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Loop Groups

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1

INTRODUCTION

A loop group LG is the group of parametrized loops in another group G, i.e. the group of maps from the circle S^1 into G. Its composition law comes from pointwise composition in G. As far as this book is concerned G will always be either a compact Lie group or its complexification. There are a number of points of view from which one might be led to study these groups.

From a purely mathematical standpoint it is natural to ask whether the rich and highly developed theory of finite dimensional Lie groups can be extended to include infinite dimensional groups. In that light the groups $\mathrm{Map}(X; G)$ of maps from a compact space X to a Lie group G are probably the most obvious examples of infinite dimensional Lie groups. Their behaviour is untypical in its simplicity. For example the exponential map $\mathfrak{g} \to G$ from the Lie algebra $\mathfrak{g}$ of G induces an exponential map $\mathrm{Map}(X; \mathfrak{g}) \to \mathrm{Map}(X; G)$ which is locally one-to-one and onto: that is not a property one can expect of infinite dimensional Lie groups in general.

In quantum field theory groups of the form $\mathrm{Map}(X; G)$—where X is usually physical space $\mathbb{R}^3$—arise in two slightly different ways, as gauge groups and as current groups. Loop groups therefore appear in the simplified model of quantum field theory in which space is taken to be one-dimensional, and also in 'string' models of elementary particles, where the particles are represented by one-dimensional extended objects. Some of the important facts in the representation theory of loop groups were first discovered by physicists, and in this book our whole approach to the subject has been coloured by quantum field theory.

Although it is intrinsically a very simple and natural group we know surprisingly little about $\mathrm{Map}(X; G)$—particularly about its representations—except when X is a circle. In that case the situation is enormously easier, and has been very fully worked out. Loop groups turn out to behave like compact Lie groups to a quite remarkable extent. The aim of this book is to give a general exposition of what is known about them, concentrating on the global and analytical aspects of the theory rather than the algebraic ones. This has not been the usual approach to the subject, and deserves some comment. Usually loop groups have been approached by way of their Lie algebras, which are (essentially) examples of what are called Kac–Moody algebras. These are the Lie algebras which can be described in terms of generators and relations in the same way as the finite dimensional semisimple algebras. The classical theory of

Cartan and Killing shows how to associate an algebra to a finite integer matrix satisfying certain conditions which include positive definiteness. Omitting the positivity condition leads to the Kac–Moody algebras. If one merely weakens 'positive–definite' to 'positive–semidefinite' one obtains a subclass of the Kac–Moody algebras which are usually called *affine* algebras. These are (up to a one-dimensional central extension) the Lie algebras of loop groups and of the twisted versions of loop groups which we shall describe presently. For Kac–Moody algebras which are not affine no description is known other than the one in terms of generators and relations, and it is a mystery in what contexts the corresponding groups may arise. They are certainly not of the form $\mathrm{Map}(X; G)$ with $\dim(X) > 1$. On the other hand when the theory of Kac–Moody algebras is developed algebraically it does not make very much difference—for many purposes at least—whether or not the algebra is affine.

A considerable stimulus was given to the theory of general Kac–Moody algebras by the discovery by Macdonald [107] in 1972 of a class of formal power series identities analogous to the Weyl denominator formula, but reducing in particular cases to such results as the Jacobi triple product identity. Kac pointed out that Macdonald's identities resulted immediately from a generalization of the Weyl character formula to Kac–Moody algebras, and the character formula has been much studied since then from various points of view. The characters turn out to be modular functions in a certain sense, though why that should be true remains another of the mysteries of the subject.

It is perfectly reasonable to conclude from all this that the groups we are studying are interesting not because they are loop groups but because they possess a certain very special combinatorial structure. From that point of view this book, which ignores all Kac–Moody algebras except the 'affine' ones, must seem rather perverse. The case for our approach is partly aesthetic. In studying loop groups we are pursuing geometry and analysis rather than algebra and combinatorics, and for some people the geometrical picture is more illuminating and attractive. On the other hand, while acknowledging that it would be very rash and optimistic to think that the theory of loop groups will tell us anything directly about more general groups of the form $\mathrm{Map}(X; G)$, it is plain that the methods and constructions we use are very basic ones and belong to the mainstream of mathematics, especially in connection with quantum field theory. It does not seem unreasonable to think they may find application elsewhere.

The first main feature of our approach is to think of loop groups as groups of operators in Hilbert space. We regard an element of LG as a multiplication operator in the Hilbert space $H = L^2(S^1; V)$ of L^2 functions on the circle with values in some finite dimensional representation V of G. We decompose H as $H_+ \oplus H_-$, where H_+ (resp. H_-) consists of the functions whose negative (resp. positive) Fourier coefficients vanish, and we study the way in which the multiplication operators behave with respect to this decomposition. The idea here comes from quantum field theory, where H is the space of solutions of a relativistic wave equation and H_+ and H_- are the solutions of positive and negative energy: we examine how an operator in H moves particles from states of positive energy to ones of negative energy and vice versa.

Our second main technique is the geometrical study of the 'fundamental homogeneous space' X of LG. This is LG/G, where G is regarded as the subgroup of LG consisting of constant loops. It can be identified with the space ΩG of *based* loops in G. It has two crucial properties. First it is a *complex manifold*, for it can be identified with the homogeneous space $LG_{\mathbb{C}}/L^+G_{\mathbb{C}}$, where $G_{\mathbb{C}}$ is the complexification of G and $L^+G_{\mathbb{C}}$ consists of the loops which are boundary values of holomorphic maps

$$\{z \in \mathbb{C} : |z| < 1\} \to G_{\mathbb{C}}.$$

The second property is that X has a stratification by complex manifolds of finite codimension, indexed by the conjugacy classes of homomorphisms $S^1 \to G$. This is precisely analogous to the subdivision of a Grassmann manifold into its Schubert cells, and also to the Bruhat decomposition of a complex semisimple group. Both properties together amount to a restatement of the Birkhoff factorization theorem of 1909, which asserts that a loop γ in $G_{\mathbb{C}}$ can be factorized $\gamma = \gamma_- . \lambda . \gamma_+$, where $\gamma_\pm$ are loops which extend holomorphically inside (resp. outside) the unit circle, and $\lambda : S^1 \to G$ is a homomorphism.

A more geometrical way of thinking of the stratification of $X = \Omega G$ is in terms of the *energy* function $\mathscr{E} : \Omega G \to \mathbb{R}$, defined by

$$\mathscr{E}(\gamma) = \frac{1}{4\pi}\int_0^{2\pi} \|\gamma'(\theta)\|^2 \, d\theta.$$

The critical points of $\mathscr{E}$ are the homomorphisms $\gamma : S^1 \to G$, and they fall into connected components according to their conjugacy class. The stratum corresponding to a conjugacy class consists of all loops which flow to it under the downwards gradient flow of $\mathscr{E}$.

Putting the two ideas together, we observe that because LG acts on the Hilbert space H it acts on the Grassmann manifold of closed subspaces of H. The orbit of the subspace H_+ under this action is a copy of the homogeneous space X. We shall constantly use this embedding of X as a subvariety of the Grassmannian.

The importance of the geometry of X in the theory of loop groups is partly as a tool in proving structural theorems such as the Birkhoff factorization theorem itself, but more fundamentally is because the irreducible representations of LG arise as spaces of holomorphic sections of line bundles on X. (Strictly speaking, we need here the closely related space $Y = LG/T$, where T is a maximal torus of G.)

The present book falls roughly into two halves, the first eight chapters being concerned with the groups and the rest with their representations. For an introduction to the representation theory we refer the reader to the beginning of Chapter 9. Indeed one can begin reading the book there, referring back to the earlier chapters only when they are needed: Chapter 8 in particular goes into rather more detail than many readers will find interesting.

We originally intended to devote part of the book to the applications of loop groups, but in the end we have not felt competent or energetic enough to do so. Loop groups arise in two dimensional quantum field theory, as has already been mentioned; more recently they have found extensive applications in connection with so-called 'completely integrable systems' of partial differential equations. The more combinatorial aspects of the theory arise in connection with the classification of certain kinds of singularities in algebraic geometry—a generalization of the classical correspondence between simple singularities and finite dimensional simple Lie algebras ([22], [140], [105]). There is the same kind of correspondence in the classification of systems of subspaces in linear algebra, generalizing Gabriel's theorem [53]. Finally, the character formula for representations of loop groups can be used as a fruitful source of combinatorial identities, as was originally pointed out by Macdonald [107].

Of all these applications only the first two, to quantum field theory and to differential equations, appear to involve the groups themselves rather than just their combinatorial presentation. The quantum field theory applications are hard to survey, as they consist of scattered instances of a number of different types. We refer the interested reader to Jacob [80], Dolan [38], Chau *et al.* [26], Witten [155]. The application to differential equations is more straightforward. It was first worked out by Sato [35], [126], although the ideas had been implicit in earlier writings on the subject (cf. especially Zakharov and Shabat [156]). There are now a number of accounts of the subject in the literature. For an expository account from the point of view of the present book we refer to [132]. The other applications are not really within our terms of reference, but we refer to Slodowy [141] for the classification of singularities, and to [85] for the generalizations of Gabriel's theory (sometimes called the theory of 'quivers'). The most extensive work in connection with combinatorial identities has been done by Lepowsky (cf. [43], [101], [103]).

Apart from applications in the strict sense we should mention one or two related matters. The central idea in the theory is the Birkhoff

factorization theorem, which was found in the course of classifying singularities of ordinary differential equations (see Section 8.2 below). The related Riemann–Hilbert problem of finding a meromorphic function with prescribed monodromies about prescribed poles has recently been given a constructive solution using the kinds of method described in this book—notably 'vertex operators'. (Cf. [127].) In a completely different direction there is the recent work of Frenkel, Lepowsky, and Meurman [50] in connection with the 'monster' group in which once again the central point is the use of vertex operators. And vertex operators are also responsible for the boson–fermion correspondence which is important in two-dimensional quantum field theory (cf. Coleman [29], Mandelstam [111], Frenkel [45], and also Section 10.7 below).

The aim of this book is expository, and we have made no attempt to give a historical account of the evolution of the ideas and theorems we describe. It would in any case be rather hard to do so objectively, for many of the ideas have been worked out by a number of people independently in somewhat different contexts, and have often remained for some time 'well known to experts' without being written down. The best course for us seems to be to list a fairly representative sample of the different approaches to the subject. The definitive treatment of the Lie algebra theory is to be found in Kac's book [86], which contains an extensive list of references. Another algebraic approach is due to Garland and Lepowsky [54], [55], [56]. Goodman and Wallach [64] have approached the subject from the point of view of Banach Lie groups. Frenkel's paper [47] treats the character theory in terms of Wiener integration on orbits. We should also mention the important papers [49], [87], [100], and [88].

The representation theory of loop groups is closely related to that of the group $\mathrm{Diff}(S^1)$ of diffeomorphisms of the circle, which acts as a group of automorphisms of every loop group. It turns out that $\mathrm{Diff}^+(S^1)$—the orientation preserving diffeomorphisms—acts projectively on all the representations of loop groups which we consider, and all the known representations of $\mathrm{Diff}^+(S^1)$ have been constructed in this way—cf. [131], [64], [61]. We shall not study $\mathrm{Diff}^+(S^1)$ systematically in this book, but we shall prove the crucial intertwining property.

We have already mentioned twisted loop groups. If α is an automorphism of G then the corresponding twisted loop group $L_{(\alpha)}G$ consists of the maps $\gamma : \mathbb{R} \to G$ such that

$$\gamma(\theta + 2\pi) = \alpha(\gamma(\theta))$$

for all θ. The theory of loop groups extends essentially without change to the twisted groups. We have made a few sporadic remarks about the twisted case, but for the most part it presents nothing new, and we have not pursued it.

Let us now outline the contents of the book systematically.

Chapter 2 is a survey of the results about finite dimensional Lie groups which we shall use. It is included simply to make the book more self-contained, and—we hope—more accessible to readers with a variety of different backgrounds.

Chapter 3 introduces infinite dimensional Lie groups, and considers what can be said about loop groups—and some related groups—from that point of view, without entering into their more specific and characteristic properties.

Chapter 4, on the other hand, is devoted to one of the most important and distinctive features of loop groups, the existence of a natural class of central extensions by the circle-group $\mathbb{T}$, or equivalently the fact that loop groups admit non-trivial projective representations. The extended loop groups play a bigger role in the theory than the loop groups themselves, and all the representations we shall construct are projective ones. In this chapter the extensions are constructed by differential-geometric methods. In fact we find all possible extensions of the group $\mathrm{Map}(X; G)$ for any compact manifold X, though the result shows that only the case $X = S^1$ is important. We notice that the extensions exist only for groups of *smooth* (rather than continuous) loops.

Chapter 5 is a brief account of the theory of the Lie algebras of loop groups, about which we have said only what is needed for our purposes. We do give the definition of a Kac–Moody algebra, and explain how loop groups provide examples of them; but we do not mention the question of their classification.

Chapter 6 considers loop groups as groups of operators in Hilbert space. We introduce the restricted general linear group $GL_{\mathrm{res}}(H)$ of a Hilbert space H with a polarization, i.e. a decomposition $H = H_+ \oplus H_-$. This group, which consists of the invertible operators in H whose off-diagonal blocks $H_\pm \to H_\mp$ are Hilbert–Schmidt, is central throughout the rest of the book. If $H_+ \oplus H_-$ is the space of positive and negative energy solutions of a relativistic wave equation then the elements of $GL_{\mathrm{res}}(H)$ are precisely the transformations of H which are 'implementable' on the corresponding Fock space. (Cf. Shale [136].)

The group $GL_{\mathrm{res}}(H)$ has a basic central extension by $\mathbb{C}^\times$ from which all the central extensions of loop groups are derived. Roughly speaking, the extension measures the extent to which the process of associating to an operator in H its $H_+ \to H_+$ component fails to be a homomorphism. We believe it becomes clear at this point that the direction to be followed if one wants to study the group $\mathrm{Map}(X; G)$ when $\dim(X) > 1$ is the theory of Connes [32]. We have said a little about that in Section 6.10.

Chapter 7 introduces the Grassmannian $\mathrm{Gr}(H)$ of a polarized Hilbert space, another fundamental concept for our approach. Its most important property is that one can define a determinant line bundle on it. This is a holomorphic bundle which is homogeneous under the central extension of $GL_{\mathrm{res}}(H)$: in fact it provides a definition of the central extension. The Grassmannian has a decomposition into Schubert varieties exactly like a finite dimensional Grassmannian: more precisely, it has a stratification by manifolds of finite codimension, and a dual decomposition into finite dimensional cells.

Chapter 8 is devoted to the geometry of the fundamental homogeneous space X of LG, and the theorems about the structure of LG which are obtained from it. Our basic tool, as we have said, is the embedding of X in the Grassmannian $\mathrm{Gr}(H)$, from which we see that X is a complex Kähler manifold. We derive the decomposition of X—and hence in particular the Birkhoff factorization theorems—from the Schubert decomposition of $\mathrm{Gr}(H)$. (We believe that this geometric point of view is more perspicuous than the usual treatment of Birkhoff's theorem by integral equations [63].) Our procedure works in the first instance for the loop group of U_n, and the general case is derived from that. Having obtained the decomposition by these methods we can then (in Section 8.9) consider the situation anew in the light of the gradient flow of the energy function. We show that the decomposition is precisely the Morse decomposition of X corresponding to the energy. In particular we obtain a very complete and appealing description of the energy flow (Proposition (8.9.8)).

In Section 8.10 we consider some of the properties X possesses simply as a complex manifold. Although it is an infinite dimensional homogeneous space it has many 'finite dimensional' properties. Not only is every holomorphic function on it constant on each connected component, but each connected component of the space of (based) maps from any compact complex manifold to X is of finite dimension. Our treatment here follows Atiyah [5].

Besides the homogeneous space $X = \Omega G$ of LG we shall see that there is a similar space X_M associated to any closed Riemann surface M. It has the homotopy type of the space of principal G-bundles on M.

The relationship between ΩG and the Grassmannian was first observed in 'scattering theory' in the sense of Lax and Phillips [99], as we explain in Section 8.12. It is also at the heart of the Bott periodicity theorem (see Section 8.8).

After Chapter 8 the remainder of the book is devoted to representation theory, and we refer to Chapter 9 for a survey of its contents. Our approach is on the one hand to imitate the finite dimensional Borel–Weil theory, and on the other hand to make use of the natural representations which are well-known from quantum field theory. The character formula is postponed until the last chapter, where we also show how the decomposition of X described in Chapter 8 leads directly to the Bernstein–Gelfand–Gelfand resolution of a representation of LG by Verma modules. This is the basis of all the more delicate algebraic and cohomological analysis of the representations into which we have refrained from entering.

This book has taken a long time to write. It began from lectures given by the second author at Berkeley at the beginning of 1982. A first draft of the book was written by the first author along the lines of the lectures, and was subsequently greatly expanded by the second author to produce the present work. Obviously we have profited greatly from the influence of many people, and we have tried to acknowledge our indebtedness at various places in the text. But we should like here to express our especial gratitude to Sir Michael Atiyah, who suggested the whole project and has constantly given us encouragement and advice, and to Daniel Quillen, who shaped our approach to the subject in 1978 by teaching us about the Grassmannian model of ΩG and its importance, and has been a continual influence since then.

We hope that some parts of the book, at least, will be of interest to readers who are not primarily concerned with loop groups. A few sections, such as the treatment of the spin representation, have indeed been written with that specifically in mind. We have tried within reason to make the sections of the book fairly independent, even at the cost of some repetition, in order to encourage readers to turn at once to the parts they find interesting. We have also deliberately written different sections with different levels of mathematical sophistication. We have tried hard to presuppose very little when dealing with the central parts of the theory.

PART I

2

FINITE DIMENSIONAL LIE GROUPS

The purpose of this chapter is to outline and assemble the basic facts about finite dimensional Lie groups which will be used in the course of the book, and to establish some notation and terminology. We make no attempt to provide proofs, referring the reader to any of a number of standard treatments, of which the nearest in spirit to our approach is probably Adams [1]. (Cf. also [20], [72], [76].) We have called attention especially, however to the complex homogeneous spaces and the Borel–Weil theorem—see Sections 2.8 and 2.9 below—as they are the basis of our approach to loop groups.

2.1 The Lie algebra

A Lie group G is a topological group which is locally like a Euclidean space $\mathbb{R}^n$. It is then automatically a differentiable manifold; in other words it can be covered by a collection of coordinate charts between which the transition functions are differentiable—in fact of class C^∞.

The tangent space $\mathfrak{g}$ to G at its identity element 1 is called the Lie algebra of G. For every vector ξ in $\mathfrak{g}$ there is a unique one-parameter subgroup $\{g_t\}_{t\in\mathbb{R}}$ in G which has ξ as its tangent vector

$$\left.\frac{\mathrm{d}g_t}{\mathrm{d}t}\right|_{t=0}$$

at the identity. The group element g_t is denoted by $\exp(t\xi)$; and the map $\xi \mapsto \exp(\xi)$ from $\mathfrak{g}$ to G is called the exponential map. It provides a one-to-one correspondence between a neighbourhood of 0 in $\mathfrak{g}$ and a neighbourhood of 1 in G, and is one of the preferred coordinate charts.

The group which will be central in this book is the unitary group U_n, which consists of all $n\times n$ complex matrices u such that $u^*u=1$. (Here u^* denotes the transpose of the complex conjugate of u.) The Lie algebra of U_n is the vector space $\mathfrak{u}_n$ of skew-Hermitian matrices (i.e. those such that $\xi^* = -\xi$), and the exponential map is given by ordinary exponentiation of matrices.

For any Lie group G there is an operation $\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ defined on $\mathfrak{g}$, denoted by $(\xi,\eta)\mapsto[\xi,\eta]$ and called the 'bracket'. It is defined by

$$[\xi,\eta] = \lim_{s,t\to 0}\frac{1}{st}\exp^{-1}\{\exp(s\xi)\exp(t\eta)\exp(-s\xi)\exp(-t\eta)\}.$$

We see that it measures, in some sense, the group's failure to be commutative. The bracket is bilinear, and has the properties

$$[\xi, \eta] = -[\eta, \xi]$$

and

$$[\xi, [\eta, \zeta]] + [\eta, [\zeta, \xi]] + [\zeta, [\xi, \eta]] = 0.$$

(The latter is called the 'Jacobi identity'.) In the case of U_n, or indeed of any other matrix group, it is easy to see that the bracket is given in terms of matrix multiplication by

$$[\xi, \eta] = \xi\eta - \eta\xi.$$

An alternative description of the Lie algebra $\mathfrak{g}$ is as the space of left-invariant vector fields on the group G. For a tangent vector to G at the identity element defines by left-translation a tangent vector at each point of G, and hence a smooth vector field. Conversely, a left-invariant vector field is completely determined by its value at $1 \in G$. From this point of view the bracket operation $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is the usual bracket of vector fields: if ξ and η are expressed in some local coordinate chart as

$$\xi = \sum_i \xi_i \frac{\partial}{\partial x_i} \quad \text{and} \quad \eta = \sum_i \eta_i \frac{\partial}{\partial x_i},$$

then in the same chart $[\xi, \eta]$ is given by

$$\sum_{i,j} \left(\xi_j \frac{\partial \eta_i}{\partial x_j} - \eta_j \frac{\partial \xi_i}{\partial x_j} \right) \frac{\partial}{\partial x_i}.$$

In other words if ξ and η are regarded as differentiation operators acting on smooth functions on G then $[\xi, \eta] = \xi\eta - \eta\xi$.

If the group G is connected then the Lie algebra $\mathfrak{g}$, together with its bracket operation, determines G completely, up to the possibility of replacing it by a locally isomorphic group. The group U_n, for example, has the same Lie algebra as its simply connected covering group $\tilde{U}_n$, which is the subgroup of the product $U_n \times \mathbb{R}$ consisting of pairs (u, t) such that $\det(u) = e^{it}$. (In other words, an element of $\tilde{U}_n$ is an element of U_n together with a choice of the logarithm of its determinant.) For any finite dimensional Lie algebra $\mathfrak{g}$ there is always a simply connected group G whose Lie algebra is $\mathfrak{g}$, and any homomorphism of Lie algebras $\mathfrak{g} \to \mathfrak{g}'$, where $\mathfrak{g}'$ is the Lie algebra of a group G', arises from a homomorphism of groups $G \to G'$.

2.2 Complex groups

If a Lie group G is a complex manifold and the composition law $G \times G \to G$ is a holomorphic map then G is called a complex Lie group.

The most obvious example is $GL_n(\mathbb{C})$, the group of all invertible $n \times n$ complex matrices. The Lie algebra of a complex Lie group is a complex vector space, and the bracket $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is complex-bilinear. Conversely, such a complex Lie algebra always arises from a complex Lie group.

Any Lie algebra $\mathfrak{g}$ has a complexification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. If $\mathfrak{g}$ is the Lie algebra of a group G then a complex group $G_{\mathbb{C}}$ corresponding to $\mathfrak{g}_{\mathbb{C}}$ which contains G as a subgroup is called a *complexification* of G. Such a complexification need not exist.

Example. The group $SL_2(\mathbb{R})$ of real 2×2 matrices of determinant one has the complexification $SL_2(\mathbb{C})$. Now $SL_2(\mathbb{C})$ is simply connected, while the fundamental group of $SL_2(\mathbb{R})$ is $\mathbb{Z}$. (A loop in $SL_2(\mathbb{R})$ has a winding number which is the winding number of its first column, which is a non-zero vector in $\mathbb{R}^2$.) Thus there is a simply connected covering group G of $SL_2(\mathbb{R})$ such that the kernel of $G \to SL_2(\mathbb{R})$ is $\mathbb{Z}$. If G possessed a complexification it would necessarily be a covering group of $SL_2(\mathbb{C})$, which is impossible because $SL_2(\mathbb{C})$ is simply connected. We can express the same thing by saying that the kernel of any homomorphism from G to a complex group must contain the subgroup $\mathbb{Z}$. (In particular, G cannot possess any faithful finite dimensional representation $G \to GL_n(\mathbb{C})$.)

If G is a compact group, however, then it does possess a complexification $G_{\mathbb{C}}$. For G can be embedded in a unitary group U_n, and $G_{\mathbb{C}}$ can be realized as a subgroup of the complexification $GL_n(\mathbb{C})$ of U_n. This group $G_{\mathbb{C}}$ is unique up to isomorphism, and we shall always refer to it as *the* complexification of G. Thus the complexification of $\mathbb{T}$ will always mean $\mathbb{C}^\times$: other possible complexifications such as $\mathbb{C}/\mathbb{Z}^2 \cong \mathbb{T} \times \mathbb{T}$ cannot arise as complex subgroups of a general linear group.

2.3 Compact Lie groups

The most important fact about compact Lie groups is the Peter–Weyl theorem, which is essentially the assertion that any compact Lie group is isomorphic to a subgroup of some unitary group U_n. This is deduced fairly easily from the most basic property of compact groups, the existence of Haar measure, a probability measure on the group which is invariant under both left and right translations.

A more obvious application of the existence of Haar measure is to show that when a compact group G acts linearly on a finite dimensional vector space V there is always a positive definite inner product on V which is invariant under G. (One can take an arbitrary inner product on V and make it invariant by averaging it with respect to the action of G.) The existence of an invariant inner product, in turn, implies that V is the orthogonal direct sum of subspaces on which G acts irreducibly.

Let us apply the preceding remark when V is the Lie algebra $\mathfrak{g}$ of G. The group G acts on $\mathfrak{g}$ by the *adjoint representation*: the adjoint action of $g \in G$ is defined as the derivative of the map $x \mapsto gxg^{-1}$ at the identity element $x = 1$. We find

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k, \tag{2.3.1}$$

where G acts irreducibly on each $\mathfrak{g}_i$. It is immediate that the $\mathfrak{g}_i$ are sub-Lie-algebras, and that $[\mathfrak{g}_i, \mathfrak{g}_j] = 0$ when $i \neq j$. If G_i is the subgroup of G corresponding to $\mathfrak{g}_i$ then G is locally isomorphic to the product $G_1 \times \ldots \times G_k$.

The groups G_i into which we have decomposed G clearly have no non-trivial connected normal subgroups. Apart from the circle group $\mathbb{T}$ such groups are usually called *simple* groups—although the terminology is not ideal as the groups can possess finite normal subgroups (necessarily contained in the centre). Thus any compact Lie group is locally isomorphic to a product of circles and simple groups. If there are no circles in the decomposition then the group is called *semisimple*.

The simple compact groups have been classified. They consist of the special unitary groups SU_n, the special orthogonal groups SO_n, the symplectic groups Sp_n, and five exceptional groups—and, of course, groups locally isomorphic to these. The traditional notation for the Lie algebras is: $\mathfrak{su}_n = A_{n-1}$, $\mathfrak{so}_{2n+1} = B_n$, $\mathfrak{sp}_n = C_n$, and $\mathfrak{so}_{2n} = D_n$. The exceptional groups are called G_2, F_4, E_6, E_7, and E_8. In all cases the subscript gives the rank (see Section 2.4).

In Chapter Four we shall need to use the following simple result whose proof can conveniently be given here.

Proposition (2.3.2). *If $\mathfrak{g}$ is the Lie algebra of a compact semisimple group G then any $\mathbb{C}$-bilinear G-invariant map*

$$B : \mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \to \mathbb{C}$$

is necessarily symmetric.

Proof. Such a map B can be equivalently regarded as a $\mathbb{C}$-linear map $\tilde{B}$ from $\mathfrak{g}_{\mathbb{C}}$ to its dual $\mathfrak{g}_{\mathbb{C}}^*$ which commutes with the action of G.

First suppose G is simple. Then $\mathfrak{g}_{\mathbb{C}}$ is irreducible as a complex representation of G, so by Schur's lemma ([1] 3.22) any two choices of $\tilde{B}$ differ only by multiplication by a complex number. On the other hand there is a choice of $\tilde{B}$ which corresponds to a symmetric map B, for $\mathfrak{g}$ possesses an invariant inner product. It follows that any choice of B is symmetric.

In general we can decompose $\mathfrak{g}$ as in (2.3.1). The factors $\mathfrak{g}_i$ are obviously all non-isomorphic as representations of G, and the same

applies to their complexifications. So by Schur's lemma the map

$$\tilde{B} : \oplus\, \mathfrak{g}_{i,\mathbb{C}} \to \oplus\, \mathfrak{g}_{i,\mathbb{C}}^*$$

must be of the form $\oplus \tilde{B}_i$, where $\tilde{B}_i : \mathfrak{g}_{i,\mathbb{C}} \to \mathfrak{g}_{i,\mathbb{C}}^*$. By the preceding argument each $\tilde{B}_i$ must be symmetric, and so $\tilde{B}$ is symmetric.

2.4 The root system

In studying the structure of a compact connected Lie group G one begins by choosing a maximal torus T. Any maximal connected abelian subgroup of G is necessarily a torus, i.e. a product $\mathbb{T}^\ell$ of circles, and any two such subgroups are conjugate in G (cf. [1] 2.32, 4.22). The dimension ℓ of a maximal torus is called the *rank* of G. In the case of the unitary group U_n the diagonal matrices form a maximal torus $\mathbb{T}^n$. (In this case it is elementary that any two maximal tori are conjugate, as any set of commuting matrices can be simultaneously diagonalized. The general case, however, is less simple.)

The group G acts linearly on its Lie algebra $\mathfrak{g}$ by the adjoint representation; this action induces, of course, a complex-linear action of G on $\mathfrak{g}_{\mathbb{C}}$. The crucial step in analysing the structure of G is to decompose the vector space $\mathfrak{g}_{\mathbb{C}}$ under the action of the maximal torus T. Any complex representation of a compact abelian group such as T breaks up as a direct sum of one-dimensional representations on each of which T acts by means of a homomorphism

$$\alpha : T \to \mathbb{T} \subset \mathbb{C}^\times.$$

Of course T acts trivially on its own complexified Lie algebra $\mathfrak{t}_{\mathbb{C}}$, but no other vectors in $\mathfrak{g}_{\mathbb{C}}$ are left fixed by T because T is a maximal abelian subgroup. We can write

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_\alpha \mathfrak{g}_\alpha, \tag{2.4.1}$$

where $\mathfrak{g}_\alpha$ is the vector subspace of $\mathfrak{g}_{\mathbb{C}}$ on which T acts by the homomorphism $\alpha : T \to \mathbb{T}$. The homomorphisms α which occur in this decomposition are called the *roots* of G. They form a finite subset of the character group $\hat{T}$ of T. A homomorphism $\alpha : T \to \mathbb{T}$ is determined by its derivative $\dot{\alpha}$ at the identity, which lies in the vector space $\mathfrak{t}^*$ of linear maps $\mathfrak{t} \to \mathbb{R}$ and is such that

$$\alpha(\exp \xi) = e^{i\dot{\alpha}(\xi)}.$$

It is usual to think of $\hat{T}$ as a lattice in $\mathfrak{t}^*$, and to write it additively. In other words we shall usually not distinguish in notation between α and $\dot{\alpha}$. Notice that if α is a root then so is $-\alpha$, for $\mathfrak{g}_{-\alpha} = \bar{\mathfrak{g}}_\alpha$. The lattice $\hat{T}$ is called the lattice of *weights* of G.

In the case of the unitary group U_n the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ consists of all $n \times n$ complex matrices, and the roots α_{ij} are indexed by the ordered pairs (i, j) with $i \neq j$ and $1 \leq i, j \leq n$. The space $\mathfrak{g}_{\alpha_{ij}}$ consists of the matrices which are zero except in the $(i, j)^{\text{th}}$ place, and α_{ij} maps the diagonal matrix with entries $(u_1, \ldots, u_n)$ to $u_i u_j^{-1} \in \mathbb{T}$, and is identified with the linear map $(\xi_1, \ldots, \xi_n) \mapsto \xi_i - \xi_j$ on $\mathbb{R}^n$.

The centre of G is contained in every maximal torus, and is obviously the intersection of the kernels of all the roots $\alpha: T \to \mathbb{T}$. From this it follows that if G is semisimple and so has a finite centre then the roots span the vector space $\mathfrak{t}^*$.

It turns out—cf. [1] 5.5—that the subspaces $\mathfrak{g}_\alpha$ in (2.4.1) are always one-dimensional. This fact enables one to give a simple description of the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ in terms of generators and relations.

Let us choose for each root α a non-zero vector e_α in $\mathfrak{g}_\alpha$. We shall assume that $e_{-\alpha} = \bar{e}_\alpha$. It is easy to see that the bracket

$$h_\alpha = -i[e_\alpha, e_{-\alpha}]$$

belongs to $\mathfrak{t}$, and cannot be zero.

It follows that the three vectors $\{e_\alpha, e_{-\alpha}, h_\alpha\}$ span a sub-Lie-algebra of $\mathfrak{g}_{\mathbb{C}}$ which is isomorphic to the complexification of the Lie algebra of SU_2. If we normalize e_α so that the relations take the form

$$[h_\alpha, e_\alpha] = 2ie_\alpha$$
$$[h_\alpha, e_{-\alpha}] = -2ie_{-\alpha}$$
$$[e_\alpha, e_{-\alpha}] = ih_\alpha,$$

corresponding to those of the matrices

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad f = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \quad h = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

then h_α is canonically determined by α, and $2\pi h_\alpha$ belongs to the kernel of the exponential map $\exp: \mathfrak{t} \to T$.

The element h_α is called the *coroot* corresponding to α. For any root $\beta: \mathfrak{t} \to \mathbb{R}$ the number $\beta(h_\alpha)$ is an integer (because $\exp(2\pi h_\alpha) = 1$), and $\alpha(h_\alpha) = 2$. The coroot h_α defines a homomorphism $\eta_\alpha: \mathbb{T} \to T$ by

$$\eta_\alpha(e^{i\theta}) = \exp(\theta h_\alpha); \tag{2.4.2}$$

this extends canonically to a homomorphism $i_\alpha: SU_2 \to G$. We shall usually think of the lattice $\check{T}$ of all homomorphisms $\mathbb{T} \to T$ as contained in the vector space $\mathfrak{t}$, just as we regard the lattice $\hat{T} = \mathrm{Hom}(T; \mathbb{T})$ as contained in $\mathfrak{t}^*$; in other words we shall usually not distinguish between the coroot h_α and the homomorphism η_α. The lattice $\check{T}$ is canonically dual to $\hat{T}$ over the integers by the composition

$$\check{T} \times \hat{T} \to \mathrm{Hom}(\mathbb{T}; \mathbb{T}) = \mathbb{Z}.$$

It is an important fact that for a simply connected group G the coroots h_α generate the lattice $\check{T}$. ([1] 5.47)

The Lie algebra $\mathfrak{g}_{\mathbb{C}}$ is clearly generated additively by the root vectors e_α together with the elements of $\mathfrak{t}$, and the relations are necessarily of the form

$$\begin{aligned} [e_\alpha, e_\beta] &= n_{\alpha\beta} e_{\alpha+\beta} && \text{if } \alpha + \beta \text{ is a root,} \\ &= ih_\alpha && \text{if } \alpha + \beta = 0, \\ &= 0 && \text{otherwise;} \\ [h, e_\alpha] &= i\alpha(h) e_\alpha. \end{aligned} \tag{2.4.3}$$

So far the elements e_α have been fixed only up to multiplication by complex numbers of modulus one. It turns out that they can be chosen so that the numbers $n_{\alpha\beta}$ are integers—in fact all $n_{\alpha\beta}$ are non-zero. Cf. [20] Chapter 8 Section 2.4.

2.5 Simply laced groups

There is a class of groups for which the relations (2.4.3) take an especially simple form: they are the *simply laced* groups. A group G is called simply laced if there is a G-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ for which all the coroots h_α have the same length. In that case we shall normalize the inner product so that $\langle h_\alpha, h_\alpha \rangle = 2$. The resulting identification $\mathfrak{t} \cong \mathfrak{t}^*$ makes h_α correspond to α. The unitary group U_n and the orthogonal group SO_{2n} are simply laced: the preferred inner product is

$$\langle A, B \rangle = -\mathrm{trace}(AB)$$

for U_n and

$$\langle A, B \rangle = -\tfrac{1}{2}\,\mathrm{trace}(AB)$$

for SO_{2n}, when the Lie algebras are identified with the algebras of skew-Hermitian (resp. skew) matrices. The exceptional groups of type E are also simply laced. In general, a compact group is simply laced if its Lie algebra has no simple factors of types B, C, F, or G.

For a simply laced group the inner product on $\mathfrak{t}$ induces an inner product on $\mathfrak{t}^*$ which is integral on the lattice $\hat{T}$:

$$\langle\ \ \rangle: \hat{T} \times \hat{T} \to \mathbb{Z}.$$

Furthermore $\langle \lambda, \lambda \rangle$ is *even* for each $\lambda \in \hat{T}$. Let us choose a bilinear form

$$B: \hat{T} \times \hat{T} \to \mathbb{Z}/2$$

such that

$$B(\lambda, \lambda) \equiv \tfrac{1}{2}\langle \lambda, \lambda \rangle \pmod 2.$$

(This form B cannot be symmetric.) Then

(i) the roots of G are precisely the set of all vectors α in $\hat{T}$ such that $\langle \alpha, \alpha \rangle = 2$, and

(ii) the first relation in the set (2.4.3) can be taken to be

$$[e_\alpha, e_\beta] = (-1)^{B(\alpha,\beta)} e_{\alpha+\beta}. \tag{2.5.1}$$

It is easy to see that different choices of B amount simply to changing the signs of some of the e_α.

2.6 The Weyl group and the Weyl chambers: positive roots

For a compact Lie group G with maximal torus T the group of all automorphisms of T which are obtained by conjugating by elements of G is called the *Weyl group* W. Thus $W \cong N(T)/T$, where $N(T)$ is the normalizer of T in G. For the unitary group U_n the Weyl group is the symmetric group S_n, which acts on the diagonal matrices by permuting their entries.

The Weyl group is a finite group of isometries of $\mathfrak{t}$: it preserves the lattice $\hat{T}$, and permutes the set of roots in $\hat{T}$. For each root α it contains an element s_α of order two represented by $\exp \tfrac{1}{2}\pi(e_\alpha + e_{-\alpha})$ in $N(T)$. The action of s_α on $\mathfrak{t}$ is given by

$$s_\alpha(\xi) = \xi - \alpha(\xi) h_\alpha; \tag{2.6.1}$$

it is the reflection in the hyperplane H_α of $\mathfrak{t}$ whose equation is $\alpha(\xi) = 0$. The reflections s_α generate W.

The elements of $\mathfrak{t}$ which do not belong to any of the root hyperplanes H_α are called *regular*. They fall into a number of connected components, called *Weyl chambers*, which are permuted simply transitively by W. It is customary to select one of the chambers C and call it the *positive* Weyl chamber. Then the roots α are classified as positive or negative according as they take positive or negative values on C, and a positive root α is called *simple* if the hyperplane H_α is a wall of C. For a semisimple group of rank ℓ there are ℓ simple roots $\alpha_1, \ldots, \alpha_\ell$, and the 3ℓ elements e_{α_i}, $e_{-\alpha_i}$, h_{α_i} generate the Lie algebra $\mathfrak{g}_{\mathbb{C}}$.

In the case of U_n we can take the positive roots to be the α_{ij} with $i < j$, and the simple roots to be $\alpha_{i,i+1}$ for $1 \leqslant i < n$.

2.7 Irreducible representations and antidominant weights

Every irreducible representation of a compact group is finite dimensional. If G acts on a finite dimensional complex vector space V then one can find a basis $\{\varepsilon_i\}$ of V with respect to which the operation of the maximal torus T is diagonal. The torus then acts on ε_i by a homomorphism $\lambda_i : T \to \mathbb{T}$ called the *weight* of ε_i. The set of weights is a finite subset of $\hat{T}$ which is invariant under W.

If the representation V is irreducible then it possesses a unique basis vector ε_1 whose weight λ_1 is dominated by the other λ_i's: one says that λ dominates μ if $\lambda - \mu$ is positive on the positive chamber. The association of λ_1 to V defines a one-to-one correspondence between the equivalence classes of irreducible representations of G and the set $\hat{T}_-$ of antidominant weights; a weight $\lambda \in \hat{T}$ is called *antidominant* if λ is dominated by $w . \lambda$ for each $w \in W$, or equivalently (from (2.6.1)) if $\lambda(h_\alpha) \leqslant 0$ for each positive root α. We can identify $\hat{T}_-$ with the set $\hat{T}/W$ of orbits of W on $\hat{T}$.

It is with considerable hesitation that we have decided, with loop groups in mind, to describe representations in terms of *lowest* weights rather than highest weights as is usual. That has led us to use the unattractive compromise term 'antidominant'. Of course λ is antidominant if and only if $-\lambda$ is dominant in the usual sense.

One method of associating an irreducible representation V_λ to a weight $\lambda \in \hat{T}_-$ is described in Section 2.9.

2.8 Complex homogeneous spaces

Much of our study of loop groups will be based on the consideration of their complex homogeneous spaces. We shall outline here the main facts about the complex homogeneous spaces of compact groups, beginning with the unitary group U_n.

The complex algebraic homogeneous spaces for U_n are the Grassmannians and flag manifolds. For each ordered partition $\mathbf{k}$ of n, i.e. $\mathbf{k} = (k_1, k_2, \ldots, k_r)$ with $k_i > 0$ and $\Sigma k_i = n$, we define the flag manifold $Fl_{\mathbf{k}}$ as the space of r-tuples $\mathbf{E} = (E_1, \ldots, E_r)$ of subspaces of $\mathbb{C}^n$ such that $E_1 \subset E_2 \subset \ldots \subset E_r$ and $\dim(E_i) = k_1 + \ldots + k_i$. If $\mathbf{k} = (k, n-k)$ then $Fl_{\mathbf{k}}$ is the *Grassmannian* $\mathrm{Gr}_k(\mathbb{C}^n)$ of all k-dimensional subspaces of $\mathbb{C}^n$. If $\mathbf{k} = (1, 1, \ldots, 1)$ we shall write $Fl(\mathbb{C}^n)$ for $Fl_{\mathbf{k}}$.

The space $Fl_{\mathbf{k}}$ is a homogeneous space under the action of U_n, and the isotropy group of its natural base-point—the flag $\mathbf{E}$ such that E_i is the subspace $\mathbb{C}^{k_1+\cdots+k_i}$ spanned by the first $k_1 + \ldots + k_i$ vectors of the standard basis of $\mathbb{C}^n$—is $U_{\mathbf{k}} = U_{k_1} \times \ldots \times U_{k_r} \subset U_n$. So $Fl_{\mathbf{k}}$ can be identified with $U_n/U_{\mathbf{k}}$. On the other hand $Fl_{\mathbf{k}}$ is also a complex algebraic variety, and is a homogeneous space of the complex group $GL_n(\mathbb{C})$. Thus

$$Fl_{\mathbf{k}} \cong U_n/U_{\mathbf{k}} \cong GL_n(\mathbb{C})/P_{\mathbf{k}},$$

where $P_{\mathbf{k}}$ is the group of upper echelon matrices of type $\mathbf{k}$. In particular

$Fl(\mathbb{C}^n) \cong U_n/T \cong GL_n(\mathbb{C})/B^+$, where T is the standard torus of U_n and B^+ is the group of upper triangular matrices.

The spaces $Fl_{\mathbf{k}}$ are, up to isomorphism, the only homogeneous spaces of U_n which are complex algebraic varieties, and they are the only compact homogeneous spaces of $GL_n(\mathbb{C})$. The subgroups $P_{\mathbf{k}}$ are the only subgroups of $GL_n(\mathbb{C})$ which contain B^+. One of the important properties of $Fl_{\mathbf{k}}$ is that it possesses a canonical decomposition into complex cells, i.e. subspaces isomorphic to some $\mathbb{C}^r$. The cells are simply the orbits of B^+ on $Fl_{\mathbf{k}}$. Their closures are usually called 'Schubert varieties' ([116] §6, [68] p. 196). For example, $\mathrm{Gr}_k(\mathbb{C}^n)$ is the union of $\binom{n}{k}$ cells $C_{\mathbf{m}}$ indexed by the sequences $\mathbf{m} = (m_1, \ldots, m_k)$ such that $1 \leq m_1 < m_2 < \ldots < m_k \leq n$. In fact

$$C_{\mathbf{m}} = \{W \subset \mathbb{C}^n : \dim(W \cap \mathbb{C}^j) = i \quad \text{when} \quad m_i \leq j < m_{i+1}\}, \quad (2.8.1)$$

and it has dimension $\Sigma\,(m_i - i)$.

The situation just described has a precise analogue for any compact Lie group G. The subgroup of the complexification $G_{\mathbb{C}}$ which plays the role of the upper triangular matrices is the standard *Borel subgroup* B^+ whose Lie algebra is spanned by $\mathfrak{t}_{\mathbb{C}}$ and the root vectors e_α corresponding to the positive roots α. We have $B^+ \cap G = T$, and $G/T \cong G_{\mathbb{C}}/B^+$.

Proposition (2.8.2). *There is a one-to-one correspondence between*
(i) *complex algebraic homogeneous spaces for G,*
(ii) *compact Kähler homogeneous spaces for $G_{\mathbb{C}}$,*
(iii) *subgroups of $G_{\mathbb{C}}$ containing B^+, and*
(iv) *subsets of the set of simple roots of G.*

For the proof see Wang [152] and Serre [133], and also [20] Chapter 4 Section 2.5. To a subset A of the set of simple roots there corresponds the homogeneous space $G_{\mathbb{C}}/P_A$, where P_A is the subgroup of $G_{\mathbb{C}}$ whose Lie algebra is generated by the Lie algebra of B^+ and by the elements $e_{-\alpha}$ for $\alpha \in A$.

Subgroups of $G_{\mathbb{C}}$ which are conjugate to one of the P_A are called *parabolic*. We have $G_{\mathbb{C}}/P_A = G/(P_A \cap G)$, and each such space has a canonical decomposition into complex cells which are the orbits of B^+: this is called the *Bruhat decomposition* ([20] Chapter 6 §2, [72] Chapter 9 §1).

In the case of $G_{\mathbb{C}}/B^+ = G/T$ one can think of the Weyl group $W = N(T)/T$ as a subset of G/T, and there is exactly one element of W in each cell: in other words the cell decomposition of G/T is $\{C_w\}_{w \in W}$, where $C_w = B^+w$. The dimension of C_w is the *length* of w, which is defined as the number of positive roots α such that $w\,.\,\alpha$ is negative.

There is also a dual cell decomposition of G/T provided by the orbits of the opposite Borel subgroup B^-, which is the complex conjugate† of B^+. The cells B^+w and B^-w have complementary dimensions, and intersect transversally in the single point w.

When $G = SO_{2n}$ the space G/T has a description as a flag manifold like that for U_n/T. It consists of all flags $E_1 \subset E_2 \subset \ldots \subset E_n \subset \mathbb{C}^{2n}$ such that $\dim(E_i) = i$ and each E_i is *isotropic* for the standard bilinear form on $\mathbb{C}^{2n}$.

2.9 The Borel–Weil theorem

The importance of the complex homogeneous spaces of G arises from their role in constructing the irreducible representations. In fact only the largest one $G/T \cong G_{\mathbb{C}}/B^+$ is needed. Every homomorphism $\lambda: T \to \mathbb{T}$ extends uniquely to a holomorphic homomorphism $\lambda: B^+ \to \mathbb{C}^\times$. It therefore defines a homogeneous holomorphic line bundle $L_\lambda = G_{\mathbb{C}} \times_{B^+} \mathbb{C}_\lambda$ on $G_{\mathbb{C}}/B^+$. (The notation $G_{\mathbb{C}} \times_{B^+} \mathbb{C}_\lambda$ means the quotient of $G_{\mathbb{C}} \times \mathbb{C}$ by the equivalence relation which identifies (gb, ξ) with $(g, \lambda(b)\xi)$ for all $b \in B^+$.) The group $G_{\mathbb{C}}$ acts on the line bundle L_λ, and hence acts on its holomorphic cross-sections.

The Borel–Weil theorem (cf. Bott [15]) is

Theorem (2.9.1).
(i) *The line bundle L_λ has no non-zero holomorphic sections unless λ is an antidominant weight.*
(ii) *If λ is an antidominant weight then the space of holomorphic sections of L_λ is an irreducible representation of G with lowest weight λ.*

It may be worth explaining briefly why the space Γ_λ of holomorphic sections of L_λ is an irreducible representation. We first observe that if Γ_λ is expressed as a sum of irreducible representations then each component contains an element of lowest weight. Now an element of lowest weight is invariant under the subgroup N^- whose Lie algebra is spanned by the e_α with $\alpha < 0$ (for acting on it with such an e_α would give an element of lower weight). So it is enough to show that L_λ cannot have two linearly independent N^--invariant sections. But N^- acts on the base $G_{\mathbb{C}}/B^+$ with an open dense orbit—the orbit of the base-point. So if s_1 and s_2 were two N^--invariant sections their ratio would have to be constant on the open orbit, and so it would be constant on all of $G_{\mathbb{C}}/B^+$.

We should also mention the relation between holomorphic line bundles on a manifold X and holomorphic maps from X to complex projective space.

† 'Complex conjugation' here means the involution of $G_{\mathbb{C}}$ whose fixed points are G: thus for $GL_n(\mathbb{C})$ it means $A \mapsto (\bar{A}^t)^{-1}$.

In one direction, suppose we have a holomorphic map $f: X \to P(V)$, where $P(V)$ denotes the projective space of rays in a vector space V. Then we can define a holomorphic line bundle L_f on X whose fibre at x is the line $f(x)$ of V. Thus L_f is a subspace of $X \times V$, and there is a map $\pi: L_f \to V$ which is linear on each fibre. A linear form $\alpha: V \to \mathbb{C}$ therefore defines by composition with π a section of the dual line bundle L_f^*, whose fibre at x is the dual of the fibre of L_f at x, and so we have a linear map $V^* \to \Gamma(L_f^*)$, where $\Gamma(L_f^*)$ denotes the space of holomorphic sections of L_f^*.

In the other direction, suppose that L is a line bundle on X, and suppose that for each $x \in X$ there is a section of L which does not vanish at x. Then there is a canonical map $f_L: X \to P(\Gamma^*)$, where Γ is the space of sections of L. One defines $f_L(x)$ as the map $\Gamma \to \mathbb{C}$ given by evaluating sections at the point x—one must choose an identification of the fibre of L at x with $\mathbb{C}$, but the choice affects $f_L(x)$ only up to multiplication by an element of $\mathbb{C}^\times$.

In the light of this reinterpretation the proof of another part of the Borel–Weil theorem is almost obvious. To prove that an irreducible representation of $G_{\mathbb{C}}$ on V arises as the sections of a line bundle on $G_{\mathbb{C}}/B^+$ it is enough to show that there is a ray Ω in V^* which is stable under B^+. For then considering the orbit of Ω gives us an equivariant map $f: G_{\mathbb{C}}/B^+ \to P(V^*)$, and hence a line bundle L_f and a map $V \to \Gamma(L_f^*)$. Any vector of highest weight in V^* defines a ray stable under B^+.

Example. As an example of the Borel–Weil theorem let us consider the irreducible representation of U_n on the k^{th} exterior power $\Lambda^k(\mathbb{C}^n)$.

Perhaps the most obvious of all holomorphic line bundles is the determinant bundle Det on the Grassmannian $\mathrm{Gr}_k(V)$ of k-dimensional subspaces of a finite dimensional vector space V. This is the bundle whose fibre at a subspace $W \subset V$ is the top exterior power $\Lambda^k(W)$. It has no non-zero holomorphic sections, but its dual Det*, whose fibre at W is the dual line $\Lambda^k(W)^*$, does possess sections. The following well-known fact will be crucial for us in Chapter 10, so we shall give a proof of it here.

Proposition (2.9.2). *The space of holomorphic sections of* Det* *on* $\mathrm{Gr}_k(V)$ *is naturally isomorphic to* $\Lambda^k(V^*)$.

Proof. A holomorphic section of Det* is the same thing as a holomorphic map $s: \mathrm{Det} \to \mathbb{C}$ which is linear on each fibre. A typical point of Det can be represented in the form $\lambda v_1 \wedge \ldots \wedge v_k$, where $\lambda \in \mathbb{C}$ and $\{v_1, \ldots, v_k\}$ is a basis for some $W \in \mathrm{Gr}_k(V)$. We can therefore define a map

$$\Lambda^k(V^*) \to \Gamma(\mathrm{Det}^*) \tag{2.9.3}$$

by

$$\alpha_1 \wedge \ldots \wedge \alpha_k \mapsto \{\lambda v_1 \wedge \ldots \wedge v_k \mapsto \langle \alpha_1 \wedge \ldots \wedge \alpha_k, \lambda v_1 \wedge \ldots \wedge v_k \rangle\},$$

where $\langle \alpha_1 \wedge \ldots \wedge \alpha_k, v_1 \wedge \ldots \wedge v_k \rangle$ is the determinant of the matrix $(\langle \alpha_i, v_j \rangle)$.

It is clear that the map (2.9.3) is injective. To prove it is surjective, let U denote the open subspace of V^k consisting of k-tuples of linearly independent vectors. There is a natural map $\pi: U \to \mathrm{Det}$. If $s: \mathrm{Det} \to \mathbb{C}$ is a section of Det* then what we must show is that the composite $s \circ \pi$ extends to a multilinear map $V^k \to \mathbb{C}$. To prove that, consider $s(\pi(v_1, v_2, \ldots, v_k))$ as a function of v_1 for fixed $v_2, \ldots, v_k$. The resulting holomorphic function f is defined on the complement of the subspace $\langle v_2, \ldots, v_k \rangle$ of V, and satisfies $f(\lambda v_1) = \lambda f(v_1)$ for every $\lambda \in \mathbb{C}^\times$. The subspace $\langle v_2, \ldots, v_k \rangle$ has codimension greater than 1 (for we can assume that $k < n$), and so f extends to a holomorphic function on all of V by Hartogs's theorem ([68] page 7). We can then expand f in a Taylor's series at the origin, and because of the condition $f(\lambda v_1) = \lambda f(v_1)$ we find that f must be a linear function on V. Treating the other variables in the same way shows that $s \circ \pi$ is multilinear.