3

GROUPS OF SMOOTH MAPS

3.1 Infinite dimensional manifolds

Before discussing infinite dimensional Lie groups we must make clear what we mean by an infinite dimensional smooth manifold, if only to emphasize that there is nothing esoteric involved in the idea. For an excellent short treatment of the subject we refer the reader to Milnor [115]. We shall follow his approach closely. A more detailed account can be found in Hamilton [70].

The manifolds we consider will be paracompact topological spaces X 'modelled on' some topological vector space E, in the sense that X is covered by an atlas of open sets $\{U_\alpha\}$ each of which is homeomorphic to an open set E_α of E by a given homeomorphism $\phi_\alpha : U_\alpha \to E_\alpha$. The vector space E will always be locally convex and complete. The transition functions between charts

$$\phi_\alpha(U_\alpha \cap U_\beta) \xrightarrow{\phi_\alpha^{-1}} U_\alpha \cap U_\beta \xrightarrow{\phi_\beta} \phi_\beta(U_\alpha \cap U_\beta)$$

are assumed to be smooth, i.e. infinitely differentiable. The meaning of 'infinitely differentiable' is as follows.

A map $f: U \to E$, where U is an open set of E, is continuously differentiable (or C^1) if the limit

$$Df(u; v) = \lim_{t\to 0} t^{-1}(f(u+tv) - f(u))$$

exists for all $u \in U$ and $v \in E$, and is continuous as a map $Df: U \times E \to E$. (Of course Df is linear in its second variable.) The second derivative, if it exists, is then the map

$$D^2f: U \times E \times E \to E$$

defined by

$$D^2f(u; v, w) = \lim_{t\to 0} t^{-1}(Df(u+tw; v) - Df(u; v)),$$

and so on.

We shall collect here some remarks about calculus on infinite dimensional manifolds.

Complex manifolds

If E is a complex topological vector space and the transition functions are holomorphic, then we have a *complex* manifold. To say that $f: U \to E$ is holomorphic, where U is an open set of E, means that f is smooth and that $Df: U \times E \to E$ is complex-linear in the second variable.

Differential forms

If U is an open set of E then a differential form of degree p on U is a smooth map

$$U \times \underbrace{E \times \ldots \times E}_{p} \to E$$

which is multilinear and alternating in the last p variables. Differential forms can then be defined in the usual way on a smooth manifold, and the usual definition of the exterior derivative, and the proof of the Poincaré lemma, apply without modification.

In order to make serious use of differential forms, however, one needs to know that for each open covering of the manifold there is a subordinate smooth partition of unity. That is true providing the following two conditions are satisfied.

(I) The vector space E has enough smooth functions, in the sense that for each open set U of E there is a non-vanishing smooth function $E \to \mathbb{R}$ which vanishes outside U.

(II) The manifold is Lindelöf, i.e. each open covering has a countable refinement.

Both of these conditions are satisfied for all the manifolds we shall consider.

De Rham's theorem holds for any manifold X which has smooth partitions of unity. The usual proof applies. In particular, if a cohomology class $c \in H^p(X; \mathbb{R})$ is represented by a Čech cocycle $\{c_{\alpha_0 \ldots \alpha_p}\}$ with respect to an open covering $\{U_\alpha\}$ of X, then c is also represented by the differential form

$$\sum_{\alpha_0,\ldots,\alpha_p} c_{\alpha_0\ldots\alpha_p} \lambda_{\alpha_0}\, d\lambda_{\alpha_1} \wedge \ldots \wedge d\lambda_{\alpha_p}, \qquad (3.1.1)$$

where $\{\lambda_\alpha\}$ is a partition of unity subordinate to $\{U_\alpha\}$.

Vector fields

There is no difficulty in defining smooth vector fields, or the bracket of two vector fields; and vector fields act as differentiation operators on functions in the usual way. One must beware, however, that vector fields on infinite dimensional manifolds do *not* in general have trajectories. We shall meet an interesting example of this phenomenon when we discuss the gradient flow of the energy function on a loop space in Chapter 8.

3.2 Groups of maps as infinite dimensional Lie groups

An infinite dimensional Lie group is a group Γ which is at the same time an infinite dimensional smooth manifold, and is such that the composition law $\Gamma\times\Gamma\to\Gamma$ and the operation of inversion $\Gamma\to\Gamma$ are given by smooth maps. The tangent space to Γ at the identity element is its Lie algebra, the bracket being defined by identifying tangent vectors at the identity element with left-invariant vector fields on Γ. If for each element ξ of the Lie algebra there is a unique one-parameter subgroup $\gamma_\xi:\mathbb{R}\to\Gamma$ such that $\gamma'_\xi(0)=\xi$, then the exponential map is defined. This is the case in all known examples.

For infinite dimensional Lie groups modelled on Banach spaces there is a well-developed theory ([20] Chapter 3) which is closely parallel to the theory of finite dimensional Lie groups. For groups modelled on more general topological vector spaces there is no such theory, and most of the standard theorems about Lie groups do not hold. We shall meet interesting examples of Lie algebras which do not correspond to any Lie group and of Lie groups whose exponential maps are not locally bijective. We hope that it will emerge all the same that the concept of a general infinite dimensional Lie group is a useful one.

Probably the simplest and most immediate example of an infinite dimensional Lie group is the group $\mathrm{Map}_{\mathrm{cts}}(X;G)$ of all continuous maps from a compact space X to a finite dimensional Lie group G. (The group law, of course, is pointwise composition in G.) The natural topology on $\mathrm{Map}_{\mathrm{cts}}(X;G)$ is the topology of uniform convergence. We see that it is a smooth manifold as follows.

If U is an open neighbourhood of the identity element in G which is homeomorphic by the exponential map to an open set $\tilde{U}$ of the Lie algebra $\mathfrak{g}$ of G, then $\mathcal{U}=\mathrm{Map}_{\mathrm{cts}}(X;U)$ is an open neighbourhood of the identity in $\mathrm{Map}_{\mathrm{cts}}(X;G)$ which is homeomorphic to the open set $\tilde{\mathcal{U}}=\mathrm{Map}_{\mathrm{cts}}(X;\tilde{U})$ of the Banach space $\mathrm{Map}_{\mathrm{cts}}(X;\mathfrak{g})$. If f is any element of $\mathrm{Map}_{\mathrm{cts}}(X;G)$, then $\mathcal{U}_f=\mathcal{U}.f$ is a neighbourhood of f which is also homeomorphic to $\tilde{\mathcal{U}}$. The sets $\mathcal{U}_f$ provide an atlas which makes $\mathrm{Map}_{\mathrm{cts}}(X;G)$ into a smooth manifold, and in fact into a Lie group: there is no difficulty at all in checking that the transition functions are smooth, or that multiplication and inversion are smooth maps.

In this book, however, we shall be concerned not with groups of continuous maps but with groups of smooth maps.

Suppose now that X is a finite dimensional compact smooth manifold, and let $\mathrm{Map}(X;G)$ denote the group of *smooth* maps $X\to G$. The case we are primarily interested in is when X is the circle S^1; then $\mathrm{Map}(X;G)$ is the *loop group* of G, which is denoted by LG. We shall think of the circle as consisting interchangeably of real numbers θ modulo 2π or of complex numbers $z=e^{i\theta}$ of modulus one.

Defining the atlas $\{\mathcal{U}_f\}$ for $\mathrm{Map}(X;G)$ just as in the continuous case, we find that the set $\tilde{\mathcal{U}}$ is an open set in the vector space $E=\mathrm{Map}(X;\mathfrak{g})$ of all smooth maps $X\to\mathfrak{g}$. The simplest way to define the topology of $\mathrm{Map}(X;G)$ is to prescribe that the sets $\mathcal{U}_f$ are open and homeomorphic to the open set $\tilde{\mathcal{U}}$ of E. The standard topology on E is the topology of uniform convergence of the functions and all their partial derivatives of all orders [70]. It makes E into a complete separable metrizable topological vector space, but not a Banach space. We shall not describe it in detail here. But when X is the circle the convergence of a sequence $\{f_k\}$ in E to f means that $(d^nf_k/d\theta^n)$ converges uniformly to $d^nf/d\theta^n$ for each n. Again there is no difficulty in seeing that $\mathrm{Map}(X;G)$ is an infinite dimensional Lie group.

For most of the purposes of this book it would make no difference if we considered, instead of smooth maps, maps of a given finite degree r of differentiability. $\mathrm{Map}(X;G)$ would then be a Banach Lie group. (We should have to interpret C^r maps in the Sobolev sense [144], otherwise the manifold would not have enough smooth functions.) No practical advantage would be gained by the change, however, and so we shall keep to smooth maps, which seem aesthetically more appealing. Thus $\mathrm{Map}(X;G)$ will always denote smooth maps, and LG will denote the smooth loop group. In the case of diffeomorphism groups, as we shall see, there is no choice but to work with smooth maps.

The Lie algebra of $\mathrm{Map}(X;G)$ is obviously $\mathrm{Map}(X;\mathfrak{g})$, and the exponential map

$$\exp:\mathrm{Map}(X;\mathfrak{g})\to\mathrm{Map}(X;G)$$

is defined, and is a local homeomorphism near the identity. One of our themes is that the loop group of a compact group G behaves surprisingly like a compact group itself, but we shall begin by pointing out a slight difference. In a compact group G every element in the identity component G^0 lies on a one-parameter subgroup, i.e. the exponential map $\mathfrak{g}\to G^0$ is surjective. This property is not inherited by $\mathrm{Map}(X;G)$.

Example. Consider LG, where $G=SU_2$. Then G is simply connected, so LG is connected. The element γ of LG defined by

$$z\mapsto\begin{pmatrix}z & 0\\ 0 & z^{-1}\end{pmatrix}$$

does not lie on any one-parameter subgroup. For if γ is $\exp(\xi)$ for some $\xi\in L\mathfrak{g}$ then ξ must commute with γ and hence must be diagonal: but there is no smooth function θ on the circle such that $e^{i\theta}=z$. Notice that this example is precisely analogous to the non-surjectivity of exp for finite dimensional non-compact groups: the element

$$\begin{pmatrix}-2 & 0\\ 0 & -\frac{1}{2}\end{pmatrix}$$

of $SL_2(\mathbb{R})$ does not lie on a one-parameter subgroup. It is easy to see, however, that when G is compact the image of the exponential map is dense in the identity component of LG. That is not true in groups like $SL_2(\mathbb{R})$.

Another obvious but important remark about groups of maps is that when G has a complexification $G_{\mathbb{C}}$ then $\text{Map}(X; G)$ has the complexification $\text{Map}(X; G_{\mathbb{C}})$. It is clear that the last group is a complex Lie group.

3.3 Diffeomorphism groups

The group of diffeomorphisms of the circle will play only a peripheral role in this book, but it is a very interesting example of an infinite dimensional Lie group.

First let us remark that for any finite dimensional compact smooth manifold X the group $\text{Diff}(X)$ of all smooth diffeomorphisms $X \to X$ is a Lie group (cf. [70], [115]). Its Lie algebra is the vector space $\text{Vect}(X)$ of all smooth vector fields on X, with the usual bracket operation, and the exponential map

$$\exp : \text{Vect}(X) \to \text{Diff}(X)$$

assigns to a vector field the unique flow that it generates. For a finite k, however, the group of k-times continuously differentiable diffeomorphisms obviously does *not* form a Lie group, for left translation is not a differentiable map. (Still more obviously, the bracket of two vector fields of class C^k is only of class C^{k-1}.)

Although the exponential map is defined for $\text{Diff}(X)$, it is far from being a local homeomorphism. There are diffeomorphisms arbitrarily close to the identity which are not on any one-parameter subgroup, and others which are on many. The following discussion is based on Omori [120]. (Cf. also Milnor [115].)

Proposition (3.3.1). *The map* $\exp: \text{Vect}(S^1) \to \text{Diff}(S^1)$ *is neither locally one-to-one nor locally surjective.*

Proof.

(i) Consider the rotation $R_{2\pi/n}$ through the angle $2\pi/n$. This belongs to the subgroup $\mathbb{T}$ of all rigid rotations in $\text{Diff}(S^1)$. The centralizer of $R_{2\pi/n}$ is the subgroup H of all diffeomorphisms which are periodic with period $2\pi/n$. So $R_{2\pi/n}$ lies on all of the one-parameter subgroups $\phi\mathbb{T}\phi^{-1}$ for $\phi \in H$. This shows that exp is not locally one-to-one.

(ii) In seeing that exp is not locally surjective, the essential point is that a one-parameter subgroup of $\text{Diff}(S^1)$ which has no stationary points is conjugate to the subgroup $\mathbb{T}$. Granting this, and observing that a diffeomorphism conjugate to a rotation has no fixed points unless it is the identity, we see that a diffeomorphism cannot be on a one-parameter subgroup if

(a) it has no fixed point,
(b) it has a point of finite order n, and
(c) it is not of order n.

Such diffeomorphisms ϕ are very plentiful, and can be chosen arbitrarily close to the identity. For example one can define $\phi(\theta) = \theta + \pi$ for $0 \leq \theta \leq \pi$, and then extend ϕ to the remainder of the circle in any way at all which does not make $\phi = R_\pi$. Alternatively, one can define

$$\phi(\theta) = \theta + \frac{2\pi}{n} + \varepsilon \sin n\theta,$$

where ε is small. Then $\phi^n(0) = 0$, but ϕ^n is not the identity, as its derivative there is $(1 + n\varepsilon)^n$.

To see that any one-parameter subgroup with no stationary points is conjugate to a group of rigid rotations it is enough to observe that any nowhere-vanishing vector field $v(\theta)\,\mathrm{d}/\mathrm{d}\theta$ can be conjugated to a constant vector field. The conjugating diffeomorphism ψ is given by

$$\psi(\theta) = k \int_0^\theta v(t)^{-1}\,\mathrm{d}t,$$

where k is chosen so that $\psi(2\pi) = 2\pi$.

Before leaving diffeomorphism groups we should point out another way in which they differ from the loop groups. The complexification of the Lie algebra $\text{Vect}(X)$ does not correspond to any Lie group. This is intuitively unsurprising, for complex vector fields on S^1 generate paths in the space of maps $S^1 \to \mathbb{C}$, and these do not form a group. A proof that there is no Lie group corresponding to $\text{Vect}_{\mathbb{C}}(S^1)$ can be given as follows.

Proposition (3.3.2). *Any homomorphism from* $\text{Diff}^+(S^1)$ *to a complex Lie group is trivial.*

Proof. The group $PSL_2(\mathbb{R})$ is contained in $\text{Diff}^+(S^1)$, for S^1 can be regarded as the real projective line. Consider the n-fold covering map $\pi: S^1 \to S^1$ given by $z \mapsto z^n$. Let G_n denote the group of diffeomorphisms ϕ which are n-fold coverings of elements $\psi \in PSL_2(\mathbb{R})$, i.e. such that $\pi \circ \phi = \psi \circ \pi$. It is easy to see that G_n is isomorphic to the n-fold covering group of $PSL_2(\mathbb{R})$: its centre consists of the rotations $R_{2\pi k/n}$ for $k = 0, 1, \ldots, n-1$. But we have pointed out in Section 2.2 that any homomorphism from G_n into a complex Lie group must factorize through $SL_2(\mathbb{R})$ or $PSL_2(\mathbb{R})$. The kernel of any homomorphism from $\text{Diff}^+(S^1)$ into a complex group must therefore contain all rotations through $2\pi k/n$ with n odd, and so must contain all rotations. Being a normal subgroup, the kernel is therefore the whole of $\mathrm{Diff}^+(S^1)$, in view of the following result.

Proposition (3.3.3). *$\mathrm{Diff}^+(S^1)$ is a simple group.*

The proof of this result, due to Herman [74], is surprisingly difficult, and we shall not give it here.

3.4 Some group-theoretic properties of Map(X; G)

In this section G will be a compact connected Lie group, and X will be a compact smooth manifold. For brevity we shall write MG for the group of smooth maps Map(X; G).

If G is semisimple then it is perfect, i.e. equal to its commutator subgroup $[G, G]$. We shall show that then the identity component M_0G is also perfect. (We cannot expect MG itself to be perfect: for example in the case of LG the group of connected components is the fundamental group $\pi_1(G)$, which is abelian.)

Proposition (3.4.1). *If G is semisimple then M_0G is perfect, and in fact $[G, M_0G] = M_0G$.*

Proof. Let us first consider the case $G = SU_2$. If

$$e_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad e_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

is the usual basis for the Lie algebra of G and $\mathbb{T}_1$, $\mathbb{T}_2$, $\mathbb{T}_3$ are the circle subgroups they generate, then the multiplication $\mathbb{T}_1 \times \mathbb{T}_2 \times \mathbb{T}_3 \to G$ is surjective. The multiplication

$$M\mathbb{T}_1 \times M\mathbb{T}_2 \times M\mathbb{T}_3 \to MG$$

is therefore surjective in a neighbourhood of the identity; and so it is enough (because the subgroups $\mathbb{T}_1, \mathbb{T}_2, \mathbb{T}_3$ are conjugate) to prove that every element of the identity component of $M\mathbb{T}_3$ belongs to $[G, M_0G]$. This last statement is true because

$$\begin{pmatrix} \phi & 0 \\ 0 & \phi^{-1} \end{pmatrix} = \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} \phi^{-\frac{1}{2}} & 0 \\ 0 & \phi^{+\frac{1}{2}} \end{pmatrix}\right].$$

(The bracket here denotes the group-theoretic commutator $[x, y] = xyx^{-1}y^{-1}$, not the Lie bracket; and $\phi^{\frac{1}{2}}$ is defined because ϕ has winding number zero.)

The result for a general semisimple group G follows at once from the particular case of SU_2. For, as has been explained in Section 2.4, one can find a finite number of homomorphisms $i_1, \ldots, i_n : SU_2 \to G$,

corresponding to the positive roots of G, such that the multiplication map

$$\Pi i_k : (SU_2)^n \to G,$$

and hence the induced map $(MSU_2)^n \to MG$, is locally surjective.

We shall now discuss the group of automorphisms of MG.

The group of diffeomorphisms of X acts on MG as a group of automorphisms. Apart from that, there are obvious pointwise automorphisms of MG arising from smooth maps $X \to A$, where A is the group of automorphisms of G. If G is simple there are essentially no others.

Proposition (3.4.2). *If G is simple then the group of automorphisms of M_0G is the semidirect product* $\mathrm{Diff}(X) \ltimes MA$.

Proof. Suppose that $\alpha : M_0G \to M_0G$ is an automorphism. Composing it with $\varepsilon_x : MG \to G$, the evaluation map at $x \in X$, gives a homomorphism $\alpha_x : M_0G \to G$. The restriction of α_x to the subgroup G of constant maps in MG must be an automorphism a_x of G, for if it were trivial then α_x would be trivial because $M_0G = [G, M_0G]$. Clearly $x \mapsto a_x$ is an element a of MA, and by replacing α with $a^{-1} \circ \alpha$ we may as well assume that a_x is the identity for each x. Then the crucial step is to see that $\alpha_x = \varepsilon_x \circ \alpha = \varepsilon_y$ for some $y \in X$. For this it is enough to consider the derivative of α_x, which is a homomorphism of Lie algebras $\dot{\alpha}_x : M\mathfrak{g} \to \mathfrak{g}$.

If U is an open set of X, let $M_U\mathfrak{g}$ denote the ideal of $M\mathfrak{g}$ consisting of elements with support in U. Because $\mathfrak{g}$ is simple and $\dot{\alpha}_x$ is surjective, $\dot{\alpha}_x \mid M_U$ must be either trivial or surjective. It follows that when $\dot{\alpha}_x$ is regarded as a distribution on X its support consists of a single point y: for if y and y' were distinct points of the support, and U and U' are disjoint neighbourhoods of y and y', then the commuting ideals $M_U\mathfrak{g}$ and $M_{U'}\mathfrak{g}$ would each map surjectively on to the non-abelian algebra $\mathfrak{g}$, which is impossible. Thus the kernel of $\dot{\alpha}_x$ contains the ideal $J_{y,k}$ of all elements of $M\mathfrak{g}$ which vanish to some order k at y. But

$$[\ldots[[J_{y,1}, J_{y,1}], J_{y,1}, \ldots, J_{y,1}] = J_{y,k}$$

(where $J_{y,1}$ occurs k times on the left), so the kernel must contain $J_{y,1}$. As $M\mathfrak{g}/J_{y,1} \cong \mathfrak{g}$, this proves that $\dot{\alpha}_x$ is evalution at y.

If $\phi : X \to X$ is defined by $\alpha_x = \varepsilon_x \circ \alpha = \varepsilon_{\phi(x)}$, then $\alpha : M_0G \to M_0G$ is given by $\alpha(f)(x) = f(\phi(x)) = (\phi^* f)(x)$. The map ϕ must be smooth because ϕ^* takes smooth functions to smooth functions, and it must be a diffeomorphism because α is an automorphism.

Remarks.

(i) The preceding result obviously does not hold if G is not simple, but the method enables one to describe all automorphisms of MG when G is semisimple. If G has a torus factor then MG contains a large vector space as a factor, and the automorphism group contains its general linear group.

(ii) The proof of (3.4.2), as has been pointed out to us by P. de la Harpe, actually proves the following result.

Proposition (3.4.3). *If G is simple then the maximal normal subgroups of M_0G are precisely the kernels of the evaluation maps $M_0G \to G$ at the points of X.*

To conclude this section let us return briefly to loop groups. The identity component A^0 of the group A of automorphisms of G consists of inner automorphisms, and $\pi_0(A) = A/A^0$ is the finite group of classes of outer automorphisms. Now LA acts as a group of automorphisms of LG, and again its identity component $(LA)^0$ consists of inner automorphisms, for any null-homotopic loop in A can be lifted to G. In fact a loop can be lifted precisely when its homotopy class belongs to the image of $\pi_1(G) \to \pi_1(A)$. The cokernel of this homomorphism is the centre Z of G, so we have

Proposition (3.4.4). *The semidirect product $\pi_0(A) \tilde{\times} Z$ is a subgroup of the group of outer automorphism classes of LG.*

The action of the centre Z is the important point. For any $g \in Z$ one can choose a smooth map $\eta : \mathbb{R} \to G$ such that $\eta(\theta + 2\pi) = g \cdot \eta(\theta)$, and then conjugation by η is the associated outer automorphism of LG.

3.5 Subgroups of LG: polynomial loops

From time to time we shall want to mention a number of subgroups of LG. The most obvious of these is the group $L_{\text{an}}G$ of real-analytic loops. If G is embedded in a unitary group U_n, so that a loop γ in G is a matrix-valued function and can be expanded in a Fourier series

$$\gamma(z) = \sum_{k=-\infty}^{\infty} \gamma_k z^k, \tag{3.5.1}$$

then the real-analytic loops are those such that the series converges in some annulus $r \leq |z| \leq r^{-1}$ with $r < 1$, i.e. such that $\|\gamma_k r^{-|k|}\|$ is bounded for all k for some $r < 1$. The natural topology on $L_{\text{an}}G$ is got by regarding it as the direct limit of the Banach Lie groups $L_{\text{an},r}G$ consisting of functions holomorphic in $r \leq |z| \leq r^{-1}$; the group $L_{\text{an},r}G$ has the topology of uniform convergence. There is no difficulty in seeing that $L_{\text{an}}G$ is a Lie group with Lie algebra $L_{\text{an}}\mathfrak{g}$. (The choice of the embedding $G \subset U_n$ was immaterial, and indeed was not really used: it was introduced above only for concreteness.)

A slightly smaller subgroup is the group $L_{\text{rat}}G$ of rational loops, i.e. loops which, when regarded as matrix-valued functions, have entries which are rational functions of z with no poles on $|z| = 1$. (A rational function means the quotient of two polynomials.) We shall not pursue the question of the appropriate topology to be put on $L_{\text{rat}}G$: let us notice only that it is a dense subgroup of LG.

The smallest subgroup we shall consider is $L_{\text{pol}}G$: the group of loops whose matrix entries are finite Laurent polynomials in z and z^{-1}, i.e. loops of the form (3.5.1) where only finitely many of the matrices γ_k are non-zero. This group is the union of the subsets $L_{\text{pol},N}G$ consisting of the loops (3.5.1) for which $\gamma_k = 0$ when $|k| > N$. Each of these subsets is naturally a compact space, and we give $L_{\text{pol}}G$ the direct limit topology. It is associated with the Lie algebra $L_{\text{pol}}\mathfrak{g}$ of all finite series

$$\sum_{k=-N}^{N} \xi_k z^k \tag{3.5.2}$$

where ξ_k belongs to the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$ and $\xi_{-k} = \bar{\xi}_k$. This vector space is the direct limit of its finite dimensional subspaces $L_{\text{pol},N}\mathfrak{g}$, and has the direct limit topology. There is of course no exponential map $L_{\text{pol}}\mathfrak{g} \to L_{\text{pol}}G$, for the exponential of a finite series (3.5.2) is usually not a finite series.

The group $L_{\text{pol}}G$ has the complexification $L_{\text{pol}}G_{\mathbb{C}}$ which consists of the loops in $G_{\mathbb{C}}$ which, *together with their inverses*, are given by finite Laurent polynomials (3.5.1). (In the case of $L_{\text{pol}}G$ we did not need to say 'together with their inverses' because for $\gamma \in LG$ we have $\gamma^{-1} = \gamma^*$, and so the inverse of a polynomial loop is automatically polynomial.) If $G = U_n$ then $L_{\text{pol}}G_{\mathbb{C}}$ is just $GL_n(\mathbb{C}[z, z^{-1}])$. In general, if G is thought of as an algebraic group, then $L_{\text{pol}}G_{\mathbb{C}}$ is the 'points of G with values in $\mathbb{C}[z, z^{-1}]$' in the sense of algebraic geometry.

It is *not* always true that $L_{\text{pol}}G$ is dense in LG. For example, if $G = \mathbb{T}$ then the only elements of $L_{\text{pol}}G$ are the loops uz^k, with $u \in \mathbb{T}$; i.e. the identity component of $L_{\text{pol}}G$ is simply the constant loops. (For the inverse of a non-constant polynomial cannot be a polynomial.) The following result is therefore a little surprising.

Proposition (3.5.3). *If G is semisimple, then $L_{\text{pol}}G$ is dense in LG.*

Proof. Let H be the closure of $L_{\text{pol}}G$ in LG, and let V be the subset of $L\mathfrak{g}$ formed by the tangent vectors ξ such that the corresponding one-parameter subgroup γ_ξ belongs to H. The essential observation is that V is a vector space. To see that it is closed under addition one uses the formula

$$\gamma_{\xi+\eta}(t) = \lim_{t \to \infty} (\gamma_\xi(t/n)\gamma_\eta(t/n))^n,$$

which holds in LG because, for a suitable neighbourhood U of the identity in G, the sequence of maps $f_n : U \times U \to G$ defined by

$$f_n(x, y) = (x^{1/n} y^{1/n})^n$$

converges in the C^∞ topology.

It is clear that V is a closed subspace of $L\mathfrak{g}$. To prove (3.5.3) it is enough to show (because the exponential map is locally surjective in LG) that $V = L\mathfrak{g}$.

Consider first the case $G = SU_2$. Then the elements

$$\xi_n = \begin{pmatrix} 0 & z^n \\ -z^{-n} & 0 \end{pmatrix} \quad \text{and} \quad \eta_n = \begin{pmatrix} 0 & iz^n \\ iz^{-n} & 0 \end{pmatrix}$$

belong to V, as the corresponding one-parameter subgroups lie in $L_{\text{pol}}G$. (For $\xi_n^2 = \eta_n^2 = -1$.) By linearity then, and because it is closed, V contains every element of the form

$$f\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + g\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

where f and g are smooth real-valued functions on the circle. But V is invariant under conjugation by constant elements of SU_2, so we must have $V = L\mathfrak{g}$.

The general case follows in the usual way, because for any semisimple G there are a finite number of homomorphisms $SU_2 \to G$ for which the images of $\mathfrak{su}_2$ in $\mathfrak{g}$ span $\mathfrak{g}$. (The argument proves that the closure of $L_{\text{pol}}G$ contains the identity component of LG; the proof is completed by observing that $L_{\text{pol}}G$ contains at least one element from each connected component of LG.)

3.6 Maximal abelian subgroups of LG

We shall show that there is a conjugacy class of maximal abelian subgroups of LG associated naturally to each conjugacy class in the Weyl group of G.

If A is any abelian subgroup of LG then for any point θ of the circle the subgroup $A(\theta)$ of G got by evaluating the loops in A at θ is abelian, and so is contained in a maximal torus of G. Thus the most obvious maximal abelian subgroup of LG is LT, where T is a maximal torus of G. More generally, if λ is a map which assigns a maximal torus $T_{\lambda(\theta)}$ of G smoothly to each point θ of the circle, then the subgroup

$$A_\lambda = \{\gamma \in LG : \gamma(\theta) \in T_{\lambda(\theta)} \text{ for all } \theta\}$$

is a maximal abelian subgroup. As all maximal tori are conjugate, the space of maximal tori can be identified with G/N, where N is the normalizer of a fixed torus T. Thus λ is a smooth map $\lambda : S^1 \to G/N$.

The conjugacy class of A_λ depends only on the homotopy class of λ. This follows easily from the homotopy lifting property of the fibration $N \to G \to G/N$: for example if λ is contractible it can be lifted to $\tilde{\lambda} : S^1 \to G$, and then $T_{\lambda(\theta)} = \tilde{\lambda}(\theta) . T . \tilde{\lambda}(\theta)^{-1}$ and $A_\lambda = \tilde{\lambda} . LT . \tilde{\lambda}^{-1}$. The fundamental group of G/N is the Weyl group $W = N/T$, for W acts freely on the simply connected space G/T, and $G/N = (G/T)/W$. The set of homotopy classes of maps $S^1 \to G/N$, with no basepoints, is therefore the set of conjugacy classes of W (see Spanier [143] p. 379), and we shall think of λ as representing such a class. An element $w \in W$ defines an automorphism α_w of T by conjugation, and the corresponding A_λ can be described as follows.

Proposition (3.6.1). *If λ corresponds to $w \in W$ then A_λ is isomorphic to the group of smooth maps $\gamma : \mathbb{R} \to T$ such that*

$$\gamma(\theta + 2\pi) = \alpha_w^{-1}(\gamma(\theta)) \tag{3.6.2}$$

for all $\theta \in \mathbb{R}$.

Proof. Suppose w is represented by $n \in N$, and let ω be an element of the Lie algebra of G such that $\exp(2\pi\omega) = n$. Then we can take

$$A_{\lambda(\theta)} = \exp(\theta\omega) . T . \exp(-\theta\omega).$$

If $\tilde{\gamma}$ belongs to A_λ then $\gamma : \mathbb{R} \to T$, defined by

$$\gamma(\theta) = \exp(-\theta\omega)\tilde{\gamma}(\theta)\exp(\theta\omega),$$

satisfies (3.6.2) and conversely.

By using the description (3.6.1) of A_λ, and considering the exact sequence of groups

$$\Omega T \to A_\lambda \to T,$$

where $A_\lambda \to T$ is evaluation at $\theta = 0$, it is easy to prove

Proposition (3.6.3). *The group of connected components $\pi_0(A_\lambda)$ of A_λ, and its fundamental group $\pi_1(A_\lambda)$, are the cokernel and kernel of the homomorphism*

$$w_* - 1 : \check{T} \to \check{T},$$

where $\check{T}$ is the lattice $\pi_1(T)$, and w_* is the action of w on $\check{T}$.

The maximal abelian subgroups A_λ we have described are not the only ones, for example if T_0 and T_1 are two different maximal tori in G then the subgroup consisting of loops γ such that $\gamma(\theta) \in T_0$ for $0 \leqslant \theta \leqslant \pi$, and $\gamma(\theta) \in T_1$ for $\pi \leqslant \theta \leqslant 2\pi$, is clearly maximal. It seems likely, however, that the A_λ exhaust all the maximal abelian subgroups in the group of real-analytic loops.

We shall make considerable use in Part II of this book of the subgroup A_λ of LU_n corresponding to the *Coxeter element* w of the Weyl group. The Weyl group of U_n is the symmetric group S_n, which permutes the entries of the diagonal matrices which form the maximal torus T of U_n. The Coxeter element is the cyclic permutation $(12\ldots n)$.

Proposition (3.6.4). *The maximal abelian subgroup of LU_n corresponding to the Coxeter element is isomorphic to $L\mathbb{T}$.*

Proof. This follows from (3.6.1). For if $\gamma:\mathbb{R}\to T$ satisfies (3.6.2) when w is the Coxeter element then each diagonal element of γ is a function $\gamma_i:\mathbb{R}\to\mathbb{T}$ which is periodic with period $2\pi n$, and the γ_i differ from each other only by translation by multiples of 2π.

We shall describe this embedding of $L\mathbb{T}$ in LU_n in a somewhat different way in Section 6.5. Its importance was first recognized by Lepowsky and Wilson [102]. (Cf. also [87].) Its Lie algebra—or, strictly, a central extension of it—is sometimes referred to as a 'principal Heisenberg subalgebra' of Lu_n.

Remark. The abelian subgroup corresponding to a general element w of S_n is easily seen to be a product of copies of $L\mathbb{T}$, one for each cycle in the permutation w.

3.7 Twisted loop groups

The abelian subgroups A_λ of LG which we have just described are examples of what are called *twisted loop groups*. If α is any automorphism of a group G then one can define

$$L_{(\alpha)}G=\{\gamma:\mathbb{R}\to G \text{ such that } \gamma(\theta+2\pi)=\alpha(\gamma(\theta))\}. \qquad (3.7.1)$$

The group $L_{(\alpha)}G$ depends (up to isomorphism) only on the class of α modulo inner automorphisms. For if

$$\beta(g)=c\alpha(g)c^{-1}$$

for some $c\in G$, then we can choose a smooth map $\lambda:\mathbb{R}\to G$ such that $\lambda(\theta+2\pi)=c\,.\,\alpha(\lambda(\theta))$, and then the map $\gamma\mapsto\tilde{\gamma}$, where

$$\tilde{\gamma}(\theta)=\lambda(\theta)\gamma(\theta)\lambda(\theta)^{-1},$$

defines an isomorphism $L_{(\alpha)}G\to L_{(\beta)}G$. This means that if G is semisimple one may as well think of α as belonging to the finite group of outer automorphism classes of G; in particular one can assume that α has finite order.

An alternative description of $L_{(\alpha)}G$ is as the group of cross-sections of a fibre bundle on S^1 with fibre G. The bundle is the quotient space of

$G\times\mathbb{R}$ by the equivalence relation which identifies (g,θ) with $(\alpha(g),\theta+2\pi)$.

The theory of twisted loop groups is exactly analogous to that of loop groups, but we shall not pursue it in this book (cf. Section 5.3). The position would be different if we had anything significant to say about groups of the form Map(X; G) for spaces X other than the circle: in that case the analogue of the twisted groups would include the groups of automorphisms of principal fibre bundles on X with structure group G—so called *gauge groups*.

4

CENTRAL EXTENSIONS

For the remainder of this book G will always denote a compact connected Lie group.

4.1 Introduction

A fundamental property of the loop group LG is the existence of interesting central extensions

$$\mathbb{T} \to \tilde{L}G \to LG$$

of LG by the circle $\mathbb{T}$. (In other words, $\tilde{L}G$ is a group containing $\mathbb{T}$ in its centre and such that the quotient group $\tilde{L}G/\mathbb{T}$ is LG.) The $\tilde{L}G$ are analogous to the finite-sheeted covering groups of a finite dimensional Lie group, in that any projective unitary representation of LG comes from a genuine representation of some $\tilde{L}G$: we recall that a projective unitary representation of a group L on a Hilbert space H is the assignment to each $\lambda \in L$ of a unitary operator $U_\lambda : H \to H$ so that

$$U_\lambda U_{\lambda'} = c(\lambda, \lambda') U_{\lambda\lambda'}$$

holds for all $\lambda, \lambda' \in L$, where $c(\lambda, \lambda')$ is a complex number of modulus 1. The function $c: L \times L \to \mathbb{T}$ is called the 'projective multiplier' or 'cocycle' of the representation.

As topological spaces the $\tilde{L}G$ are fibre bundles over LG with the circle as fibre. Except for the product extension $LG \times \mathbb{T}$ they are non-trivial fibre bundles: that is to say $\tilde{L}G$ is not homeomorphic to the cartesian product $LG \times \mathbb{T}$, and there is no continuous cross-section $LG \to \tilde{L}G$. In fact the group extension $\tilde{L}G$ is completely determined by its topological type as a fibre bundle, and every circle bundle on LG can be made into a group extension. It is interesting that the behaviour of $\mathrm{Map}(X; G)$ when $\dim(X) > 1$ is completely different. There are often non-trivial circle bundles on $\mathrm{Map}(X; G)$, but if X is simply connected only the flat ones can be made into groups. (That follows from Propositions (4.2.8) and (4.5.6) below.)

When G is a simple and simply connected group there is a *universal* central extension among the $\tilde{L}G$, i.e. one of which all the others are quotient groups. This is analogous to the universal covering group of a finite dimensional group. Any central extension E of LG by any abelian group A arises from the universal extension $\tilde{L}G$ by a homomorphism

$\theta: \mathbb{T} \to A$, in the sense that $E = \tilde{L}G \times_{\mathbb{T}} A$. (The last notation denotes the quotient group of $\tilde{L}G \times A$ by the subgroup consisting of all elements $\{(z, -\theta(z)) : z \in \mathbb{T}\}$.) When G is simply connected but not simple there is still a universal central extension, but, as we shall see, it is an extension of LG by the homology group $H_3(G; \mathbb{T})$, a torus whose dimension is the number of simple factors in G.

The group LG has a complexification $LG_{\mathbb{C}}$. The extensions $\tilde{L}G$ also have complexifications $\tilde{L}G_{\mathbb{C}}$, which are extensions of $LG_{\mathbb{C}}$ by $\mathbb{C}^\times$. We shall postpone the construction of the complexifications, however, until Chapter 6.

It is worth noticing that the central extensions of LG are closely related to its natural affine action on the space of *connections* in the trivial principal G-bundle on the circle. (See (4.3.3).)

This chapter ends with an appendix discussing the cohomology of the space LG and of the Lie algebra $L\mathfrak{g}$.

4.2 The Lie algebra extensions

On the level of Lie algebras the extensions can be defined and classified very simply: they correspond precisely to invariant symmetric bilinear forms on $\mathfrak{g}$. As a vector space $\tilde{L}\mathfrak{g}$ is $L\mathfrak{g} \oplus \mathbb{R}$, and the bracket is given by

$$[(\xi, \lambda), (\eta, \mu)] = ([\xi, \eta], \omega(\xi, \eta)) \tag{4.2.1}$$

for $\xi, \eta \in L\mathfrak{g}$ and $\lambda, \mu \in \mathbb{R}$, where $\omega: L\mathfrak{g} \times L\mathfrak{g} \to \mathbb{R}$ is the bilinear map

$$\omega(\xi, \eta) = \frac{1}{2\pi} \int_0^{2\pi} \langle \xi(\theta), \eta'(\theta) \rangle \, d\theta \tag{4.2.2}$$

and $\langle \ , \ \rangle$ is a symmetric invariant form on the Lie algebra $\mathfrak{g}$. Let us recall that if $\mathfrak{g}$ is semisimple then *every* invariant bilinear form on $\mathfrak{g}$ is symmetric. (See (2.3.2).)

For the formula (4.2.1) to define a Lie algebra, ω must be skew—which is clear by integrating by parts in (4.2.2)—and must satisfy the 'cocycle condition'

$$\omega([\xi, \eta], \zeta) + \omega([\eta, \zeta], \xi) + \omega([\zeta, \xi], \eta) = 0. \tag{4.2.3}$$

This condition follows from the Jacobi identity in the Lie algebra $L\mathfrak{g}$ and the fact that the inner product on $\mathfrak{g}$ is invariant:

$$\langle [\xi, \eta], \zeta \rangle = \langle \xi, [\eta, \zeta] \rangle.$$

One of the first things to notice about the cocycle ω is that it is invariant under the action of the group $\mathrm{Diff}^+(S^1)$ of orientation-preserving diffeomorphisms of the circle, i.e.

$$\omega(f^*\xi, f^*\eta) = \omega(\xi, \eta),$$

for $f \in \mathrm{Diff}^+(S^1)$. (Here $f^*\xi(\theta)$ denotes $\xi(f(\theta))$.) This means that $\mathrm{Diff}^+(S^1)$ acts as a group of automorphisms of the extended Lie algebra. We shall see later that it also acts on the group extension. It is important that the extension singles out a particular orientation of S^1: orientation-reversing diffeomorphisms can act on $\tilde{L}\mathfrak{g}$ only by reversing the kernel $\mathbb{R}$.

There are essentially no other cocycles on $L\mathfrak{g}$ than the ω given by (4.2.2). To make this precise, notice that ω is invariant under conjugation by constant loops, i.e. $\omega(\xi, \eta) = \omega(g\xi, g\eta)$ for $g \in G$, where $g\xi$, $g\eta$ are the adjoint action of g on ξ, η. There is no point in considering cocycles which are not invariant in this way. Indeed, for any cocycle α, the cocycle $g \,.\, \alpha$ defines the same extension as α, where $g \,.\, \alpha$ is defined by $g \,.\, \alpha(\xi, \eta) = \alpha(g^{-1}\xi, g^{-1}\eta)$. So the extension defined by α is also given by the invariant cocycle

$$\int_G g.\alpha \; dg$$

obtained by averaging α over the compact group G. (Notice that the cocycle identity (4.2.3) expresses precisely that the cohomology class of the cocycle does not change under an infinitesimal conjugation.)

Then we have

Proposition (4.2.4). *If $\mathfrak{g}$ is semisimple then the only continuous G-invariant cocycles on the Lie algebra $L\mathfrak{g}$ are those given by* (4.2.2).

Remark. One cannot omit 'semisimple' here. For example, if $G = \mathbb{T}$ then *any* skew bilinear form on the vector space $L\mathfrak{g} = L\mathbb{R}$ is a cocycle. But if we require the cocycles to be invariant under $\mathrm{Diff}^+(S^1)$ then 'semisimple' is not needed, for $L\mathbb{R}/\mathbb{R}$ is an irreducible representation of $\mathrm{Diff}^+(S^1)$, and so it is easy to see that the only bilinear form on $L\mathbb{R}$ which is invariant under $\mathrm{Diff}^+(S^1)$ is, up to a scalar multiple,

$$(\xi, \eta) \mapsto \int_{S^1} \xi \, d\eta.$$

Proof of (4.2.4). Any cocycle $\alpha: L\mathfrak{g} \times L\mathfrak{g} \to \mathbb{R}$ can be extended to a complex bilinear map $\alpha: L\mathfrak{g}_{\mathbb{C}} \times L\mathfrak{g}_{\mathbb{C}} \to \mathbb{C}$. An element $\xi \in L\mathfrak{g}_{\mathbb{C}}$ can be expanded in a Fourier series $\Sigma\, \xi_k z^k$, with $\xi_k \in \mathfrak{g}_{\mathbb{C}}$. By continuity α is completely determined by its values on elements of the form $\xi_k z^k$. Let us write $\alpha_{p,q}(\xi, \eta) = \alpha(\xi z^p, \eta z^q)$ for $\xi, \eta \in \mathfrak{g}_{\mathbb{C}}$. Then $\alpha_{p,q}$ is a G-invariant bilinear map $\mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \to \mathbb{C}$, which is necessarily symmetric, and $\alpha_{p,q} = -\alpha_{q,p}$.

The cocycle identity (4.2.3) translates into the statement

$$\alpha_{p+q,r} + \alpha_{q+r,p} + \alpha_{r+p,q} = 0 \tag{4.2.5}$$

for all p, q, r. Putting $q = r = 0$ we find $\alpha_{p,0} = 0$ for all p. Putting $r = -p - q$ we find

$$\alpha_{p+q,-p-q} = \alpha_{p,-p} + \alpha_{q,-q},$$

whence

$$\alpha_{p,-p} = p\alpha_{1,-1}.$$

Putting $r = n - p - q$ in (4.2.5) we find

$$\alpha_{n-p-q,p+q} = \alpha_{n-p,p} + \alpha_{n-q,q},$$

whence

$$\alpha_{n-k,k} = k\alpha_{n-1,1}.$$

This imples that $\alpha_{p,q} = 0$ if $p + q \neq 0$, for

$$n\alpha_{n-1,1} = \alpha_{0,n} = 0.$$

Returning to $\xi = \Sigma\, \xi_p z^p$ and $\eta = \Sigma\, \eta_q z^q$, we have

$$\begin{aligned}\alpha(\xi, \eta) &= \sum p\alpha_{1,-1}(\xi_p, \eta_{-p}) \\ &= \frac{i}{2\pi} \int_0^{2\pi} \alpha_{1,-1}(\xi(\theta), \eta'(\theta))\, d\theta,\end{aligned}$$

which is of the form (4.2.2).

Proposition (4.2.4) determines the universal central extension of $L\mathfrak{g}$. We can reformulate it in the following way. For any finite dimensional Lie algebra $\mathfrak{g}$ there is a universal invariant symmetric bilinear form

$$\langle\ ,\ \rangle_K : \mathfrak{g} \times \mathfrak{g} \to K \tag{4.2.6}$$

from which every $\mathbb{R}$-valued form arises by a unique linear map $K \to \mathbb{R}$. (Of course K is simply the dual of the space of all $\mathbb{R}$-valued forms.) The cocycle ω_K given by

$$\omega_K(\xi, \eta) = \frac{1}{2\pi} \int_0^{2\pi} \langle \xi(\theta), \eta'(\theta) \rangle_K \, d\theta \tag{4.2.7}$$

defines an extension of $L\mathfrak{g}$ by K, which by Proposition (4.2.4) is the universal central extension of $L\mathfrak{g}$ when $\mathfrak{g}$ is semisimple. For semisimple groups K can be identified with $H_3(\mathfrak{g}; \mathbb{R})$, because a bilinear form $\langle\ ,\ \rangle$ on $\mathfrak{g}$ gives rise to an invariant skew 3-form

$$(\xi, \eta, \zeta) \mapsto \langle \xi, [\eta, \zeta] \rangle,$$

and all elements of $H^3(\mathfrak{g}; \mathbb{R})$ are so obtained. When $\mathfrak{g}$ is simple then $K = \mathbb{R}$.

Extensions of Map(X; g)

Before leaving the subject of Lie algebra extensions, it is worth pointing out that very little extra work is needed to determine all central extensions of Map(X; g) for any smooth manifold X. We shall indicate briefly a proof of the following result, which is a very simple case of a general theory of Loday and Quillen [104] relating the cohomology of Lie algebras to Connes's cohomology [33]. We shall content ourselves with the case of a simple algebra g. There is then an essentially unique inner product $\langle\ ,\ \rangle$.

Proposition (4.2.8). *If* g *is simple then the kernel of the universal central extension of* Map(X; g) *is the space* $K = \Omega^1(X)/d\Omega^0(X)$ *of 1-forms on X modulo exact 1-forms. The extension is defined by the cocycle*

$$(\xi, \eta) \mapsto \langle \xi, d\eta \rangle. \tag{4.2.9}$$

Equivalently, the extensions of Map(X; g) *by* $\mathbb{R}$ *correspond to the one-dimensional* closed currents C *on* X, *the cocycle being given by integrating* (4.2.9) *over* C.

Before proving this let us remark that from one point of view it is a disappointing result, as it tells us that there are no 'interesting' extensions of Map(X; g) when $\dim(X) > 1$. More precisely, if $f: S^1 \to X$ is any smooth loop in X one can always obtain an extension of Map(X; g) by pulling back the universal extension of Lg by f. Proposition (4.2.8) asserts that any extension is a weighted linear combination of extensions of this form. The first 'interesting' cohomology class of Map(X; g), for a compact $(n-1)$-dimensional manifold X, is in dimension n, and is defined by the cocycle

$$(\xi_1, \ldots, \xi_n) \mapsto P(\xi_1, d\xi_2, \ldots, d\xi_n),$$

where P is an invariant polynomial of degree n on g.

Proof of (4.2.8). Let us write Map(X; g) as $A \otimes$ g, where A is the ring of smooth functions on X. Any G-invariant real-valued bilinear form on $A \otimes$ g must be of the form

$$(f \otimes \xi, g \otimes \eta) \mapsto \alpha(f \otimes g)\langle \xi, \eta \rangle,$$

where $\alpha: A \otimes A \to \mathbb{R}$ is linear. Such an α can be identified with a distribution with compact support on $X \times X$. The cocycle condition translates into the statement that α vanishes on functions of the form

$$fg \otimes h + gh \otimes f + hf \otimes g, \tag{4.2.10}$$

where f, g, h are smooth functions on X. This means that $\alpha(f \otimes g) = 0$ when f and g have disjoint support, for then $fg = 0$ and one can find h so that $fh = f$ and $gh = 0$. Thus the distribution α has support along the diagonal. Proposition (4.2.8) is the assertion that $\alpha(f \otimes g)$ depends only on the 1-form fdg. This in turn reduces to two facts:

(i) $\alpha(f \otimes 1) = 0$ for all f; and

(ii) $\alpha \mid I^2 = 0$, where I is the ideal of functions in $A \otimes A$ which vanish on the diagonal.

Both of these facts follow directly from (4.2.10), for I is generated additively by functions of the form $f \otimes g - fg \otimes 1$.

Extensions of Vect(S^1)

Another calculation that fits in very naturally at this point is that for the Lie algebra Vect(S^1) of smooth vector fields on the circle, i.e. the Lie algebra of the group Diff(S^1). A complex-linear 2-cocycle

$$\alpha: \mathrm{Vect}_{\mathbb{C}}(S^1) \times \mathrm{Vect}_{\mathbb{C}}(S^1) \to \mathbb{C},$$

where $\mathrm{Vect}_{\mathbb{C}}(S^1) = \mathrm{Vect}(S^1) \otimes \mathbb{C}$, is determined by the numbers $\alpha_{p,q} = \alpha(L_p, L_q)$, where $L_n = e^{in\theta}(d/d\theta)$. We have

$$[L_n, L_m] = i(m-n)L_{n+m}.$$

The cocycle identity for (L_0, L_p, L_q) shows that the cohomology class of α is not changed by rotation, and so we can (by averaging) assume that α is itself invariant. Then $\alpha_{p,q} = 0$ unless $p + q = 0$. If we write $\alpha_{p,-p} = \alpha_p$, and notice that $\alpha_{-p} = -\alpha_p$, then the cocycle identity gives

$$(p + 2q)\alpha_p - (2p + q)\alpha_q = (p - q)\alpha_{p+q}.$$

This determines all the α_p in terms of α_1 and α_2. The general solution is $\alpha_p = \lambda p^3 + \mu p$. But $\alpha_p = p$ is a coboundary, so the value of μ is unimportant. We have proved

Proposition (4.2.11). *The most general central extension of Vect(S^1) by* $\mathbb{R}$ *is described by the cocycle* α, *where*

$$\alpha\left(e^{in\theta}\frac{d}{d\theta}, e^{im\theta}\frac{d}{d\theta}\right) = i\lambda n(n^2-1) \quad \text{if} \quad n + m = 0,$$
$$= 0 \quad \text{if} \quad n + m \neq 0,$$

for some $\lambda \in \mathbb{R}$.

The representing cocycle given here is characterized by the fact that it is invariant under rotation and vanishes on the subalgebra $\mathfrak{sl}_2(\mathbb{R})$ of Vect(S^1).

4.3 The coadjoint action of LG on $\tilde{L}$g, and its orbits

In this section $\tilde{L}$g denotes the extension of Lg by $\mathbb{R}$ associated to an invariant bilinear form $\langle\ ,\ \rangle$ on g by the formula (4.2.2). We shall use the form $\langle\ ,\ \rangle$, to define a form on $L\mathfrak{g}$, again denoted $\langle\ ,\ \rangle$, by

$$\langle \xi, \eta\rangle = \frac{1}{2\pi}\int_0^{2\pi} \langle \xi(\theta), \eta(\theta)\rangle \, d\theta.$$

Because we are dealing with a central extension, the adjoint action of $\tilde{L}\mathfrak{g}$ on itself is really an action of $L\mathfrak{g}$, given by

$$\eta \,.\, (\xi, \lambda) = ([\eta, \xi], \omega(\eta, \xi)). \qquad (4.3.1)$$

Proposition (4.3.2). *The adjoint action of $L\mathfrak{g}$ on $\tilde{L}\mathfrak{g}$ comes from an action of LG given by*

$$\gamma \,.\, (\xi, \lambda) = (\gamma.\xi,\ \lambda - \langle \gamma^{-1}\gamma', \xi\rangle).$$

Here $\gamma \,.\, \xi$ denotes the adjoint action of $\gamma \in LG$ on $\xi \in L\mathfrak{g}$.

Proof. One has only to check that the desired formula does define a group action, and that its derivative at $\gamma = 1$ is given by (4.3.1). Both verifications are straightforward.

Now let us consider the dual $(\tilde{L}\mathfrak{g})^*$ of $\tilde{L}\mathfrak{g}$. This fits into the exact sequence

$$(L\mathfrak{g})^* \to (\tilde{L}\mathfrak{g})^* \to \mathbb{R},$$

on which the group LG acts. If we identify $(\tilde{L}\mathfrak{g})^*$ with $(L\mathfrak{g})^* \oplus \mathbb{R}$, and use the form $\langle\ ,\ \rangle$ on $L\mathfrak{g}$ to map $L\mathfrak{g}$ into $(L\mathfrak{g})^*$, then we have

Proposition (4.3.3). *The coadjoint action of LG on $(\tilde{L}\mathfrak{g})^*$ is given by*

$$\gamma \,.\, (\phi, \lambda) = (\gamma \,.\, \phi + \lambda\gamma'\gamma^{-1}, \lambda).$$

The reason for being interested in this coadjoint action is the heuristic principle due to Kirillov [92] that the irreducible unitary representations of a group Γ correspond roughly to the orbits of its coadjoint action. A little more precisely, the correspondence is with those orbits which satisfy an integrality condition (*C*) which is described below.

Let us assume that the inner product on $\mathfrak{g}$ is positive-definite. Then $L\mathfrak{g}$ is identified with a dense subspace of $(L\mathfrak{g})^*$ which we shall call the 'smooth part' of the dual. We can describe the orbits of the action of LG on this in the following way.

For each smooth element $(\phi, \lambda) \in (\tilde{L}\mathfrak{g})^*$ with $\lambda \neq 0$ we can find a unique smooth path $f: \mathbb{R} \to G$ by solving the differential equation

$$f'f^{-1} = -\lambda^{-1}\phi \qquad (4.3.4)$$

with the initial condition $f(0) = 1$. Because ϕ is periodic in θ we have

$$f(\theta + 2\pi) = f(\theta) \,.\, M_\phi,$$

where $M_\phi = f(2\pi)$. If (ϕ, λ) is transformed by $\gamma \in LG$ then f is changed

to $\tilde{f}$, where

$$\tilde{f}(\theta) = \gamma(\theta)f(\theta)\gamma(0)^{-1}. \qquad (4.3.5)$$

Thus M_ϕ is changed to $\gamma(0)M_\phi\gamma(0)^{-1}$. In fact (4.3.4) defines a bijection between $L\mathfrak{g} \times \{\lambda\}$ and the space of maps f such that $f(0) = 1$ and $f(\theta + 2\pi) = f(\theta) \,.\, M$ for some $M \in G$. From this we can read off

Proposition (4.3.6).

(i) *If G is simply connected and $\lambda \neq 0$ then the orbits of LG on the smooth part of $(L\mathfrak{g})^* \times \{\lambda\} \subset (\tilde{L}\mathfrak{g})^*$ correspond precisely to the conjugacy classes of G under the map $(\phi, \lambda) \mapsto M_\phi$.*

(ii) *The stabilizer of (ϕ, λ) in LG is isomorphic to the centralizer Z_ϕ of M_ϕ in G by the map $\gamma \mapsto \gamma(0)$; and γ stabilizes (ϕ, λ) if and only if $\gamma(\theta) = f(\theta)\gamma(0)f(\theta)^{-1}$.*

According to Kirillov's idea, the irreducible unitary representations of a group Γ correspond to the coadjoint orbits Ω with the property

(C) if the stabilizer of $\Phi \in \Omega$ is the subgroup H of Γ then Φ is the derivative of a character of the identity component of H.

To apply the principle in our case we need to know that $\tilde{L}\mathfrak{g}$ is the Lie algebra of an extension $\tilde{L}G$ of LG by the circle $\mathbb{T}$. The conditions under which $\tilde{L}G$ exists will be determined in the following sections. Granting its existence for the present, we find at once that if an orbit belonging to $(L\mathfrak{g})^* \times \{\lambda\}$ is allowable then λ must be an *integer*. Then an orbit in the smooth part of the dual corresponds to the conjugacy class of an element $g \in G$, which we can assume to belong to a given maximal torus T. If we choose

$$\xi \in \mathfrak{t} \subset \mathfrak{g} \subset L\mathfrak{g} \subset (L\mathfrak{g})^*$$

so that $\exp(\lambda^{-1}\xi) = g$ then (ξ, λ) belongs to the orbit. If g is sufficiently generic then its centralizer in G is T, and the condition (*C*) clearly amounts to the requirement that $\xi \in \mathfrak{t} \subset \mathfrak{t}^*$ belongs to the lattice $\hat{T}$. In fact it is not hard to check that this is true in any case. On the other hand (ξ, λ) and $(\tilde{\xi}, \lambda)$ belong to the same orbit if $\tilde{\xi} = w \,.\, \xi + \lambda\eta$ for some $\eta \in \check{T}$ and some w in the Weyl group W of G. Thus we have

Proposition (4.3.7). *If λ is a non-zero integer then the coadjoint orbits in the smooth part of $(L\mathfrak{g})^* \times \{\lambda\}$ which satisfy the condition (C) correspond to the orbits of the affine Weyl group*† *$W_{\text{aff}} = W \tilde{\times} \check{T}$ on the lattice $\hat{T}$, where $(w, \eta) \in W_{\text{aff}}$ acts on $\hat{T}$ by*

$$\xi \mapsto w \,.\, \xi + \lambda\eta.$$

We shall see later—in Chapters 9 and 11—that if $\lambda > 0$ these orbits

† For a discussion of the affine Weyl group we refer to Section 5.1.

correspond exactly to the representations of $\tilde{L}G$ of positive energy. Furthermore it is worth observing that an orbit belongs to the smooth part of $(\tilde{L}\mathfrak{g})^*$ if it is stable under the rotation action of $\mathbb{T}$ on $(\tilde{L}\mathfrak{g})^*$: for if (ϕ, λ) belongs to such a stable orbit then one must have

$$\frac{d\phi}{d\theta} = [\eta, \phi] + \lambda\eta'$$

for some $\eta \in L\mathfrak{g}$, which implies that ϕ is smooth. That fits well with Kirillov's viewpoint, because representations of positive energy are stable under rotations.

4.4 The group extensions when G is simply connected

The Lie algebra extensions that we have been describing do not all correspond to Lie groups. For that to be true, a certain integrality condition must be satisfied. The Lie algebra cocycle ω is a skew form on the tangent space to LG at the identity; it therefore defines a left-invariant 2-form ω on LG, and the cocycle condition (4.2.3) translates into the fact that this differential form is closed.

Theorem (4.4.1).

(i) *If G is simply connected then the Lie algebra extension*

$$\mathbb{R} \to \tilde{L}\mathfrak{g} \to L\mathfrak{g}$$

defined by a cocycle ω corresponds to a group extension

$$\mathbb{T} \to \tilde{L}G \to LG$$

if and only if the differential form $\omega/2\pi$ represents an integral *cohomology class on LG, i.e. its integral over every 2-cycle in LG is an integer.*

(ii) *In that case the group extension $\tilde{L}G$ is completely determined by ω, and there is a unique action of* $\text{Diff}^+(S^1)$ *on $\tilde{L}G$ which covers its action on LG.*

(iii) *If $\lambda\omega$ is not integral for any non-zero real number λ, then $\tilde{L}\mathfrak{g}$ does not correspond to any Lie group.*

(iv) *The cocycle ω defined by the formula* (4.2.2) *satisfies the integrality condition if and only if $\langle h_\alpha, h_\alpha \rangle$ is an even integer for each coroot h_α of G. (See Section* 2.4.)

Remark. The part of the extension $\tilde{L}G$ over the subgroup G of constant loops is *canonically* isomorphic to $G \times \mathbb{T}$, as there are no non-trivial homomorphisms $G \to \mathbb{T}$. We shall therefore often think of G as a subgroup of $\tilde{L}G$.

Let us notice at once that (4.4.1) (iii) is an immediate consequence of

(4.4.1) (i). For if there is any Lie group at all which corresponds to $\tilde{L}\mathfrak{g}$ it will be an extension of LG by either $\mathbb{R}$ or $\mathbb{T}$, and an extension by $\mathbb{R}$ gives rise to an extension by $\mathbb{T}$. The multiplier λ corresponds to the different ways of identifying the Lie algebra of $\mathbb{T}$ with $\mathbb{R}$. This result, together with Theorem (4.4.1) (iv), provides us with a class of Lie algebras which do not correspond to any Lie group, for as soon as G has more than one simple factor a generic invariant inner product on $\mathfrak{g}$ is not a multiple of one which satisfies the integrality conditions.

The 'if' part of Theorem (4.4.1) (i) is deduced from the following quite general result, which will be proved in the next section. We shall return to the 'only if' part in Proposition (4.5.6).

Proposition (4.4.2). *Suppose that a Lie group Γ acts smoothly on a connected and simply connected manifold X, leaving invariant an integral closed 2-form $\omega/2\pi$ on X. (Both Γ and X may be infinite dimensional.) Then there is an extension $\tilde{\Gamma}$ of Γ by $\mathbb{T}$ canonically associated to (X, ω), and for any point $x \in X$ the associated extension of Lie algebras can be represented by the cocycle*

$$(\xi, \eta) \mapsto \omega(\xi_x, \eta_x),$$

where ξ_x denotes the tangent vector at $x \in X$ corresponding to the action of the infinitesimal element ξ of Γ.

The group $\tilde{\Gamma}$ in this proposition can be described quite explicitly. The integral closed 2-form ω allows us to associate to each piecewise smooth loop ℓ in X an element $C(\ell)$ of $\mathbb{T}$ by

$$C(\ell) = \exp i \int_\sigma \omega,$$

where σ is a piece of surface in X bounded by ℓ. (If σ and σ' are two such surfaces then $\int_\sigma \omega$ and $\int_{\sigma'} \omega$ differ by a multiple of 2π because $\omega/2\pi$ is integral; so $C(\ell)$ is well defined.) The assignment $\ell \mapsto C(\ell)$ has the three properties:

(H1) *independence of parametrization,* i.e. $C(\ell) = C(\ell \circ \phi)$ when $\phi: S^1 \to S^1$ is any piecewise smooth map of degree 1;

(H2) *additivity,* i.e. $C(p * r^{-1}) = C(p * q^{-1})C(q * r^{-1})$, when p, q, r are three paths from x_0 to x_1, and $p * q^{-1}$ denotes the loop obtained by performing p followed by the reverse of q; and

(H3) *Γ-invariance,* i.e. $C(\gamma . \ell) = C(\ell)$ for any $\gamma \in \Gamma$.

Any map $\ell \mapsto C(\ell)$ with these three properties defines a central extension $\tilde{\Gamma}$ of Γ by $\mathbb{T}$, as follows. We choose a base-point x_0 in X. Then an element of $\tilde{\Gamma}$ is represented by a triple (γ, p, u), where $\gamma \in \Gamma$, $u \in \mathbb{T}$, and p is a path in X from x_0 to $\gamma . x_0$. Two triples (γ, p, u) and

(γ', p', u') are regarded as equivalent if $\gamma = \gamma'$ and $u = C(p' * p^{-1}) . u'$. The composition in $\tilde{\Gamma}$ is given by

$$(\gamma_1, p_1, u_1) . (\gamma_2, p_2, u_2) = (\gamma_1\gamma_2, p_1 * \gamma_1 . p_2, u_1u_2).$$

It is easy to check that $\tilde{\Gamma}$ is a well-defined group.

The description of $\tilde{\Gamma}$ just given is very convenient and we shall often make use of it in this chapter. It is not, however, the best way to see that $\tilde{\Gamma}$ is a Lie group, or to understand its global topology; for that reason we shall give a different proof of (4.4.2) in the next section.

To obtain the desired central extensions of LG we can apply Proposition (4.4.2) with $\Gamma = X = LG$, for if G is simply connected then LG is simply connected too. Indeed as a space LG is the product $G \times \Omega G$, where G is the subgroup of constant loops and ΩG is the subgroup of loops γ such that $\gamma(1) = 1$. Thus

$$\pi_1(LG) \cong \pi_1(G) \oplus \pi_1(\Omega G) \cong \pi_1(G) \oplus \pi_2(G).$$

It is a classical theorem that $\pi_2(G) = 0$ for any compact Lie group: we shall assume this for the present, but a proof will be given in Section 8.6.

The explicit construction of $\tilde{L}G$ makes clear that $\mathrm{Diff}^+(S^1)$ acts on it, for $\mathrm{Diff}^+(S^1)$ acts on $X = LG$ preserving the 2-form ω and leaving fixed the identity element. That proves part of (4.4.1) (ii). We shall postpone to the next section the proof that there is no other extension of LG with the Lie algebra cocycle ω.

Now let us turn to (4.4.1) (iv). There is a so-called 'transgression' homomorphism

$$\tau : H^3(G) \to H^2(\Omega G), \tag{4.4.3}$$

where the cohomology has either real or integer coefficients, defined as the composite

$$H^3(G) \to H^3(S^1 \times \Omega G) \to H^2(\Omega G),$$

where the first map is induced by the evaluation $S^1 \times \Omega G \to G$, and the second is integration over S^1. (Cf. [18] p. 247.) When G is simply connected the transgression τ is an isomorphism: it reduces to the transpose of the obvious isomorphism $\pi_2(\Omega G) \to \pi_3(G)$ when one uses the Hurewicz isomorphisms $\pi_2(\Omega G) \cong H_2(\Omega G)$ and $\pi_3(G) \cong H_3(G)$. Thus (4.4.1) (iv) is obtained by putting together the following two results.

Proposition (4.4.4). *Let σ denote the left-invariant 3-form on G whose value at the identity element is given by*

$$\sigma(\xi, \eta, \zeta) = \langle [\xi, \eta], \zeta \rangle.$$

Then the transgression $\tau(\sigma)$ is cohomologous to the invariant form $\omega/2\pi$ on ΩG.

Proposition (4.4.5). *The skew form σ of (4.4.4) defines an integral cohomology class on the simply connected group G if and only if $\langle h_\alpha, h_\alpha \rangle \in 2\mathbb{Z}$ for each coroot h_α of G.*

Proof of (4.4.5). For each root α there is a homomorphism $i_\alpha : SU_2 \to G$ which on the diagonal matrices in SU_2 induces the coroot h_α—see Section 2.4. When the form σ is pulled back to $SU_2 \cong S^3$ by i_α one obtains $\frac{1}{2}\langle h_\alpha, h_\alpha \rangle \sigma_0$, where σ_0 is the invariant 3-form on SU_2 with integral 1. Thus (4.4.5) follows from the fact that the maps i_α generate $\pi_3(G)$. It is enough to prove this for simple groups G, and we can also replace G by any locally isomorphic group. One has then to show that for suitable α the homogeneous space $G/i_\alpha(SU_2)$ is 3-connected. That is obvious for the classical groups. We refer to Bott [14] for a simple proof by Morse theory which works in all cases. In fact α can always be taken to be the highest root.

Proof of (4.4.4). When the form $2\pi\sigma$ is pulled back to $S^1 \times \Omega G$ its value at the point (θ, γ) on the triple of tangent vectors $(\delta\theta, \delta_1\gamma, \delta_2\gamma)$ is

$$\frac{1}{4\pi} \langle \gamma(\theta)^{-1}\gamma'(\theta), [\xi_1(\theta), \xi_2(\theta)] \rangle \delta\theta,$$

where $\xi_i(\theta) = \gamma(\theta)^{-1}\delta_i\gamma(\theta)$. This is to be integrated over S^1 and compared with

$$\omega_\gamma(\delta_1\gamma, \delta_2\gamma) = \frac{1}{2\pi} \int_0^{2\pi} \langle \xi_1(\theta), \xi_2'(\theta) \rangle \, d\theta.$$

Consider the 1-form β on ΩG given by

$$\beta_\gamma(\delta\gamma) = \frac{1}{4\pi} \int_0^{2\pi} \langle \gamma(\theta)^{-1}\gamma'(\theta), \gamma(\theta)^{-1}\delta\gamma(\theta) \rangle \, d\theta.$$

A simple calculation shows that $d\beta = \tau(2\pi\sigma) - \omega$.

If $\mathfrak{g}$ is a simple algebra then all invariant inner products on it are proportional, and so there is a smallest one satisfying the integrality condition (4.4.5). We shall call this the *basic inner product*, and the associated extension the *basic central extension* of LG. If G is simply laced then the basic inner product is the one discussed in Section 2.5 for which $\langle h_\alpha, h_\alpha \rangle = 2$ for every coroot. In general it is characterized by the property that $\langle h_\alpha, h_\alpha \rangle = 2$ when α is the highest root. The Killing form [1] on $\mathfrak{g}$ satisfies the integrality condition, so it is an integer multiple of the basic form. We shall obtain a formula for the integer in Section 14.5. When G is simply laced it is the Coxeter number ([20] Chapter 6, Section 1.11) of G.

The basic central extension is universal.

Proposition (4.4.6). *If G is simply connected and simple then the extension $\tilde{L}G$ associated to the basic inner product is itself simply connected. It is the unique simply connected extension of LG by $\mathbb{T}$, and it is the universal central extension in the category of Lie groups. Furthermore $\pi_2(\tilde{L}G)=0$.*

Proof. We calculate $\pi_1(\tilde{L}G)$ and $\pi_2(\tilde{L}G)$ from the homotopy exact sequence of the circle bundle $\mathbb{T}\to\tilde{L}G\to LG$. This gives us

$$0\to\pi_2(\tilde{L}G)\to\pi_2(LG)\to\pi_1(\mathbb{T})\to\pi_1(\tilde{L}G)\to 0.$$

The map $\pi_2(LG)\to\pi_1(\mathbb{T})=\mathbb{Z}$ is defined by evaluating the first Chern class of the bundle on 2-spheres in LG; and the basic extension is defined so that the first Chern class generates H^2 of the 2-sphere corresponding to the highest root. Thus $\pi_1(\tilde{L}G)=0$; and because we know $\pi_2(LG)\cong\pi_3(G)\cong\mathbb{Z}$ it follows that $\pi_2(\tilde{L}G)=0$ too.

To prove the universality, let $A\to E\to LG$ be an arbitrary central extension. The corresponding Lie algebra extension can be defined by a skew form

$$\omega_A: L\mathfrak{g}\times L\mathfrak{g}\to\mathfrak{a},$$

where $\mathfrak{a}$ is the Lie algebra of A. Because $\tilde{L}\mathfrak{g}$ is universal (see (4.2.4)) we know that $\omega_A=\phi\circ\omega$, where ω is the basic cocycle and $\phi:\mathbb{R}\to\mathfrak{a}$ is some map. It follows that if we pull E back to $\tilde{L}G$, i.e. we form $\tilde{E}=\tilde{L}G\times_{LG}E$, the subgroup of $\tilde{L}G\times E$ consisting of pairs (x, y) such that x and y have the same image in LG, then the resulting Lie algebra extension of $\tilde{L}\mathfrak{g}$ by $\mathfrak{a}$ is trivial. But we shall see in the next section that an extension of a simply connected group such as $\tilde{L}G$ is trivial if its Lie algebra cocycle is trivial. So $\tilde{E}$ is a trivial extension of $\tilde{L}G$, and its splitting map gives us the desired homomorphism $\tilde{L}G\to E$.

Let us notice that the induced homomorphism $\mathbb{T}\to A$ can be determined as the image of the generator of $\pi_2(LG)$ in $\pi_1(A)$, for $\pi_1(A)\cong\mathrm{Hom}(\mathbb{T};A)$.

4.5 Circle bundles, connections, and curvature

The main object of this section is to prove Proposition (4.4.2), but we shall begin by summarizing the facts about circle bundles, connections, and curvature that we shall make use of. This material is all well known (cf. [27], [68]), at least in the finite dimensional case. At the end of the summary we shall give brief proofs of the essential points.

Suppose that $\pi: Y\to X$ is a smooth principal fibre bundle whose fibre is a circle and whose base X is a possibly infinite dimensional manifold. This means that the group $\mathbb{T}$ acts freely on Y, the fibres are its orbits, and X is the orbit-space $Y/\mathbb{T}$. A *connection* in the bundle is a prescription

which decomposes the tangent space T_yY at a point $y\in Y$ as

$$\mathbb{R}\oplus T_y^{\mathrm{horiz}}Y,$$

where $\mathbb{R}$ is the tangent space along the fibre and $T_y^{\mathrm{horiz}}Y$—the 'horizontal' tangent vectors—is a replica of $T_{\pi(y)}X$. The decomposition is required to be invariant under the action of $\mathbb{T}$ on Y.

A connection tells one how a path in X can be lifted to a horizontal path in Y with a prescribed starting point. If one lifts a closed path in X the lifted path in Y will in general fail to close. The gap between its ends corresponds to an element of $\mathbb{T}$ called the *holonomy* around the path. The *curvature* of the connection measures the holonomy around infinitesimally small closed paths: it is the closed 2-form ω on X whose value on a pair of tangent vectors ξ, η at a point of X is the infinitesimal holonomy around the parallelogram spanned by ξ and η. The curvature defines an element of the cohomology group $H^2(X;\mathbb{R})$ which depends only on the topological type of the bundle. Moreover $\omega/2\pi$ is an *integral* class—i.e. its integral over any 2-cocycle in X is an integer—and it comes from a well-defined element of $H^2(X;\mathbb{Z})$ called the (first) *Chern class* of the bundle. The Chern class describes the topological type of the circle bundle completely, and any element of $H^2(X;\mathbb{Z})$ arises from a bundle. If X is simply connected then the natural map

$$i: H^2(X;\mathbb{Z})\to H^2(X;\mathbb{R})$$

is injective, and the topological type is completely determined by the class of $\omega/2\pi$. In general the kernel of i corresponds to the *flat* bundles, i.e. those which can be given a connection with curvature zero.

Analytically a connection can be described in three ways.

(i) One can give the map $\xi\mapsto\tilde{\xi}$ which to each vector field ξ on X assigns the corresponding horizontal $\mathbb{T}$-invariant vector field $\tilde{\xi}$ on Y. From this point of view the curvature is given by

$$\omega(\xi,\eta)=[\tilde{\xi},\tilde{\eta}]-[\xi,\eta]^{\sim}. \tag{4.5.1}$$

(The right hand side of this equation is a $\mathbb{T}$-invariant vertical vector field on Y; but we can identify it with a real-valued function on X.)

(ii) One can give the $\mathbb{T}$-invariant 1-form α on Y which assigns to a tangent vector to Y its vertical component, a real number. The restriction of α to each fibre is the standard 1-form $d\theta$. The derivative $d\alpha$ is $\mathbb{T}$-invariant and vanishes on vertical vectors, so $d\alpha=\pi^*\omega$ for a unique closed 2-form ω on X, which is the curvature.

(iii) One can introduce local trivializations of Y. That is, X is covered by open sets $\{U_a\}$, and the part of Y over U_a is identified with $U_a\times\mathbb{T}$. Then the connection is described in U_a by the 1-form $\alpha_a=s_a^*\alpha$ which is obtained by pulling back the form α described above by any section

$s_a: U_a \to Y$ which is constant in terms of the local trivialization $U_a \times \mathbb{T}$. In U_a the curvature ω is described by $d\alpha_a = \omega$. If the transition functions of the bundle are

$$f_{ab}: U_a \cap U_b \to \mathbb{T}$$

(i.e. the point $(x, \rho) \in U_a \times \mathbb{T}$ is the same point as $(x, f_{ab}(x)\rho) \in U_b \times \mathbb{T}$), then

$$\alpha_b = \alpha_a + if_{ab}^{-1}\, df_{ab}. \tag{4.5.2}$$

That completes our summary. We shall now prove the essential result.

Proposition (4.5.3). *Let X be a connected and simply connected manifold.*

(i) *If ω is a closed 2-form on X such that $\omega/2\pi$ represents an integral cohomology class then there is a circle bundle on X with a connection whose curvature is ω.*

(ii) *If Y and Y' are circle bundles on X with connections α and α' which have the same curvature ω then there is an isomorphism $\psi: Y \to Y'$ such that $\psi^*\alpha' = \alpha$. Furthermore ψ is unique up to composition with the action of an element of $\mathbb{T}$.*

Proof.

(i) One way of expressing the condition that $\omega/2\pi$ is integral is to say that there is an integral Čech cocycle $\{\nu_{abc}\}$ defined with respect to an open covering $\{U_a\}$ of X such that $\omega = d\beta + 2\pi\nu$, for some 1-form β, where ν is the 2-form associated to $\{\nu_{abc}\}$ by means of a smooth partition of unity $\{\lambda_a\}$ subordinate to $\{U_a\}$—see (3.1.1). (Thus ν_{abc} is an integer defined when $U_a \cap U_b \cap U_c$ is non-empty, and we can assume that it is skew with respect to changing the order of a, b, c.)

Let us construct a bundle Y on X by means of the transition functions $\{f_{ab}\}$, where

$$f_{ab}(x) = \exp 2\pi i \sum_c \nu_{abc}\lambda_c.$$

The coherence of the f_{ab} follows from the cocycle condition

$$\nu_{bcd} - \nu_{acd} + \nu_{abd} - \nu_{abc} = 0.$$

A connection in this bundle is defined by the 1-forms

$$\alpha_a = 2\pi \sum_{b,c} \nu_{abc}\lambda_b\, d\lambda_c,$$

its curvature is the 2-form $2\pi\nu$:

$$2\pi\nu = 2\pi \sum_{a,b,c} \nu_{abc}\lambda_a\, d\lambda_b \wedge d\lambda_c.$$

To obtain a connection with curvature ω we simply add the 1-form β to each α_a.

(ii) Suppose that Y and Y' are defined with respect to the same open covering $\{U_a\}$ of X by transition functions $\{f_{ab}\}$ and $\{f'_{ab}\}$. A map $\psi: Y \to Y'$ will be given locally by functions $\psi_a: U_a \to \mathbb{T}$ such that

$$\psi_b(x) f_{ab}(x) = f'_{ab}(x)\psi_a(x) \quad \text{for} \quad x \in U_a \cap U_b. \tag{4.5.4}$$

The condition $\psi^*\alpha' = \alpha$ is expressed by

$$\alpha'_a = \alpha_a + i\psi_a^{-1}\, d\psi_a. \tag{4.5.5}$$

It is permissible to assume that each set U_a is contractible, and each intersection $U_a \cap U_b$ connected. Then we can find a function $\phi_a: U_a \to \mathbb{R}$ such that $d\phi_a = \alpha'_a - \alpha_a$. From (4.5.2) we then find that

$$d(e^{-i\phi_b} f_{ab}) = d(f'_{ab} e^{-i\phi_a})$$

in $U_a \cap U_b$. In other words

$$e^{-i\phi_b} f_{ab} = f'_{ab} e^{-i\phi_a} e^{i\mu_{ab}},$$

where $\mu_{ab} \in \mathbb{R}$ is constant. Because X is simply connected (and $\{\mu_{ab}\}$ is a Čech 1-cocycle) we can find numbers m_a such that $\mu_{ab} = m_b - m_a$. Then the functions $\psi_a = e^{-i(\phi_a + m_a)}$ satisfy both (4.5.4) and (4.5.5).

As to the uniqueness of ψ, or equivalently of the functions ψ_a, it follows from (4.5.4) that any two possible choices differ by multiplication by a global function $g: X \to \mathbb{T}$. The equations (4.5.5) then show that g is constant.

Proof of (4.4.2). We can now prove (4.4.2) very simply. We first construct a circle bundle Y on X with a connection α with curvature ω. For each $\gamma \in \Gamma$ we can pull back Y by the map $\gamma: X \to X$. The resulting bundle γ^*Y has a connection α_γ whose curvature is $\gamma^*\omega = \omega$. We now define $\tilde{\Gamma}$ as the group of all pairs (γ, ψ) with $\gamma \in \Gamma$ and $\psi: Y \to \gamma^*Y$ an isomorphism such that $\psi^*\alpha_\gamma = \alpha$. By Proposition (4.5.3) there is a circle of possible choices of ψ for each γ. Ths isomorphism ψ can equally well be regarded as a map $\psi: Y \to Y$ which covers the action of γ on X and satisfies $\psi^*\alpha = \alpha$. In other words $\tilde{\Gamma}$ is simply the group of all fibre-preserving maps $\psi: Y \to Y$ which preserve α and cover an element γ of Γ. Such a map is completely determined by γ and $\psi(y_0)$, where y_0 is an arbitrary base-point in Y. Thus as a manifold $\tilde{\Gamma}$ is the fibre product $\Gamma \times_X Y$, the pull-back of Y by the map $\Gamma \to X$ which takes γ to $\gamma(\pi(y_0))$.

To identify the description of $\tilde{\Gamma}$ just given with the description by triples (γ, p, u) given after the statement of (4.4.2) we associate to (γ, p, u) the unique automorphism of Y which preserves α and maps y_0 to the point obtained by parallel transport of y_0 along p. (We assume

$\pi(y_0)=x_0$.) We leave it to the reader to check that this defines a group isomorphism.

Proposition (4.4.2) proves one half of (4.4.1) (i). The other half, asserting that if a cocycle ω on Lg corresponds to a group extension then $\omega/2\pi$ is integral, is now fairly obvious, but we shall restate it in the form:

Proposition (4.5.6). *If an extension of Lie groups*

$$\mathbb{T}\to\tilde{\Gamma}\to\Gamma$$

corresponds to the Lie algebra cocycle ω on the Lie algebra of Γ, then $\omega/2\pi$, regarded as a left-invariant 2-form on Γ, represents the Chern class of the bundle $\tilde{\Gamma}$, and is therefore integral.

Proof. This follows at once from the first method of describing a connection. To obtain the cocycle ω we must choose a vector space decomposition of the Lie algebra $\mathrm{Lie}(\tilde{\Gamma})$ of $\tilde{\Gamma}$:

$$\mathrm{Lie}(\tilde{\Gamma})\cong\mathbb{R}\oplus\mathrm{Lie}(\Gamma).$$

This induces a decomposition of the tangent space at each point of $\tilde{\Gamma}$, i.e. a connection in $\tilde{\Gamma}\to\Gamma$. The splitting map $\mathrm{Lie}(\Gamma)\to\mathrm{Lie}(\tilde{\Gamma})$ can be identified with the horizontal lifting $\xi\mapsto\tilde{\xi}$ of left-invariant vector fields; so by the formula (4.5.1) the curvature ω is given by the same expression

$$\omega(\xi,\eta)=[\tilde{\xi},\tilde{\eta}]-[\xi,\eta]^{\sim}$$

which defines the Lie algebra cocycle.

The argument we have just given suffices also to complete the two remaining proofs—of (4.4.6) and (4.4.1) (ii)—which were postponed from the preceding section. First, a central extension $A\to E\to\Gamma$ of a simply connected group Γ is trivial if the induced extension of Lie algebras is trivial. For then the principal bundle E has a flat connection, and we can define a map $\Gamma\to E$ which takes γ to the end-point of the horizontal lift of any path in Γ from the identity to γ. The map is a homomorphism because the connection is E-invariant. That completes the proof of (4.4.6).

To complete (4.4.1) (ii) we must show that two extensions E and E' of LG by $\mathbb{T}$ are isomorphic if their Lie algebra cocycles coincide. To do that we form the 'difference' extension $\mathbb{T}\to E''\to LG$, i.e. we pull E' back to E, and then pass to the quotient by the image of the homomorphism $\mathbb{T}\to E\times_{LG}E'$ which takes u to (u,u^{-1}). The Lie algebra extension corresponding to E'' is trivial, and so E'' itself is trivial, and $E\cong E'$.

4.6 The group extensions when G is semisimple but not simply connected

If G is a semisimple group we can write $G=\tilde{G}/Z$, where $\tilde{G}$ is the simply connected covering group of G, and $Z\cong\pi_1(G)$ is a finite subgroup of the centre of $\tilde{G}$.

We have seen that an integral bilinear form $\langle\ ,\ \rangle$ on $\mathfrak{g}$ gives rise to a unique central extension $\tilde{L\tilde{G}}$ of $L\tilde{G}$. Because $\tilde{G}$ can be identified canonically with a subgroup of $\tilde{L\tilde{G}}$ we can regard Z as a subgroup of $\tilde{L\tilde{G}}$. In fact Z belongs to the centre of $\tilde{L\tilde{G}}$, because its adjoint action on $\tilde{L\mathfrak{g}}$ is trivial. (That follows from (4.3.2).) Thus we have an extension

$$\mathbb{T}\to(\tilde{L\tilde{G}})/Z\to(LG)^0, \tag{4.6.1}$$

where $(LG)^0\cong(L\tilde{G})/Z$ is the identity component of LG. But (4.6.1) is usually *not* the restriction of an extension of the whole group LG. To understand this we observe that the form $\langle\ ,\ \rangle$ on $\mathfrak{g}$ induces a pairing

$$c:Z\times Z\to\mathbb{T} \tag{4.6.2}$$

in the following way. Let $\tilde{T}$ be a maximal torus of $\tilde{G}$. Given $z_1,z_2\in Z$, choose ζ_1,ζ_2 in the Lie algebra of $\tilde{T}$ so that $\exp(2\pi\zeta_i)=z_i$. Then define

$$c(z_1,z_2)=e^{2\pi i\langle\zeta_1,\zeta_2\rangle}.$$

The pairing is independent of the choices made.

Lemma (4.6.3). *The extension* (4.6.1) *is not the restriction of an extension of LG unless the pairing c is trivial.*

Proof. Consider the automorphism A_λ of $L\mathfrak{g}$ induced by conjugating by an element λ of LG which does not belong to the identity component. We can lift A_λ *uniquely* to an automorphism $\tilde{A}_\lambda$ of $\tilde{L\mathfrak{g}}$: a simple calculation (cf. (4.3.2)) shows that

$$\tilde{A}_\lambda(\xi,a)=(A_\lambda\xi,a-\langle\lambda^{-1}\lambda',\xi\rangle). \tag{4.6.4}$$

Let us apply this when $\xi\in\mathfrak{t}$ and $\exp(2\pi\xi)\in Z$, and λ is the loop $\theta\mapsto\exp(\theta\zeta)$ in G for some $\zeta\in\mathfrak{t}$ such that $\exp(2\pi\zeta)\in Z=\pi_1(G)$. We find

$$\tilde{A}_\lambda(\xi,0)=(\xi,-\langle\xi,\zeta\rangle).$$

Because the exponential map of $(\tilde{LG})^0$ takes $(2\pi\xi,0)$ to 1 we see that $\tilde{A}_\lambda$ cannot induce an automorphism of $(\tilde{LG})^0$ unless $\langle\xi,\zeta\rangle\in\mathbb{Z}$. That proves (4.6.3).

In the other direction, however, we can assert

Lemma (4.6.5). *The conjugation action of LG on $\tilde{L\tilde{G}}$ lifts uniquely to an* *action on $\tilde{L}\tilde{G}$, and then the action of $Z \cong \pi_1(G)$ on the centre $Z \times \mathbb{T}$ of $\tilde{L}\tilde{G}$ is given by* (4.6.2).

Proof. We first notice that the lift, if it exists, will be unique. In fact $\mathrm{Aut}(\tilde{L}\tilde{G})$ can be identified with a subgroup of $\mathrm{Aut}(L\tilde{G})$ because $L\tilde{G}$ is a perfect group (see (3.4.1)) and there are no non-trivial homomorphisms $L\tilde{G} \to \mathbb{T}$.

The extension $\tilde{L}\tilde{G}$ is defined by a 2-form ω on $L\tilde{G}$. The same calculation that gave us (4.6.4) shows that if c_λ denotes conjugation by $\lambda \in LG$ we have

$$c_\lambda^* \omega = \omega - \mathrm{d}\beta, \tag{4.6.6}$$

where β is the left-invariant 1-form given by

$$\beta(\xi) = \langle \lambda^{-1}\lambda', \xi \rangle. \tag{4.6.7}$$

Let us think of elements of $\tilde{L}\tilde{G}$ as triples (γ, p, u) as in Section 4.4. Then $c_\lambda : L\tilde{G} \to L\tilde{G}$ is covered by the automorphism

$$(\gamma, p, u) \mapsto (c_\lambda \gamma, c_\lambda p, e^{i\beta \cdot p} u), \tag{4.6.8}$$

where $\beta \, . \, p$ denotes the integral of β along p.

We can now describe a class of extensions of LG for semisimple groups G.

Proposition (4.6.9). *For any integral inner product $\langle \ , \ \rangle$ on $\mathfrak{g}$ there is a group $\tilde{L}G$ whose identity component is $\tilde{L}\tilde{G}$ and whose group of components is $Z \cong \pi_1(G)$. It is an extension of LG by $\mathbb{T} \times Z$, and the conjugation action of the group of components on $\mathbb{T} \times Z$ is given by* (4.6.2).

The extension $\tilde{L}G$ is not determined uniquely by $\langle \ , \ \rangle$, but is unique up to the addition of an arbitrary extension of $\pi_1(G)$ by $\mathbb{T}$.

Remark. An extension of $\pi_1(G)$ by $\mathbb{T}$ clearly gives an extension of LG by $\mathbb{T}$, and this can be added—in the usual sense of addition for group extensions—to $\tilde{L}G$ without changing the identity component.

Proof. Let us choose a maximal torus T in G, and let $\Lambda = \check{T} = \mathrm{Hom}(\mathbb{T}; T)$. Thus Λ is a subgroup of LG. The part Λ_0 of Λ which is contained in the identity component of LG can be identified with a subgroup of $L\tilde{G}$. Let $\tilde{\Lambda}_0$ be the extension of Λ_0 by $\mathbb{T}$ induced by $\tilde{L}\tilde{G}$. Suppose that $\tilde{\Lambda}_0$ can be extended to an extension $\tilde{\Lambda}$ of Λ. Then we can form the semidirect product $\tilde{\Lambda} \tilde{\times} \tilde{L}\tilde{G}$, where $\tilde{\Lambda}$ acts on LG through Λ, which acts on $\tilde{L}\tilde{G}$ by (4.6.5). But the antidiagonal map $\tilde{\Lambda}_0 \to \tilde{\Lambda} \tilde{\times} \tilde{L}\tilde{G}$ (i.e. $\lambda \mapsto (\lambda, \lambda^{-1})$) embeds $\tilde{\Lambda}_0$ as a normal subgroup, and we can define $\tilde{L}G$ as the quotient group $(\tilde{\Lambda} \tilde{\times} \tilde{L}\tilde{G})/\tilde{\Lambda}_0$.

Returning to the existence of $\tilde{\Lambda}$, we observe that an extension $\tilde{\Lambda}$ of a free abelian group Λ by $\mathbb{T}$ is completely determined by the commutator map $(x, y) \mapsto xyx^{-1}y^{-1}$, which is a skew biadditive map $\Lambda \times \Lambda \to \mathbb{T}$. This can be lifted to a skew biadditive map $\Lambda \times \Lambda \to \mathbb{R}$. Conversely any such map $b : \Lambda \times \Lambda \to \mathbb{R}$ defines an extension of Λ by $\mathbb{T}$ with the cocycle

$$(\lambda, \mu) \mapsto e^{\frac{1}{2} i b(\lambda, \mu)}.$$

Thus $\tilde{\Lambda}$ can be obtained by extending arbitrarily the skew map which defines $\tilde{\Lambda}_0$; the ambiguity is precisely an arbitrary extension of $\Lambda/\Lambda_0 = \pi_1(G)$.

Remark. The extension $\tilde{L}G$ is most appealing when G is simply laced and $\langle \ , \ \rangle$ is the basic inner product. Then the pairing c of (4.6.2) is nondegenerate, and the centre of $\tilde{L}G$ is exactly $\mathbb{T}$. We shall determine the induced extension of Λ_0 in this case in Section 4.8.

4.7 The basic central extension of LU_n

We shall not bother to discuss the extensions of LG when G is not semisimple except in the particular case $G = U_n$. The extension of LU_n which we shall now describe plays a central role in the theory.

Proposition (4.7.1).

(i) *There is a canonical extension $\tilde{L}U_n$ of LU_n by $\mathbb{T}$ whose Lie algebra cocycle is given by* (4.2.2), *where $\langle \ , \ \rangle$ is the basic inner product on $\mathfrak{u}_n$.*

(ii) *The subgroup U_n of constant loops is identified canonically with a subgroup of $\tilde{L}U_n$.*

(iii) *The centre of the identity component of $\tilde{L}U_n$ is $\mathbb{T} \times \mathbb{T}$, where the first $\mathbb{T}$ is the kernel of the extension and the second is the centre of the canonical copy of U_n. Conjugation by a loop of winding number k transforms $\mathbb{T} \times \mathbb{T}$ by*

$$(u, v) \mapsto (uv^{-k}, v).$$

(iv) *The natural action of $\mathrm{Diff}^+(S^1)$ on $\tilde{L}\mathfrak{u}_n$ comes from a unique action of the double covering of $\mathrm{Diff}^+(S^1)$ on $\tilde{L}U_n$. (But see Remark* (4.7.2) *below.)*

Remark. The extension $\tilde{L}U_n$ is determined completely by the Lie algebra extension, but only up to non-canonical isomorphism. (For any group Γ the group of automorphisms of Γ which are the identity on Γ^0 and Γ/Γ^0 is $\mathrm{Hom}(\Gamma/\Gamma^0; Z)$, where Z is the centre of Γ^0.) Furthermore there is an infinite dimensional space—namely $\mathrm{Hom}(L\mathfrak{u}_n; \mathbb{R})$—of automorphisms of $\tilde{L}\mathfrak{u}_n$ which induce the identity on $L\mathfrak{u}_n$. These facts make the study of $\tilde{L}U_n$ quite confusing, especially where the action of $\mathrm{Diff}^+(S^1)$ is concerned.

Proof. We shall be brief, as the details are much the same as we have encountered in the preceding sections.

We construct the desired extension of the identity component $(LU_n)^0$ by applying Proposition (4.4.2) to the simply connected space $X = (LU_n)^0/T$, where T is the standard maximal torus of U_n. Then we observe that LU_n is a semidirect product $\mathbb{Z} \tilde{\times} (LU_n)^0$, where the copy of $\mathbb{Z}$ is generated by any loop λ of winding number 1. We can define $\tilde{L}U_n = \mathbb{Z} \tilde{\times} (\tilde{L}U_n)^0$ providing we can lift the conjugation action of λ on $(LU_n)^0$ to an automorphism of $(\tilde{L}U_n)^0$. The argument of (4.6.5) shows that the lift is possible, and also proves assertion (iii). The essential point is that we can choose λ to be a loop in the abelian subgroup T, so that conjugation by λ does act on X. The construction depends on the choice of λ, but there is a canonical choice.

To investigate the action of $\text{Diff}^+(S^1)$ on $\tilde{L}U_n$ we introduce the simply connected covering group $\mathcal{D}$ of $\text{Diff}^+(S^1)$. This can be realized as the group of diffeomorphisms $\phi: \mathbb{R} \to \mathbb{R}$ such that $\phi(\theta + 2\pi) = \phi(\theta) + 2\pi$. To prove that the action of $\mathcal{D}$ on LU_n lifts to $\tilde{L}U_n$ it is enough to construct an extension of the semidirect product $\mathcal{D} \tilde{\times} LU_n$ by $\mathbb{T}$ which restricts to $\tilde{L}U_n$ over LU_n and is trivial over $\mathcal{D}$. That can be done by first constructing the extension over the connected component $\mathcal{D} \tilde{\times} (LU_n)^0$ by applying our standard procedure to the homogeneous space $Y = \mathcal{D} \tilde{\times} (LU_n)^0/T$, on which the usual 2-cocycle of Lu_n defines an invariant integral 2-form. Then one must lift the conjugation action of λ on $\mathcal{D} \tilde{\times} (LU_n)^0$. The equation (4.6.6) is still valid on the larger space, where now β is the invariant form defined by the linear map

$$\left(f\frac{\mathrm{d}}{\mathrm{d}\theta}, \xi\right) \mapsto \langle \lambda'\lambda^{-1}, \tfrac{1}{2}f\lambda'\lambda^{-1} + \xi \rangle$$

on the Lie algebra of $\mathcal{D} \tilde{\times} (LU_n)^0$. Thus the argument is as before.

Finally we must calculate the action on $\tilde{L}U_n$ of the central element τ of $\mathcal{D}$ defined by $\tau(\theta) = \theta + 2\pi$. It is enough to calculate the conjugation action of λ on the element $(\tau, p, 1)$ of the extension of $\mathcal{D} \tilde{\times} (LU_n)^0$, where p is the obvious path in Y from the base-point to τ. If we assume that λ is a homomorphism $\mathbb{T} \to T$ then the path p is left fixed by the conjugation, and so the formula (4.6.8) shows that $(\tau, p, 1)$ is multiplied by the element $e^{i\beta \cdot p}$ of the centre. Because $\langle \lambda'\lambda^{-1}, \lambda'\lambda^{-1} \rangle = 1$ we find that $e^{i\beta \cdot p} = -1$. This means that τ acts on the components of odd winding number in $\tilde{L}U_n$ by multiplication by -1. That completes the proof of (4.7.1).

Remark (4.7.2). The action of $\mathcal{D}$ on $\tilde{L}U_n$ is completely determined by its action on the Lie algebra $\tilde{L}u_n$ (for the kernel of $\text{Aut}(\tilde{L}U_n) \to \text{Aut}((\tilde{L}U_n)^0)$ is abelian). But there are automorphisms of $\tilde{L}u_n$ which do not extend to $\tilde{L}U_n$, and so the action of $\mathcal{D}$ on $\tilde{L}U_n$ can be changed without affecting its action on $\tilde{L}u_n$ *up to isomorphism*. Thus despite Proposition (4.7.1) (iv) it *is* possible to lift the action of $\text{Diff}^+(S^1)$ from LU_n to $\tilde{L}U_n$. One way is to

make $\phi \in \mathcal{D}$ act on an element $\tilde{\gamma} \in \tilde{L}U_n$ above $\gamma \in LU_n$ by

$$(\phi, \tilde{\gamma}) \mapsto A_\phi(\tilde{\gamma}) \cdot e^{ia(\phi, f)} \tag{4.7.3}$$

where

$$e^{if(\theta)} = \det \gamma(\theta),$$

$$a(\phi, f) = \frac{1}{4\pi}\int_0^{2\pi} (f(\phi^{-1}\theta) - f(\theta))\,\mathrm{d}\theta, \tag{4.7.4}$$

and A_ϕ is the action of $\phi \in \mathcal{D}$ described in Proposition (4.7.1). If ϕ is the translation $\theta \mapsto \theta + \alpha$ then

$$a(\phi, f) = -\tfrac{1}{2}\alpha\Delta_f,$$

where Δ_f is the winding number of $\det \gamma$. This means that the central element $\tau \in \mathcal{D}$ acts trivially.

We shall leave it to the reader to check that this action of $\text{Diff}^+(S^1)$ is unique up to automorphisms of $\tilde{L}U_n$. In Section 6.8 we shall see that the particular choice (4.7.3) arises naturally.

The case $n = 1$

The basic central extension of $LU_1 = L\mathbb{T}$ can be described by an explicit cocycle, which we give here for future reference. We first observe that any element of $L\mathbb{T}$ can be written in the form e^{if}, where f is a smooth function such that

$$f(\theta + 2\pi) = f(\theta) + 2\pi\Delta_f$$

for some integer Δ_f which is the winding number of e^{if}. We shall write $\hat{f}$ for the average of f on the interval $[0, 2\pi]$, i.e.

$$\hat{f} = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,\mathrm{d}\theta.$$

Proposition (4.7.5). *The basic central extension of $L\mathbb{T}$ is defined by the cocycle c, where*

$$c(e^{if}, e^{ig}) = e^{iS(f,g)}$$

and

$$S(f, g) = \frac{1}{4\pi}\int_0^{2\pi} f(\theta)g'(\theta)\,\mathrm{d}\theta + \tfrac{1}{2}\hat{f}\Delta_g + \tfrac{1}{2}\Delta_f(\hat{g} - g(0)).$$

The action on $\tilde{L}\mathbb{T}$ of a diffeomorphism $\phi \in \text{Diff}^+(S^1)$ can be taken to be

$$(e^{if}, u) \mapsto (e^{if\circ\phi^{-1}}, ue^{i(\Delta_f - 1)a(\phi, f)}),$$

where $a(\phi, f)$ is as in (4.7.4), and $\tilde{L}\mathbb{T}$ is identified with $L\mathbb{T} \times \mathbb{T}$ as a set. In particular, the rotation R_α through the angle α acts by

$$(e^{if}, u) \mapsto (R_\alpha e^{if}, ue^{-\frac{1}{2}i\alpha\Delta_f(\Delta_f - 1)}).$$

Proof. We leave it to the reader to check that c—which is a cocycle because S is bilinear— does define an extension of $L\mathbb{T}$ with the properties described in (4.7.1).

4.8 The restriction of the extension to LT

If T is a maximal torus of G then the lattice $\Lambda = \mathrm{Hom}(\mathbb{T}; T)$—which we usually denote by $\check{T}$—is a subgroup of LG, and we can consider the restriction to it of an extension $\tilde{L}G$. We know how to describe the resulting extension $\tilde{\Lambda}$ of Λ by $\mathbb{T}$ only when G is a simply laced group, but the result in that case is quite striking, and is the basis for the construction of the basic representation of LG by 'vertex operators' (see Chapter 13). Let us choose representatives ε_λ in $\tilde{\Lambda}$ for the elements $\lambda \in \Lambda$, so that a general element of $\tilde{\Lambda}$ can be written $u\varepsilon_\lambda$, with $u \in \mathbb{T}$.

Proposition (4.8.1). *If G is a simply laced and simply connected group then the representatives ε_λ can be chosen so that the multiplication in $\tilde{\Lambda}$ is given by*

$$\varepsilon_\lambda \, . \, \varepsilon_\mu = (-1)^{b(\lambda,\mu)} \varepsilon_{\lambda+\mu},$$

where $b : \Lambda \times \Lambda \to \mathbb{Z}/2$ is any bilinear form such that

$$b(\lambda, \lambda) \equiv \tfrac{1}{2}\langle \lambda, \lambda \rangle \pmod 2,$$

and $\langle \ , \ \rangle$ is the form on $\mathfrak{g}$ which defines $\tilde{L}G$.

Remark. If $\langle \ , \ \rangle$ is the basic form on $\mathfrak{g}$ then this multiplication formula is very reminiscent of the bracket relations (2.5.1) for the generators of $\mathfrak{g}_{\mathbb{C}}$:

$$[\mathrm{e}_\lambda, \mathrm{e}_\mu] = (-1)^{b(\lambda,\mu)} \mathrm{e}_{\lambda+\mu}.$$

Proof. Because any extension of Λ by $\mathbb{T}$ which is abelian is trivial an extension is completely determined by its commutator map $(\lambda, \mu) \mapsto \varepsilon_\lambda \varepsilon_\mu \varepsilon_\lambda^{-1} \varepsilon_\mu^{-1}$, which is a skew biadditive map

$$\Lambda \times \Lambda \to \mathbb{T}.$$

Thus what we have to show is that $\varepsilon_\lambda \varepsilon_\mu \varepsilon_\lambda^{-1} \varepsilon_\mu^{-1} = (-1)^{\langle \lambda,\mu \rangle}$. If we choose paths p_λ and p_μ in LG from the identity to λ and μ, and use the description of $\tilde{L}G$ given in Section 4.4 then the task is to show that

$$\int_\sigma \omega \equiv \pi\langle \lambda, \mu \rangle \quad (\text{modulo } 2\pi), \tag{4.8.2}$$

where σ is any piece of surface bounded by the two paths $p_\lambda * \lambda.p_\mu$ and $p_\mu * \mu.p_\lambda$ from 1 to $\lambda + \mu$. It is enough to prove the formula when λ and μ are positive coroots, for the coroots span Λ. We also may as well assume that $\langle \ , \ \rangle$ is the basic inner product on $\mathfrak{g}$.

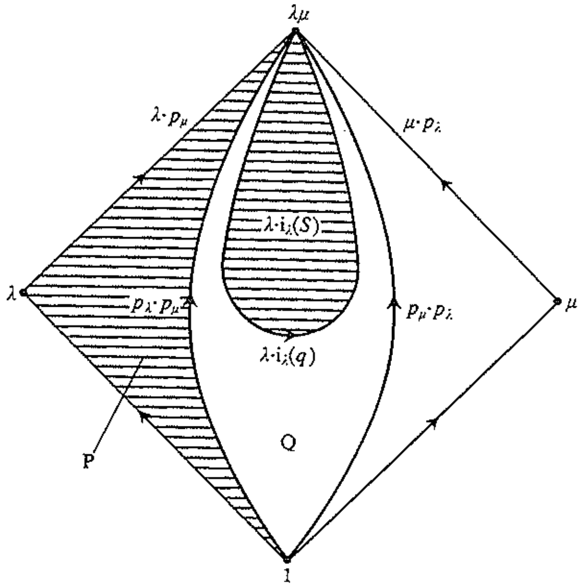

Fig. 1

If λ is a coroot there is a canonical homomorphism $i_\lambda : SU_2 \to G$ whose restriction to the diagonal matrices is λ. Let us define the path p_λ from 1 to λ as $i_\lambda \circ p$, where $p : [0, \pi] \to LSU_2$ is given by

$$p(t)(z) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \quad \text{for} \quad 0 \leq t \leq \frac{\pi}{2},$$

and

$$= \begin{pmatrix} -z \cos t & \sin t \\ -\sin t & -z^{-1} \cos t \end{pmatrix} \quad \text{for} \quad \frac{\pi}{2} \leq t \leq \pi.$$

Now consider the piece of surface in LG given by

$$P : \{(s, t) : 0 \leq s \leq t \leq \pi\} \to LG,$$

where $P(s, t) = p_\lambda(t) p_\mu(s)$. Observing that

$$p(t)^{-1} p'(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{when} \quad 0 \leq t \leq \frac{\pi}{2},$$

and

$$= \begin{pmatrix} 0 & -z^{-1} \\ z & 0 \end{pmatrix} \quad \text{when} \quad \frac{\pi}{2} \leq t \leq \pi,$$

we find that $P^*\omega = 0$. As P is bounded by the paths $p_\lambda * \lambda.p_\mu$ and $p_\lambda \cdot p_\mu$ we conclude that it is enough to integrate ω over a piece of surface bounded by $p_\lambda \cdot p_\mu$ and $p_\mu \cdot p_\lambda$. If $\langle \lambda, \mu \rangle = 0$ then $p_\lambda \cdot p_\mu = p_\mu \cdot p_\lambda$, and so the formula (4.8.2) holds in that case.

We next consider the surface

$$Q = \{(s, t) : 0 \leqslant s \leqslant t \leqslant \pi\} \to LG$$

given by $Q(s, t) = p_\lambda(s)p_\mu(t)p_\lambda(t-s)$. Once again a rather tedious check shows that $Q^*\omega = 0$. The surface Q is bounded by $p_\lambda \cdot p_\mu$, $p_\mu \cdot p_\lambda$, and the path

$$s \mapsto p_\lambda(s)p_\mu(\pi)p_\lambda(\pi - s).$$

But conjugation by

$$i_\mu\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$$

normalizes $i_\lambda(SU_2)$, and corresponds to the map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & bu^{\langle \lambda, \mu \rangle} \\ cu^{-\langle \lambda, \mu \rangle} & d \end{pmatrix}$$

on SU_2. So

$$p_\lambda(s)p_\mu(\pi)p_\lambda(\pi - s) = p_\mu(\pi)i_\lambda(q(s)),$$

where q is some loop in SU_2. The only interesting case is when $\langle \lambda, \mu \rangle = -1$. Then we find

$$q(s) = \begin{pmatrix} z\cos 2s & \sin 2s \\ -\sin 2s & z^{-1}\cos 2s \end{pmatrix} \quad \text{if} \quad 0 \leqslant s \leqslant \frac{\pi}{2},$$
$$= \begin{pmatrix} z\cos 2s & -z\sin 2s \\ z^{-1}\sin 2s & z^{-1}\cos 2s \end{pmatrix} \quad \text{if} \quad \frac{\pi}{2} \leqslant s \leqslant \pi.$$

Because $i_\lambda^*\omega$ is the standard 2-form ω on LSU_2 we are finally reduced to proving that the integral of ω over a piece of surface in LSU_2 bounded by the loop q is $\pm\pi$. A suitable surface is given by

$$S : \{(\xi, \eta, \zeta) : \xi^2 + \eta^2 + \zeta^2 = 1, \ \eta \geqslant 0\} \to LSU_2,$$

where

$$S(\xi, \eta, \zeta) = \begin{pmatrix} z(\xi + i\eta) & \zeta \\ -\zeta & z^{-1}(\xi - i\eta) \end{pmatrix} \quad \text{for} \quad \zeta \geqslant 0,$$
$$= \begin{pmatrix} z(\xi + i\eta) & -z\zeta \\ z^{-1}\zeta & z^{-1}(\xi - i\eta) \end{pmatrix} \quad \text{for} \quad \zeta \leqslant 0.$$

The calculation of the integral presents no problems.

We can now write down an explicit cocycle which describes the restriction of the extension $\tilde{L}G$ to LT. An element of LT is of the form $\exp f$, where $f : \mathbb{R} \to \mathfrak{t}$ is a map such that

$$\Delta_f = \frac{1}{2\pi}(f(\theta + 2\pi) - f(\theta))$$

is constant and belongs to the lattice Λ. We shall write $\hat{f} \in \mathfrak{t}$ for the average value of f on the interval $[0, 2\pi]$.

Proposition (4.8.3). *If G is a simply laced and simply connected group then the extension of LT induced by $\tilde{L}G$ is given by the cocycle c, where*

$$c(f, g) = (-1)^{b(\Delta_f, \Delta_g)}e^{iS(f,g)},$$

and

$$S(f, g) = \frac{1}{4\pi}\int_0^{2\pi} \langle f(\theta), g'(\theta) \rangle \, d\theta + \tfrac{1}{2}\langle \hat{f}, \Delta_g \rangle + \tfrac{1}{2}\langle \Delta_f, \hat{g} - g(0) \rangle.$$

The bilinear form b on Λ is as in (4.8.1).

Proof. We first observe that c is well-defined as a map $LT \times LT \to \mathbb{T}$. It is a cocycle because it is bimultiplicative, and so it does define some extension of LT.

Over the identity component of LT the desired extension is completely described by its Lie algebra cocycle—see the discussion of $\tilde{L}U_n$ in Section 4.7. The cocycle c clearly induces the correct cocycle on the Lie algebra. Furthermore c describes the correct extension of the lattice $\Lambda \subset LT$, in view of Proposition (4.8.1), although the coset representative ε_λ of (4.8.1) has now been replaced by $e^{\frac{1}{2}i\pi\langle \lambda, \lambda \rangle}\varepsilon_\lambda$. To complete the proof it is enough to check that c gives the correct adjoint action of Λ on the Lie algebra $\tilde{L}\mathfrak{t}$. The adjoint action was calculated in Proposition (4.3.2), and is easily seen to agree with that given by c.

Remark. The cocycle of (4.8.3) is not invariant under the action of $\mathrm{Diff}^+(S^1)$ on LT, and one cannot choose the coset representatives to make it so. If $\varepsilon(f) \in \tilde{L}T$ is a representative of $f : \mathbb{R} \to \mathfrak{t}$ chosen so that (4.8.3) holds then we have

$$\phi^*\varepsilon(f) = e^{\frac{1}{2}i\langle a(\phi, f), \Delta_f \rangle}\varepsilon(\phi^* f), \qquad (4.8.4)$$

where $\phi \in \mathrm{Diff}^+(S^1)$, and

$$a(\phi, f) = \frac{1}{4\pi}\int_0^{2\pi} (f(\phi(\theta)) - f(\theta)) \, d\theta.$$

In particular if ϕ is rotation through the angle α then

$$\phi^*\varepsilon(f) = e^{\frac{1}{2}i\alpha\langle \Delta_f, \Delta_f \rangle}\varepsilon(\phi^* f). \qquad (4.8.5)$$

Among all the cocycles describing the same extension $\tilde{L}T$ of LT the complicated-looking expression of (4.8.3) was chosen so as to make the formula (4.8.5) as simple as possible.

4.9 The inner product on $\mathbb{R} \oplus \tilde{L}\mathfrak{g}$

A nondegenerate invariant bilinear form $\langle\ ,\ \rangle$ on $\mathfrak{g}$ induces one on $L\mathfrak{g}$ by the formula

$$\langle \xi, \eta \rangle = \frac{1}{2\pi}\int_0^{2\pi} \langle \xi(\theta), \eta(\theta)\rangle \, d\theta. \tag{4.9.1}$$

There cannot, however, be a nondegenerate invariant form on the extended algebra $\tilde{L}\mathfrak{g}$—at least if $\mathfrak{g}$ is semisimple—because the centre of a Lie algebra is orthogonal to its commutator subalgebra with respect to any invariant form, and $[\tilde{L}\mathfrak{g}, \tilde{L}\mathfrak{g}] = \tilde{L}\mathfrak{g}$. In view of this it is often useful to notice that the Lie algebra of the semidirect product $\mathbb{T} \tilde{\times} \tilde{L}G$—where $\mathbb{T}$ is the group of rigid rotations of S^1, which acts naturally on $\tilde{L}G$—*does* possess a canonical nondegenerate invariant bilinear form whenever the form $\langle\ ,\ \rangle$ on $\mathfrak{g}$ which defines the extension $\tilde{L}G$ is nondegenerate.

If we identify the Lie algebra $\tilde{L}\mathfrak{g}$ with $L\mathfrak{g} \oplus \mathbb{R}$ as in Section 4.2, and identify the Lie algebra of $\mathbb{T}$ with $\mathbb{R}$ by $a \leftrightarrow a\frac{d}{d\theta}$, then the Lie algebra of $\mathbb{T} \tilde{\times} \tilde{L}G$ is $\mathbb{R} \oplus L\mathfrak{g} \oplus \mathbb{R}$ with the bracket given by

$$[(x_1, \xi_1, y_1), (x_2, \xi_2, y_2)] = (0, [\xi_1, \xi_2] + x_1\xi_2' - x_2\xi_1', \langle \xi_1, \xi_2' \rangle). \tag{4.9.2}$$

Here $\langle \xi_1, \xi_2' \rangle$ is as in (4.9.1), i.e. it is the cocycle $\omega(\xi_1, \xi_2)$. We define a bilinear form on the algebra by

$$\langle (x_1, \xi_1, y_1), (x_2, \xi_2, y_2) \rangle = \langle \xi_1, \xi_2 \rangle - x_1y_2 - y_1x_2. \tag{4.9.3}$$

It is immediate that this form is invariant under the adjoint action of the Lie algebra on itself, and hence under the adjoint action of the identity component of the group $\tilde{L}G$. This has a very useful computational corollary, which cannot be obtained so simply by other means.

Proposition (4.9.4). *If γ belongs to the identity component of LG then the adjoint action of γ on $\mathbb{R} \oplus \tilde{L}\mathfrak{g}$ is given by*

$$\gamma \cdot (x, \xi, y) = (x, \gamma \cdot \xi - x\gamma'\gamma^{-1}, y - \langle \gamma^{-1}\gamma', \xi \rangle + \tfrac{1}{2}x\langle \gamma^{-1}\gamma', \gamma^{-1}\gamma' \rangle).$$

Proof. We know from (4.3.2) that the formula is true when $x = 0$. We also know a priori that the right-hand side must be of the form $(x, \ldots, \ldots)$. The formula above is then easily seen to be the only possible one which preserves the bilinear form (4.9.3).

In practice we are more often interested in the coadjoint action of γ on the dual space

$$\mathbb{R} \oplus (\tilde{L}\mathfrak{g})^* \cong \mathbb{R} \oplus (L\mathfrak{g})^* \oplus \mathbb{R}.$$

The most important case is when γ is the homomorphism $\theta \mapsto \exp(\theta\eta)$ defined by $\eta \in \check{T} \subset \mathfrak{t}$, which represents an element of the translation part of the affine Weyl group.

Proposition (4.9.5). *The element $\eta \in \check{T} \subset W_{\text{aff}}$ acts on $\mathbb{R} \oplus \mathfrak{t}^* \oplus \mathbb{R}$ by*

$$(n, \lambda, h) \mapsto (n + \langle \lambda, \eta \rangle + \tfrac{1}{2}h\|\eta\|^2, \lambda + h\eta, h),$$

where we have identified $\mathfrak{t}$ and $\mathfrak{t}^*$ by using the inner product.

Note. When γ does not belong to the identity component of LG its action on $\mathbb{R} \oplus \tilde{L}\mathfrak{g}$ involves choices, as we saw in Section 4.7. In that case the action does not necessarily preserve the inner product. Thus for $\tilde{L}U_n$, if we identify $\check{T}$ with $\mathbb{Z}^n$ in the obvious way, the formula corresponding to (4.9.5) is

$$(n, \lambda, h) \mapsto (n + \langle \lambda, \eta \rangle + \tfrac{1}{2}h\Sigma\eta_i(\eta_i - 1), \lambda + h\eta, h), \tag{4.9.6}$$

which is not orthogonal. The proof of (4.9.6) will be given later as Proposition (6.8.7).

4.10 Extensions of Map(X; G)

For a general compact manifold X the group $\text{Map}(X; G)$ is neither connected nor simply connected. (If $G = U_n$ and $n > \frac{1}{2}\dim X$ then the group of components of $\text{Map}(X; G)$ is the group $K^{-1}(X)$ of Atiyah and Hirzebruch [3], and its fundamental group is $\tilde{K}^0(X)$.) Let $m(X; G)$ denote the simply connected covering group of $\text{Map}(X; G)$, which one should think of as formed from $\text{Map}(X; G)$ by killing successively π_0 and π_1.

Proposition (4.10.1).

(i) *If G is simply connected then there is a canonical central extension $\tilde{m}(X; G)$ of $m(X; G)$ such that $\pi_2(\tilde{m}(X; G)) = 0$.*

(ii) *The group $\tilde{m}(X; G)$ is the universal central extension of $m(X; G)$, and its Lie algebra is the universal central extension of $Map(X; \mathfrak{g})$ which was described in Section 4.2.*

(iii) *If G is simple then the kernel of the extension is $\Omega^1(X)/\Omega^1_{\mathbb{Z}}(X)$, the space of 1-forms on X modulo the 1-forms which have integral periods.*

Proof. We may as well suppose that G is simple. Let A denote the desired kernel $\Omega^1(X)/\Omega^1_{\mathbb{Z}}(X)$. We observe that the Lie algebra of A is the vector space $\Omega^1(X)/d\Omega^0(X)$ of (4.2.8). Using (4.4.2) it is enough for us to give a closed 2-form ω on $\tilde{m}(X; G)$ with values in $\Omega^1(X)/d\Omega^0(X)$ such that the integral of ω over every 2-cycle in $\tilde{m}(X; G)$ belongs to $\Omega^1_{\mathbb{Z}}(X)$. The 2-cocycle ω of $\text{Map}(X; \mathfrak{g})$ defined (see (4.2.9)) by $\omega(\xi, \eta) = \langle \xi, d\eta \rangle$ provides such a form on $\tilde{m}(X; G)$. The integrality condition holds because for any smooth map $\gamma: S^1 \to X$ the real-valued 2-form $\int_\gamma \omega$ on $\tilde{m}(X; G)$ is the pull-back of an integral form on LG.

The universality of $\tilde{m}(X; G)$, and the vanishing of π_2, are proved exactly as for loop groups.

The Mickelsson–Faddeev extension

Apart from its central extensions there is another—in principle more elementary—extension of Map$(X; G)$ which has recently aroused interest in quantum field theory. It was introduced by Mickelsson [112], Faddeev [41], and others. (Cf. also Singer [138] and Zumino [157].)

Let us first observe that if Γ is any Lie group and $c \in H^2(\Gamma; \mathbb{Z})$ is any cohomology class which is invariant under left-translation by elements of Γ—this is automatically the case if Γ is connected—then one can find a unique smooth circle bundle Y on Γ whose Chern class is c. Associated to Y there is an extension $\tilde{\Gamma}$ of Γ by the abelian group of smooth maps $\Gamma \to \mathbb{T}$. An element of $\tilde{\Gamma}$ is a smooth $\mathbb{T}$-equivariant map $\tilde{\gamma}: Y \to Y$ which covers the left-translation $\gamma: \Gamma \to \Gamma$ by some $\gamma \in \Gamma$. If ω is a closed 2-form on Γ representing the class c then the Lie algebra extension corresponding to $\tilde{\Gamma}$ is defined by the cocycle

$$(\xi, \eta) \mapsto \omega(\xi, \eta)$$

with values in the vector space of smooth real-valued functions on Γ. (Here elements ξ, η of the Lie algebra of Γ are regarded as left-invariant vector fields on Γ.) The extension of Γ so defined is not a central extension: Γ acts in the natural way on the kernel Map$(\Gamma; \mathbb{T})$.

When Γ is Map$(X; G)$ for some n-dimensional manifold X we can obtain an extension of this form by choosing any element of $H^{n+2}(G; \mathbb{Z})$, pulling it back to $X \times \text{Map}(X; G)$, and then integrating over X.

Now suppose that P is any principal Γ-bundle. For any left-invariant $c \in H^2(\Gamma; \mathbb{Z})$ we can find a 2-form ω on P whose restriction to each fibre of P is closed and represents the class c. The preceding discussion can be generalized to give

Proposition (4.10.2). *Suppose that Γ is connected and simply connected. Then there is an extension of Γ by Map$(P; \mathbb{T})$ naturally associated to c. The extension of Lie algebras is defined by $(\xi, \eta) \mapsto \omega(\xi_P, \eta_P)$, where ξ_P and η_P are the vector fields on P associated to ξ, η in the Lie algebra of Γ.*

Proof. Following the method of Section 4.4, it is enough for us to define a map

$$C: (\text{loops in } \Gamma) \to \text{Map}(P; \mathbb{T})$$

which has the properties (H1) and (H2) of Section 4.4 and is equivariant with respect to Γ. Given a loop ℓ in Γ and a point $p \in P$ we define $C_p(\ell) \in \mathbb{T}$ as $\exp(\mathrm{i} \int_\sigma \omega)$, where σ is any piece of surface in the fibre $\Gamma . p$ whose boundary is the loop $\ell . p$.

Remarks

(i) The hypothesis that Γ is simply connected in (4.10.2) is unnecessary if c is the transgression of an element of $H^3(B\Gamma; \mathbb{Z})$, where $B\Gamma$ is the classifying space of Γ.

(ii) If P is connected and simply connected then the kernel Map$(P; \mathbb{T})$ has the homotopy type of a circle, and homotopically the extension is simply the circle bundle corresponding to $c \in H^2(\Gamma; \mathbb{Z})$.

The case of interest in quantum field theory is when P is the contractible space of connections in a principal G-bundle on an orientable 3-manifold X, and $\Gamma = \text{Map}(X; G)$. One constructs $c \in H^2(\Gamma; \mathbb{Z})$ by starting from the element of $H^5(G; \mathbb{Z})$ defined by an invariant trilinear form F on $\mathfrak{g}$. If the G-bundle on X is trivial, so that $P = \Omega^1(X; \mathfrak{g})$, then the Lie algebra cocycle associates to $\xi, \eta \in \text{Map}(X; \mathfrak{g})$ the function

$$A \mapsto \int_X F(A, \mathrm{d}\xi, \mathrm{d}\eta)$$

on P.

4.11 Appendix: The cohomology of LG and its Lie algebra

For a compact group G the cohomology $H^*(G; \mathbb{R})$ with real coefficients can be calculated by de Rham's theorem. If $g . \omega$ denotes the left-translate of a closed form ω on G by $g \in G$ then the averaged form $\int_G g . \omega \, \mathrm{d}g$ represents the same cohomology class as ω, for a cohomology class is unchanged by translation. It follows that the cohomology can be calculated from the cochain complex of left-invariant forms on G, i.e. from the cochain complex of the Lie algebra $\mathfrak{g}$. In other words we have

$$H^*(G; \mathbb{R}) \cong H^*(\mathfrak{g}; \mathbb{R}).$$

It is well known that this cohomology is an exterior algebra on ℓ odd-dimensional generators, where ℓ is the rank of G. The generators correspond to the generators of the algebra of invariant polynomial functions on $\mathfrak{g}$ (which themselves form a polynomial algebra on ℓ generators ([20] Chapter 5, Section 5.3)) in the following way. If P is a polynomial of degree k, regarded as a symmetric multilinear function

$$\mathfrak{g} \times \ldots \times \mathfrak{g} \to \mathbb{R},$$

then one can define a skew multilinear function S of $2k-1$ variables by $S(\xi_1, \ldots, \xi_{2k-1}) =$

$$\sum_\pi (-1)^\pi P([\xi_{\pi_1}, \xi_{\pi_2}], [\xi_{\pi_3}, \xi_{\pi_4}], \ldots, [\xi_{\pi_{2k-3}}, \xi_{\pi_{2k-2}}], \xi_{\pi_{2k-1}}), \quad (4.11.1)$$

where the sum is over all permutations π of $\{1, 2, \ldots, 2k-1\}$.

If $G = U_n$ then the generators of the ring of invariant polynomials can be taken to be $P_1, P_2, \ldots, P_n$, where $P_k(A) = \text{trace}(A^k)$.

It is a fairly easy result of algebraic topology [16] that the cohomology $H^*(\Omega G; \mathbb{R})$ of the space of based loops on a simply connected group G is a polynomial algebra on the even dimensional classes obtained by transgressing the generators of $H^*(G; \mathbb{R})$, i.e. by pulling them back to $S^1 \times \Omega G$ by the evaluation map, and then integrating over S^1. The class so obtained from (4.11.1) is the $(2k-2)$-form on ΩG whose value at $\gamma \in \Omega G$ on tangent vectors represented by $\xi_1, \ldots, \xi_{2k-2} \in \Omega \mathfrak{g}$ is

$$\frac{1}{2\pi}\int_0^{2\pi} S(\xi_1(\theta), \xi_2(\theta), \ldots, \xi_{2k-2}(\theta), \gamma(\theta)^{-1}\gamma'(\theta))\, d\theta. \quad (4.11.2)$$

This form is naturally defined on LG. The cohomology $H^*(LG)$ is simply the tensor product $H^*(G) \otimes H^*(\Omega G)$, because $LG \cong G \times \Omega G$ as a space.

The differential form (4.11.2) is evidently not left-invariant, and we have no reason to expect that the cohomology of LG can be represented by left-invariant forms. Nevertheless we have

Proposition (4.11.3). *The $(2k-2)$-form (4.11.2) on LG is cohomologous to a rational multiple of the left-invariant form obtained by making skew the map*

$$(\xi_1, \ldots, \xi_{2k-2}) \mapsto \frac{1}{2\pi}\int_0^{2\pi} P([\xi_1, \xi_2], \ldots, [\xi_{2k-5}, \xi_{2k-4}], \xi_{2k-3}, \xi'_{2k-2})\, d\theta.$$

Corollary (4.11.4). *The natural map*

$$H^*(L\mathfrak{g}; \mathbb{R}) \to H^*(LG; \mathbb{R})$$

is surjective.

Remarks. Actually the map of (4.11.4) is an isomorphism. We shall prove that in Section 14.6. (Cf. also Kumar [97].) The result should be contrasted with our discovery in Section 4.2 that $H^2(\text{Map}(X; \mathfrak{g}))$ is vastly larger that $H^2(\text{Map}(X; G))$ when $\dim(X) > 1$. Quillen has pointed out to us that the class in $H^{2k-d-1}(\text{Map}(X; G))$ which is obtained by pulling back the class (4.11.1) by the evaluation map $X \times \text{Map}(X; G) \to G$ and integrating it over a cycle of dimension d in X can be represented by a left-invariant form if $k > d$, but usually not otherwise.

Proof of (4.11.3). Let us introduce some more convenient notation, as follows. When we pull back the Maurer–Cartan 1-form $g^{-1}\,dg$ on G (with values in $\mathfrak{g}$) by the evaluation map $S^1 \times LG \to G$ we shall write the resulting form as $\xi + \eta$, where ξ vanishes on tangent vectors in the S^1-direction and η vanishes along LG. (Thus η is $\gamma(\theta)^{-1}\gamma'(\theta)\, d\theta$ at $(\theta, \gamma) \in S^1 \times LG$.) In this notation the forms of (4.11.2) and (4.11.3) are

obtained (up to rational multiples) by integrating over S^1 the forms

$$\Theta = P([\xi, \xi], \ldots, [\xi, \xi], \eta)$$

and

$$\Phi = P([\xi, \xi], \ldots, [\xi, \xi], \xi, d'\xi),$$

respectively on $S^1 \times LG$. (We write d′ and d″ for differentiation of forms in the S^1 and LG directions respectively.)

Because $d(g^{-1}\,dg) = -\frac{1}{2}[g^{-1}\,dg, g^{-1}\,dg]$ on G we find

$$d'\eta = -\tfrac{1}{2}[\eta, \eta],$$

$$d''\xi = -\tfrac{1}{2}[\xi, \xi],$$

and

$$d'\xi + d''\eta = -[\xi, \eta].$$

Now consider the form $\Psi = P([\xi, \xi], \ldots, [\xi, \xi], \xi, \eta)$ on $S^1 \times LG$. We have $d''[\xi, \xi] = 0$, so

$$d''\Psi = -\tfrac{1}{2}P([\xi, \xi], \ldots, [\xi, \xi], \eta) + P([\xi, \xi], \ldots, [\xi, \xi], \xi, d'\xi)$$
$$+ P([\xi, \xi], \ldots, [\xi, \xi], \xi, [\xi, \eta]).$$

Using the invariance of the polynomial P, and the fact that $[[\xi, \xi], \xi] = 0$ because of the Jacobi identity, the third term on the right-hand-side is equal to Θ, so that we have

$$d\Psi = \tfrac{1}{2}\Theta + \Phi.$$

Integrating this relation over S^1 gives the desired result.