this reduces to

$$\frac{1}{4\pi}\gamma(\theta)\gamma''(\theta),$$

from which (8.9.11) follows by taking the trace and integrating by parts.

8.10 ΩG and holomorphic bundles

The factorization theorems for loops give us a description of points of ΩG as holomorphic bundles on the Riemann sphere $S^2=\mathbb{C}\cup\infty$. (We shall write $S^2=D_0\cup D_\infty$, where $D_0=\{z:|z|\leqslant 1\}$ and $D_\infty=\{z:|z|\geqslant 1\}$.)

Proposition (8.10.1). *A point of ΩG is the same thing as an isomorphism class of pairs (P,τ), where P is a holomorphic principal $G_{\mathbb{C}}$-bundle on S^2, and τ is a trivialization of P over D_∞. The stratum to which (P,τ) belongs—in the sense of (8.6.5)—is simply the isomorphism class of P.*

Here a 'trivialization over D_∞' means a smooth cross-section of $P\mid D_\infty$ which is holomorphic over the interior of D_∞.

Proof of (8.10.1). Given (P,τ) we choose a trivialization σ of $P\mid D_0$. The transition function between σ and τ over the intersection $S^1=D_0\cap D_\infty$ is an element of $LG_{\mathbb{C}}$. But σ is indeterminate up to multiplication by an element of $L^+G_{\mathbb{C}}$, so we have an element of $LG_{\mathbb{C}}/L^+G_{\mathbb{C}}=\Omega G$.

Conversely, given $\gamma\in LG$, we can choose a factorization $\gamma=\gamma_-\,.\,\lambda\,.\,\gamma_+$, where $\lambda:S^1\to G$ is a homomorphism. Because λ extends to a holomorphic map $\lambda:\mathbb{C}^\times\to G_{\mathbb{C}}$ it defines a holomorphic bundle P_λ on S^2 (see Section 8.2(ii)) which is canonically trivial over $S^2-\{\infty\}$ and $S^2-\{0\}$. We assign to γ the pair (P_λ,τ), where $\tau=\gamma_+.\tau_\infty$, and τ_∞ is the canonical trivialization of $P_\lambda\mid D_\infty$.

A complex manifold X is completely described by giving the set of holomorphic maps $M\to X$ for every complex manifold M. So ΩG is completely described as a complex manifold by the following simple generalization of (8.10.1), which was pointed out to us by Atiyah (see [5]).

Proposition (8.10.2). *A holomorphic map $M\to\Omega G$, for any complex manifold M, is the same thing as an isomorphism class of pairs (P,τ), where P is a holomorphic principal $G_{\mathbb{C}}$-bundle on $M\times S^2$, and τ is a trivialization of $P\mid M\times D_\infty$.*

Proof. First suppose that we are given (P,τ). By Proposition (8.10.1) we know that $m\mapsto(P,\tau)\mid(m\times S^2)$ defines a map $f:M\to\Omega G$. We must show that f is holomorphic. But for any $m\in M$ we can suppose that P is trivial over a set of the form $V\times\tilde{D}_0$, where V is a neighbourhood of m in M and $\tilde{D}_0$ is $\{z\in S^2:|z|\leqslant r\}$ for some $r>1$. Then $f\mid V$ can be represented by the transition function between τ and a trivialization over $V\times\tilde{D}_0$. This is a smooth map

$$V\times\{z:1\leqslant|z|\leqslant r\}\to G_{\mathbb{C}}$$

which is holomorphic for $1<|z|<r$. Its restriction to $V\times S^1$ is therefore a holomorphic map $V\to LG_{\mathbb{C}}$, as we want.

Conversely, to obtain a pair (P,τ) from each map $f:M\to\Omega G$ it is enough to define a pair (P,τ) over $\Omega G\times S^2$ itself. The definition of P as a smooth bundle is clear: it is constructed by attaching trivial bundles on $\Omega G\times D_0$ and $\Omega G\times D_\infty$ by the clutching function given by the evaluation map $\varepsilon:\Omega G\times S^1\to G$. The resulting bundle has a canonical trivialization over $\Omega G\times D_\infty$. If we had taken ΩG to be the space of *real-analytic* loops instead of smooth ones it would be clear that P was a holomorphic bundle, and the proof would be complete: for in that case ε would extend to a holomorphic map $W_0\cap W_\infty\to G_{\mathbb{C}}$, where W_0 and W_∞ are suitable open neighbourhoods of $\Omega G\times D_0$ and $\Omega G\times D_\infty$ in $\Omega G\times S^2$. But to treat the *smooth* loops ΩG requires more care.

We recall from (8.6.3) that ΩG is covered by open sets U_λ such that $U_\lambda \cong \Sigma_\lambda \times C_\lambda$ as a complex manifold, and $C_\lambda \subset \Omega_{\text{pol}} G$. Thus C_λ is a space of holomorphic maps $\mathbb{C}^\times \to G_\mathbb{C}$. Composing the projection $U_\lambda \to C_\lambda$ with the evaluation map gives us a holomorphic map $U_\lambda \times \mathbb{C}^\times \to G_\mathbb{C}$ which defines a holomorphic bundle P_λ on $U_\lambda \times S^2$. Proposition (8.6.3) implies that P_λ is *canonically* isomorphic to $P \mid U_\lambda \times S^2$ as a smooth bundle. But the bundles P_λ fit together to define a holomorphic bundle on $\Omega G \times S^2$. That is the case because the canonical smooth isomorphism between the restrictions of P_λ and P_μ to $(U_\lambda \cap U_\mu) \times S^2$ is easily seen to be holomorphic when $|z|<1$ and when $|z|>1$, and is therefore holomorphic everywhere.

If M is a compact manifold then the trivialization τ of $P \mid M \times D_\infty$ is unique up to the action of a single element of $L^+G_\mathbb{C}$, for any holomorphic map $M \to L^+G_\mathbb{C}$ is constant. That gives us

Proposition (8.10.3). *If M is a compact complex manifold with a base point m_0 then the set of base-point-preserving holomorphic maps $M \to \Omega G$ can be identified with the set of isomorphism classes of $G_\mathbb{C}$-bundles P on $M \times S^2$ which are trivial over $m_0 \times S^2$ and $M \times D_\infty$.*

This result is the starting point of a circle of interesting ideas for which we refer the reader to Atiyah [5]. We shall content ourselves with pointing out one immediate corollary.

Proposition (8.10.4). *If M is a compact complex manifold with a base-point then each connected component of the space of base-point-preserving holomorphic maps $M \to \Omega G$ is finite dimensional.*

This follows from (8.10.3), because the space of all holomorphic bundles of a given topological type on a compact manifold is finite dimensional. (See Mumford and Fogarty [119].)

Proposition (8.10.4) reveals a striking difference between ΩG or $\text{Gr}^{(n)}(H)$ and more familiar infinite dimensional complex manifolds such as the projective space $P(H)$ or the Grassmannian $\text{Gr}(H)$ of Chapter 7. For example, if $v \in H$ represents the base-point in $P(H)$ then

$$(z_0, z_1) \mapsto [z_0 v + z_1 w]$$

is a family of base-point preserving holomorphic maps $S^2 \to P(H)$ which is parametrized by the space of non-zero vectors w orthogonal to v. A similar family can be defined for $\text{Gr}(H)$.

8.11 The homogeneous space associated to a Riemann surface: the moduli spaces of vector bundles

Throughout the last three chapters we have always polarized the Hilbert space $H = L^2(S^1; \mathbb{C})$ as $H_+ \oplus H_-$, where H_+ is the space of boundary values of holomorphic functions in the disc $|z|<1$. A rather natural generalization of this procedure is to replace the disc by some other Riemann surface whose boundary is a circle.

Suppose then that X is a compact Riemann surface with a distinguished point x_∞ and a given local parameter around x_∞. We shall write the local parameter as z^{-1}: thus z is a holomorphic map from a neighbourhood of x_∞ to a neighbourhood of ∞ in the Riemann sphere. We shall assume that $z(x_\infty) = \infty$, and that z is an isomorphism between a neighbourhood of x_∞ and the region $|z| > \frac{1}{2}$ on the Riemann sphere. The standard circle S^1 can then be identified with the circle $|z| = 1$ around x_∞ on X. We shall denote the part of X where $|z|>1$ by X_∞, and the complement of the region where $|z| \geq 1$ by X_0. Thus

$$\bar{X}_0 \cap \bar{X}_\infty = S^1.$$

Associated to these data there is a subspace $H_X^{(n)}$ of $H^{(n)} = L^2(S^1; \mathbb{C}^n)$ analogous to $H_+^{(n)}$. It is the closed subspace of $H^{(n)}$ consisting of the boundary values of holomorphic maps $X_0 \to \mathbb{C}^n$.

Proposition (8.11.1). *The space $H_X^{(n)}$ belongs to* Gr, *and has virtual dimension $-ng$, where g is the genus of X.*

We shall postpone the proof of this, and shall go on to characterize the orbit of $H_X^{(n)}$ under the complex loop group $LGL_n(\mathbb{C})$ in analogy with the description of $\text{Gr}^{(n)}$ in (8.3.1) as $\{W \in \text{Gr}(H^{(n)}) : zW \subset W\}$.

Let A_X denote the ring of rational functions on X which are holomorphic everywhere except for a pole of arbitrary order at x_∞. Using the local parameter z we think of A_X as a ring of functions of z. As such it acts by multiplication operators on the Hilbert space $H^{(n)}$, and hence on $\text{Gr}(H^{(n)})$.

Definition (8.11.2).

$$\text{Gr}^{(n),X} = \{W \in \text{Gr}(H^{(n)}) : A_X . W \subset W\}.$$

If X is the Riemann sphere then A_X is the polynomial ring $\mathbb{C}[z]$, and then $\text{Gr}^{(n),X}$ coincides with $\text{Gr}^{(n)}$. We shall prove that $\text{Gr}^{(n),X}$ is always a homogeneous space of $LGL_n(\mathbb{C})$.

Let us recall, to begin with, that A_X is a finitely generated algebra which is filtered by the order of the pole at x_∞. It contains, up to a scalar multiple, exactly one element of each sufficiently large degree. More precisely, if $A_X^{(d)}$ is the vector space of functions in A_X with poles of orders $\leq d$, then the Weierstrass 'gap' theorem ([68] p. 273) asserts that $A_X^{(d)}$ has dimension $d+1-g$, where g is the genus of X, providing $d \geq 2g$.

Proposition (8.11.3). $\text{Gr}^{(n),X}$ *is the orbit of $H_X^{(n)}$ under $LGL_n(\mathbb{C})$.*

Proof. Suppose that W belongs to $\text{Gr}^{(n),X}$. The subspace W^{fin} of functions of finite order in W, which is dense in W by Proposition (7.3.2), is a module over A_X. Like A_X the module W^{fin} has an increasing filtration $\{W^{(k)}\}$ by the order of the pole at x_∞. We know that $\dim(W^{(k)}/W^{(k-1)}) = n$ for all large k, and it follows that W^{fin} is a finitely generated A_X-module. It is obviously torsion-free, and hence projective (because A_X is a Dedekind ring). This means that it is the module of algebraic sections of an algebraic vector bundle E on $X - \{x_\infty\}$ whose fibre E_x at x is $W^{\text{fin}}/\mathfrak{a}_x W^{\text{fin}}$, where $\mathfrak{a}_x = \{f \in A_X : f(x) = 0\}$. As a holomorphic bundle E is necessarily trivial, for there are no non-trivial holomorphic vector bundles on an affine curve. Thus there are holomorphic sections $w_1, \ldots, w_n$ in W whose values at any point $x \in X - \{x_\infty\}$ span the fibre E_x. In particular the matrix $(w_1(z), \ldots, w_n(z))$ is invertible at each point z of the circle, and defines a loop $\gamma : S^1 \to GL_n(\mathbb{C})$ such that $\gamma H_X^{(n)} = W$.

The stabilizer of $H_X^{(n)}$ in $LGL_n(\mathbb{C})$ is obviously the group $L_X^+ GL_n(\mathbb{C})$ of loops which are the boundary values of holomorphic maps $X_0 \to GL_n(\mathbb{C})$. So the preceding proposition gives us

$$\mathrm{Gr}^{(n),X} \cong LGL_n(\mathbb{C})/L_X^+ GL_n(\mathbb{C}). \tag{8.11.4}$$

The proof of the proposition shows also that a point of $\mathrm{Gr}^{(n),X}$ can be identified with an isomorphism class of pairs (E, α), where E is a holomorphic vector bundle on X and α is a trivialization of $E \mid X_\infty$ which extends smoothly to $\bar{X}_\infty$. The natural action of $L^- GL_n(\mathbb{C})$ on $\mathrm{Gr}^{(n),X}$ permutes the trivializations α transitively, so we have the following generalization of the Birkhoff factorization theorem.

Proposition (8.11.5). *The set of double cosets*

$$L^- GL_n(\mathbb{C}) \backslash LGL_n(\mathbb{C}) / L_X^+ GL_n(\mathbb{C})$$

is the set of isomorphism classes of n-dimensional holomorphic vector bundles on X.

It is in fact more convenient and usual to consider the double coset space $L_1^- \backslash L / L_X^+$. This is the set of isomorphism classes of bundles E with a chosen identification of the fibre at x_∞ with $\mathbb{C}^n$. It is a better space than $L^- \backslash L / L_X^+$ because L_1^- acts freely on a dense open set $\mathcal{U}$ of $L/L_X^+ = \mathrm{Gr}_X^{(n)}$, and is a contractible group. The quotient space, $L_1^- \backslash \mathcal{U}$, which is homotopy equivalent to $\mathcal{U}$, is the *moduli space* of n-dimensional bundles in the sense of Mumford [119]. (Cf. also Atiyah and Bott [7].)

It is interesting to consider the homotopy type of the group L_X^+ and the homogeneous space $\mathrm{Gr}_X^{(n)}$. Any loop γ belonging to L_X^+ has winding number zero, i.e. is contractible to a point, for if $\det(\gamma)$ is the boundary value of a holomorphic function in X_0 then its winding number is the number of zeros of $\det(\gamma)$ in X_0. On the other hand L_X^+ is not connected, for $\det(\gamma)$ has a well-defined integral winding number around each non-trivial loop in X_0. In fact the group of components of L_X^+ is given by

$$\pi_0(L_X^+) \cong \mathbb{Z}^{2g} \cong \mathrm{Hom}(\pi_1(X_0); \mathbb{Z}) \cong H^1(X; \mathbb{Z}),$$

where g is the genus of X. This is the group of homotopy classes of maps $X_0 \to \mathbb{C}^\times$, or equivalently of maps $X_0 \to GL_n(\mathbb{C})$. An even stronger result is true, namely

Proposition (8.11.6).

(i) *The group $L_X^+ GL_n(\mathbb{C})$ is homotopy equivalent to the group of continuous maps $X_0 \to GL_n(\mathbb{C})$.*

(ii) *The space $\mathrm{Gr}_X^{(n)}$ is homotopy equivalent to the space of base-point preserving maps $X \to BGL_n(\mathbb{C})$, i.e. to the space of topological $\mathbb{C}^n$-bundles on X (with the fibre at x_∞ identified with $\mathbb{C}^n$).*

In part (ii) of this proposition $BGL_n(\mathbb{C})$ denotes the classifying space of the group $GL_n(\mathbb{C})$. Thinking of the space of maps $X \to BGL_n(\mathbb{C})$ as the 'space' of vector bundles on X can be justified from various points of view. The essential point is that for a given bundle E on X the space of pairs (f, α), where $f : X \to BGL_n(\mathbb{C})$ and $\alpha : E \to f^* E_{\text{univ}}$ is an isomorphism between E and the pull-back of the universal bundle E_{univ} on $BGL_n(\mathbb{C})$, is contractible. This implies that the space of maps $\mathrm{Map}(X; BGL_n)$ is homotopy equivalent to the 'space' or 'realization' of the category of vector bundles on X in the sense of [128]. More explicitly, the space of maps has one connected component for each isomorphism class of bundles on X, and the component corresponding to a bundle E has the homotopy type of $B\,\mathrm{Aut}(E)$, the classifying space of the 'gauge group' $\mathrm{Aut}(E)$ of all automorphisms of E. For more information about this space we refer to Atiyah and Bott [7].

Assertion (ii) above follows immediately from (i). For L/L_X^+ is homotopy equivalent to the fibre of $BL_X^+ \to BL$, i.e. to the fibre of

$$\mathrm{Map}(X_0; BGL_n) \to \mathrm{Map}(S^1; BGL_n).$$

The cofibration sequence $S^1 \to X_0 \to X$ shows that this is $\mathrm{Map}(X; BGL_n)$.

Proof of (i). In order not to go too far afield we shall content ourselves with indicating how the statement follows simply from the results of [130].

The proof is by induction on n. Let us first consider the case $n = 1$. We have to prove that $\mathrm{Hol}(X_0; \mathbb{C}^\times)$ is homotopy equivalent to $\mathrm{Map}(X_0; \mathbb{C}^\times)$, where Hol denotes the holomorphic maps which extend smoothly to $\bar{X}_0$, and Map denotes continuous maps. (It is permissible, and more convenient, to replace $\mathrm{Map}(X_0; \mathbb{C}^\times)$ by $\mathrm{Map}(\bar{X}_0; \mathbb{C}^\times)$.) The exact sequence of groups

$$0 \to \mathbb{Z} \to \mathrm{Hol}(X_0; \mathbb{C}) \xrightarrow{\exp} \mathrm{Hol}(X_0; \mathbb{C}^\times) \to H^1(X_0; \mathbb{Z}) \tag{8.11.7}$$

shows that each connected component of $\mathrm{Hol}(X_0; \mathbb{C}^\times)$ has the homotopy type of a circle, as is also the case for the continuous maps. It is therefore enough to prove that the right hand map in (8.11.7) is surjective. That is true because its cokernel is $H^1(X_0; \mathcal{O})$, where $\mathcal{O}$ is the sheaf of germs of holomorphic functions on X_0. This group vanishes because X_0 is a Stein manifold.

For the inductive step we consider the holomorphic fibration

$$GL_{n-1,1} \to GL_n \to P^{n-1},$$

where $GL_{n-1,1}$ is the group of echelon matrices which is the stabilizer of a one-dimensional subspace of $\mathbb{C}^n$, and P^{n-1} is $(n-1)$-dimensional complex projective space. To prove that $\mathrm{Hol}(X_0; GL_n)$ is homotopy equivalent to $\mathrm{Map}(X_0; GL_n)$ it is enough to show that the sequence

$$\mathrm{Hol}(X_0; GL_{n-1,1}) \to \mathrm{Hol}(X_0; GL_n) \to \mathrm{Hol}(X_0; P^{n-1}) \qquad (8.11.8)$$

is a fibration (e.g. to show that it has local cross-sections) and also that the inclusions

$$\mathrm{Hol}(X_0; GL_{n-1,1}) \to \mathrm{Map}(\bar{X}_0; GL_{n-1,1})$$

and

$$\mathrm{Hol}(X_0; P^{n-1}) \to \mathrm{Map}(\bar{X}_0; P^{n-1}) \qquad (8.11.9)$$

are equivalences; for the sequence of spaces of continuous maps analogous to (8.11.8) is trivially a fibration. Now $GL_{n-1,1}$ is isomorphic to $GL_{n-1} \times \mathbb{C}^\times \times \mathbb{C}^{n-1}$ as a complex manifold, so $\mathrm{Hol}(X_0; GL_{n-1,1})$ is equivalent to the product

$$\mathrm{Hol}(X_0; GL_{n-1}) \times \mathrm{Hol}(X_0; \mathbb{C}^\times),$$

and hence to $\mathrm{Map}(X_0; GL_{n-1,1})$ by the inductive hypothesis. On the other hand it is proved in [130] that the map (8.11.9) is an equivalence. (Strictly speaking, the proof in [130] applies to the holomorphic maps $f: X_0 \to P^{n-1}$ which extend *real-analytically* to $\bar{X}_0$, for it supposes that the homogeneous coordinates $f_1, \ldots, f_n$ of f have only finitely many zeros. But it is easy to see that the space of sets of n subsets $S_1, \ldots, S_n$ of $\bar{X}_0$ which have empty intersection and no points of accumulation in X_0 is homotopy equivalent to the space $Q^{(n)}(\bar{X}_0)$ of [130] by the map $S_i \mapsto S_i \cap \bar{X}_0^\varepsilon$, where $\bar{X}_0^\varepsilon$ is obtained from $\bar{X}_0$ by deleting a collar of small width ε around the boundary.)

It remains to explain why (8.11.8) is a fibration. The second map is surjective because a holomorphic map $X_0 \to P^{n-1}$ is the same thing as a holomorphic line sub-bundle L of $X \times \mathbb{C}^n$. To lift the map to GL_n is to find an isomorphism between the exact sequence

$$L \to X_0 \times \mathbb{C}^n \to (X_0 \times \mathbb{C}^n)/L$$

and the trivial sequence

$$X_0 \times \mathbb{C} \to X_0 \times \mathbb{C}^n \to X_0 \times \mathbb{C}^{n-1}.$$

This can be done because X_0 is a Stein manifold. In view of the surjectivity and the fact that the total space of the bundle is a group, it is enough to prove that (8.11.8) has a local cross-section near a constant map $X_0 \to P^{n-1}$. But that is obvious because $GL_n \to P^{n-1}$ has holomorphic cross-sections.

Proof of (8.11.1). We conclude this section by giving the omitted proof of Proposition (8.11.1). We shall actually prove a slightly more general result.

Proposition (8.11.10). *Let E be an n-dimensional holomorphic vector bundle on X with a given trivialization in a neighbourhood of X_∞. Let W be the closed subspace of $H^{(n)} = L^2(S^1; \mathbb{C}^n)$ consisting of functions which are the boundary values of holomorphic sections of E over X_0. Then W belongs to $\mathrm{Gr}_\omega(H^{(n)})$, and its virtual dimension is*

$$\dim H^0(X; \mathscr{E}) - \dim H^1(X; \mathscr{E}) - n,$$

where $\mathscr{E}$ is the sheaf of holomorphic sections of E. In fact $H^0(X; \mathscr{E})$ and $H^1(X; \mathscr{E})$ are respectively the kernel and cokernel of the orthogonal projection $W \to zH_+$.

Proof. We observe first that the projection $\mathrm{pr}: W \to H_-$ factorizes

$$W \xrightarrow{R_{\rho^{-1}}} H^{(n)} \xrightarrow{\mathrm{pr}} H_- \xrightarrow{R_\rho} H_-,$$

for some ρ such that $0 < \rho < 1$. Here R_ρ is the operator of Section 7.6 such that $R_\rho z^k = \rho^{-k} z^k$; the operator $R_{\rho^{-1}}: W \to H^{(n)}$ is bounded because it assigns to the boundary value of a holomorphic section ϕ of $\mathscr{E}$ over X the function $z \mapsto \phi(\rho z)$, i.e. the value of ϕ on a circle slightly inside the boundary of X_0. The operator $R_\rho: H_- \to H_-$ is compact, so the projection $\mathrm{pr}: W \to H_-$ is compact. It follows that the projection $W \to H_+$ has closed image.

Now let U_0 and U_∞ be open sets of X slightly larger than $\bar{X}_0$ and $\bar{X}_\infty$. Because U_0 and U_∞ are Stein manifolds the kernel and cokernel of the map

$$\mathscr{E}(U_0) \oplus \mathscr{E}(U_\infty) \to \mathscr{E}(U_0 \cap U_\infty)$$

taking (ϕ_0, ϕ_∞) to $\phi_0 - \phi_\infty$ can be identified with $H^0(X; \mathscr{E})$ and $H^1(X; \mathscr{E})$. Passing to the direct limit as U_0 and U_∞ shrink towards $\bar{X}_0$ and $\bar{X}_\infty$ we find that the same groups are the kernel and cokernel of

$$W^{\mathrm{an}} \oplus zH_-^{\mathrm{an}} \to (H^{(n)})^{\mathrm{an}},$$

and hence of

$$\mathrm{pr}: W^{\mathrm{an}} \to zH_+^{\mathrm{an}}. \qquad (8.11.11)$$

(Here W^{an} denotes the set of real-analytic functions in W, and so on.) The kernel of (8.11.11) is the same as that of $\mathrm{pr}: W \to zH_+$, for an element of the kernel of the latter is the common boundary value of holomorphic functions in X_0 and X_∞. But we know that $W \to zH_+$ has closed image, so its cokernel must also coincide with the cokernel of (8.11.11). This essentially completes the proof: it remains only to observe that W belongs to Gr_ω because it is of the form $R_\rho \tilde{W}$ for some $\rho < 1$, where $\tilde{W}$ is the analogue of W constructed from the circle $|z| = \rho$ on X.

8.12 Appendix: Scattering theory

The Grassmannian interpretation of loop groups arises in the approach to 'scattering theory' developed by Lax and Phillips [99]. We shall give a very brief account of its role there.

Suppose that we are studying the solutions of a wave equation

$$\frac{\partial^2 \psi}{\partial t^2} - \frac{\partial^2 \psi}{\partial x^2} + \rho(x)\psi = 0, \tag{8.12.1a}$$

where ψ is a complex-valued function of x and t, and ρ is a positive real-valued function independent of t which vanishes outside a finite interval. We think of the equation as describing waves which are scattered by an obstacle described by ρ. Intuitively it seems plausible that if a solution ψ is fairly well localized in space at time $t = 0$ then after a long period the solution will (in the sense of its energy, to be defined presently) be concentrated mainly in the region where ρ is zero. That is to say, we expect that an arbitrary solution of (8.12.1a) will, for large positive t, be close to a definite solution of the 'unperturbed equation'

$$\frac{\partial^2 \psi}{\partial t^2} - \frac{\partial^2 \psi}{\partial x^2} = 0. \tag{8.12.1b}$$

We expect that the same thing will be true also for large negative t.

Now let V be the vector space of solutions of (8.12.1a) and V_0 that of (8.12.1b): to begin with we shall consider solutions which have compact support in x for each t. We expect that there will be two isomorphisms $T_\pm : V \to V_0$ which assign to a solution ψ the solutions of the unperturbed equation to which ψ is asymptotic as $t \to \pm\infty$. The composite

$$S = T_+ \circ T_-^{-1} : V_0 \to V_0$$

is called the *scattering matrix* of the original equation; from one point of view it obviously gives a good description of the behaviour of the solutions. (It would not be reasonable to expect T_+ and T_- to be isomorphisms if the equation (8.12.1a) had 'bound states', i.e. if the operator $-\partial^2/\partial x^2 + \rho$ had negative eigenvalues; but that is excluded by the positivity of ρ.)

Because ρ is independent of t there is a one-parameter group of transformations $\{U_t\}$ of V defined by time-translation. This action preserves the energy-norm

$$\|\psi\|_\rho^2 = \frac{1}{2}\int_{-\infty}^{\infty} \left\{ \left(\frac{\partial \psi}{\partial t}\right)^2 + \left(\frac{\partial \psi}{\partial x}\right)^2 + \rho\psi^2 \right\} dx;$$

here the integral is taken along any line $t = \text{constant}$. We can complete V using this norm to get a Hilbert space H with a unitary group of transformations; and V_0 can be completed similarly to get H_0. The transformations $T_\pm$ are isometries, and they clearly commute with time translations; so S is a unitary transformation which commutes with time translation.

A solution of (8.12.1b) can be analysed by making a Fourier transformation in t:

$$\psi(t, x) = \int_{-\infty}^{\infty} \psi_\omega(x) e^{i\omega t}\, d\omega. \tag{8.12.2}$$

Here ψ_ω belongs to the two-dimensional space of solutions of

$$\psi_\omega'' + \omega^2 \psi_\omega = 0,$$

which can be identified with $\mathbb{C}^2$ by mapping ψ_ω to $(\psi_\omega(0), \psi_\omega'(0))$. Thus H_0 can be identified with the Hilbert space $L^2(\mathbb{R}; \mathbb{C}^2)$ in such a way that the time translation U_t in H_0 is given by multiplication by the function $e^{i\omega t}$, where ω is the coordinate in $\mathbb{R}$. By a simple variant of Proposition (6.1.1) we know that the unitary transformations of H_0 which commute with all U_t are the group of measurable maps $\mathrm{Map}_{\mathrm{meas}}(\mathbb{R}; U_2)$. The scattering matrix S is therefore an element of this group, which is a kind of loop group. (If S corresponds to a map $\sigma: \mathbb{R} \to U_2$ then $\sigma(\omega)$ describes the scattering of waves of frequency ω. As very high-frequency waves are comparatively unaffected by the potential ρ we shall have $\sigma(\omega) \to 1$ as $\omega \to \pm\infty$, which justifies our regarding σ as a loop.)

The relevance of this discussion to the Grassmannian model of loops comes from the theorem that to give an isomorphism between a Hilbert space H with a unitary group $\{U_t\}$ and a standard space $L^2(\mathbb{R}; K)$ with its multiplication group $\{e^{i\omega t}\}$ is precisely the same thing as to prescribe what is called an *outgoing subspace* in H. (Here K is an unspecified auxiliary Hilbert space.) The standard outgoing subspace H_0^+ in H_0 is the closure of the solutions ψ such that $\psi(t, 0) = 0$ for $t < 0$. When H_0 is identified by the Fourier transform (8.12.2) with the $\mathbb{C}^2$-valued functions of ω the space H_0^+ consists of the boundary values of functions holomorphic in the half-plane $\mathrm{Im}(\omega) < 0$.

Definition (8.12.3). *An outgoing subspace in a Hilbert space H with a* *one-parameter unitary group $\{U_t\}$ is a closed subspace W of H such that*

(i) $U_t(W) \subset W$ *when* $t \geq 0$,

(ii) $\bigcap_{t \geq 0} U_t(W) = 0$,

(iii) $\bigcup_{t \leq 0} U_t(W)$ *is dense in* H.

The basic theorem of Lax–Phillips scattering theory [99] is that when the data $\{H, \{U_t\}, W\}$ are given one can construct a Hilbert space K and a canonical isomorphism of the data with the standard data $\{H_0 = L^2(\mathbb{R}; K), \{e^{i\omega t}\}, H_0^+\}$. In other words, instead of giving the two maps $T_\pm : H \to H_0$ it is equally informative simply to prescribe two subspaces $W_\pm$ in H. Intuitively, W_+, which is mapped by T_+ to H_0^+, consists of 'outgoing waves', while W_- consists of 'incoming waves', and is mapped to $(H_0^+)^\perp$ by T_-.

We shall not prove the theorem here. The variant of it which is directly related to loop groups is that where the continuous group $\{U_t\}_{t \in \mathbb{R}}$ is replaced by a discrete group $\{u^k\}_{k \in \mathbb{Z}}$. This is very easy to prove. The standard model is then the space $H_0 = L^2(S^1; K)$, the group is generated by multiplication by z, and H_0^+ has the meaning which is usual in this book. When $\{H, u, W\}$ is given one can determine K as $W \ominus u(W)$. The theorem reduces essentially to the following.

Proposition (8.12.4). *Let K be a Hilbert space, and $U(K)$ its unitary group. Then the measurable loop group $\Omega_{\text{meas}} U(K) = L_{\text{meas}} U(K)/U(K)$ can be identified with the set of closed subspaces W of $L^2(S^1; K)$ such that*

(i) $zW \subset W$,

(ii) $\bigcap_{k \geq 0} z^k W = 0$,

(iii) $\bigcup_{k \leq 0} z^k W$ *is dense in* H.

The proof of (8.3.2) included a proof of this result. In fact the hardest step in proving (8.3.2) was to show that a space $W \in \text{Gr}(H^{(n)})$ satisfies the conditions of (8.12.4).

Remark. We have pointed out that there is no simple model for $\Omega_{\text{cts}} U_n$. The present result, however, shows that there is a—not very explicit—model for the space of measurable loops.

PART II