If $G = U_n$ then the generators of the ring of invariant polynomials can be taken to be $P_1, P_2, \ldots, P_n$, where $P_k(A) = \text{trace}(A^k)$.

It is a fairly easy result of algebraic topology [16] that the cohomology $H^*(\Omega G; \mathbb{R})$ of the space of based loops on a simply connected group G is a polynomial algebra on the even dimensional classes obtained by transgressing the generators of $H^*(G; \mathbb{R})$, i.e. by pulling them back to $S^1 \times \Omega G$ by the evaluation map, and then integrating over S^1. The class so obtained from (4.11.1) is the $(2k-2)$-form on ΩG whose value at $\gamma \in \Omega G$ on tangent vectors represented by $\xi_1, \ldots, \xi_{2k-2} \in \Omega\mathfrak{g}$ is

$$\frac{1}{2\pi}\int_0^{2\pi} S(\xi_1(\theta), \xi_2(\theta), \ldots, \xi_{2k-2}(\theta), \gamma(\theta)^{-1}\gamma'(\theta))\, \mathrm{d}\theta. \quad (4.11.2)$$

This form is naturally defined on LG. The cohomology $H^*(LG)$ is simply the tensor product $H^*(G) \otimes H^*(\Omega G)$, because $LG \cong G \times \Omega G$ as a space.

The differential form (4.11.2) is evidently not left-invariant, and we have no reason to expect that the cohomology of LG can be represented by left-invariant forms. Nevertheless we have

Proposition (4.11.3). *The $(2k-2)$-form (4.11.2) on LG is cohomologous to a rational multiple of the left-invariant form obtained by making skew the map*

$$(\xi_1, \ldots, \xi_{2k-2}) \mapsto \frac{1}{2\pi}\int_0^{2\pi} P([\xi_1, \xi_2], \ldots, [\xi_{2k-5}, \xi_{2k-4}], \xi_{2k-3}, \xi'_{2k-2})\, \mathrm{d}\theta.$$

Corollary (4.11.4). *The natural map*

$$H^*(L\mathfrak{g}; \mathbb{R}) \to H^*(LG; \mathbb{R})$$

is surjective.

Remarks. Actually the map of (4.11.4) is an isomorphism. We shall prove that in Section 14.6. (Cf. also Kumar [97].) The result should be contrasted with our discovery in Section 4.2 that $H^2(\text{Map}(X; \mathfrak{g}))$ is vastly larger that $H^2(\text{Map}(X; G))$ when $\dim(X) > 1$. Quillen has pointed out to us that the class in $H^{2k-d-1}(\text{Map}(X; G))$ which is obtained by pulling back the class (4.11.1) by the evaluation map $X \times \text{Map}(X; G) \to G$ and integrating it over a cycle of dimension d in X can be represented by a left-invariant form if $k > d$, but usually not otherwise.

Proof of (4.11.3). Let us introduce some more convenient notation, as follows. When we pull back the Maurer–Cartan 1-form $g^{-1}\,\mathrm{d}g$ on G (with values in $\mathfrak{g}$) by the evaluation map $S^1 \times LG \to G$ we shall write the resulting form as $\xi + \eta$, where ξ vanishes on tangent vectors in the S^1-direction and η vanishes along LG. (Thus η is $\gamma(\theta)^{-1}\gamma'(\theta)\,\mathrm{d}\theta$ at $(\theta, \gamma) \in S^1 \times LG$.) In this notation the forms of (4.11.2) and (4.11.3) are

obtained (up to rational multiples) by integrating over S^1 the forms

$$\Theta = P([\xi, \xi], \ldots, [\xi, \xi], \eta)$$

and

$$\Phi = P([\xi, \xi], \ldots, [\xi, \xi], \xi, \mathrm{d}'\xi),$$

respectively on $S^1 \times LG$. (We write d′ and d″ for differentiation of forms in the S^1 and LG directions respectively.)

Because $\mathrm{d}(g^{-1}\,\mathrm{d}g) = -\frac{1}{2}[g^{-1}\,\mathrm{d}g, g^{-1}\,\mathrm{d}g]$ on G we find

$$\mathrm{d}'\eta = -\tfrac{1}{2}[\eta, \eta],$$
$$\mathrm{d}''\xi = -\tfrac{1}{2}[\xi, \xi],$$

and

$$\mathrm{d}'\xi + \mathrm{d}''\eta = -[\xi, \eta].$$

Now consider the form $\Psi = P([\xi, \xi], \ldots, [\xi, \xi], \xi, \eta)$ on $S^1 \times LG$. We have $\mathrm{d}''[\xi, \xi] = 0$, so

$$\mathrm{d}''\Psi = -\tfrac{1}{2}P([\xi, \xi], \ldots, [\xi, \xi], \eta) + P([\xi, \xi], \ldots, [\xi, \xi], \xi, \mathrm{d}'\xi) + P([\xi, \xi], \ldots, [\xi, \xi], \xi, [\xi, \eta]).$$

Using the invariance of the polynomial P, and the fact that $[[\xi, \xi], \xi] = 0$ because of the Jacobi identity, the third term on the right-hand-side is equal to Θ, so that we have

$$\mathrm{d}\Psi = \tfrac{1}{2}\Theta + \Phi.$$

Integrating this relation over S^1 gives the desired result.

5

THE ROOT SYSTEM: KAC–MOODY ALGEBRAS

In general it is a feature of our approach to loop groups that it does not involve any detailed analysis of the structure of the Lie algebras of the groups. These algebras are examples of what are called *Kac–Moody Lie algebras,*† and there is a very extensive literature devoted to their study. (Cf. Kac [86], Macdonald [109], Helgason [72].) In this chapter we have attempted to explain fairly briefly how loop groups fit into that context. The material in Sections 5.1 and 5.2 will be used later in classifying the representations of loop groups, but the Kac–Moody theory proper which is sketched in Section 5.3 will not be referred to again.

5.1 The root system and the affine Weyl group

We have explained in Chapter 2 that the crucial step in studying the Lie algebra $\mathfrak{g}$ of a compact Lie group G is to decompose the complexification $\mathfrak{g}_{\mathbb{C}}$ under the adjoint action of a maximal torus T of G. One has

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha} \mathfrak{g}_{\alpha},$$

where $\mathfrak{t}_{\mathbb{C}}$ is the complexified Lie algebra of T and $\mathfrak{g}_{\alpha}$ is the one-dimensional subspace of $\mathfrak{g}_{\mathbb{C}}$ where T acts by the homomorphism $\alpha: T \to \mathbb{T}$. The homomorphisms α which occur in the decomposition are called the *roots* of G. They form a finite subset of the lattice $\hat{T} = \mathrm{Hom}(T; \mathbb{T})$.

The most obvious decomposition of the complexified algebra $L\mathfrak{g}_{\mathbb{C}}$ of a loop group is into its Fourier components:

$$L\mathfrak{g}_{\mathbb{C}} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_{\mathbb{C}} \,.\, z^k.$$

This is the decomposition into eigenspaces of the action of the circle $\mathbb{T}$ which rotates the loops bodily. The rotation action commutes with the adjoint action of *constant* loops, so we can decompose $L\mathfrak{g}_{\mathbb{C}}$ further according to the action of a maximal torus T of G:

$$L\mathfrak{g}_{\mathbb{C}} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{t}_{\mathbb{C}} z^k \oplus \bigoplus_{(k,\alpha)} \mathfrak{g}_{\alpha} z^k. \tag{5.1.1}$$

† The algebras were first studied, almost simultaneously, by Kantor [89], Kac [82], and Moody [117].

The pieces in this decomposition are indexed by homomorphisms $\mathbb{T} \times T \to \mathbb{T}$, i.e. by elements of $\mathbb{Z} \times \hat{T}$. Once again the homomorphisms (k, α) which occur (with α possibly 0) are called the *roots* of LG.

We can reformulate what has just been said by introducing the group $\mathbb{T} \tilde{\times} LG$, the semidirect product of $\mathbb{T}$ and LG in which $\mathbb{T}$ acts on LG by rotating the loops. The centralizer of $\mathbb{T}$ in $\mathbb{T} \tilde{\times} LG$ is $\mathbb{T} \times G$, and so $\mathbb{T} \times T$ is a maximal abelian subgroup of $\mathbb{T} \tilde{\times} LG$. The complexified Lie algebra of $\mathbb{T} \tilde{\times} LG$ decomposes as

$$(\mathbb{C} \oplus \mathfrak{t}_{\mathbb{C}}) \oplus \bigoplus_{k \neq 0} \mathfrak{t}_{\mathbb{C}} . z^k \oplus \bigoplus_{(k,\alpha)} \mathfrak{g}_{\alpha} z^k,$$

according to the characters of $\mathbb{T} \times T$.

In the finite dimensional case the roots of G are permuted by the Weyl group. This is the group of automorphisms of T which arise from conjugation in G, i.e. $W = N(T)/T$, where $N(T) = \{n \in G : nTn^{-1} = T\}$ is the normalizer of T in G. If n is an element of $N(T)$ then $n . \mathfrak{g}_{\alpha} = \mathfrak{g}_{n\alpha}$, where $n\alpha : T \to \mathbb{T}$ is given by $n\alpha(t) = \alpha(n^{-1}tn)$.

In exactly the same way the infinite set of roots of LG is permuted by $W_{\mathrm{aff}} = N(\mathbb{T} \times T)/(\mathbb{T} \times T)$, where $N(\mathbb{T} \times T)$ denotes the normalizer in $\mathbb{T} \tilde{\times} LG$. The group W_{aff} is called the *affine Weyl group*, for reasons we shall explain in a moment. Its structure is described by the following proposition. We shall denote the 'coweight' lattice of G by $\check{T}$: it is the lattice of all homomorphisms $\mathbb{T} \to T$. (See Section 2.4.)

Proposition (5.1.2). *W_{aff} is the semidirect product of $\check{T}$ by W, the Weyl group of G.*

Proof. The lattice $\check{T}$ is a subgroup of LG, and obviously centralizes T. On the other hand, if R_u is the operation of rotating by u (i.e. $R_u \in \mathbb{T} \subset \mathbb{T} \tilde{\times} LG$) then for any $f \in LG$ we have

$$f . R_u . f^{-1} = R_u . \phi, \tag{5.1.3}$$

where $\phi(z) = f(uz)f(z)^{-1}$. If f is a homomorphism $\mathbb{T} \to T$ then $\phi(z)$ is the constant $f(u) \in T$, and so $\check{T} \subset N(\mathbb{T} \times T)$. Conversely, if $f \in LG$ belongs to $N(\mathbb{T} \times T)$ then $f(uz)f(z)^{-1}$ must be constant as a function of z for each u, which implies that $z \mapsto f(z)f(1)^{-1}$ is a homomorphism $\mathbb{T} \to T$. Furthermore $f(1)$ must belong to the normalizer N of T in G. It follows that $N(\mathbb{T} \times T)$ is in G, and this proves (5.1.2).

In the finite dimensional theory one usually thinks of the lattice $\hat{T}$, and hence the roots, as lying in the real vector space $\mathfrak{t}^*$, identifying a homomorphism $\alpha : T \to \mathbb{T}$ with the linear map $\dot{\alpha} : \mathfrak{t} \to \mathbb{R}$ such that $\alpha = e^{i\dot{\alpha}}$. One can think of the roots of LG similarly as linear forms on the Lie algebra $\mathbb{R} \times \mathfrak{t}$ of $\mathbb{T} \times T$. But it is more convenient to regard them as

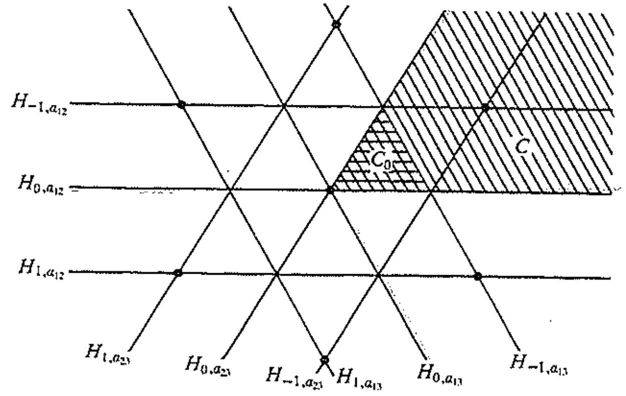

Fig. 2

affine-linear functions on $\mathfrak{t}$, identifying $\mathfrak{t}$ with the affine hyperplane $1 \times \mathfrak{t}$ in $\mathbb{R} \times \mathfrak{t}$. For this reason, and also to distinguish them from the roots of G, the roots of LG are often called *affine roots*. The group W_{aff} acts linearly on $\mathbb{R} \times \mathfrak{t}$: the action of W is obvious, and from (5.1.3) the action of $\lambda \in \check{T}$ is given by

$$\lambda \,.\, (x, \xi) = (x, \xi + x\lambda).$$

Thus W_{aff} preserves the hyperplane $1 \times \mathfrak{t}$, and $\lambda \in \check{T}$ acts on it by translation by the vector $\lambda \in \check{T} \subset \mathfrak{t}$.

If $\alpha \neq 0$, the affine root (k, α), regarded as an affine-linear function on $\mathfrak{t}$, is determined up to sign by the affine hyperplane

$$H_{k,\alpha} = \{\xi \in \mathfrak{t} : \langle \alpha, \xi \rangle = -k\}$$

in $\mathfrak{t}$ where it vanishes. This collection of hyperplanes is called the *diagram* of LG. The picture when $G = SU_3$ is shown in Fig. 2.

The connected components in $\mathfrak{t}$ of the complement of the hyperplanes $H_{k,\alpha}$ are called the *alcoves* of the diagram. Recall that the components of the complement of the $H_{0,\alpha}$ (which form the diagram of G) are called the *chambers*. Each chamber C contains a unique alcove C_0 whose closure contains the origin. If one chooses a chamber C then the roots of G are called positive or negative according to their sign on C. The set

$$\{\xi \in \mathfrak{t} : 0 < \alpha(\xi) < 1 \text{ for all positive roots } \alpha\}$$

is the corresponding alcove C_0. An affine root is called positive or negative according to its sign on C_0. The positive affine roots corresponding to the walls of C_0 are called the *simple* affine roots.

If G is semisimple then it is well-known that each chamber C is a simplicial cone, bounded by ℓ hyperplanes $H_{0,\alpha_1}, \ldots, H_{0,\alpha_\ell}$ corresponding to the simple roots $\alpha_1, \ldots, \alpha_\ell$ of G. Here ℓ is the dimension of $\mathfrak{t}$, called the *rank* of G. If G is a simple group then it has a highest root $\alpha_{\ell+1}$ (the highest weight of the adjoint representation) which dominates all the other positive roots. In that case the alcove C_0 is an ℓ-dimensional simplex bounded by $H_{0,\alpha_1}, \ldots, H_{0,\alpha_\ell}$ and $H_{1,-\alpha_{\ell+1}}$, and LG has $\ell+1$ simple roots $(0, \alpha_1)$, $(0, \alpha_2), \ldots, (0, \alpha_\ell)$, $(1, -\alpha_{\ell+1})$. In general C_0 is a product of q simplexes, one for each simple factor of G, and LG has $k+q$ simple roots, namely $(0, \alpha_i)$ for $i = 1, \ldots, \ell$, and $(1, -\alpha_i)$ for $i = \ell+1, \ldots, \ell+q$, where $\alpha_{\ell+1}, \ldots, \alpha_{\ell+q}$ are the highest weights occurring in the adjoint representation.

The finite group W is known to be generated by reflections in the hyperplanes $H_{0,\alpha}$, and acts simply transitively on the set of chambers. Corresponding statements hold for W_{aff}.

Proposition (5.1.4). *If G is simply connected, then*

(i) *W_{aff} is generated by reflections in the hyperplanes $H_{k,\alpha}$, and*
(ii) *W_{aff} acts simply transitively on the set of alcoves.*

Proof.

(i) We know that the reflection s_α in $H_{0,\alpha}$ belongs to W. It is given by

$$s_\alpha(\xi) = \xi - \alpha(\xi)h_\alpha,$$

where h_α is the coroot corresponding to α. (See Section 2.4.) Now $\alpha(h_\alpha) = 2$, so the point $-\frac{1}{2}kh_\alpha$ belongs to $H_{k,\alpha}$. The reflection $s_{k,\alpha}$ in $H_{k,\alpha}$ is therefore

$$\begin{aligned} s_{k,\alpha}(\xi) &= (\xi + \tfrac{1}{2}kh_\alpha) - \alpha(\xi + \tfrac{1}{2}kh_\alpha)h_\alpha - \tfrac{1}{2}kh_\alpha \\ &= s_\alpha(\xi) - kh_\alpha. \end{aligned} \qquad (5.1.5)$$

But h_α belongs to the lattice $\check{T}$, so $s_{k,\alpha} \in W \tilde{\times} \check{T} = W_{\text{aff}}$.

Conversely, it is enough to show that the translation t_α by $h_\alpha \in \mathfrak{t}$ belongs to the group generated by the $s_{k,\alpha}$, for the coroots h_α generate $\check{T}$. But from (5.1.5) we have

$$t_\alpha = s_{1,-\alpha} s_{0,\alpha}.$$

(ii) (Cf. Bourbaki [20] Chapter 5, §3.1.) Let A be an arbitrary alcove. We must show that $\gamma A = C_0$ for some $\gamma \in W_{\text{aff}}$. Choose a point a in A. The orbit of a under W_{aff} is a locally finite subset S in $\mathfrak{t}$. We must show that S meets C_0. Choose a point $c \in C_0$ and a point $b \in S$ whose distance from c is minimal. If $b \notin C_0$ then b must be separated from c by at least one wall H of C_0. Reflecting b in H will produce a point of the orbit S closer to c than b is: a contradiction. So W_{aff} acts transitively on the alcoves. Conversely, an element of W_{aff} is completely determined by the alcove to which it takes C_0: that follows at once from the corresponding fact about the action of W on the chambers.

Remark (5.1.6). The proof of (ii) actually shows that W_{aff} is generated by the reflections in the hyperplanes corresponding to the *simple* affine roots.

5.2 Generators and relations

We can now describe the Lie algebra $L\mathfrak{g}_{\mathbb{C}}$, or more precisely its universal central extension, in terms of generators and relations. For a finite dimensional semisimple algebra $\mathfrak{g}$, if one chooses for each root α a non-zero element e_α in the root-space $\mathfrak{g}_\alpha$ (see Section 2.4) then $\mathfrak{g}_{\mathbb{C}}$ is generated by the e_α, and even by $e_j = e_{\alpha_j}$ and $f_j = e_{-\alpha_j}$ for $j = 1, \ldots, \ell$, where the α_j are the simple roots. In fact $[e_j, f_j]$ is a multiple of the coroot $h_j = h_{\alpha_j} \in \mathfrak{t}$, and when the e_j and f_j are normalized so that $[e_j, f_j] = ih_j$ the following is a complete set of relations defining $\mathfrak{g}_{\mathbb{C}}$:

$$\begin{aligned} [e_j, f_j] &= ih_j \\ [e_j, f_k] &= 0 \quad \text{if} \quad j \neq k \\ [h_j, e_k] &= ia_{jk}e_k \\ [h_j, f_k] &= -ia_{jk}f_k \\ (\operatorname{ad} e_j)^{1-a_{jk}}e_k &= 0 \\ (\operatorname{ad} f_j)^{1-a_{jk}}f_k &= 0. \end{aligned} \tag{5.2.1}$$

Here 'ad x' means the operation $y \mapsto [x, y]$, and the a_{jk} are integers forming an $\ell \times \ell$ matrix called the *Cartan matrix* of $\mathfrak{g}$. This matrix determines the structure of $\mathfrak{g}$ completely. For a proof of this, see Serre [134].

Now let us turn to loop groups. Let us choose elements e_j, f_j in $L\mathfrak{g}_{\mathbb{C}}$ corresponding to the simple affine roots. In the notation of Section 5.1 we can take e_j and f_j to be the usual elements of $\mathfrak{g}_{\mathbb{C}} \subset L\mathfrak{g}_{\mathbb{C}}$ when $1 \leq j \leq \ell$, and $e_j = ze_{-\alpha_j}$, $f_j = z^{-1}e_{\alpha_j}$ for $\ell < j \leq \ell + q$. (Here q is the number of simple factors in $\mathfrak{g}$.)

Proposition (5.2.2). *If $\mathfrak{g}$ is semisimple then $L_{\text{pol}}\mathfrak{g}_{\mathbb{C}}$ is generated by the elements e_j, f_j corresponding to the simple affine roots.*

Proof. We may as well assume that $\mathfrak{g}$ is simple. Then the e_j and f_j for $j \leq \ell$ generate $\mathfrak{g}_{\mathbb{C}}$. But $e_{\ell+1} = ze_{-\alpha}$, where α is the highest root of $\mathfrak{g}$. Because the adjoint representation is irreducible, all of $z\mathfrak{g}_{\mathbb{C}}$ can be obtained by applying elements of $\mathfrak{g}_{\mathbb{C}}$ to $ze_{-\alpha}$, and so $z\mathfrak{g}_{\mathbb{C}}$ is contained in the algebra generated by the e_j and f_j. But then $z^2e_{-\alpha}$ is a multiple of $[zh_j, ze_{-\alpha}]$, where $ih_j = [e_j, f_j]$ and $j \leq \ell$ is chosen so that $\alpha(h_j) \neq 0$. This gives us $z^2\mathfrak{g}_{\mathbb{C}}$. And so on.

We can now check at once that the set of relations (5.2.1), where j and k run from 1 to $\ell + q$, hold in $L\mathfrak{g}_{\mathbb{C}}$. The $(\ell + q) \times (\ell + q)$ Cartan matrix is given by

$$a_{jk} = \beta_k(h_{\beta_j})$$

where $\beta_j = \alpha_j$ if $j \leq \ell$ and $\beta_j = -\alpha_j$ if $j > \ell$. Because only ℓ of the β_i are linearly independent, we see that the Cartan matrix has rank ℓ.

Although the relations (5.2.1) hold in $L_{\text{pol}}\mathfrak{g}_{\mathbb{C}}$ they do not define it. It is a theorem of Gabber and Kac [52]—which we shall not prove in this book—that the relations define the universal central extension $\tilde{L}_{\text{pol}}\mathfrak{g}_{\mathbb{C}}$ of $L_{\text{pol}}\mathfrak{g}_{\mathbb{C}}$ by $K_{\mathbb{C}} \cong \mathbb{C}^q$ which is described by the cocycle ω_K of (4.2.7). We shall content ourselves here with pointing out that the relations (5.2.1) do hold in $\tilde{L}_{\text{pol}}\mathfrak{g}_{\mathbb{C}}$. To see that, we identify $\tilde{L}_{\text{pol}}\mathfrak{g}_{\mathbb{C}}$ with $L_{\text{pol}}\mathfrak{g}_{\mathbb{C}} \oplus K_{\mathbb{C}}$, and define elements $\tilde{e}_j$, $\tilde{f}_j$, $\tilde{h}_j$ of $\tilde{L}_{\text{pol}}\mathfrak{g}_{\mathbb{C}}$ by

$$\begin{aligned} \tilde{e}_j &= (e_j, 0) \\ \tilde{f}_j &= (f_j, 0) \\ \tilde{h}_j &= (h_j, 0) \quad \text{for} \quad j = 1, \ldots, \ell, \\ &= (h_j, -\tfrac{1}{2}\langle h_j, h_j \rangle_K) \quad \text{for} \quad j = \ell + 1, \ldots, \ell + q. \end{aligned}$$

It is easy to check that the $\tilde{e}_j$, $\tilde{f}_j$, $\tilde{h}_j$ satisfy the relations (5.2.1). Furthermore the elements generate $\tilde{L}_{\text{pol}}\mathfrak{g}_{\mathbb{C}}$, because the inner products $\langle h_j, h_j \rangle_K$, for $\ell < j \leq \ell + q$, span $K_{\mathbb{C}}$.

With the preceding formulae in mind it is natural, whenever we are studying a central extension $\tilde{L}G$ of LG defined by a bilinear form $\langle\ ,\ \rangle$ on $\mathfrak{g}$, to associate to each affine root $\boldsymbol{\alpha} = (k, \alpha)$ with $\alpha \neq 0$ an *affine coroot* $h_{\boldsymbol{\alpha}}$ in $\tilde{L}\mathfrak{g}$ defined by

$$h_{\boldsymbol{\alpha}} = (h_\alpha, -\tfrac{1}{2}k\langle h_\alpha, h_\alpha \rangle). \tag{5.2.3}$$

Taking $h_{\boldsymbol{\alpha}}$ together with $e_{\boldsymbol{\alpha}} = (z^k e_\alpha, 0)$ and $e_{-\boldsymbol{\alpha}} = (z^{-k}e_{-\alpha}, 0)$ we then have a copy of the Lie algebra of SU_2 embedded in $\tilde{L}\mathfrak{g}_{\mathbb{C}}$, and we can exponentiate to obtain a homomorphism

$$i_{\boldsymbol{\alpha}}: SU_2 \to \tilde{L}G. \tag{5.2.4}$$

The argument of the proof of (3.5.3) clearly implies the following.

Proposition (5.2.5). *If G is simply connected then the $\ell + q$ subgroups $i_{\boldsymbol{\alpha}}(SU_2)$ corresponding to the simple affine roots generate $\tilde{L}_{\text{pol}}G$.*

Example. The group $L_{\text{pol}}SU_2$ is generated by the subgroup SU_2 of constant loops, and the copy of SU_2 consisting of the elements

$$\begin{pmatrix} a & bz \\ -\bar{b}z^{-1} & \bar{a} \end{pmatrix}$$

with $|a|^2+|b|^2=1$. The latter subgroup is the transform of the former by the outer automorphism of LSU_2 corresponding to the non-trivial element of the centre of SU_2. (Cf. (3.4.4).)

5.3 Kac–Moody Lie algebras

It is well known that the Cartan matrix $A=(a_{ij})$ of any semisimple Lie algebra $\mathfrak{g}_{\mathbb{C}}$ satisfies the following two conditions (see [20] Chapter 6, where a_{ij} is written $n(\alpha_i, \alpha_j)$).

(C1) $a_{ij}\in\mathbb{Z}$ for all i, j; $a_{ii}=2$ for all i; $a_{ij}\leqslant 0$ if $i\neq j$; $a_{ij}=0$ whenever $a_{ji}=0$.

(C2) A is positive definite, in the sense that all the principal minors of A are >0.

Conversely, given any $\ell\times\ell$ matrix A satisfying (C1), the relations (5.2.1) define an abstract complex Lie algebra $\mathfrak{g}'(A)$, which is essentially the *Kac–Moody Lie algebra* defined by the Cartan matrix A. If A satisfies (C2) as well, then $\mathfrak{g}'(A)$ will be finite dimensional and semisimple, but if (C2) does not hold, $\mathfrak{g}'(A)$ will be infinite dimensional. It is natural to ask what can be said about $\mathfrak{g}'(A)$ in this infinite dimensional case. It turns out that, after modifying $\mathfrak{g}'(A)$ slightly in a way which corresponds to passing from LG to the semidirect product $\mathbb{T}\tilde{\times}LG$, one obtains an algebra to which much of the finite dimensional structure theory can be carried over. As it is not our purpose in this book to give a systematic exposition of Kac–Moody Lie algebras, we shall only describe the beginning of the theory, referring the reader to the excellent survey article of Macdonald [109] or the book of Kac [86] for further details.

The first thing to notice is that the elements h_i are linearly independent and generate a maximal abelian subalgebra $\mathfrak{h}'$ of $\mathfrak{g}'(A)$. The analogy with the finite dimensional case now suggests that one should define the simple roots $\alpha_j'\in\mathfrak{h}'^*$ of $\mathfrak{g}'(A)$ by the formula $\alpha_j'(h_i)=a_{ij}$. Unfortunately, if A is not invertible, the resulting simple roots will be linearly dependent. This problem can be overcome by passing to a semidirect product $\mathfrak{g}(A)=\mathfrak{d}\oplus\mathfrak{g}'(A)$, where $\mathfrak{d}$ is a space of derivations of $\mathfrak{g}'(A)$ defined as follows. Let $\mathfrak{d}'$ be spanned by the derivations d_i, $i=1,\ldots,\ell$, defined by $d_i(e_j)=\delta_{ij}e_j$, and $d_i(f_j)=-\delta_{ij}f_j$. Then $\mathrm{ad}(\mathfrak{h}')$ is a subspace of $\mathfrak{d}'$, since $\mathrm{ad}(h_i)=\sum a_{ij}d_j$; let $\mathfrak{d}$ be a complementary subspace, so that $\dim\mathfrak{d}=\mathrm{corank}\ A$. Then $\mathfrak{g}(A)=\mathfrak{d}\oplus\mathfrak{g}'(A)$ is the Kac–Moody Lie algebra with Cartan matrix A; it is independent of the choice of $\mathfrak{d}$ up to isomorphism.

With this modification, $\mathfrak{h}=\mathfrak{d}\oplus\mathfrak{h}'$ is a maximal abelian subalgebra of $\mathfrak{g}(A)$, and the simple roots $\boldsymbol{\alpha}_i$ can be defined so that $[h, e_i]=\boldsymbol{\alpha}_i(h)e_i$ for all $h\in\mathfrak{h}$. Then $\boldsymbol{\alpha}_j(h_i)=a_{ij}$ and the $\boldsymbol{\alpha}_i$ are linearly independent. The remaining roots are defined in the obvious way.

To proceed further we need a complex-valued invariant symmetric bilinear form on $\mathfrak{g}(A)$. Unfortunately $\mathfrak{g}(A)$ may not have such a form; in fact a necessary and sufficient condition for this is that the Cartan matrix is *symmetrizable*, which means that there is an invertible diagonal matrix D such that DA is symmetric. When it exists, the form is non-degenerate, and it is unique up to a constant factor if A is indecomposable, i.e. cannot be written as a non-trivial direct sum of two other matrices. (If A is decomposable then $\mathfrak{g}(A)$ is the direct product of the corresponding subalgebras.)

If there is an invariant form $\langle\ ,\ \rangle$ on $\mathfrak{g}(A)$ then its restriction to $\mathfrak{h}$ is non-degenerate. Relative to the corresponding symmetric bilinear form on $\mathfrak{h}^*$ one has $\langle\boldsymbol{\alpha}_i,\boldsymbol{\alpha}_i\rangle>0$ for all i. In general, $\langle\boldsymbol{\alpha},\boldsymbol{\alpha}\rangle$ is real for each root $\boldsymbol{\alpha}$, but is not always positive unless $\mathfrak{g}(A)$ is finite dimensional; the roots $\boldsymbol{\alpha}$ for which $\langle\boldsymbol{\alpha},\boldsymbol{\alpha}\rangle>0$ are called real, the others imaginary.

The Weyl group W of $\mathfrak{g}(A)$ can now be defined as the group of isometries of $\mathfrak{h}$ generated by the reflections in the planes $H_{\alpha_i}=\ker\boldsymbol{\alpha}_i$. Obviously W takes real roots to real roots; in fact, the real roots are precisely the W-orbit of the simple roots.

We must now explain where loop algebras fit into this picture. The Cartan matrix of a Lie algebra $\tilde{L}\mathfrak{g}_{\mathbb{C}}$ satisfies (C1) and

(C2′) $\det A=0$ and all the proper principal minors of A are >0.

(see [86]). Conversely, the Kac–Moody Lie algebras corresponding to Cartan matrices satisfying (C1) and (C2′) are usually called *affine Lie algebras*. Thus $\tilde{L}\mathfrak{g}_{\mathbb{C}}$ is an affine Lie algebra, or, more precisely, the semidirect product $\mathbb{C}^q\oplus\tilde{L}\mathfrak{g}_{\mathbb{C}}$, where q is the number of simple factors in $\mathfrak{g}_{\mathbb{C}}$. (If $\mathfrak{g}_{\mathbb{C}}=\mathfrak{g}_{1,\mathbb{C}}\oplus\ldots\oplus\mathfrak{g}_{q,\mathbb{C}}$ then $\mathbb{C}^q\oplus\tilde{L}\mathfrak{g}_{\mathbb{C}}$ means the product of the algebras $\mathbb{C}\oplus\tilde{L}\mathfrak{g}_{i,\mathbb{C}}$, where the factor $\mathbb{C}$ is generated by the obvious derivation $-i\dfrac{\mathrm{d}}{\mathrm{d}\theta}=z\dfrac{\mathrm{d}}{\mathrm{d}z}$ of $\tilde{L}\mathfrak{g}_{i,\mathbb{C}}$.)

Only about half of the affine Lie algebras arise in this way. The remainder are the Lie algebras of the twisted loop groups introduced in Section 3.7. In the algebraic context one chooses an outer automorphism α of $\mathfrak{g}_{\mathbb{C}}$ of finite order k, so that $k=1$, 2 or 3, and replaces $L\mathfrak{g}_{\mathbb{C}}$ by its subalgebra $L_{(\alpha)}\mathfrak{g}_{\mathbb{C}}$ consisting of the loops $f\in L\mathfrak{g}_{\mathbb{C}}$ which are equivariant:

$$f(\varepsilon^{-1}z)=\alpha(f(z)), \tag{5.3.1}$$

where ε is a primitive k^{th} root of unity.

All affine Lie algebras admit an invariant symmetric bilinear form, and their root systems are well understood (see Helgason [72] Chapter X §5).

We have already discussed the root system of $\tilde{L}\mathfrak{g}_{\mathbb{C}}$ in the preceding section, and the twisted algebras present little extra difficulty.

If $G_{\mathbb{C}}$ is a simply connected Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$, then any automorphism of $\mathfrak{g}_{\mathbb{C}}$ lifts uniquely to an automorphism of $G_{\mathbb{C}}$ and one can define the twisted loop group $L_{(\alpha)}G_{\mathbb{C}}$ by the same formula (5.3.1), where f is now interpreted as an element of $LG_{\mathbb{C}}$. Thus every affine Lie algebra comes from a Lie group. The extent to which this is true for an arbitrary Kac–Moody Lie algebra is an open question. Certainly no concrete realizations of the groups are known. (See Tits [145], [146] for the current state of affairs.)

6

LOOP GROUPS AS GROUPS OF OPERATORS IN HILBERT SPACE

In this chapter we shall study the natural embedding of the loop group of $GL_n(\mathbb{C})$ in the restricted general linear group of Hilbert space. This embedding will play a fundamental role throughout the rest of the book.

6.1 Loops as multiplication operators

Let $H^{(n)}$ denote the Hilbert space $L^2(S^1; \mathbb{C}^n)$ of square-summable $\mathbb{C}^n$-valued functions on the circle. The group $L_{\text{cts}}GL_n(\mathbb{C})$ of continuous maps $S^1 \to GL_n(\mathbb{C})$ acts on $H^{(n)}$ by multiplication operators: if γ is a matrix-valued function on the circle, we denote the corresponding multiplication operator by M_γ.

The norm $\|M_\gamma\|$ of the operator M_γ is $\|\gamma\|_\infty$, the supremum of $\|\gamma(\theta)\|$ for $\theta \in S^1$. It follows that $\gamma \mapsto M_\gamma$ embeds the Banach Lie group $L_{\text{cts}}GL_n(\mathbb{C})$ as a closed subgroup (with the induced topology) of the Banach Lie group $GL(H^{(n)})$ of all invertible bounded operators in $H^{(n)}$, with the operator-norm topology. We recall that $GL(H^{(n)})$ is an open subset of the Banach algebra $\mathscr{B}(H^{(n)})$ of all bounded operators in $H^{(n)}$ [34].

The operators M_γ all commute with M_z, the operation of multiplication by the scalar-valued function $z = e^{i\theta}$ on the circle. Indeed $L_{\text{cts}}GL_n(\mathbb{C})$ is not far from being the commutant of M_z in $GL(H^{(n)})$.

Theorem (6.1.1). *The commutant of M_z in $GL(H^{(n)})$ is the group $L_{\text{meas}}GL_n(\mathbb{C})$ of bounded measurable maps $S^1 \to GL_n(\mathbb{C})$.*

To say here that γ is bounded means that both $\|\gamma(\theta)\|$ and $\|\gamma(\theta)^{-1}\|$ are bounded outside a set of measure zero.

Proof. If $A \in GL(H^{(n)})$ commutes with M_z, let $\phi_i = A\varepsilon_i \in H^{(n)}$, where ε_i is the i^{th} basis vector of $\mathbb{C}^n$, identified with the corresponding constant function in $H^{(n)}$. Thinking of the ϕ_i as taking values which are n-component column vectors, we put $\phi_1, \phi_2, \ldots, \phi_n$ side by side to form a measurable matrix-valued function ϕ. Then $A = M_\phi$. For we can approximate any $f \in H^{(n)}$ by elements of the form $\sum p_i(z)\varepsilon_i$ where each p_i is a polynomial in z and z^{-1}; and because A commutes with M_z we have

$$A\left(\sum p_i(z)\varepsilon_i\right) = \sum p_i(z)\phi_i = M_\phi\left(\sum p_i(z)\varepsilon_i\right).$$

It follows that $A = M_\phi$, and that $\|\phi(\theta)\|$ and $\|\phi(\theta)^{-1}\|$ are essentially bounded.

6.2 The restricted general linear group of Hilbert space

To obtain more refined results we must introduce a restricted general linear group. (This group was first studied by Shale [136].) It is defined for a Hilbert space H which is equipped with a *polarization*, i.e. a decomposition $H = H_+ \oplus H_-$ as the orthogonal sum of two closed subspaces. The decomposition can be conveniently given by the unitary operator $J: H \to H$ which is $+1$ on H_+ and -1 on H_-. The restricted general linear group consists of the operators which are fairly close to preserving the decomposition $H = H_+ \oplus H_-$.

Definition (6.2.1). *$GL_{res}(H)$ is the subgroup of $GL(H)$ consisting of operators A such that the commutator $[J, A]$ is a Hilbert–Schmidt operator.*

We recall (cf. [125]) that an operator $T: H_1 \to H_2$ is Hilbert–Schmidt if for some (and hence every) complete orthonormal sequence $\{e_i\}$ in H_1 the sequence $\sum \|Te_i\|^2$ converges. The Hilbert–Schmidt norm $\|T\|_2$ is then $(\sum \|Te_i\|^2)^{\frac{1}{2}}$. The Hilbert–Schmidt operators in H form a two-sided ideal $\mathcal{I}_2(H)$ in $\mathcal{B}(H)$—it follows from this that $GL_{res}(H)$ really is a group—and they are themselves a Hilbert space under the norm $\| \ \|_2$.

The definition of $GL_{res}(H)$ can be reformulated as follows. If an element A of $GL(H)$ is written as a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{6.2.2}$$

with respect to the decomposition $H = H_+ \oplus H_-$, then A belongs to $GL_{res}(H)$ if and only if b and c are Hilbert–Schmidt operators.

For yet another formulation, we introduce the Banach algebra $\mathcal{B}_J(H)$ of all bounded operators $A: H \to H$ such that $[J, A]$ is Hilbert–Schmidt. The norm $\| \ \|_J$ is defined by

$$\|A\|_J = \|A\| + \|[J, A]\|_2.$$

The group $GL_{res}(H)$ is the group of units of $\mathcal{B}_J(H)$: we give it the topology defined by $\| \ \|_J$, and it is then a complex Banach Lie group.

We shall also define the restricted unitary group.

Definition (6.2.3). *$U_{res}(H)$ is the subgroup of $GL_{res}(H)$ consisting of unitary operators.*

$U_{res}(H)$ is a real Banach Lie group. The standard polar decomposition (cf. [125]) of operators in Hilbert space shows that $GL_{res}(H)$ is the topological product of $U_{res}(H)$ and the contractible space of positive definite elements.† $GL_{res}(H)$ is the complexification of $U_{res}(H)$.

If the operator A of (6.2.2) belongs to $GL_{res} = GL_{res}(H)$ then its components a and d are Fredholm operators, i.e. they have finite dimensional kernels (null spaces) and cokernels. For an operator is Fredholm if it is invertible modulo compact operators, and the invertibility of A implies that a and d are invertible modulo Hilbert–Schmidt operators, which are compact. (An account of Fredholm operators from a topologist's point of view can be found in the appendix to Atiyah [3]. In Douglas [40] there is a detailed discussion from the point of view of operator theory.) A Fredholm operator a has an integer invariant $\chi(a)$, its *index*, defined by

$$\chi(a) = \dim \ker(a) - \dim \operatorname{coker}(a).$$

This is invariant under continuous deformation, and divides up the space of Fredholm operators into its connected components. It follows that the group GL_{res} is disconnected into pieces characterized by the integer $\chi(a)$. (Notice that $\chi(a) = -\chi(d)$, because $a \oplus d$ can be deformed linearly through Fredholm operators to the invertible operator A.) In fact two elements A_1 and A_2 are in the same connected component if $\chi(a_1) = \chi(a_2)$: that follows from the following much more precise result, which shows that GL_{res} has the homotopy type of the space which topologists call $\mathbb{Z} \times BU$.

Proposition (6.2.4). *The map $A \mapsto a$ from $GL_{res}(H)$ to the space* $\mathrm{Fred}(H_+)$ *of Fredholm operators in H_+ is a homotopy equivalence.*

Proof. Consider the map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a \\ c \end{pmatrix}$$

which assigns to an element of GL_{res} its first column. This is a map onto an open subset $\mathcal{F}$ of $\mathrm{Fred}(H_+) \times \mathcal{I}_2(H_+; H_-)$, where $\mathcal{I}_2(H_+; H_-)$ is the space of Hilbert–Schmidt operators $H_+ \to H_-$. On the other hand $\mathcal{F}$ is also the homogeneous space $GL_{res}/\mathcal{B}$, where $\mathcal{B}$ is the subgroup of all elements of the form

$$\begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix}.$$

This subgroup is contractible, for it is the semidirect product of $GL(H_-)$ and the vector space $\mathcal{I}_2(H_-; H_+)$. (We recall—see Kuiper [96]—that the general linear group of Hilbert space is contractible.) So $GL_{res} \to \mathcal{F}$ is a homotopy equivalence.

† One must check that taking the positive square-root of positive elements of $\mathcal{B}_J(H)$ is well-defined and continuous.

But the projection $\mathscr{F} \to \mathrm{Fred}(H_+)$ is also a homotopy equivalence, for the inverse image of an element a is the contractible open set of $\mathscr{I}_2(H_+; H_-)$ consisting of all operators c such that $c \mid \ker(a)$ is injective. (A map with contractible fibres is a homotopy equivalence providing certain local conditions are satisfied. (See Dold [39].) These hold automatically for a projection from an open set of a Banach space.)

6.3 The map $LGL_n(\mathbb{C}) \to GL_{\mathrm{res}}(H^{(n)})$

Returning to loop groups, we have seen that the continuous loops in $GL_n(\mathbb{C})$ can be regarded as a subgroup of $GL(H^{(n)})$. The smooth loops are contained in $GL_{\mathrm{res}}(H^{(n)})$. In defining the restricted group we shall always decompose $H^{(n)} = L^2(S^1; \mathbb{C}^n)$ as $H_+^{(n)} \oplus H_-^{(n)}$, where

$$\begin{aligned} H_+^{(n)} &= \{\text{functions whose negative Fourier coefficients vanish}\} \\ &= \Big\{ f \in H^{(n)} : f(\theta) = \sum_{k \geq 0} f_k e^{ik\theta} \text{ with } f_k \in \mathbb{C}^n \Big\}, \\ &= \{ f \in H^{(n)} : f \text{ is the boundary value of a} \\ &\qquad \text{function holomorphic in } |z| < 1\}, \end{aligned}$$

and

$$\begin{aligned} H_-^{(n)} &= (H_+^{(n)})^\perp \\ &= \Big\{ f \in H^{(n)} : f(\theta) = \sum_{k<0} f_k e^{ik\theta} \Big\}. \end{aligned}$$

In other words, we have decomposed $H^{(n)}$ essentially into the positive and negative eigenspaces of the infinitesimal rotation operator $-i\, d/d\theta$.

Proposition (6.3.1). *If $\gamma : S^1 \to GL_n(\mathbb{C})$ is continuously differentiable, then the multiplication operator M_γ belongs to $GL_{\mathrm{res}}(H^{(n)})$.*

We shall give two proofs of this, as both are instructive.

First Proof. Let us write γ as a Fourier series

$$\gamma(\theta) = \sum_{k \in \mathbb{Z}} \gamma_k e^{ik\theta},$$

where the γ_k are $n \times n$ matrices. With respect to the obvious orthonormal basis of $H^{(n)}$ the operator M_γ is represented by a $\mathbb{Z} \times \mathbb{Z}$ matrix (M_{pq}) whose entries are $n \times n$ matrices. In fact $M_{pq} = \gamma_{p-q}$. We must show that the $(H_+^{(n)} \to H_-^{(n)})$ and $(H_-^{(n)} \to H_+^{(n)})$ components of M_γ are Hilbert-Schmidt, in other words that

$$\sum_{p \geq 0, q < 0} \|M_{pq}\|^2 < \infty \quad \text{and} \quad \sum_{p<0, q \geq 0} \|M_{pq}\|^2 < \infty.$$

This is equivalent to

$$\sum_{k \in \mathbb{Z}} (|k| + 1)\, \|\gamma_k\|^2 < \infty,$$

which is certainly true if γ is differentiable, as the square of the L^2 norm of the derivative γ' is $\sum k^2 \|\gamma_k\|^2$.

Second Proof. The operator J defining the decomposition $H_+^{(n)} \oplus H_-^{(n)}$ is a singular integral operator

$$(Jf)(\theta) = \frac{1}{2\pi} PV \int_0^{2\pi} K(\theta, \phi) f(\phi)\, d\phi$$

whose kernel K is given by

$$\begin{aligned} K(\theta, \phi) &= \sum_{k \geq 0} e^{ik(\theta - \phi)} - \sum_{k<0} e^{ik(\theta-\phi)} \\ &= 1 + i \cot \tfrac{1}{2}(\theta - \phi). \end{aligned} \tag{6.3.2}$$

Here PV denotes the 'principal value' of the integral, i.e.

$$\lim_{\varepsilon \to 0} \left(\int_0^{\theta - \varepsilon} + \int_{\theta+\varepsilon}^{2\pi} \right).$$

(J is the analogue for the circle of the *Hilbert transform* $\tilde{J}$ of functions on the line (cf. [158] Vol. 2, p. 243) defined by

$$(\tilde{J}f)(x) = \frac{1}{2\pi i} PV \int_{-\infty}^{\infty} \frac{f(y)}{x - y}\, dy. \Big)$$

The commutator $[M_\gamma, J]$ is therefore described by the kernel $K(\theta, \phi)(\gamma(\theta) - \gamma(\phi))$, and is Hilbert–Schmidt when the kernel is square-summable, i.e. when

$$\int_0^{2\pi} \int_0^{2\pi} \frac{\|\gamma(\theta) - \gamma(\phi)\|^2}{\sin^2 \frac{1}{2}(\theta - \phi)}\, d\theta\, d\phi < \infty.$$

That is certainly true if γ is continuously differentiable, for then the integrand is a continuous function on $S^1 \times S^1$.

The loop group $LGL_n(\mathbb{C})$ is of course not a topological subgroup of $GL_{\mathrm{res}}(H)$: it has a much finer topology than its image. In fact the topology on $LGL_n(\mathbb{C})$ induced by $GL_{\mathrm{res}}(H)$ can be described in the following way.

If $\gamma = \sum \gamma_k z^k$ is a matrix-valued function on S^1 then the Hilbert–Schmidt norm of the commutator $[M_\gamma, J]$ is

$$\|\gamma\|_{2,\frac{1}{2}} = \Big\{ \sum |k|\, \|\gamma_k\|^2 \Big\}^{\frac{1}{2}}.$$

This is commonly known as the Sobolev norm corresponding to '$\frac{1}{2}$-differentiable' functions [144]. Let A denote the Banach algebra of measurable matrix-valued functions γ on S^1 such that

$$|||\gamma||| = \|\gamma\|_\infty + \|\gamma\|_{2,\frac{1}{2}} < \infty.$$

(Here $\|\gamma\|_\infty$ denotes the L^∞ norm.) The group $GL_n(A)$ will be denoted by $L_{\frac{1}{2}}GL_n(\mathbb{C})$. It is a Banach Lie group. Clearly we have

Proposition (6.3.3). *$L_{\frac{1}{2}}GL_n(\mathbb{C})$ is the commutant of the multiplication operator M_z in $GL_{\text{res}}(H^{(n)})$.*

The group $L_{\frac{1}{2}}GL_n(\mathbb{C})$ is of interest to us because it is the largest loop group to which much of the theory of this book applies: in particular it is the largest group for which the crucial central extension can be constructed and the basic irreducible representation defined. On the other hand it is hard to describe its elements explicitly. It contains all loops of class $C^{(1)}$, but it neither contains nor is contained in the group of continuous loops, and the smooth loops are *not* dense in it. We shall now give some examples to illustrate these facts.

Examples.

(i) Piecewise smooth loops belong to $L_{\frac{1}{2}}$ if and only if they are continuous: the typical discontinuous example is

$$\sum_{k>0} \frac{\sin k\theta}{k} = \tfrac{1}{2}(\pi - \theta) \quad \text{for} \quad 0 < \theta < 2\pi.$$

(ii) The function

$$f = \sum_{k>1} \frac{\cos k\theta}{k \log k}$$

satisfies $\|f\|_{2,\frac{1}{2}} < \infty$ but is not bounded near $\theta = 0$ (because $\sum 1/k \log k = \infty$: see Zygmund [158] Chapter 5 §1), and hence not continuous.

(iii) The function

$$g = \sum_{k>1} \frac{\sin k\theta}{k(\log k)^{\frac{1}{2}}}$$

is continuous, but $\|g\|_{2,\frac{1}{2}} = \infty$.

(iv) The function e^{if}, where f is as in (ii), belongs to $L_{\frac{1}{2}}\mathbb{T}$, but is not continuous. To see that e^{if} belongs to $L_{\frac{1}{2}}\mathbb{T}$ we begin with e^a, where

$$a = -\sum_{k>1} \frac{\sin k\theta}{k \log k}.$$

This function a is bounded and continuous and belongs to the Banach algebra A, so e^a belongs to $L_{\frac{1}{2}}\mathbb{C}^\times$. But e^a has the unique factorization $e^a = e^{if}.e^h$, where $h(z) = -i\sum_{k>1} z^k/k$ is the boundary value of a function

holomorphic for $|z| < 1$. We shall see later—from Proposition (8.3.5)—that this implies that e^{if} belongs to $L_{\frac{1}{2}}\mathbb{T}$.

6.4 Bott periodicity

The inclusion $LGL_n(\mathbb{C}) \to GL_{\text{res}}(H^{(n)})$ is essentially the map known to algebraic topologists as the inverse of the 'Bott periodicity' map. Bott's theorem asserts that its restriction to the subgroup $\Omega GL_n(\mathbb{C})$ of 'based' loops, i.e. those such that $\gamma(0) = 1$, induces an isomorphism of homotopy groups π_i for $i < 2n - 1$. Because $\pi_i \Omega GL_n(\mathbb{C}) \cong \pi_{i+1} GL_n(\mathbb{C})$, while, as we shall see in Section 6.6,

$$\pi_i GL_{\text{res}} \cong \pi_{i-1} GL_n(\mathbb{C}) \tag{6.4.1}$$

when $i < 2n + 1$, this means that

$$\pi_i GL_n(\mathbb{C}) \cong \pi_{i+2} GL_n(\mathbb{C})$$

when $i < 2n - 2$. We shall return to this subject in Section 8.8. Meanwhile let us notice the interesting fact that the inverse Bott map (ordinarily defined only up to homotopy) has been realized as a homomorphism of infinite dimensional Lie groups.

6.5 The isomorphism $H^{(n)} \cong H$ and the embedding $L\mathbb{T} \to LU_n$

Although it seems at first sight a very unnatural thing to do, we shall find it surprisingly useful to identify the Hilbert space $H^{(n)} = L^2(S^1; \mathbb{C}^n)$ with the standard Hilbert space $H = H^{(1)} = L^2(S^1; \mathbb{C})$ by means of the obvious lexicographic correspondence between their orthonormal bases: if $\{\varepsilon_i : 1 \leq i \leq n\}$ is the standard basis of $\mathbb{C}^n$, we make $\varepsilon_i z^k \in H^{(n)}$ correspond to $z^{nk+i-1} \in H$. More invariantly, given a vector-valued function with components $(f_1, f_2, \ldots, f_n)$, we associate to it the scalar-valued function $\tilde{f}$ given by

$$\tilde{f}(\zeta) = f_1(\zeta^n) + \zeta f_2(\zeta^n) + \ldots + \zeta^{n-1} f_n(\zeta^n). \tag{6.5.1a}$$

Conversely, given $\tilde{f} \in H$, we obtain $(f_i) \in H^{(n)}$ by

$$f_{i+1}(z) = \frac{1}{n} \sum_{\zeta} \zeta^{-i} \tilde{f}(\zeta), \tag{6.5.1b}$$

where ζ runs through the n^{th} roots of z.

The isomorphism $H^{(n)} \cong H$ is an isometry. It makes continuous functions correspond to continuous ones, and also preserves all other reasonable classes of functions, for example: smooth, real analytic, rational, polynomial. Furthermore the decomposition $H^{(n)} = H_+^{(n)} \oplus H_-^{(n)}$ corresponds to the decomposition $H = H_+ \oplus H_-$.

The multiplication operator M_z on $H^{(n)}$ corresponds to M_{z^n} on H.

Identifying the commutant of M_{z^n} on H with $L_{\text{meas}}GL_n(\mathbb{C})$ by (6.1.1), and noticing that it must contain the commutant of M_z on H, we have an inclusion

$$L_{\text{meas}}\mathbb{C}^\times \subset L_{\text{meas}}GL_n(\mathbb{C}),$$

inducing

$$L\mathbb{C}^\times \subset LGL_n(\mathbb{C}),$$
$$L\mathbb{T} \subset LU_n,$$

and so on. The last inclusion has already been described in Proposition (3.6.4).

6.6 The central extension of $GL_{\text{res}}(H)$

We shall now define a central extension of GL_{res} by the multiplicative group $\mathbb{C}^\times$ of non-zero complex numbers. The motivation for the definition will become clear in Chapter 7.

We begin by recalling a few facts about traces and determinants for operators in Hilbert space. For proofs and further details we refer the reader to Simon [137].

(i) An operator $T: H_1 \to H_2$ (where H_1 and H_2 are Hilbert spaces) is of *trace class* if it is of the form

$$Tv = \sum \lambda_k \langle u_k, v\rangle w_k,$$

where $\{u_k\}$ and $\{w_k\}$ are orthonormal families in H_1 and H_2, and $\sum |\lambda_k| < \infty$. The *trace norm* $\|T\|_1$ of T is then $\sum |\lambda_k|$, and the *trace* of T, defined if $H_1 = H_2$, is given by

$$\text{tr}(T) = \sum \lambda_k \langle u_k, w_k\rangle.$$

(ii) The operators of trace class in $\mathscr{B}(H)$ form a two-sided ideal $\mathscr{I}_1(H)$ contained in the ideal $\mathscr{I}_2(H)$ of Hilbert–Schmidt operators. The product of two Hilbert–Schmidt operators is of trace class.

(iii) An operator $A: H \to H$ has a determinant (by definition) if and only if $A - 1$ is of trace class. If A has a determinant it is invertible if and only if $\det(A) \neq 0$. If A_1 and A_2 have determinants, then so does A_1A_2, and $\det(A_1A_2) = \det(A_1)\det(A_2)$.

To obtain the central extension of GL_{res} we begin by constructing an extension

$$\mathscr{T} \to \mathscr{E} \to GL^0_{\text{res}} \qquad (6.6.1)$$

of the identity component of GL_{res} by the group $\mathscr{T}$ of all invertible operators $q: H_+ \to H_+$ which have a determinant. (The topology of $\mathscr{T}$ is defined by using the trace-norm as a metric.) The extension $\mathscr{E}$ is of interest in its own right. We shall see that it is a *contractible* Banach Lie group. The exact sequence of homotopy groups associated to the fibration (6.6.1) shows that

$$\pi_i(GL^0_{\text{res}}) \cong \pi_{i-1}(\mathscr{T}).$$

The last group is well-known to coincide with $\pi_{i-1}(GL_n(\mathbb{C}))$ when $i-1 < 2n$—see, for example, Palais [121]—and this gives us the isomorphism (6.4.1) already mentioned.

The definition of $\mathscr{E}$ is very simple. The identity component of GL_{res} consists of the operators

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

such that the Fredholm operator a has index zero. Because a has index zero one can add to it an operator t of finite rank so that $q = a + t$ is invertible. We define $\mathscr{E}$ as a subgroup of $GL_{\text{res}} \times GL(H_+)$:

$$\mathscr{E} = \{(A, q) \in GL_{\text{res}} \times GL(H_+) : a - q \text{ is of trace class}\}.$$

We give it, however, not the subgroup topology, but that induced by its embedding

$$(A, q) \mapsto (A, a - q)$$

as an open set of $GL_{\text{res}} \times \mathscr{I}_1(H_+)$. It is then a Banach Lie group. The motivation for the definition of $\mathscr{E}$ will appear, as we have said, in Chapter Seven, but it is at any rate clear that $\mathscr{E}$ is an extension of GL^0_{res} by $\mathscr{T}$.

Proposition (6.6.2). *The group $\mathscr{E}$ is contractible.*

Proof. Consider the diagram

$$\begin{array}{ccc} \mathscr{E} & \to & GL(H_+) \times \mathscr{I}_1(H) \\ \downarrow & & \downarrow \\ GL^0_{\text{res}} & \to & \text{Fred}^0(H_+), \end{array}$$

where the upper horizontal map is $(A, q) \mapsto (q, aq^{-1} - 1)$,
the lower horizontal map is $A \mapsto a$,
the right-hand vertical map is $(q, t) \mapsto (1 + t)q$, and
$\text{Fred}^0(H_+)$ denotes the Fredholm operators of index 0 in H_+.

Both vertical maps are fibrations with the group $\mathscr{T}$ as fibre, and the diagram is cartesian (i.e. the fibration on the left is the pull-back of that on the right). We know from (6.2.4) that the lower horizontal map is a homotopy equivalence; it follows that the upper one is too. But $GL(H_+) \times \mathscr{I}_1(H)$ is contractible.

We can use the determinant homomorphism $\det: \mathcal{T} \to \mathbb{C}^\times$ to obtain from $\mathscr{E}$ an extension—obviously a central extension—of GL^0_{res} by $\mathbb{C}^\times$. This extension is simply $\mathscr{E}/\mathcal{T}_1$, where $\mathcal{T}_1$ is the kernel of det. We wish, however, to have an extension of all of GL_{res}, not just of its identity component. Now GL_{res} is the semidirect product of its identity component by $\mathbb{Z}$, where we can take for $\mathbb{Z}$ the subgroup generated by any element in the ± 1 components, for example a shift operator $\sigma: H \to H$ which embeds H_+ in itself with codimension one. The automorphism $A \mapsto \sigma A \sigma^{-1}$ of GL^0_{res} is covered by the endomorphism $\tilde{\sigma}$ of $\mathscr{E}$ defined by

$$(A, q) \mapsto (\sigma A \sigma^{-1}, q_\sigma),$$

where

$$q_\sigma = \begin{cases} \sigma q \sigma^{-1} & \text{on} \quad \sigma(H_+) \\ 1 & \text{on} \quad H_+ \ominus \sigma(H_+). \end{cases}$$

The endomorphism $\tilde{\sigma}$ induces $q \mapsto q_\sigma$ on the normal subgroup $\mathcal{T}$ of $\mathscr{E}$, and is not an automorphism. Indeed, though we shall not give the proof here, there is *no* automorphism of $\mathscr{E}$ which covers $A \mapsto \sigma A \sigma^{-1}$, and so there is no extension of GL_{res} by $\mathcal{T}$ which restricts to $\mathscr{E}$. On the other hand, because $\det(q_\sigma) = \det(q)$, the endomorphism of $\mathscr{E}$ does induce an automorphism of $\mathscr{E}/\mathcal{T}_1$, and we can form the semidirect product $\mathbb{Z} \tilde{\times} (\mathscr{E}/\mathcal{T}_1)$ to obtain a central extension $GL^\sim_{\text{res}}$ of GL_{res} by $\mathbb{C}^\times$. This is the extension we have been seeking.

Remark (6.6.3). It is easy to check that if σ is *any* element of GL_{res} such that $\sigma(H_+) \subset H_+$, and $\tilde{\sigma} \in GL^\sim_{\text{res}}$ is a representative of σ, then the above formula

$$\tilde{\sigma} \,.\, (A, q) \,.\, \tilde{\sigma}^{-1} = (\sigma A \sigma^{-1}, q_\sigma)$$

always holds in $GL^\sim_{\text{res}}$.

There is no continuous cross-section of $GL^\sim_{\text{res}} \to GL_{\text{res}}$ (indeed it follows from (6.6.2) that its Chern class is the universal first Chern class in $H^2(\mathbb{Z} \times BU)$), and so the extension cannot be described by a continuous cocycle. But there is a cross-section of $\mathscr{E} \to GL_{\text{res}}$ defined in the subset U of GL_{res} where a is invertible: it is given by $A \mapsto \tilde{A} = (A, a)$. In that region, which is a dense open subset of the identity component of GL_{res}, we have

Proposition (6.6.4). *If $A_1 A_2 = A_3$ in GL_{res}, and A_1, A_2, A_3 all belong to U, then*

$$\tilde{A}_1 \tilde{A}_2 = c(A_1, A_2) \tilde{A}_3$$

in $GL^\sim_{\text{res}}$, where

$$c(A_1, A_2) = \det(a_1 a_2 a_3^{-1}).$$

Notice that the operators a_1, a_2, a_3 do not themselves have determinants, but only the combination $a_1 a_2 a_3^{-1}$.

The main practical utility of Proposition (6.6.4) is that it enables us to read off the Lie algebra cocycle of the extension $GL^\sim_{\text{res}}$. If an extension of a Lie group Γ is defined by a smooth cocycle $c: \Gamma \times \Gamma \to K$ then the corresponding Lie algebra cocycle is

$$(\xi, \eta) \mapsto D_1 D_2 c(\xi, \eta) - D_1 D_2 c(\eta, \xi),$$

where $D_1 D_2 c$ denotes the mixed second partial derivative of c at the identity. The Lie algebra of GL_{res} consists of all bounded operators A such that $[J, A]$ is Hilbert–Schmidt. As usual we shall write them

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where b and c are Hilbert–Schmidt. We find

Proposition (6.6.5). *The Lie algebra cocycle corresponding to the extension $GL^\sim_{\text{res}}$ is given by*

$$\begin{aligned} (A_1, A_2) \mapsto & \operatorname{trace}([a_1, a_2] - a_3) \\ &= \operatorname{trace}(c_1 b_2 - b_1 c_2) \\ &= \tfrac{1}{4}\operatorname{trace}(J[J, A_1][J, A_2]), \end{aligned} \tag{6.6.6}$$

where $A_3 = [A_1, A_2]$.

To conclude this section let us notice that it is natural to define $U^\sim_{\text{res}}$ as the intersection of $GL^\sim_{\text{res}}$ with $U(H) \times U(H_+)$. Then $U^\sim_{\text{res}}$ is an extension of $U_{\text{res}} = U_{\text{res}}(H)$ by $\mathbb{T}$, and its complexification is GL_{res}.

6.7 The central extension of $LGL_n(\mathbb{C})$

We can use the homomorphism $M: LGL_n(\mathbb{C}) \to GL_{\text{res}}$ to pull back the extension $GL^\sim_{\text{res}}$. We obtain a central extension $\tilde{L}_{\mathbb{C}}$ of $LGL_n(\mathbb{C})$ by $\mathbb{C}^\times$. It is a complex Lie group, because the homomorphism M is holomorphic. The subgroup LU_n of $LGL_n(\mathbb{C})$ maps into the unitary group U_{res}, so the extension of U_{res} by $\mathbb{T}$ pulls back to an extension $\tilde{L}$ of LU_n by $\mathbb{T}$; and $\tilde{L}_{\mathbb{C}}$ is obviously a complexification of $\tilde{L}$.

Proposition (6.7.1). *The extension $\tilde{L}$ of LU_n induced by $U^\sim_{\text{res}}$ is the basic extension constructed in Section* 4.7.

Proof. We saw in Section 4.7 that the basic extension of LU_n is characterized by its Lie algebra cocycle. We shall calculate the Lie algebra cocycle of $\tilde{L}$.

Let ξ and η be elements of Lu_n, and A_1, A_2, A_3 be the operators on

$H^{(n)}$ corresponding to ξ, η, and $[\xi, \eta]$. In view of (6.6.5) we must show that

$$\operatorname{trace}\{[a_1, a_2] - a_3\} = i\omega(\xi, \eta), \qquad (6.7.2)$$

where

$$\omega(\xi, \eta) = \frac{1}{2\pi}\int_0^{2\pi} \langle \xi(\theta), \eta'(\theta)\rangle \, d\theta,$$

and $\langle \ , \ \rangle$ is the standard inner product on $\mathfrak{u}_n$ given by $\langle X, Y\rangle = -\operatorname{trace}(XY)$.

By linearity we can suppose that $\xi = Xz^k$ and $\eta = Yz^m$, with $X, Y \in \mathfrak{gl}_n(\mathbb{C})$. If $k + m \neq 0$ then the left-hand-side of (6.7.2) is zero because the matrices of $[a_1, a_2]$ and a_3 have no diagonal entries; and $\omega(\xi, \eta) = 0$ also. If $m = -k$, on the other hand, then both operators $[a_1, a_2]$ and a_3 preserve each of the subspaces $\mathbb{C}^n . z^q$ of which $H_+^{(n)}$ is the sum (for $q \geq 0$). If $q \geq k$ then $[a_1, a_2]$ and a_3 coincide on $\mathbb{C}^n . z^q$. If $q < k$ then $[a_1, a_2]$ acts as $-YX$ on $\mathbb{C}^n . z^q$, while a_3 acts as $[X, Y]$. The left-hand-side of (6.7.2) is therefore $-k . \operatorname{trace}(XY)$, i.e. $k\langle X, Y\rangle$. The right-hand-side is the same.

Remark. Although it is not needed for the proof of (6.7.1) it is instructive to use the remark (6.6.3) to calculate explicitly the effect of conjugation by a loop γ of winding number m on the centre of the identity component $\tilde{L}^0$ of $\tilde{L}$. The centre of $\tilde{L}^0$ is canonically $\mathbb{T} \times \mathbb{T}$, where the first $\mathbb{T}$ is the scalar matrices in $U_n \subset LU_n$, and the second $\mathbb{T}$ belongs to the extension. An element u of the first $\mathbb{T}$ can be represented by $(u, u) \in \mathscr{E} \subset GL_{\text{res}} \times GL(H_+)$. We can assume that M_γ maps H_+ into itself with codimension m. Then the automorphism of $\mathscr{E}$ corresponding to M_γ takes (u, u) to (u, v), where $v: H_+ \to H_+$ is multiplication by u on $M_\gamma H_+$, and is the identity on the m-dimensional space $H_+ \ominus M_\gamma H_+$. So $\det(vu^{-1}) = u^{-m}$, as we want.

We can now prove that *all* of the extensions

$$\mathbb{T} \to \tilde{L}G \to LG$$

considered in Chapter 4 possess complexifications

$$\mathbb{C}^\times \to \tilde{L}G_{\mathbb{C}} \to LG_{\mathbb{C}}.$$

It is enough to consider the case where G is simply connected and simple. If we choose a unitary representation ρ of G on $\mathbb{C}^n$, then by pulling back the extension $\tilde{L}_{\mathbb{C}}$ of $LGL_n(\mathbb{C})$ found above we get an extension $\tilde{L}_\rho G_{\mathbb{C}}$ which corresponds to the trace form $\langle \ , \ \rangle_\rho$ of ρ on $\mathfrak{g}$:

$$\langle \xi, \eta\rangle_\rho = -\operatorname{trace}(\rho(\xi)\rho(\eta)).$$

Now any integral invariant form on $\mathfrak{g}$ is an integral multiple of the basic form $\langle \ , \ \rangle$ (see Section 4.4). The extension $\tilde{L}G_{\mathbb{C}}$ corresponding to $\langle \ , \ \rangle$ can be constructed as the simply connected covering group $\tilde{L}G_{\mathbb{C}}$ of $\tilde{L}_\rho G_{\mathbb{C}}$; groups corresponding to other forms are quotients of $\tilde{L}G_{\mathbb{C}}$ by finite subgroups of its centre.

6.8 Embedding Diff$^+(S^1)$ in $U_{\text{res}}(H)$

The group of diffeomorphisms of the circle acts on $H^{(n)} = L^2(S^1; \mathbb{C}^n)$. In fact one can make it act in more than one way. We shall choose to make the action *unitary*, i.e. to regard the elements of $H^{(n)}$ as $\frac{1}{2}$-densities on S^1. Thus a diffeomorphism $f: S^1 \to S^1$ acts on functions $\xi: S^1 \to \mathbb{C}^n$ by $\xi \mapsto f . \xi$, where

$$(f . \xi)(\theta) = \xi(g(\theta)) . |g'(\theta)|^{\frac{1}{2}}, \qquad (6.8.1)$$

and g is the inverse of f.

Proposition (6.8.2). $\quad \operatorname{Diff}^+(S^1) \subset U_{\text{res}}(H^{(n)})$.

Proof. This can be proved by either of the methods of (6.3.1). It is done by the first method in [131], so here we shall sketch the alternative argument. We represent J by the kernel K of (6.3.2). From (6.8.1) the kernel representing the action of f is $\delta(g(\theta) - \phi)g'(\theta)^{\frac{1}{2}}$, where δ is the Dirac δ-function. The kernel of the commutator $[f, J]$ is therefore

$$\int_0^{2\pi} \{\delta(g(\theta) - \psi)g'(\theta)^{\frac{1}{2}}K(\psi, \phi) - K(\theta, \psi)\delta(g(\psi) - \phi)g'(\psi)^{\frac{1}{2}}\} \, d\psi.$$

This reduces to

$$g'(\theta)^{\frac{1}{2}}K(g(\theta), \phi) - K(\theta, f(\phi))f'(\phi)^{\frac{1}{2}}. \qquad (6.8.3)$$

From (6.3.2) we see that K is a smooth function of both its variables except on the diagonal, where

$$K(\theta, \phi) = 2i/(\theta - \phi) + (\text{smooth function}).$$

Inserting this in (6.8.3), we find that the kernel of $[f, J]$ is continuous (indeed smooth) everywhere, and hence that $[f, J]$ is Hilbert–Schmidt.

There is, however, an important difference between $LGL_n(\mathbb{C})$ and $\operatorname{Diff}^+(S^1)$ in relation to GL_{res}. The former maps smoothly into GL_{res}, whereas the inclusion of $\operatorname{Diff}^+(S^1)$ is not even continuous. Indeed the norm topology of $GL(H^{(n)})$—and hence â fortiori that of GL_{res}—induces the discrete topology on $\operatorname{Diff}^+(S^1)$. (To see that, observe that for any diffeomorphism f except the identity there is a unit vector $\xi \in H^{(n)} = L^2(S^1; \mathbb{C}^n)$ such that $\langle \xi, f^*\xi\rangle = 0$: take ξ to have support in a small neighbourhood of a point of S^1 not fixed by f.) If one attempts to calculate formally the homomorphism of Lie algebras induced by

$\mathrm{Diff}^+(S^1) \to GL_{\mathrm{res}}$ then one finds that vector fields on S^1 correspond to *unbounded* operators in $H^{(n)}$: consider, for example, $\mathrm{d}/\mathrm{d}\theta$.

Despite this, the central extension of $\mathrm{Diff}^+(S^1)$—considered as an abstract group—induced by $GL^{\sim}_{\mathrm{res}}$ *is* indeed a Lie group, and its Lie algebra cocycle can be calculated exactly as was done in (6.7.1) above, ignoring the unboundedness of the operators. The formal calculation is carried out in [131]; the reason for its validity is that the *composite* map

$$\mathrm{Diff}^+(S^1) \to U_{\mathrm{res}} \to U_{\mathrm{res}}/(U_+ \times U_-), \tag{6.8.4}$$

where $U_+ \times U_- = U(H_+) \times U(H_-)$ is the commutant of J in U_{res}, is smooth, and the 2-cocycle of $\mathfrak{u}_{\mathrm{res}}$, regarded as an invariant form on U_{res}, actually comes from $U_{\mathrm{res}}/(U_+ \times U_-)$. To see that (6.8.4) is smooth we observe first that the map $A \mapsto [A, J]$ defines a smooth immersion from a neighbourhood of the base-point in $U_{\mathrm{res}}/(U_+ \times U_-)$ into the space of Hilbert–Schmidt operators, and then that when f is a diffeomorphism the commutator $[f, J]$ is represented by a smooth kernel which depends smoothly on f.

We shall record the result here.

Proposition (6.8.5). *The central extension of* $\mathrm{Diff}^+(S^1)$ *induced by* $GL^{\sim}_{\mathrm{res}}(H^{(n)})$ *is trivial over* $SL_2(\mathbb{R})$, *and has the Lie algebra cocycle*

$$(\xi, \eta) \mapsto \frac{n}{12\pi} \int_0^{2\pi} (\xi'''(\theta) + \xi'(\theta))\eta(\theta)\, \mathrm{d}\theta$$

for $\xi = \xi(\theta)\,\mathrm{d}/\mathrm{d}\theta$, $\eta = \eta(\theta)\,\mathrm{d}/\mathrm{d}\theta$ *in* $\mathrm{Vect}(S^1)$.

Because $\mathrm{Diff}^+(S^1)$ is contained in U_{res} it acts on the extension $U^{\sim}_{\mathrm{res}}$ by conjugation, and hence on the subgroup $\tilde{L}U_n$ which covers $LU_n \subset GL_{\mathrm{res}}$. This way of seeing that $\mathrm{Diff}^+(S^1)$ acts on $\tilde{L}U_n$ is simpler and more natural than the one given in Section 4.7.

Proposition (6.8.6). *The action of* $\mathrm{Diff}^+(S^1)$ *on* $\tilde{L}U_n$ *induced by the embedding in* $GL^{\sim}_{\mathrm{res}}(H^{(n)})$ *is given by the formula* (4.7.3).

Proof. From the discussion in Section 4.7 we know that it is enough to check the action on the Lie algebra $\tilde{L}\mathfrak{u}_n$, and therefore even enough to find the action of $\xi \in \mathrm{Vect}(S^1)$ on $\tilde{\eta} \in \tilde{L}\mathfrak{u}_n$. In notation corresponding to that of (6.7.2) we must show that

$$\mathrm{trace}\{[a_1, a_2] - a_3\} = -\frac{1}{4\pi} \int_0^{2\pi} \xi(\theta)\mathrm{trace}\, \eta'(\theta)\, \mathrm{d}\theta.$$

This is proved by a calculation exactly like the proof of (6.7.2): when the action of the vector field $\xi = \sum \xi_k e^{ik\theta}\,\mathrm{d}/\mathrm{d}\theta$ is written as a $\mathbb{Z} \times \mathbb{Z}$ matrix whose entries are $n \times n$ blocks, the $(p, q)^{\text{th}}$ entry is $-\frac{1}{2}\mathrm{i}(p+q)\xi_{p-q}\mathbf{1}_n$, where $\mathbf{1}_n$ is the $n \times n$ identity matrix.

We can also prove the following explicit formula which will be useful later. (It agrees with (4.7.5) when $n = 1$.)

Proposition (6.8.7). *If* $\gamma: S^1 \to U_n$ *is the homomorphism* $\theta \mapsto \exp(\theta\xi)$, *and* $\tilde{\gamma}$ *is a representative of* γ *in* $\tilde{L}U_n$, *then the action on* $\tilde{\gamma}$ *of the rotation* R_α *through the angle* α *is given by*

$$R_\alpha \tilde{\gamma} = e^{-\frac{1}{2}\mathrm{i}\alpha(\|\xi\|^2 - m)} \,.\, \tilde{\gamma} \,.\, \gamma(\alpha)^{-1},$$

where m is the winding number of γ.

Proof. It is enough to consider the case when $\gamma H_+ \subset H_+$, and then we can use Remark (6.6.3) to calculate $\tilde{\gamma} R_\alpha \tilde{\gamma}^{-1}$ in $GL^{\sim}_{\mathrm{res}}$. One finds

$$\tilde{\gamma} R_\alpha \tilde{\gamma}^{-1} = R_\alpha \,.\, \gamma(\alpha) \,.\, u^{-1},$$

where u is the determinant of the action of R_α on $H_+ \ominus \gamma H_+$. To calculate u we can assume that γ is the diagonal loop $z \mapsto \mathrm{diag}(z^{k_1}, \ldots, z^{k_n})$. Then

$$u = \exp - \mathrm{i}\alpha \sum \tfrac{1}{2} k_j(k_j - 1) = \exp - \tfrac{1}{2}\mathrm{i}\alpha(\|\xi\|^2 - m).$$

6.9 Other polarizations of H: replacing the circle by the line, and the introduction of 'mass'

In two-dimensional quantum field theory one is interested primarily in functions defined on the line $\mathbb{R}$, which represents physical space, rather than on the circle. One can identify $\mathbb{R} \cup \infty$ with S^1 by stereographic projection, i.e.

$$e^{i\theta} \in S^1 \leftrightarrow 2 \tan \tfrac{1}{2}\theta \in \mathbb{R}, \tag{6.9.1}$$

where we choose $\theta \in (-\pi, \pi]$. The Hilbert space $H = L^2(S^1; \mathbb{C})$ is then isometrically isomorphic to $H^{\mathbb{R}} = L^2(\mathbb{R}; \mathbb{C})$ by the correspondence

$$\phi \in H \leftrightarrow \tilde{\phi} \in H^{\mathbb{R}},$$

where

$$\tilde{\phi}(2 \tan \tfrac{1}{2}\theta) = \phi(\theta)\cos \tfrac{1}{2}\theta. \tag{6.9.2}$$

The natural polarization of $H^{\mathbb{R}}$ is given by the positive and negative eigenspaces of $-\mathrm{i}\,\mathrm{d}/\mathrm{d}x$ on $\mathbb{R}$, i.e. $H^{\mathbb{R}} = H^{\mathbb{R}}_+ \oplus H^{\mathbb{R}}_-$, where $H^{\mathbb{R}}_+$ consists of the functions f whose Fourier transform $\hat{f}$, given by

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-\mathrm{i}x\xi}\, \mathrm{d}x,$$

vanishes for $\xi < 0$. Under the isomorphism $H \cong H^{\mathbb{R}}$ of (6.9.2) we find

$$H^{\mathbb{R}}_+ \leftrightarrow e^{\frac{1}{2}\mathrm{i}\theta} H_+,$$

where $e^{\frac{1}{2}i\theta}$ is an L^∞ function on the circle with a jump discontinuity at $\theta = \pm\pi$. The Fourier series of $e^{\frac{1}{2}i\theta}$ is $\sum a_k e^{ik\theta}$, with

$$a_k = 2 \,.\, (-1)^{k+1}/(2k-1)\pi.$$

Because $\sum |k a_k^2|$ does not converge, the subspace $H_+^{\mathbb{R}}$ does not belong to $\mathrm{Gr}(H)$. Conjugation by $e^{\frac{1}{2}i\theta}$ therefore does not define an automorphism of $GL_{\mathrm{res}}(H)$, and the groups $GL_{\mathrm{res}}(H)$ and $GL_{\mathrm{res}}(H_{\mathbb{R}})$ defined with respect to the natural polarizations are not mapped to each other by the isomorphism $H \cong H^{\mathbb{R}}$ of (6.9.2), even though they are isomorphic groups.

The group $L\mathbb{T}$ acts on $H^{\mathbb{R}}$ by the correspondence (6.9.1), and similarly LU_n acts on $H^{\mathbb{R},(n)} = L^2(\mathbb{R};\mathbb{C}^n)$. But the embedding $LU_n \to GL_{\mathrm{res}}(H^{\mathbb{R},(n)})$ so obtained is not interestingly different from the standard $LU_n \to GL_{\mathrm{res}}(H^{(n)})$, for conjugation by $e^{\frac{1}{2}i\theta}$ induces the identity on the image of LU_n.

More interesting is to polarize the space $H^\Delta = L^2(\mathbb{R};\mathbb{C}^2)$ according to the positive and negative parts of the spectrum of the operator

$$D_m = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\frac{\mathrm{d}}{\mathrm{d}x} + \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}$$

where m is some positive number. One should think of H^Δ as the space of solutions ψ of the Dirac equation with mass m

$$\left\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{\partial}{\partial t} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\frac{\partial}{\partial x}\right\}\psi = im\psi \tag{6.9.3}$$

for functions $\psi: \mathbb{R}^2 \to \mathbb{C}^2$; the decomposition $H^\Delta = H_+^\Delta \oplus H_-^\Delta$ is then the decomposition according to the spectrum of the 'energy' operator $-i(\partial/\partial t)$. As usual we can form the tensor product $H^{\Delta,(n)} = H^\Delta \otimes \mathbb{C}^n$, and the loop group LU_n acts on this by multiplication operators. (We are again using the identification (6.9.1).)

Proposition (6.9.4). *The action of LU_n on $H^{\Delta,(n)}$ induces an embedding $i_m: LU_n \to GL_{\mathrm{res}}(H^{\Delta,(n)})$.*

Proof. If we replace functions on $\mathbb{R}$ by their Fourier transforms then D_m becomes the multiplication operator on $H^\Delta \otimes \mathbb{C}^n$ by the matrix-valued function

$$\begin{pmatrix} \xi & m \\ m & -\xi \end{pmatrix} \otimes 1.$$

The operator J corresponding to the polarization is therefore multiplication by the function $J(\xi) \otimes 1$, where

$$J(\xi) = \frac{1}{E(\xi)}\begin{pmatrix} \xi & m \\ m & -\xi \end{pmatrix} \otimes 1,$$

and $E(\xi) = +\sqrt{(\xi^2 + m^2)}$. (Notice that $J(\xi)^2 = 1$.) If $\gamma: \mathbb{R} \to U_n$ is an element of LU_n it is enough for us to show that the commutator $[M_{\gamma-\gamma(\infty)}, J]$ is Hilbert–Schmidt. Let $\hat{\gamma}$ be the Fourier transform of $\gamma - \gamma(\infty)$. The commutator is represented by the kernel (see the second proof of Proposition (6.3.1))

$$(J(\xi) - J(\eta)) \otimes \hat{\gamma}(\xi - \eta). \tag{6.9.5}$$

Now the trace of the matrix $(J(\xi) - J(\eta))^2$ is

$$a(\xi, \eta) = \frac{4(E(\xi)E(\eta) - \xi\eta - m^2)}{E(\xi)E(\eta)},$$

and it is easy to show that

$$\int_{\mathbb{R}} a(\eta + \zeta, \eta)\,\mathrm{d}\eta \le C\,|\zeta|, \tag{6.9.6}$$

where C is some constant which depends only on m. The Hilbert–Schmidt norm of the commutator (6.9.5) is therefore dominated by

$$\int_{\mathbb{R}} |\zeta|\,|\hat{\gamma}(\zeta)|^2\,\mathrm{d}\zeta < \infty,$$

as we want.

The embedding i_m gives us nothing new when $m = 0$, for then H^Δ breaks into two independent copies of $H^{\mathbb{R}}$ on which we have used the standard polarization and its opposite. It follows that the central extension of GL_{res} restricts trivially to LU_n. This remains true in general.

Proposition (6.9.7). *The central extension of $GL_{\mathrm{res}}(H^{\Delta,(n)})$ is trivial over $i_m(LU_n)$.*

Proof. We use the formula (6.6.6) to calculate the Lie algebra cocycle. Suppose that two elements of Lu_n are represented by matrix-valued functions f and g on $\mathbb{R}$. We can suppose that f and g are square-summable, for the value of the cocycle is not changed by replacing them by $f - f(\infty)$ and $g - g(\infty)$. Then the cocycle is

$$\frac{1}{4}\iint \mathrm{trace}\{J(\xi)(J(\xi) - J(\eta))\hat{f}(\xi - \eta)(J(\eta) - J(\xi))\hat{g}(\eta - \xi)\}\,\mathrm{d}\xi\,\mathrm{d}\eta$$

$$= \frac{1}{4}\iint \mathrm{trace}\{J(\xi)(J(\xi) - J(\eta))^2\}\langle \hat{f}(\xi - \eta), \hat{g}(\eta - \xi)\rangle\,\mathrm{d}\xi\,\mathrm{d}\eta.$$

But this vanishes, because

$$J(\xi)(J(\xi) - J(\eta))^2 = 2J(\xi) - J(\eta) - J(\xi)J(\eta)J(\xi)^{-1},$$

which has trace 0.

Before mentioning our final variant in this direction we should point out that the main interest of the Dirac polarization of H^Δ is not for constructing representations of LU_n. More important is that it leads to a new representation of the so-called 'canonical commutation relations'. We shall prove the basic result here, but we shall postpone discussing its significance till Chapter 10.

Let M_f denote the operator on $H^\Delta = L^2(\mathbb{R}; \mathbb{C}^2)$ given by multiplication by

$$\begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix}$$

where $f:\mathbb{R}\to\mathbb{C}$ is a smooth function which is constant outside a finite interval (but perhaps with different values at the two ends of the line). The commutator $[J, M_f]$ is Hilbert–Schmidt. (We proved this above when $f(+\infty)=f(-\infty)$, but the argument works in general.) Thus the operators M_f form an abelian subalgebra $\mathscr{F}$ of the Lie algebra $\mathfrak{gl}_{\text{res}}(H^\Delta)$, and the central extension of $\mathfrak{gl}_{\text{res}}$ is trivial when restricted to $\mathscr{F}$.

Now let N_f denote the operator on H^Δ given by

$$\begin{pmatrix} f & 0 \\ 0 & -f \end{pmatrix},$$

where $f:\mathbb{R}\to\mathbb{C}$ is smooth with compact support. Again the commutator $[J, N_f]$ is Hilbert–Schmidt: its kernel is

$$(J(\xi)A - AJ(\eta))\hat{f}(\xi-\eta),$$

where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and the calculation is essentially as before, except that because $J(\xi)A - AJ(\eta)$ does not vanish when $\xi=\eta$ we need f to vanish at $\pm\infty$. (The estimate (6.9.6) is replaced by

$$\int_{\mathbb{R}} \text{trace}(J(\eta+\zeta)A - AJ(\eta))^2 \, d\eta \leqslant C_1 + C_2|\zeta|.)$$

The operators N_f form an abelian subalgebra $\mathscr{F}'$ of $\mathfrak{gl}_{\text{res}}(H^\Delta)$, and the central extension restricts trivially to it.

Let us now think of $\mathscr{F}$ and $\mathscr{F}'$ as subalgebras of the extension $\tilde{\mathfrak{gl}}_{\text{res}}$, and if f is a smooth function with compact support let us introduce the notation

$$\Phi(f) = 2^{-\frac{1}{2}} M_F$$
$$\dot{\Phi}(f) = 2^{-\frac{1}{2}} N_f$$

where $F(x) = \int_{-\infty}^{x} f(y)\,dy$. The motivation for this notation is that we have

$$-i\dot{\Phi}(f) = [D_m, \Phi(f)] \tag{6.9.8}$$

as operators on H^Δ. The remarkable result is

Proposition (6.9.9). *In* $\tilde{\mathfrak{gl}}_{res}$ *the operators* $\Phi(f)$ *and* $\dot{\Phi}(f)$ *satisfy the 'canonical commutation relations', i.e.*

$$[\Phi(f), \Phi(g)] = [\dot{\Phi}(f), \dot{\Phi}(g)] = 0,$$

and

$$[\dot{\Phi}(f), \Phi(g)] = -i\int_{\mathbb{R}} f(x)g(x)\,dx.$$

Proof. The formula (6.6.6) for the cocycle tells us that $[\dot{\Phi}(f), \Phi(g)]$ is given by

$$\frac{1}{8}\int\int \text{trace}\{J(\xi)(J(\xi)A - AJ(\eta))(J(\eta)-J(\xi))\hat{f}(\xi-\eta)\hat{G}(\eta-\xi)\}\,d\xi\,d\eta.$$

But

$$\begin{aligned}\text{trace}\{J(\xi)(J(\xi)A - AJ(\eta))(J(\eta)-J(\xi))\} &= 2\,\text{trace}\,A(J(\eta)-J(\xi)) \\ &= 4\left[\frac{\eta}{E(\eta)} - \frac{\xi}{E(\xi)}\right] \\ &= 4b(\xi,\eta), \text{ say.}\end{aligned}$$

Because $\xi/E(\xi)\to\pm1$ as $\xi\to\pm\infty$ it is clear that

$$\int_{\mathbb{R}} b(\eta+\zeta, \eta)\,d\eta = -2\zeta,$$

so the commutator is

$$\int_{\mathbb{R}} \zeta\hat{f}(\zeta)\hat{G}(-\zeta)\,d\zeta = -i\int_{\mathbb{R}} f(x)g(x)\,dx.$$

We conclude this section with one last subgroup of $U_{\text{res}}(H^\Delta)$. It is the group $\Lambda\mathbb{T}$ of smooth maps $\gamma:\mathbb{R}\to\mathbb{T}$ which have compact support—i.e. $\gamma(x)=1$ when $|x|$ is large. We make it act on H^Δ by associating to γ the multiplication operator

$$\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix}.$$

It follows from the preceding discussion that this belongs to $U_{\text{res}}(H^\Delta)$, and that the induced central extension of $\Lambda\mathbb{T}$ is the basic one (i.e. the one coming from the inclusion $\Lambda\mathbb{T}\subset L\mathbb{T}$).

More generally, we can embed ΛU_n in $U_{\text{res}}(H^{\Delta,(n)})$ in a precisely analogous way. The interest of these embeddings is that Carey and Ruijsenaars [24] have shown that when $m > 0$ the standard representation of U_{res} restricts to give a type III factor representation of ΛU_n. (Cf. Section 10.7.)

6.10 Generalizations to other groups of maps

In the definition (6.2.1) of GL_{res} the ideal $\mathcal{I}_2 = \mathcal{I}_2(H)$ of Hilbert–Schmidt operators can be replaced by any other symmetrically normed two-sided ideal $\mathcal{I}$ (cf. Simon [137]). Let us denote the resulting group by $GL_{\mathcal{I}}$. The biggest such $\mathcal{I}$ is the ideal $\mathcal{K}$ of compact operators; and for any $p \geqslant 1$ there is the ideal $\mathcal{I}_p$ consisting of operators T such that $(T^*T)^{p/2} \in \mathcal{I}_1$. All the groups $GL_{\mathcal{I}}$ have properties very similar to GL_{res}. Their homotopy type is independent of $\mathcal{I}$ (see Palais [121]), and there is an extension

$$\mathcal{T}_{\mathcal{I}^2} \to \mathcal{E}_{\mathcal{I}} \to GL_{\mathcal{I}},$$

where $\mathcal{E}_{\mathcal{I}}$ is contractible, and $\mathcal{T}_{\mathcal{I}^2}$ is the group of invertible operators belonging to $1 + \mathcal{I}^2$. But only if $\mathcal{I}^2 \subset \mathcal{I}_1$, i.e. if $\mathcal{I} \subset \mathcal{I}_2$, is there a determinant homomorphism $\mathcal{T}_{\mathcal{I}^2} \to \mathbb{C}^\times$ enabling one to construct a central extension by $\mathbb{C}^\times$. In other words, unless $\mathcal{I} \subset \mathcal{I}_2$ the basic 2-dimensional cohomology class of the space $GL_{\mathcal{I}}$ cannot be represented by a left-invariant differential form.

Now let us consider how far the theory of this chapter can be generalized from the loop group LGL_n to the group $\text{Map}(X; GL_n)$, where X is a compact smooth manifold.

It is easy to find embeddings

$$\text{Map}(X; GL_n) \to GL_{\mathcal{K}},$$

and their classification is an interesting question in algebraic topology. If X is of odd dimension $d = 2m - 1$ and is a 'spin manifold'—i.e. it is orientable and satisfies the additional mild global condition that its second Stiefel-Whitney class vanishes—then there is a complex vector bundle E on X called the bundle of 'spinors'. The fibres of E have dimension 2^{m-1}. There is also a self-adjoint first order differential operator D, the Dirac operator, which acts on the space of sections of E. If H is the space of L^2 sections of E then we can write $H = H_+ \oplus H_-$, where H_+ (resp. H_-) is spanned by the eigenfunctions of D with positive (resp. negative) eigenvalues. The group $\text{Map}(X; \mathbb{C}^\times)$ acts by multiplication operators on H, and similarly $\text{Map}(X; GL_n)$ acts on $H \otimes \mathbb{C}^n$, the space of sections of $E \otimes \mathbb{C}^n$. This action defines an embedding of $\text{Map}(X; GL_n)$ in $GL_{\mathcal{K}}(H \otimes \mathbb{C}^n)$.

More generally, an embedding

$$i_{(E,J)} : \text{Map}_{\text{cts}}(X; GL_n) \to GL_{\mathcal{K}}(H \otimes \mathbb{C}^n) \qquad (6.10.1)$$

is defined by any pair (E, J), where E is a vector bundle on X and J is a self-adjoint operator in the Hilbert space H of L^2 sections of E such that

(i) $J^2 = 1$, and
(ii) $[J, M_f]$ is compact for every continuous function f on X.

If $n > \frac{1}{2} \dim X$ then the group of connected components of $\text{Map}(X; GL_n)$ is, almost by definition, the generalized cohomology group $K^{-1}(X)$ of Atiyah and Hirzebruch [3]. This group is closely related to the classical cohomology

$$H^{\text{odd}}(X; \mathbb{Z}) = \bigoplus_{k \text{ odd}} H^k(X; \mathbb{Z}),$$

and becomes isomorphic to it when tensored with the rationals. Passing to connected components in (6.10.1) gives a homomorphism

$$\text{ind}_{(E,J)} : K^{-1}(X) \to \mathbb{Z}. \qquad (6.10.2)$$

With a little more care it is not hard to show that (E, J) defines an element $\sigma_{(E,J)}$ of the generalized homology group $K_{-1}(X)$, and that $\text{ind}_{(E,J)}$ is the natural pairing with $\sigma_{(E,J)}$. Furthermore, every element of $K_{-1}(X)$ arises in this way.

Among the embeddings $i_{(E,J)}$ the one defined by the Dirac operator is basic: its class $\sigma_{(E,J)}$ is the fundamental class of the manifold X, and the corresponding map (6.10.2) is the 'Gysin' map in K-theory. If X is a sphere, then $i_{(E,J)}$ is the Bott periodicity map.

The preceding statements are a rapid summary of the easy part of an extensive theory which has been developed by Atiyah [4], Kasparov [90], and Connes [32]. A slightly different way of looking at the same material is found in the work of Brown, Douglas, and Fillmore [23], who prove that elements of $K_{-1}(X)$ can be identified with isomorphism classes of algebra extensions

$$\mathcal{K} \to A \to C(X),$$

where $C(X)$ is the algebra of continuous complex-valued functions on X. Such an extension of algebras clearly defines a group extension

$$\mathcal{T}_{\mathcal{K}} \to GL_n(A) \to GL_n(C(X)) = \text{Map}_{\text{cts}}(X; GL_n).$$

This is the extension got by pulling back $\mathcal{E}_{\mathcal{K}}$ by $i_{(E,J)}$.

But from the point of view of the present book the group $GL_{\mathcal{K}}$ is not of very much use, because nothing is known about its representations. To embed $\text{Map}(X; GL_n)$ in $GL_{\mathcal{I}_2}$ we must use pairs (E, J) such that $[J, M_f]$ is Hilbert–Schmidt when f is smooth. In the language of Brown, Douglas, and Fillmore we must study extensions

$$\mathcal{I}_1 \to A \to \mathcal{D}(X)$$

of the algebra of smooth functions by the ideal of trace-class operators.

These extensions were studied by Helton and Howe [73]. They correspond to elements of $H_1(X;\mathbb{Z})$, which is a canonical subgroup of $K_{-1}(X)$. The corresponding extensions are the 'uninteresting' ones which we found in Chapter 4.

The best way to think of the situation is probably in terms of pseudo-differential operators [144]. In practice J will be given by a pseudo-differential operator of order zero. The commutator $[J, M_f]$, when f is a smooth function, will then be an operator of order -1. On a manifold of dimension d such an operator belongs to the ideal $\mathcal{I}_r$ if $r > d$. It will thus not normally be Hilbert–Schmidt if $d > 1$.

Example. Let us consider the polarization corresponding to the Dirac operator on a torus X of odd dimension $d = 2m - 1$, i.e. $X = \mathbb{R}^d/2\pi\mathbb{Z}^d$. The spin bundle on X is a trivial bundle whose fibre $\Delta \cong \mathbb{C}^N$ (where $N = 2^{m-1}$) is an irreducible module for the Clifford algebra C_d generated by elements $e_1, \ldots, e_d$ such that $e_i^2 = 1$ and $e_ie_j = -e_je_i$ when $i \neq j$. The Dirac operator on the space H of maps $X \to \Delta$ is

$$D = -i \sum e_j \frac{\partial}{\partial \theta_j}.$$

If we expand the functions in Fourier series, so that H is identified with $\ell^2(\mathbb{Z}^d; \Delta)$ then D becomes the multiplication operator

$$\{f_p\} \mapsto \{pf_p\}.$$

(Here $p \in \mathbb{Z}^d$, and $pf_p \in \Delta$ is got by acting with $p \in \mathbb{R}^d \subset C_d$ on $f_p \in \Delta$.) The corresponding polarization operator J is multiplication by $p/\|p\|$. The commutator $[J, M_f]$, where M_f is multiplication by the scalar-valued function $f = \sum f_p e^{i\langle p, \theta\rangle}$, is represented by the kernel

$$(p, q) \mapsto f_{p-q} \cdot \{p/\|p\| - q/\|q\|\} \tag{6.10.3}$$

on $\mathbb{Z}^d \times \mathbb{Z}^d$. Now $p/\|p\| - q/\|q\|$ is a self-adjoint operator on Δ whose square is

$$2\left(1 - \frac{\langle p, q\rangle}{\|p\|\,\|q\|}\right) = 4\sin^2\frac{\phi}{2},$$

where ϕ is the angle between p and q. If $p - q$ is held fixed then $4\sin^2(\phi/2)$ decays like $\|p\|^{-2}$ as $p \to \infty$. The kernel (6.10.3) is therefore square-summable only if $\dim(X) = 1$. In general it belongs to the Schatten class $\mathcal{I}_r$ when $r > \dim(X)$.

7

THE GRASSMANNIAN OF HILBERT SPACE AND THE DETERMINANT LINE BUNDLE

Because we are studying loop groups by regarding them as groups of operators in Hilbert space we shall need to have a rather detailed knowledge of the structure of the Grassmannian of Hilbert space. This chapter is devoted to that subject. The most important part is the construction of the determinant line bundle in Section 7.7, and the reader interested in that can omit everything between Sections 7.1 and 7.7 except for the definition of an 'admissible basis' in Section 7.5.

7.1 The definition of Gr(H)

Suppose that H is a separable Hilbert space with a given polarization $H = H_+ \oplus H_-$: we assume that H_+ and H_- are infinite dimensional orthogonal closed subspaces. We shall study the Grassmannian of closed subspaces of H which are 'comparable' in size with H_+. Before giving the formal definition of this class of subspaces, let us explain that they are a completion of the class of subspaces W which are *commensurable* with H_+, i.e. those such that $W \cap H_+$ has finite codimension in both W and H_+. They may, however, have zero intersection with H_+: for example the graph W_T of every Hilbert–Schmidt operator $T: H_+ \to H_-$ is included, but W_T is commensurable with H_+ only if T is of finite rank.

Definition (7.1.1). Gr(H) *is the set of all closed subspaces W of H such that*

(i) *the orthogonal projection* $\mathrm{pr}_+: W \to H_+$ *is a Fredholm operator, and*

(ii) *the orthogonal projection* $\mathrm{pr}_-: W \to H_-$ *is a Hilbert–Schmidt operator.*

Fredholm and Hilbert–Schmidt operators have been discussed already in Section 6.2. We recall that a bounded operator is Fredholm if its kernel and cokernel are finite dimensional.

Another way of stating the definition (7.1.1) is: W belongs to Gr(H) if it is the image of an operator $w: H_+ \to H$ such that $\mathrm{pr}_+ \circ w$ is Fredholm and $\mathrm{pr}_- \circ w$ is Hilbert–Schmidt. As the sum of a Fredholm operator and a Hilbert–Schmidt operator is Fredholm, we see that if W belongs to Gr(H) then so does the graph of every Hilbert-Schmidt operator $W \to W^\perp$. These graphs form the subset U_W of Gr(H) consisting of all W' for which the orthogonal projection $W' \to W$ is an isomorphism: it is in one-to-one correspondence with the Hilbert space $\mathscr{I}_2(W; W^\perp)$ of Hilbert–Schmidt operators $W \to W^\perp$. In fact

Proposition (7.1.2). *$\mathrm{Gr}(H)$ is a Hilbert manifold modelled on $\mathscr{I}_2(H_+; H_-)$.*

Before proving this we need one further observation. The group $GL_{\mathrm{res}}(H)$ introduced in Section 6.2 acts on the set $\mathrm{Gr}(H)$. We have

Proposition (7.1.3). *The subgroup $U_{\mathrm{res}}(H)$ of $GL_{\mathrm{res}}(H)$ acts transitively on $\mathrm{Gr}(H)$, and the stabilizer of H_+ is $U(H_+) \times U(H_-)$.*

Proof of (7.1.3). Suppose $W \in \mathrm{Gr}(H)$; we shall find $A \in U_{\mathrm{res}}(H)$ such that $A(H_+) = W$. Let $w: H_+ \to H$ be an isometry with image W, and $w^\perp: H_- \to H$ an isometry with image $W^\perp$. Then

$$w \oplus w^\perp : H_+ \oplus H_- \to H_+ \oplus H_-$$

is a unitary transformation A such that $A(H_+) = W$. We write it

$$A = \begin{pmatrix} w_+ & w_+^\perp \\ w_- & w_-^\perp \end{pmatrix}.$$

Because W belongs to $\mathrm{Gr}(H)$ we know that w_+ is Fredholm and w_- is Hilbert–Schmidt. But because A is unitary it follows that $w_+^\perp$ is Hilbert–Schmidt also (for $w_+^* w_+^\perp + w_-^* w_-^\perp = 0$), and so A belongs to $U_{\mathrm{res}}(H)$.

The assertion about the stabilizer of H_+ is obvious.

Proof of (7.1.2). Suppose that U_{W_0} and U_{W_1} are the subsets of $\mathrm{Gr}(H)$ described above corresponding to the Hilbert spaces $I_0 = \mathscr{I}_2(W_0; W_0^\perp)$ and $I_1 = \mathscr{I}_2(W_1; W_1^\perp)$. Let $U_{W_0} \cap U_{W_1}$ correspond to I_{01} in I_0 and I_{10} in I_1. We must show that I_{01} and I_{10} are open sets, and that the 'change of coordinates' $I_{01} \to I_{10}$ is smooth.

Let the matrix of the identity transformation

$$W_0 \oplus W_0^\perp \to W_1 \oplus W_1^\perp$$

be

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{7.1.4}$$

(i.e. a is a map $W_0 \to W_1$, etc.) From the proof of (7.1.3) we know that a and d are Fredholm, and b and c are Hilbert–Schmidt. Suppose that $W \in \mathrm{Gr}(H)$ is simultaneously the graph of $T_0: W_0 \to W_0^\perp$ and $T_1: W_1 \to W_1^\perp$. Then the operators

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ T_0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ T_1 \end{pmatrix} q$$

from W_0 to $W_1 \oplus W_1^\perp$ must coincide, where q is some isomorphism

$W_0 \to W_1$. We conclude that

$$T_1 = (c + dT_0)(a + bT_0)^{-1}. \tag{7.1.5}$$

Thus T_1 is a holomorphic function of T_0 in the open set

$$I_{01} = \{T_0 \in I_0 : a + bT_0 \text{ is invertible}\}.$$

For a subspace W of H which is commensurable with H_+ it is natural to define the *virtual dimension* of W relative to H_+ as

$$\dim(W/W \cap H_+) - \dim(H_+/W \cap H_+).$$

The generalization of this for an arbitrary $W \in \mathrm{Gr}(H)$ is the *index* of the perpendicular projection $\mathrm{pr}_+: W \to H_+$, i.e.

$$\text{virt.dim}\, W = \dim(\ker \mathrm{pr}_+) - \dim(\operatorname{coker} \mathrm{pr}_+).$$

Equivalently,

$$\text{virt.dim}\, W = \dim(W \cap H_-) - \dim(W^\perp \cap H_+).$$

The virtual dimension separates $\mathrm{Gr}(H)$ into disconnected pieces. In fact the subspaces with a given virtual dimension form a connected set; we shall see presently, for example, that the spaces of virtual dimension zero are the closure of the coordinate patch consisting of the graphs of all Hilbert-Schmidt operators $H_+ \to H_-$. Notice also that if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

belongs to $GL_{\mathrm{res}}(H)$, then

$$\text{virt.dim}\, A(W) = \text{virt.dim}\, W + \chi(a),$$

where $\chi(a)$ is the index of the Fredholm operator a.

To proceed further we shall introduce an orthonormal basis in H. That amounts to identifying H with the space $L^2(S^1; \mathbb{C})$ with its natural basis $\{z^k\}_{k\in\mathbb{Z}}$. (As usual $z = e^{i\theta}$.) We then have a collection of special points $\{H_S\}$ in $\mathrm{Gr}(H)$: H_S is just the closed subspace spanned by z^s for $s \in S$, where S is a subset of $\mathbb{Z}$ which has finite difference from the positive integers $\mathbb{N}$ (i.e. S is bounded below, and contains all sufficiently large integers). We shall write $\mathscr{S}$ for the collection of such sets S. Notice that

$$\text{virt.dim}\, H = \operatorname{card}(S - \mathbb{N}) - \operatorname{card}(\mathbb{N} - S).$$

We shall call this number the *virtual cardinal* of S.

Proposition (7.1.6). *For any $W \in \mathrm{Gr}(H)$ there is a set $S \in \mathscr{S}$ such that the orthogonal projection $W \to H_S$ is an isomorphism. In other words the sets $\{U_S\}_{S\in\mathscr{S}}$, where $U_S = U_{H_S}$, form an open covering of $\mathrm{Gr}(H)$.*

Proof. Because the projection $W \to H_+$ has finite dimensional kernel one