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Isocrystals with additional structure. II *

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Abstract. Let F be a p-adic field, let L be the completion of a maximal unramified extension of F, and let σ be the Frobenius automorphism of L over F. For any connected reductive group G over F one denotes by $B(G)$ the set of σ-conjugacy classes in $G(L)$ (elements x, y in $G(L)$ are said to be σ-conjugate if there exists g in $G(L)$ such that $g^{-1}\kappa\sigma(g) = y$. One of the main results of this paper is a concrete description of the set $B(G)$ (previously this was known only in the quasi-split case).

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Let F be a p-adic field, and let G be a connected reductive group over F. We write L for the completion of the maximal unramified extension F^{un} of F in some algebraic closure $\overline{F}$ of F. We write σ for the Frobenius automorphism of L over F; it induces an automorphism of $G(L)$ which we also denote by σ. We say that two elements x, y in $G(L)$ are σ-conjugate if there exists $g \in G(L)$ such that $g^{-1}x\sigma(g) = y$, and we write $B(G)$ for the set of σ-conjugacy classes in $G(L)$.

In case F is $\mathbb{Q}_p$ the set $B(G)$ can be identified with the set of isomorphism classes of isocrystals with G-structure. For example when G is GL_n, the set $B(G)$ can be identified with the set of isomorphism classes of n-dimensional isocrystals, a set that can be easily described using the classification (due to Dieudonné and Manin) of the simple objects in the category of isocrystals.

The set $B(G)$ turns up naturally when one studies Shimura varieties over finite fields [LR], [K5], and also plays a role in recent work of Rapoport and Zink [RZ] on period spaces for p-divisible groups and Shimura varieties over p-adic fields. Thus it is of interest to have a concrete description of $B(G)$ for any connected reductive group G.

For quasi-split groups such a description is given in [K]. The first step is to associate to any element $b \in G(L)$ a homomorphism $\nu\colon \mathbb{D} \to G$ over L, where $\mathbb{D}$ denotes the diagonalizable group over F with character group $\mathbb{Q}$. The conjugacy class of ν under $G(L)$ depends only on the class of b in $B(G)$, and this conjugacy class of homomorphisms is fixed by σ. Let B be a Borel subgroup (over F) in the quasi-split group G, let T be a maximal F-torus in B, and let A be the maximal F-split torus in T. Let $\mathfrak{A}$ denote the real vector space obtained by tensoring the

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cocharacter group of A with $\mathbb{R}$, and let $\overline{C}$ denote the closed Weyl chamber in $\mathfrak{A}$ determined by B. The homomorphism ν is conjugate under $G(L)$ to a unique element $\bar{\nu} \in \overline{C}$ (a homomorphism from $\mathbb{D}$ to A determines a point in the obvious $\mathbb{Q}$-subspace of $\mathfrak{A}$). Following [RR] we refer to the map $b \mapsto \bar{\nu}$ from $B(G)$ to $\overline{C}$ as the Newton map.

The Newton point $\bar{\nu}$ determines a parabolic subgroup $P = MN$ of G over F. We write $B(G)_P$ for the subset of $B(G)$ consisting of all elements for which the associated parabolic subgroup is equal to P. The first main result of [K] is a description of the subset $B(G)_G$ (elements in this subset are said to be basic). The second main result of [K] is a description of $B(G)_P$ in terms of basic elements in $B(M)$, where M is a Levi component for P (see 5.1 for a precise statement).

One of the main results of this paper is a description of $B(H)$ for any inner form H of the quasi-split group G. In fact it is best to introduce a set $B_s(G)$, which, loosely speaking, is the disjoint union of the sets $B(H)$ as H ranges through the inner forms of G (this point of view is suggested by work of Adams and Vogan [AV] on representations of inner forms of real groups). It turns out that $B_s(G)$ has a description (see 5.3) that is entirely analogous to the one for $B(G)$ given in [K].

We continue to let H denote an inner form of G. In Section 6 we introduce a subset $B(H,\mu)$ of $B(H)$. Here μ denotes a dominant coweight of the maximal torus T in the quasi-split group G. Pairs (H,μ) as above arise naturally in the study of Shimura varieties. Indeed, to get a (tower of) Shimura varieties one needs to start with a connected reductive group $H_{\mathbb{Q}}$ over $\mathbb{Q}$ and a miniscule coweight μ_0 of $H_{\mathbb{Q}}$ over $\mathbb{C}$. We assume that F is $\mathbb{Q}_p$ and that $H_{\mathbb{Q}}$ is a $\mathbb{Q}$-form of H. Let $E \subset \mathbb{C}$ be the Shimura field (the field of definition of the conjugacy class of μ_0). Fix an embedding ι of E in $\bar{\mathbb{Q}}_p$. Then there is a unique dominant coweight μ of T that is 'conjugate' to μ_0. Thus we obtain a pair (H,μ) as above with μ miniscule. Given the conjectural interpretation of our Shimura variety X as a moduli space of motives with H-structure, we expect the special fiber of any natural integral model of X to decompose as a disjoint union of pieces indexed by the set $B(H,\mu)$.

In Section 6 we find all pairs (H,μ) for which the set $B(H,\mu)$ has a unique element. For pairs (H,μ) arising from Shimura varieties it seems plausible that $B(H,\mu)$ has a unique element if and only if the Shimura variety admits p-adic uniformization at the place of E determined by ι. The results in Section 6 support Rapoport's idea [R] that p-adic uniformization occurs only in very special circumstances and always involves products of Drinfeld's spaces Ω^d.

The last main result of this paper is Proposition 13.4. It is too technical to discuss in this introduction, but it is probably worthwhile to mention that this proposition will be needed in order to prove that the transfer factors of [KS] for unramified cyclic base change (use the Frobenius element as generator for the cyclic Galois group) have the form given in Section 7 of [K3]. The point is that the transfer factor in [K3] involves the groups $B(T)$ while the one in [KS] involves hypercohomology groups. In order to compare the two it is necessary to introduce a hypercohomology variant $B(T \to U)$ of $B(T)$ and prove a number of results about it; this is done in Sections 7–13. In particular a duality theorem for $B(T \to U)$ is proved in Section 11, a valuation mapping from $B(T \to U)$ onto a finitely generated abelian group is defined in Section 12, and an important compatibility between the duality theorem and the valuation mapping is proved in 12.6 (this compatibility is needed to prove Proposition 13.4).

One last point deserves mention as well. Although the set $B(G)$ can be defined for any linear algebraic group G over F, it is not the 'right' set unless G is connected. For disconnected groups one should use instead the variant $\mathbf{B}(G)$ defined in 1.4. The first three sections of the paper develop the elementary properties of $\mathbf{B}(G)$ and also serve as a review of $B(G)$. Following Rapoport and Zink [RZ], in 3.3 we give a more natural definition of the group J appearing in [K] (the group J was introduced by Langlands in the appendix to [L]).

It is a pleasure to acknowledge the influence of M. Rapoport, with whom I have had many stimulating conversations on the material in Sections 1–6.

The following notation is used throughout this paper. We denote by $\mathrm{Int}(x)$ the inner automorphism $y \mapsto xyx^{-1}$. For an abelian group X we denote by $X_{\mathbb{R}}$ the group $X \otimes_{\mathbb{Z}} \mathbb{R}$. For a connected reductive group G we denote by G_{der} the derived group of G, by G_{sc} the simply connected cover of G_{der}, and by G_{ad} the adjoint group of G.

1. Preliminaries

1.1. The following notation will be used throughout this paper. Let p be a prime number and let F be a p-adic field (a finite extension of $\mathbb{Q}_p$). Let

$$\mathrm{val}\colon F^\times \to \mathbb{Z}$$

be the usual valuation on F, normalized so that uniformizing elements have valuation 1. Let $\mathfrak{o}$ denote the valuation ring of F, let $\mathfrak{p}$ denote its maximal ideal, let k denote the residue field $\mathfrak{o}/\mathfrak{p}$, and let q denote the number of elements in k.

Let $\overline{F}$ be an algebraic closure of F, let F^{un} denote the maximal unramified extension of F in $\overline{F}$, let L denote the completion of F^{un}, and let $\overline{L}$ be an algebraic closure of L containing $\overline{F}$. The Frobenius automorphism σ of F^{un} over F (which induces $x \mapsto x^q$ on the residue field of F^{un}) extends continuously to an automorphism (also denoted σ) of L over F. Let Γ denote the Galois group of $\overline{F}$ over F, and let W_F denote the Weil group of $\overline{F}$ over F (the subgroup of Γ consisting of all elements in Γ whose restriction to F^{un} is an integral power of σ). Let I_F denote the inertia subgroup $\mathrm{Gal}(\overline{F}/F^{\mathrm{un}})$ of Γ. We will often abbreviate W_F, I_F to W, I. Of course I is also a subgroup of W, and we regard W as a topological group in the usual way, by requiring that I, with the Krull topology, be an open subgroup of W. Thus we have an exact sequence of topological groups

$$1 \to I \to W \to \langle\sigma\rangle \to 1, \qquad (1.1.1)$$

where $\langle\sigma\rangle$ denotes the infinite cyclic group generated by σ (we give $\langle\sigma\rangle$ the discrete topology). It is not difficult to see that $\overline{L} = L \otimes_{F^{\mathrm{un}}} \overline{F}$. Thus I is also equal to $\mathrm{Gal}(\overline{L}/L)$, and we may regard elements of W (or even Γ) as automorphisms of $\overline{L}$ over F. Note that the fixed field of W in $\overline{L}$ is F. For any finite Galois extension K of F in $\overline{F}$ there is an exact sequence

$$1 \to W_K \to W_F \to \mathrm{Gal}(K/F) \to 1. \tag{1.1.2}$$

1.2. Let A be a group on which W acts. We assume that the W-group A is *discrete*, by which we mean that the stabilizer of any element of A is open in W. This is equivalent to the condition that the action map

$$W \times A \to A$$

be continuous when A is given the discrete topology. By a 1-cocycle of W in A we mean a continuous map $\tau \mapsto a_\tau$ from W to A (give A the discrete topology) satisfying the usual 1-cocycle condition

$$a_{\tau\rho} = a_\tau \tau(a_\rho) \quad \text{for all } \tau, \rho \in W.$$

Note that an abstract 1-cocycle a_τ is continuous if and only if there exists an open normal subgroup N of W such that $a_\tau = 1$ for all $\tau \in N$, in which case a_τ is the inflation to W of an (abstract) 1-cocycle of W/N in A^N. If a_τ is a 1-cocycle of W in A and b is an element of A, then $b^{-1}a_\tau\tau(b)$ is a 1-cocycle of W in A and is said to be cohomologous to a_τ. We define $H^1(W, A)$ to be the quotient of the set of 1-cocycles of W in A by the equivalence relation of being cohomologous. Then

$$H^1(W, A) = \varinjlim_{N} H^1(W/N, A^N),$$

where N runs over the directed set of open normal subgroups of W.

1.3. Let A be a W-subgroup of a discrete W-group B. Then there is an exact sequence of pointed sets

$$1 \to A^W \to B^W \to (B/A)^W \xrightarrow{\partial} H^1(W, A) \to H^1(W, B) \tag{1.1.3}$$

and if A is normal in B, this exact sequence can be prolonged by adding $H^1(W, B/A)$ at the right end. Of course the map ∂ sends $\bar{b} \in (B/A)^W$, represented by $b \in B$, to the class of the 1-cocycle $\tau \mapsto b^{-1}\tau(b)$.

1.4. Since the fixed field of W in $\overline{L}$ is F, the fixed point set of W in $X(\overline{L})$ is $X(F)$ for any scheme X over F. Let G be a linear algebraic group over F. Then $G(\overline{L})$ is a discrete W-group and $G(\overline{L})^W = G(F)$. We define a pointed set $\mathbf{B}(G)$ by

$$\mathbf{B}(G) := H^1(W, G(\overline{L})).$$

We define another pointed set as follows. Let $B(G)$ be the quotient of $G(L)$ by the equivalence relation σ-conjugacy (two elements $x, y \in G(L)$ are said to be

σ-conjugate if there exists $g \in G(L)$ such that $y = g^{-1}x\sigma(g)$). Clearly $B(G)$ can be identified with the pointed set

$$H^1(\langle\sigma\rangle, G(L)) = H^1(W/I, G(\overline{L})^I),$$

which can be identified (by inflation) with a subset of $\mathbf{B}(G)$; in this way we will always view $B(G)$ as a subset of $\mathbf{B}(G)$. Of course there is an exact sequence of pointed sets

$$1 \to B(G) \to \mathbf{B}(G) \to H^1(L, G) \tag{1.4.1}$$

(here, as always, we denote the Galois cohomology set $H^1(\mathrm{Gal}(\overline{L}/L), G(\overline{L}))$ by $H^1(L, G)$). If G is connected, then $H^1(L, G)$ is trivial [St], and the sets $B(G)$, $\mathbf{B}(G)$ are equal. For disconnected groups $B(G)$, $\mathbf{B}(G)$ need not coincide, and it is $\mathbf{B}(G)$ that is the more useful notion.

The inflation maps for the surjections $W \to \mathrm{Gal}(K/F)$ appearing in (1.1.2) yield injections

$$H^1(K/F, G(K)) \to \mathbf{B}(G)$$

for every finite Galois extension K of F in $\overline{F}$, and these fit together to give an injection

$$H^1(F, G) \hookrightarrow \mathbf{B}(G). \tag{1.4.2}$$

1.5. Let

$$1 \to G_1(F)G_2(F)G_3 \to 1$$

be an exact sequence of linear algebraic groups over F. Then

$$1 \to G_1(F) \to G_2(F) \to G_3(F) \to \mathbf{B}(G_1) \to \mathbf{B}(G_2) \to \mathbf{B}(G_3)$$

is an exact sequence of pointed sets. The group $G_3(F)$ acts on $\mathbf{B}(G_1)$ in the following way. Let $g_3 \in G_3(F)$ and let $\mathbf{g}_1 \in \mathbf{B}(G_1)$. Pick a 1-cocycle x_τ of W in $G_1(\overline{L})$ lying in the class $\mathbf{g}_1$, and pick an element $g_2 \in G_2(\overline{L})$ mapping to g_3 under $G_2 \to G_3$. Then the action

$$G_3(F) \times \mathbf{B}(G_1) \to \mathbf{B}(G_1)$$

sends the pair $(g_3, \mathbf{g}_1)$ to the class of the 1-cocycle $g_2 x_\tau \tau(g_2)^{-1}$. It is easy to see that the orbits of the action of $G_3(F)$ on $\mathbf{B}(G_1)$ coincide with the fibers of the map

$$\mathbf{B}(G_1) \to \mathbf{B}(G_2).$$

1.6. Let F' be a finite extension of F in $\overline{F}$. Let G be a linear algebraic group over F', and let RG denote the F-group obtained from G by Weil's restriction of scalars. Then there is a Shapiro bijection

$$\mathbf{B}(RG) \simeq \mathbf{B}(G). \tag{1.6.1}$$

2. σ-L-spaces

2.1. As in [K] we use the terminology *σ-L-space* to refer to a pair (V, Φ) consisting of a finite dimensional vector space V over L and a σ-linear bijection $\Phi\colon V \to V$ (thus $\Phi(\alpha v) = \sigma(\alpha)\Phi(v)$ for all $\alpha \in L$, $v \in V$). There is an obvious tensor product on the category *σ-L-spaces* of all such objects, and in fact *σ-L-spaces* is a Tannakian category over F.

Of course in the special case that F is $\mathbb{Q}_p$ the category *σ-L-spaces* is just the category of isocrystals. Just as for isocrystals the category *σ-L-spaces* is semi-simple, and there is a natural bijection from $\mathbb{Q}$ to the set of isomorphism classes of simple objects in *σ-L-spaces*. Thus every object in *σ-L-spaces* has an isotypic decomposition

$$V = \bigoplus_{r \in \mathbb{Q}} V_r,$$

and, as for isocrystals, we refer to V_r as the part of V having *slope* r. If V_1, V_2 are isotypic of slopes r_1, r_2 respectively, then $V_1 \otimes V_2$ is isotypic of slope $r_1 + r_2$. If V is a simple object of slope r, then its endomorphism ring is a central division algebra over F whose Hasse invariant is the element $-r$ of $\mathbb{Q}/\mathbb{Z}$. Note that in this paper we normalize the Hasse invariant in the same way that Serre does in the appendix to Section 1 of [S2]. This is also the normalization used in Section 2.6 of [K], so that for consistency the homomorphisms

$$\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$$

is Section 3 of [K] should all be replaced by their negatives (this inconsistency in [K] affects none of the results of that paper).

2.2. There is a second way to look at σ-L-spaces. By a *W_F-$\overline{L}$-space* we mean a finite dimensional $\overline{L}$-vector space V equipped with a semilinear action of the Weil group W_F for which V is a *discrete* W_F-module in the sense of 1.2 (semilinear means that $\tau(\alpha v) = \tau(\alpha)\tau(v)$ for all $\tau \in W_F$, $\alpha \in \overline{L}$, $v \in V$). The category *W_F-$\overline{L}$-spaces* of all such objects has an obvious tensor product. There is an obvious $\otimes$-functor $V \mapsto \overline{L} \otimes_L V$ from *σ-L-spaces* to *W_F-$\overline{L}$-spaces*, the action of W_F on $\overline{L} \otimes_L V$ being given by the formula

$$\tau(\alpha \otimes v) = \tau(\alpha) \otimes \Phi^j(v)$$

for all $\alpha \in \overline{L}$, $v \in V$ and $\tau \in W_F$ mapping to $\sigma^j \in \langle \sigma \rangle$. There is an obvious $\otimes$-functor $V \mapsto V^I$ (invariants of inertia) from *W_F-$\overline{L}$-spaces* to *σ-L-spaces*, and by the usual Galois descent theory for $\overline{L}/L$ this functor is quasi-inverse to the previous one. Thus both functors are $\otimes$-equivalences of $\otimes$-categories. We say that a simple object in **W_F-$\overline{L}$-spaces** has slope r if the corresponding simple object in *σ-L-spaces* has slope r.

Let F' be a finite extension of F in $\overline{F}$. Then $W_{F'}$ is an open subgroup of finite index in W_F, and any W_F-$\overline{L}$-space V can be viewed as a $W_{F'}$-$\overline{L}$-space by restricting the action of W_F to $W_{F'}$. If V is isotypic of slope r as W_F-$\overline{L}$-space, then it is isotypic of slope $r[F' : F]$ as $W_{F'}$-$\overline{L}$-space.

3. σ-L-spaces with G-structure

3.1. Now let G be a linear algebraic group over F and let g_τ be a 1-cocycle of W_F in $G(\overline{L})$ (see 1.2). For any representation

$$\rho\colon G \to GL(V)$$

of G on a finite dimensional vector space V over F we get a W_F-$\overline{L}$-space structure on $\overline{L} \otimes_F V$ by letting $\tau \in W_F$ act by the τ-linear automorphism

$$\rho(g_\tau) \circ (\tau \otimes \mathrm{id}_V).$$

In this way we get an F-linear functor β from **Rep**(G) to W_F-$\overline{L}$-*spaces* sending V to $\overline{L} \otimes_F V$, and this functor is a $\otimes$-functor in an obvious way (we denote by **Rep**(G) the Tannakian category of representations of G on finite dimensional F-vector spaces). The Tannakian category W_F-$\overline{L}$-*spaces* has an obvious fiber functor ω over $\overline{L}$ (forget the W_F-action). The Tannakian category **Rep**(G) also has an obvious fiber functor ω_G over $\overline{L}$, namely the functor $V \mapsto \overline{L} \otimes_F V$. Therefore there is an obvious $\otimes$-isomorphism from $\omega \circ \beta$ to ω_G (namely the identity map on $\overline{L} \otimes_F V$).

We can turn this around. Suppose that (β, α) is a pair consisting of a $\otimes$-functor β from **Rep**(G) to W_F-$\overline{L}$-*spaces* and a $\otimes$-isomorphism α from $\omega \circ \beta$ to ω_G. Then for every representation V of G the isomorphism α allows us to view $\beta(V)$ as a discrete semilinear W_F-module structure on $\overline{L} \otimes_F V$. Thus for each $\tau \in W_F$ there is a uniquely determined linear automorphism $g_\tau(V)$ of V such that the action of τ on $\overline{L} \otimes_F V$ is given by

$$g_\tau(V) \circ (\tau \otimes \mathrm{id}_V).$$

There is a unique element $g_\tau \in G(\overline{L})$ such that

$$\rho(g_\tau) = g_\tau(V)$$

for every representation (ρ, V) of G, and $\tau \mapsto g_\tau$ is a 1-cocycle of W_F in $G(\overline{L})$.

The two constructions above are inverse to each other, so that we get a bijection from the set of 1-cocycles of W_F in $G(\overline{L})$ to the set of $\otimes$-isomorphism classes of pairs (β, α) as above. Now suppose that we are given an exact $\otimes$-functor β from **Rep**(G) to W_F-$\overline{L}$-*spaces* (we include F-linearity in the definition of $\otimes$-functor). Then $\omega \circ \beta$ is a fiber functor on **Rep**(G) over $\overline{L}$, so there exists a $\otimes$-isomorphism α from $\omega \circ \beta$ to ω_G. This isomorphism is well-defined up to a $\otimes$-automorphism of ω_G, or, in other words, up to an element of $G(\overline{L})$. Associated to (β, α) is a

1-cocycle of W_F in $G(\overline{L})$, and changing α by an element of $G(\overline{L})$ replaces the 1-cocycle by a cohomologous one. In this way we get a bijection from the set of $\otimes$-isomorphism classes of exact $\otimes$-functors β as above to the set $\mathbf{B}(G)$ defined in 1.4.

3.2. Let G and g_τ be as above. Let $\mathbb{D}$ be the diagonalizable group over F whose character group $X^*(\mathbb{D})$ is $\mathbb{Q}$ (with trivial Galois action). Then, just as in [K, 4.2], we get from g_τ a homomorphism $\nu : \mathbb{D} \to G$ over $\overline{L}$. Indeed, as we saw above, for any representation (ρ, V) of G the 1-cocycle g_τ turns $\overline{L} \otimes_F V$ into a W_F-$\overline{L}$-space, so that $\overline{L} \otimes_F V$ acquires a $\mathbb{Q}$-grading (its slope decomposition), which can also be thought of as a homomorphism $\nu_\rho : \mathbb{D} \to \mathrm{GL}(V)$ over $\overline{L}$. The desired homomorphism $\nu : \mathbb{D} \to G$ over $\overline{L}$ is the unique one such that

$$\nu_\rho = \rho \circ \nu \quad \text{for all } (\rho, V).$$

Let x be an element of $G(\overline{L})$. It is clear that replacing g_τ by the cohomologous 1-cocycle $x g_\tau \tau(x)^{-1}$ replaces ν by $\mathrm{Int}(x) \circ \nu$.

Let F' be a finite extension of F in $\overline{F}$. Then the restriction of g_τ to $W_{F'}$ is a 1-cocycle of $W_{F'}$ in $G(\overline{L})$ and therefore determines a homomorphism $\nu' : \mathbb{D} \to G$ over $\overline{L}$. It follows from 2.2 that

$$\nu' = \nu^{[F':F]}.$$

We claim that ν is trivial if and only if the cohomology class of g_τ lies in the image of the natural injection

$$H^1(F, G) \to \mathbf{B}(G).$$

Indeed, if g_τ comes from $H^1(F, G)$, then there exists a finite Galois extension F' of F in $\overline{F}$ such that the restriction of g_τ to $W_{F'}$ is cohomologous to the trivial 1-cocycle. Therefore $\nu^{[F':F]}$ is trivial, which implies that ν itself is trivial. Conversely, if ν is trivial, then for every representation (ρ, V) of G the W_F-$\overline{L}$-space $V \otimes_F \overline{L}$ has slope 0. Therefore

$$V \mapsto (V \otimes_F \overline{L})^W$$

is a fiber functor on **Rep**(G) over F, and it follows that the functor

$$\beta : \mathbf{Rep}(G) \to W_F\text{-}\overline{L}\text{-}\textit{spaces}$$

determined by g_τ is $\otimes$-isomorphic to one of the form

$$V \mapsto \omega(V) \otimes_F \overline{L},$$

where ω is a fiber functor on **Rep**(G) over F. To such a fiber functor corresponds an element of $H^1(F, G)$, and it is immediate that this element maps to the class of g_τ in $\mathbf{B}(G)$.

3.3. We continue with G and g_τ as above. We will again denote the Weil group W_F simply by W. Since G is defined over F, the action of W on $\overline{L}$ induces an action of W on $G(\overline{L})$, which we refer to as the *standard* action. The 1-cocycle g_τ determines a *twisted* action of W on $G(\overline{L})$; for $\tau \in W$ this twisted action τ^* is related to the standard action τ by

$$\tau^* = \mathrm{Int}(g_\tau) \circ \tau. \tag{3.3.1}$$

We want to define a linear algebraic group J over F such that $J(F)$ is equal to $G(\overline{L})^W$ (the fixed point subgroup of the twisted W-action on $G(\overline{L})$).

First let us define the functor J that we wish to represent by a linear algebraic group; here we are following Rapoport-Zink [RZ, 1.12]. For any F-algebra R there is a natural action of W on $R \otimes_F \overline{L}$. This yields an action of W on $G(R \otimes_F \overline{L})$, and again the 1-cocycle g_τ determines a twisted action of W on $G(R \otimes_F \overline{L})$ (use (3.3.1), as before). We define the functor J by

$$J(R) := G(R \otimes_F \overline{L})^W. \tag{3.3.2}$$

When R is an $\overline{L}$-algebra, the canonical $\overline{L}$-algebra homomorphism

$$R \otimes_F \overline{L} \to R$$

induces an injection

$$J(R) = G(R \otimes_F \overline{L})^W \hookrightarrow G(R)$$

(the injectivity of this map follows from Appendix A and the discussion below). When R is $\overline{L}$ itself, the injection

$$J(\overline{L}) \hookrightarrow G(\overline{L}) \tag{3.3.3}$$

is W-equivariant for the standard W-action on $J(\overline{L})$ and the twisted W-action on $G(\overline{L})$. Moreover the injections $J(R) \to G(R)$ defined above for each $\overline{L}$-algebra R identify $J_{\overline{L}}$ with a closed subgroup scheme of $G_{\overline{L}}$, namely the centralizer in G of the homomorphism $\nu : \mathbb{D} \to G$ defined in 3.2. In particular (3.3.3) identifies $J(\overline{L})$ with the $\overline{L}$-points of the centralizer of ν in G.

In order to define J we only need the functor β from **Rep**(G) to W_F-$\overline{L}$-*spaces* determined by g_τ; the choice of $\otimes$-isomorphism α (of fiber functors over $\overline{L}$) needed to determine a particular 1-cocycle serves to identify $J(R)$ with $G(R \otimes_F \overline{L})^W$. We proceed as follows. Let R be an F-algebra. For any Tannakian category $\mathcal{T}$ over F we write $\mathcal{T}^R$ for the category whose objects are the same as those in $\mathcal{T}$ and whose morphisms are given by

$$\mathrm{Hom}_{\mathcal{T}^R}(X, Y) := \mathrm{Hom}_{\mathcal{T}}(X, Y) \otimes_F R.$$

Then there is an obvious structure of R-linear $\otimes$-category on $\mathcal{T}^R$, and there is an obvious $\otimes$-functor

$$\mathcal{T} \to \mathcal{T}^R$$

(given on objects by the identity map). We denote by β^R the composition of β and the functor described above from W_F-$\overline{L}$-*spaces* to W_F-$\overline{L}$-*spaces*^{R}. We then define $J_\beta(R)$ to be the group of $\otimes$-automorphisms of the $\otimes$-functor β^R (in particular $J_\beta(F)$ is the group of $\otimes$-automorphisms of β itself).

It follows from Appendix A that J_β is representable by an affine group scheme over F, and that a choice α of $\otimes$-isomorphism of fiber functors over $\overline{L}$ determines an isomorphism over $\overline{L}$ from J_β to the centralizer in G of the homomorphism $\nu\colon \mathbb{D} \to G$. Let g_τ be the 1-cocycle associated to β and α. It remains to show that

$$J_\beta(R) = G(R \otimes_F \overline{L})^W.$$

By definition an element $x \in J_\beta(R)$ is given by a compatible family of elements

$$x_V \in \bigl(\mathrm{End}_{W_F,\overline{L}}(\overline{L} \otimes_F V) \otimes_F R\bigr)^\times,$$

one for each representation V of G, where compatible means functorial in V as well as compatible with all finite tensor products. It is obvious that for any W_F-$\overline{L}$-space U and any F-vector space T we have

$$U^W \otimes_F T = \bigl(U \otimes_{\overline{L}} (\overline{L} \otimes_F T)\bigr)^W$$

(to prove this choose a basis for T). Applying this to the W_F-$\overline{L}$-space $\mathrm{End}_{\overline{L}}(\overline{L} \otimes_F V)$ and the F-algebra R (again V is a representation of G), we see that

$$\begin{aligned}\mathrm{End}_{W_F,\overline{L}}(\overline{L} \otimes_F V) \otimes_F R &= \bigl(\mathrm{End}_{\overline{L}}(\overline{L} \otimes_F V) \otimes_{\overline{L}} (\overline{L} \otimes_F R)\bigr)^W \\ &= \bigl(\mathrm{End}_{R \otimes_F \overline{L}}(R \otimes_F \overline{L} \otimes_F V)\bigr)^W\end{aligned}$$

(the second equality follows from the finite dimensionality of V). Therefore x is a compatible family of elements

$$x_V \in \mathrm{Aut}_{R \otimes_F \overline{L}}(R \otimes_F \overline{L} \otimes_F V)^W.$$

Since a compatible family of elements of

$$\mathrm{Aut}_{R \otimes_F \overline{L}}(R \otimes_F \overline{L} \otimes_F V)$$

is the same as an element of

$$G(R \otimes_F \overline{L}),$$

we conclude that

$$J_\beta(R) = G(R \otimes_F \overline{L})^W$$

(it is easy to see that the W-action is the twisted one described earlier.

3.4. We continue with G and g_τ as above. We let $\nu\colon \mathbb{D} \to G$ be the homomorphism over $\overline{L}$ determined by g_τ (as in 3.2), and we let J be the F-group obtained from g_τ

(as in 3.3). As in 3.3 we identify J over $\overline{L}$ with the centralizer in G of ν. Since the slope decomposition of any W_F-$\overline{L}$-space is stable under W, the homomorphism ν satisfies

$$\text{Int}(g_\tau) \circ \tau(\nu) = \nu \quad \text{for all } \tau \in W. \tag{3.4.1}$$

Since $\mathbb{D}$ is abelian, the homomorphism ν factors through J (and even through the center of J), yielding a homomorphism

$$\nu \colon \mathbb{D} \to J$$

defined over F (use (3.4.1) to see that it is defined over F).

Let x_τ be a 1-cocycle of W in $J(\overline{L})$. Then $g'_\tau := x_\tau g_\tau$ is a 1-cocycle of W in $G(\overline{L})$, and the map $x_\tau \mapsto g'_\tau$ on 1-cocycles induces a map

$$\mathbf{B}(J) \xrightarrow{\cdot g_\tau} \mathbf{B}(G). \tag{3.4.2}$$

Let $\nu' \colon \mathbb{D} \to G$ be the homomorphism over $\overline{L}$ associated to g'_τ, and let $\mu \colon \mathbb{D} \to J$ be the homomorphism over $\overline{L}$ associated to x_τ. Note that since ν is central in J, the product of μ and ν is a well-defined homomorphism $\mathbb{D} \to J$, and we claim that

$$\nu' = \mu\nu. \tag{3.4.3}$$

To check this, pick any faithful representation V of G. Put $\overline{V} := V \otimes_F \overline{L}$. Then g_τ turns $\overline{V}$ into a W_F-$\overline{L}$-space, which we still denote by $\overline{V}$. Of course g'_τ also turns $\overline{V}$ into a W_F-$\overline{L}$-space, which we denote by $\overline{V}'$. Now put

$$U := \text{End}_{W_F,\overline{L}}(\overline{V})$$

an F-vector space. The group J acts on U by left multiplication (note that $J(F)$ is a subgroup of $\text{Aut}_{W_F,\overline{L}}(\overline{V})$, and, more generally, that for any F-algebra R the group $J(R)$ is a subgroup of $\text{Aut}_{W_F,\overline{L}\otimes R}(\overline{V} \otimes_F R)$). Put

$$\begin{aligned} \overline{U} &:= U \otimes_F \overline{L} \\ &= \text{End}_{\mathbb{D}}(\overline{V}), \end{aligned}$$

and use the 1-cocycle x_τ to turn $\overline{U}$ into a W_F-$\overline{L}$-space. The natural evaluation map

$$\text{End}_{\mathbb{D}}(\overline{V}) \otimes_{\overline{L}} \overline{V} \to \overline{V},$$

sending $f \otimes v$ to $f(v)$, yields a surjective map

$$\overline{U} \otimes_{\overline{L}} \overline{V} \to \overline{V}'$$

of W_F-$\overline{L}$-spaces. Let $f_0 \in \overline{U}$ be the identity endomorphism of $\overline{V}$, and let $v \in \overline{V}$. We write v' instead of v when we regard v as an element of $\overline{V}'$. Then for $x \in \mathbb{D}(\overline{L})$ we have

$$\begin{aligned} \nu'(x)v' &= \nu'(x) f_0(v) \\ &= (\mu(x) f_0)(\nu(x) v) \\ &= \mu(x)\nu(x)v', \end{aligned}$$

which shows that $\nu' = \mu\nu$, as desired.

3.5. We continue to use the same notation. Restricting the map (3.4.2) to the subset $H^1(F, J)$ of $\mathbf{B}(J)$, we get a map

$$H^1(F, J) \xrightarrow{\cdot g_\tau} \mathbf{B}(G). \tag{3.5.1}$$

We claim that this map is injective and that its image is the set of classes of 1-cocycles g'_τ (of W in $G(\overline{L})$) for which the associated homomorphism $\nu' : \mathbb{D} \to G$ is conjugate under $G(\overline{L})$ to ν. The analogous result for $B(G)$ appears in [RR].

It is clear from 3.2 and 3.4 that (3.5.1) maps $H^1(F, J)$ into the subset of $\mathbf{B}(G)$ described above. Now suppose that g'_τ is a 1-cocycle such that ν' is conjugate to ν; we must show that the class of g'_τ lies in the image of (3.5.1). Replacing g'_τ by a cohomologous 1-cocycle, we may assume that $\nu' = \nu$. For $\tau \in W$ define $x_\tau \in G(\overline{L})$ by $g'_\tau = x_\tau g_\tau$. Applying (3.4.1) to both g_τ and g'_τ, we see that $\text{Int}(x_\tau) \circ \nu = \nu$, which means that x_τ lies in $J(\overline{L})$. Moreover x_τ is a 1-cocycle of W in $J(\overline{L})$, and from (3.4.3) we see that the homomorphism $\mu : \mathbb{D} \to J$ associated to x_τ is trivial. Thus (see 3.2) the class of x_τ in $\mathbf{B}(J)$ lies in the subset $H^1(F, J)$, and this shows that the class of g'_τ in $\mathbf{B}(G)$ lies in the image of (3.5.1), as desired.

It remains to check that (3.5.1) is injective. Suppose that x_τ, y_τ are 1-cocycles of W in $J(\overline{L})$ arising as the restrictions of 1-cocycles of Γ in $J(\overline{F})$, and suppose further that h is an element of $G(\overline{L})$ such that

$$y_\tau g_\tau = h x_\tau g_\tau \tau(h)^{-1}.$$

It follows from this equation that

$$\nu = \text{Int}(h) \circ \nu;$$

thus $h \in J(\overline{L})$ and

$$y_\tau = h x_\tau (g_\tau \tau(h) g_\tau^{-1})^{-1},$$

which shows that y_τ is cohomologous to x_τ.

3.6. Let G be a linear algebraic group over F. Let N be the unipotent radical of G. We claim that the natural map

$$\mathbf{B}(G) \to \mathbf{B}(G/N) \tag{3.6.1}$$

is a bijection. Choosing a Levi factor M in G (see [BS, 5.1]), so that $G = MN$, we see immediately that (3.6.1) is surjective. Now we show that (3.6.1) is injective. Let g_τ, g'_τ be 1-cocycles of W in $G(\overline{L})$ whose images in $(G/N)(\overline{L})$ are cohomologous; we must show that g_τ, g'_τ are cohomologous. Without loss of generality we may assume that the images of g_τ and g'_τ in $(G/N)(\overline{L})$ are equal. Let $\nu, \nu' : \mathbb{D} \to G$ be the homomorphisms associated to g_τ, g'_τ respectively. Replacing g_τ by $n g_\tau \tau(n)^{-1}$ for suitable $n \in N(\overline{L})$ (note that this does not change the image of g_τ in $(G/N)(\overline{L})$), we may assume that ν factors through M, and in the same way we may assume that ν' also factors through M. Since g_τ, g'_τ have the same image in $G/N \simeq M$, it follows that ν' and ν are equal. Therefore (see 3.5) there exists a 1-cocycle x_τ of W in $J(\overline{L})$ such that

$$g'_\tau = x_\tau g_\tau.$$

Since g'_τ and g_τ have the same image in G/N, the 1-cocycle x_τ takes values in the unipotent radical of J. Thus we are reduced to proving that $\mathbf{B}(U)$ is trivial for any unipotent group U. Since every homomorphism $\nu\colon \mathbb{D} \to U$ is trivial, we see that the natural map

$$H^1(F,U) \to \mathbf{B}(U)$$

is bijective. It is well-known that $H^1(F,U)$ is trivial, and this concludes the proof.

4. $\mathbf{B}(H)$ for connected reductive H

Let H be a connected reductive group over F. In this case there is more to be said about the objects ν, J appearing in the previous section. The results in 4.6–4.18 will be used in the next section. We also need to review the notion of basic elements in $\mathbf{B}(H)$. Since H is connected, the sets $\mathbf{B}(H)$ and $B(H)$ coincide (see 1.4), and therefore the results of [K] are valid for $\mathbf{B}(H)$.

4.1. Choose a quasi-split group G over F and a Γ-stable $G_{\mathrm{ad}}(\overline{F})$-orbit Ψ of $\overline{F}$-isomorphisms

$$\psi\colon G \to H.$$

Thus, for any $\psi \in \Psi$ and any $\tau \in \Gamma$ the automorphism $\psi^{-1} \circ \tau(\psi)$ of G over $\overline{F}$ is inner. In other words Ψ consists of a $G_{\mathrm{ad}}(\overline{F})$-orbit in the set of inner twistings $G \to H$.

Choose a maximal split torus A in G, let T be the centralizer in G of A (a maximal torus of G since G is quasi-split), and let B be a Borel subgroup of G that contains T and is defined over F. Let N_0 denote the unipotent radical of B. Put

$$\mathfrak{A}_{\mathbb{Q}} = X_*(A) \otimes_{\mathbb{Z}} \mathbb{Q},$$
$$\mathfrak{A} = X_*(A) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Let $\overline{C}$ denote the closed chamber

$$\{x \in \mathfrak{A} \mid \langle \alpha, x\rangle \geqslant 0 \text{ for every root } \alpha \text{ of } A \text{ in } \mathrm{Lie}(N_0)\}$$

in $\mathfrak{A}$, and let $\overline{C}_{\mathbb{Q}}$ denote its intersection with the rational subspace $\mathfrak{A}_{\mathbb{Q}}$ of $\mathfrak{A}$. It is an easy consequence of standard facts about root systems (see [K1, Lemma 1.1.3] for example) that there is a canonical bijection from $\overline{C}_{\mathbb{Q}}$ to the set of Γ-fixed points in the set of $G(\overline{L})$-conjugacy classes of $\overline{L}$-homomorphisms $\mathbb{D} \to G$ (view $\mathfrak{A}_{\mathbb{Q}}$ as $\mathrm{Hom}(\mathbb{D}, A)$ in order to obtain a homomorphism $\mathbb{D} \to G$ from an element of $\overline{C}_{\mathbb{Q}}$).

4.2. Let h_τ be a 1-cocycle of W in $H(\overline{L})$. Let $\nu : \mathbb{D} \to H$ be the $\overline{L}$-homomorphism associated to h_τ in 3.2. It is obvious from (3.4.1) that the $H(\overline{L})$-conjugacy class of the homomorphism ν is fixed by W (and hence by Γ). Composing ν with the inverse of any inner twisting $\psi \in \Psi$, we get an $\overline{L}$-homomorphism

$$\psi^{-1} \circ \nu \colon \mathbb{D} \to G,$$

whose $G(\overline{L})$-conjugacy class is fixed by Γ and independent of the choice of ψ in Ψ. Let $\bar{\nu}$ be the element of $\overline{C}_{\mathbb{Q}}$ corresponding to the $G(\overline{L})$-conjugacy class of $\psi^{-1} \circ \nu$ under the bijection mentioned above. Clearly $\bar{\nu}$ depends only on the class of h_τ in $\mathbf{B}(H)$. Following [RR] we call this map

$$\mathbf{B}(H) \to \mathfrak{A} \tag{4.2.1}$$

(sending the class of h_τ to $\bar{\nu}$) the *Newton map*. We refer to $\bar{\nu}$ as the *Newton point* of h_τ. Of course the Newton map takes values in the subset $\overline{C}_{\mathbb{Q}}$ of $\mathfrak{A}$.

4.3. Let h_τ be a 1-cocycle of W in $H(\overline{L})$, let $\nu : \mathbb{D} \to H$ be the associated $\overline{L}$-homomorphism, and let $\bar{\nu} \in \overline{C}_{\mathbb{Q}}$ be the Newton point of h_τ. In 3.3 we used the 1-cocycle h_τ to define a linear algebraic group J over F; recall that $J_{\overline{L}}$ can be identified with the centralizer in H of ν. Let M denote the centralizer in G of $\bar{\nu}$ (view $\bar{\nu}$ as a homomorphism $\mathbb{D} \to G$ factoring through A). Since $\bar{\nu}$ is defined over F, so is M.

We claim that J is an inner form of M. Indeed, let Ψ_J be the set of elements ψ in Ψ such that $\psi \circ \bar{\nu} = \nu$. It is evident that Ψ_J is non-empty, and that it forms a single orbit under the action of the group $\overline{M}(\overline{L})$, where $\overline{M}$ denotes the image of M in G_{ad}. Let Ψ'_J denote the set of $\overline{F}$-isomorphisms

$$\psi'_J \colon M \to J$$

for which there exists $\psi_J \in \Psi_J$ whose restriction to M is ψ'_J. Then Ψ'_J is non-empty and forms a single orbit under $M_{\mathrm{ad}}(\overline{F})$. Moreover Ψ'_J is Γ-stable. Indeed, it is enough to show that Ψ'_J is stable under W, and for this one uses that $\bar{\nu}$ is defined over F, that ν is fixed by the twisted W-action (see (3.4.1)) and that the injection (3.3.3) is W-equivariant. Therefore (J, Ψ'_J) is an inner form of M.

4.4. Let h_τ be a 1-cocycle of W in $H(\overline{L})$. As in [K] we say that h_τ is *basic* if the associated homomorphism $\nu \colon \mathbb{D} \to H$ over $\overline{L}$ factors through the center of H. In this case the centralizer of ν in H is H itself, so that J is an inner form of H (and of G). If h_τ is basic, then so is every cohomologous 1-cocycle; we say that a class in $\mathbf{B}(H)$ is basic if it consists of basic 1-cocycles, and we denote by $\mathbf{B}(H)_b$ the set of basic elements in $\mathbf{B}(H)$.

Let $\widehat{G}$ be a (connected) Langlands dual group for G, and let $Z(\widehat{G})$ denote its center. The Galois group Γ acts on $Z(\widehat{G})$, and the fixed point subgroup $Z(\widehat{G})^\Gamma$ is a diagonalizable group over $\mathbb{C}$. Recall [K, 5.6] that there is a canonical bijection

$$\mathbf{B}(H)_b \simeq X^*(Z(\widehat{G})^\Gamma) \tag{4.4.1}$$

between $\mathbf{B}(H)_b$ and the character group of $Z(\widehat{G})^\Gamma$ (of course we have used the canonical Γ-equivariant isomorphism between $Z(\widehat{H})$ and $Z(\widehat{G})$).

Let A_G denote the maximal split torus in the center of G. Then any element $\nu \in X_*(A_G)$ determines a homomorphism

$$\nu \colon \widehat{G} \to \mathbb{C}^\times$$

of algebraic groups, which we may restrict to $Z(\widehat{G})^\Gamma$, obtaining an element in $X^*(Z(\widehat{G})^\Gamma)$. In this way we get a homomorphism

$$X_*(A_G) \to X^*(Z(\widehat{G})^\Gamma), \tag{4.4.2}$$

and by tensoring with $\mathbb{R}$ we get from (4.4.2) an isomorphism

$$\mathfrak{A}_G := X_*(A_G)_\mathbb{R} \simeq X^*(Z(\widehat{G})^\Gamma)_\mathbb{R}. \tag{4.4.3}$$

It follows from [K, 4.4, 5.8] that the restriction to $\mathbf{B}(H)_b$ of the Newton map is equal to the composition of (4.4.1), the natural map

$$X^*(Z(\widehat{G})^\Gamma) \to X^*(Z(\widehat{G})^\Gamma)_\mathbb{R},$$

and the isomorphism (4.4.3) (we view $\mathfrak{A}_G$ as a subspace of $\mathfrak{A}$).

4.5. Let Z be the center of G. Note that Ψ allows us to identify Z with the center of H. There is an obvious action of the abelian group $\mathbf{B}(Z)$ on $\mathbf{B}(H)$ (the product of 1-cocycles z_τ in $Z(\overline{L})$ and h_τ in $H(\overline{L})$ is defined to be the 1-cocycle $\tau \mapsto z_\tau h_\tau$ in $H(\overline{L})$).

It is clear that the stabilizer in $\mathbf{B}(Z)$ of the base point in $\mathbf{B}(H)$ is

$$\ker[\mathbf{B}(Z) \to \mathbf{B}(H)], \tag{4.5.1}$$

and this group coincides with

$$\ker[H^1(F, Z) \to H^1(F, H)]$$

since the homomorphism $\mu \colon \mathbb{D} \to Z$ associated to an element in (4.5.1) must be trivial.

Now let h_τ be any 1-cocycle of W in $H(\overline{L})$, and let $\mathbf{h}$ denote its class in $\mathbf{B}(H)$. Let J be the F-group associated to h_τ in 3.3. We claim that the stabilizer in $\mathbf{B}(Z)$ of $\mathbf{h}$ is also the subgroup (4.5.1). Let z_τ be a 1-cocycle of W in $Z(\overline{L})$. Then the class of z_τ stabilizes $\mathbf{h}$ if and only if there exists $x \in H(\overline{L})$ such that

$$z_\tau h_\tau = x h_\tau \tau(x)^{-1}. \tag{4.5.2}$$

It follows from this equation (use (3.4.3), noting that Z can be identified with a subgroup of J) that

$$\mu\nu = \mathrm{Int}(x) \circ \nu, \tag{4.5.3}$$

where μ (respectively, ν) is the homomorphism $\mathbb{D} \to Z$ (respectively, $\mathbb{D} \to H$) associated to z_τ (respectively, h_τ). Projecting the equation (4.5.3) into the quotient of H by its derived group H_{der}, we see that μ factors through $Z \cap H_{\mathrm{der}}$, a finite group. We conclude that μ is trivial; looking back at (4.5.3), we now see that x centralizes ν and hence is an element of $J(\overline{L})$. Rewriting (4.5.2) as

$$\begin{aligned} z_\tau &= x \cdot h_\tau \tau(x)^{-1} h_\tau^{-1} \\ &= x \cdot \tau^*(x)^{-1}, \end{aligned}$$

we now see that the stabilizer in $\mathbf{B}(Z)$ of $\mathbf{h}$ is

$$\ker[\mathbf{B}(Z) \to \mathbf{B}(J)],$$

which is also equal to

$$\ker[H^1(F, Z) \to H^1(F, J)]. \tag{4.5.4}$$

Let $\bar{\nu} : \mathbb{D} \to G$ be the image of h_τ under the Newton map, and let M be the centralizer of $\bar{\nu}$ in G, a Levi subgroup of G. It is well-known that

$$H^1(F, M) \to H^1(F, G)$$

is injective (this is true for any field F, not just p-adic fields). Therefore the group

$$\ker[H^1(F, Z) \to H^1(F, M)]$$

is equal to

$$\ker[H^1(F, Z) \to H^1(F, G)].$$

It is a special property of p-adic fields that the group

$$\ker[H^1(F, Z) \to H^1(F, G)]$$

is equal to

$$\ker[H^1(F, Z) \to H^1(F, H)]$$

for any inner form H of G. Applying this to the inner forms M, J as well, we see that (4.5.4) coincides with (4.5.1), as desired.

The special property of p-adic fields stated above can be proved easily using the methods in [K2, Sect. 1]. Indeed, if the derived group of G is simply connected, then both

$$\ker[H^1(F, Z) \to H^1(F, G)]$$

and

$$\ker[H^1(F, Z) \to H^1(F, H)]$$

coincide with

$$\ker[H^1(F, Z) \to H^1(F, D)],$$

where

$$D = G/G_{\text{der}} = H/H_{\text{der}}.$$

Using z-extensions as in the proof of Theorem 1.2 in [K2], one reduces the general case to the special case just treated.

4.6. The group $H_{\text{ad}}(F)$ acts on H by F-automorphisms and therefore acts on $\mathbf{B}(H)$. We claim that this action is in fact trivial (it is obvious that $H(F)$ acts trivially on $\mathbf{B}(H)$, but since

$$H(F) \to H_{\text{ad}}(F)$$

need not be surjective, it is not obvious that $H_{\text{ad}}(F)$ acts trivially on $\mathbf{B}(H)$). Let $\bar{x} \in H_{\text{ad}}(F)$ and pick $x \in H(\overline{F})$ representing $\bar{x}$. Then

$$z_\tau := x^{-1}\tau(x) \tag{4.6.1}$$

is a 1-cocycle of Γ in $Z(\overline{F})$. The action of $\bar{x} \in H_{\text{ad}}(F)$ on $\mathbf{B}(H)$ takes the 1-cocycle h_τ of W in $H(\overline{L})$ into the 1-cocycle

$$\tau \mapsto x h_\tau x^{-1},$$

and this 1-cocycle is cohomologous to

$$\tau \mapsto h_\tau z_\tau.$$

Equation (4.6.1) shows that the class of z_τ in $H^1(F, Z)$ lies in

$$\ker[H^1(F, Z) \to H^1(F, H)]. \tag{4.6.2}$$

It follows from 4.5 that any element in (4.6.2) acts trivially on $\mathbf{B}(H)$. Therefore $h_\tau z_\tau$ is cohomologous to h_τ, as desired.

4.7. Let $X \to S$ be a map of sets, and let $s \in S$. We write X_s for the fiber of $X \to S$ over s. Recall that a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ S & \xrightarrow{f} & T \end{array}$$

of sets and maps is said to be *cartesian* if the natural map from X to the fiber product $S \times_T Y$ is an isomorphism (equivalently, if for every $s \in S$ the natural map $X_s \to Y_{f(s)}$ is bijective).

Consider a commutative diagram of the following type

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ S & \xrightarrow{f} & T & \longrightarrow & U. \end{array}$$

Let us denote by (L) (respectively, (R)) the left-hand (respectively, right-hand) square, and let us denote by (LR) the outer rectangle. If (L) and (R) are cartesian, then so is (LR). If (LR) and (R) are cartesian, then so is (L). If (LR) and (L) are cartesian and if f is surjective, then (R) is cartesian.

4.8. We say that a homomorphism $f : H \to H'$ from H to another connected reductive group H' over F is an *ad-isomorphism* if f maps the center of H into the center of H' and the induced map $H_{\text{ad}} \to H'_{\text{ad}}$ is an isomorphism (in which case $H_{\text{sc}} \to H'_{\text{sc}}$ is also an isomorphism).

4.9. Recall from [K3, Sect. 6] (see also 7.5) that there is a canonical map

$$\mathbf{B}(H) \to X^*(Z(\widehat{H})^\Gamma), \tag{4.9.1}$$

and that the restriction of (4.9.1) to $\mathbf{B}(H)_b$ coincides with the bijection (4.4.1) (after identifying $Z(\widehat{H})$ with $Z(\widehat{G})$). As Borovoi [B] has observed, the Γ-module $X^*(Z(\widehat{H}))$ can be identified with

$$\text{cok}[X_*(T_{\text{sc}}) \to X_*(T)]$$

for any maximal torus T in H (where T_{sc} denotes the inverse image of T in H_{sc}), and since this cokernel is easily seen to be functorial in H, it follows that the construction $H \mapsto Z(\widehat{H})$ is functorial in H for all connected reductive H and all F-homomorphisms $H \to H'$. It is easy to see that the maps (4.9.1) are functorial in H as well. Thus, an F-homomorphism $f : H \to H'$ gives rise to a commutative square

$$\begin{array}{ccc} \mathbf{B}(H) & \longrightarrow & \mathbf{B}(H') \\ \downarrow & & \downarrow \\ X^*(Z(\widehat{H})^\Gamma) & \longrightarrow & X^*(Z(\widehat{H}')^\Gamma). \end{array} \tag{4.9.2}$$

PROPOSITION 4.10. *If f is an ad-isomorphism, then the commutative square* (4.9.2) is cartesian.

We will prove the proposition in 4.17, after proving Lemmas 4.15 and 4.16. At the moment we are concerned with two useful corollaries of the proposition.

COROLLARY 4.11. *The set $\mathbf{B}(H)$ is the fiber product of $\mathbf{B}(H_{\text{ad}})$ and $X^*(Z(\widehat{H})^\Gamma)$ over $X^*(\widehat{Z}_{\text{sc}}^\Gamma)$, where $\widehat{Z}_{\text{sc}}$ denotes the center of $(\widehat{H})_{\text{sc}}$.*

To prove the corollary take $H' = H_{\text{ad}}$ in the proposition.

COROLLARY 4.12. *Let $\lambda \in X^*(Z(\widehat{H})^\Gamma)$ and let h be a basic 1-cocycle of W in $H(\overline{L})$ whose class $\mathbf{h}$ in $\mathbf{B}(H)$ maps to λ under the bijection* (4.4.1). Let J^h denote the inner form (see 4.4) of H determined by h, and let J_{sc}^h denote the simply connected cover of its derived group; thus J_{sc}^h is an inner form of H_{sc}. Then the composed map

$$\mathbf{B}(J_{\text{sc}}^h) \to \mathbf{B}(J^h) \xrightarrow{\cdot h} \mathbf{B}(H)$$

(see (3.4.2)) induces a bijection

$$\mathbf{B}(J_{\text{sc}}^h) \simeq \mathbf{B}(H)_\lambda,$$

where $\mathbf{B}(H)_\lambda$ denotes the fiber over λ of

$$\mathbf{B}(H) \to X^*(Z(\widehat{H})^\Gamma).$$

Consequently $\mathbf{B}(H)$ can be written as the disjoint union

$$\mathbf{B}(H) = \coprod_{\mathbf{h} \in \mathbf{B}(H)_b} \mathbf{B}(J_{\text{sc}}^h).$$

To prove Corollary 4.12 one begins by applying the proposition to the ad-isomorphism $H_{\text{sc}} \to H$ in order to conclude that there is a natural bijection

$$\mathbf{B}(H_{\text{sc}}) \simeq \ker[\mathbf{B}(H) \to X^*(Z(\widehat{H})^\Gamma)].$$

This is the special case of the corollary in which λ is trivial. Now consider the diagram

$$\begin{array}{ccc}
\mathbf{B}(J^h) & \xrightarrow{\cdot h} & \mathbf{B}(H) \\
\downarrow & & \downarrow \\
X^*(Z(\widehat{H})^\Gamma) & \xrightarrow{\cdot \lambda} & X^*(Z(\widehat{H})^\Gamma).
\end{array}$$

This diagram commutes (use z-extensions to reduce to the case in which H_{der} is simply connected), and both horizontal arrows are obviously bijections. It follows that

$$\begin{aligned}\mathbf{B}(H)_\lambda &\simeq \ker[\mathbf{B}(J^h) \to X^*(Z(\widehat{H})^\Gamma)] \\ &\simeq \mathbf{B}(J^h_{\text{sc}}),\end{aligned}$$

as desired.

4.13. Corollary 4.12 gives an even clearer picture of $\mathbf{B}(H)$ when it is combined with the following observation: for any simply connected group H the Newton map (4.2.1)

$$\mathbf{B}(H) \to \mathfrak{A}$$

is injective. More generally, for any connected reductive group H the map

$$\mathbf{B}(H) \to \mathfrak{A} \times X^*(Z(\widehat{H})^\Gamma) \tag{4.13.1}$$

is injective (the first component of the map (4.13.1) is the Newton map and the second component is the map (4.9.1)).

Indeed, let $\mathbf{h}$, $\mathbf{h}'$ be two elements in $\mathbf{B}(H)$ having the same image under the Newton map. Pick a 1-cocycle h lying in the class $\mathbf{h}$ and let J be the group associated to $\mathbf{h}$ in 3.3. In 3.5 we saw that $\mathbf{h}'$ lies in the image of

$$H^1(F, J) \xrightarrow{\cdot h} \mathbf{B}(H). \tag{4.13.2}$$

Let M be the Levi subgroup of G associated to h in 4.3 (recall that J is an inner form of M). The diagram

$$\begin{array}{ccc} \mathbf{B}(J) & \xrightarrow{\cdot h} & \mathbf{B}(H) \\ \downarrow & & \downarrow \\ X^*(Z(\widehat{M})^\Gamma) & \xrightarrow{\cdot \lambda} & X^*(Z(\widehat{G})^\Gamma) \end{array}$$

commutes, where λ denotes the image of $\mathbf{h}$ in $X^*(Z(\widehat{G})^\Gamma)$, and $\cdot\lambda$ denotes the map obtained by composing the restriction map

$$X^*(Z(\widehat{M})^\Gamma) \xrightarrow{\text{res}} X^*(Z(\widehat{G})^\Gamma)$$

and the map from $X^*(Z(\widehat{G})^\Gamma)$ to itself given by multiplication by λ. As before one proves the commutativity of the diagram above by using z-extensions to reduce to the case in which H_{der} is simply connected (which implies that G_{der}, M_{der}, J_{der} are simply connected as well).

Now suppose that **h**, **h**′ also have the same image in $X^*(Z(\widehat{H})^\Gamma) = X^*(Z(\widehat{G})^\Gamma)$. Pick an element $x \in H^1(F, J)$ that maps to **h**′ under (4.13.2). It follows that the image of x under

$$H^1(F,J) \to X^*(Z(\widehat{M})^\Gamma) \xrightarrow{\text{res}} X^*(Z(\widehat{G})^\Gamma) \tag{4.13.3}$$

is trivial. Recall from [K2, 1.2] that

$$H^1(F,J) \simeq X^*\big(\pi_0(Z(\widehat{M})^\Gamma)\big),$$

where $\pi_0(Z(\widehat{M})^\Gamma)$ denotes the group of connected components of $Z(\widehat{M})^\Gamma$. Moreover $Z(\widehat{G})^\Gamma$ meets every connected component of $Z(\widehat{M})^\Gamma$, since $[Z(\widehat{M})/Z(\widehat{G})]^\Gamma$ is connected (reduce to the case in which $Z(\widehat{G})$ is trivial, and then note that $Z(\widehat{M})$ is a torus whose character group has a basis permuted by Γ). Therefore the composed map (4.13.3) is injective, and we conclude that x is trivial. It follows that $\mathbf{h} = \mathbf{h}'$, and this completes the proof that (4.13.1) is injective.

4.14. It remains to prove Proposition 4.10. We say that an ad-isomorphism f is good if the conclusion of Proposition 4.10 holds for f; our goal is to prove that every ad-isomorphism is good. To this end we must first prove two lemmas.

LEMMA 4.15. *Let $f : H \to H'$ be a surjective ad-isomorphism whose kernel Z is a torus such that $H^1(F, Z)$ is trivial. Then f is good.*

Consider the exact sequence

$$1 \to Z \to H \xrightarrow{f} H' \to 1,$$

as well as the associated exact sequence

$$1 \to Z(\widehat{H}') \to Z(\widehat{H}) \to \widehat{Z} \to 1.$$

We must show that the square

$$\begin{array}{ccc} \mathbf{B}(H) & \longrightarrow & \mathbf{B}(H') \\ \downarrow & & \downarrow \\ X^*(Z(\widehat{H})^\Gamma) & \longrightarrow & X^*(Z(\widehat{H}')^\Gamma) \end{array}$$

is cartesian. Let $\mathbf{h}' \in \mathbf{B}(H')$ and let λ' denote its image in $X^*(Z(\widehat{H}')^\Gamma)$. We must show that

$$\mathbf{B}(H)_{\mathbf{h}'} \simeq X^*(Z(\widehat{H})^\Gamma)_{\lambda'}$$

(as before X_s denotes the fiber of $X \to S$ over $s \in S$). The group $\mathbf{B}(Z)$ acts transitively on $\mathbf{B}(H)_{\mathbf{h}'}$. Since Z is a subgroup of the center of H the discussion in 4.5 shows that the stabilizer in $\mathbf{B}(Z)$ of any point in $\mathbf{B}(H)$ is equal to the stabilizer of the base point in $\mathbf{B}(H)$, namely

$$\ker[\mathbf{B}(Z) \to \mathbf{B}(H)],$$

which is also equal to

$$\ker[H^1(F, Z) \to H^1(F, H)].$$

Since $H^1(F, Z)$ is trivial by hypothesis, we see that $\mathbf{B}(Z)$ acts simply transitively on $\mathbf{B}(H)_{\mathbf{h}'}$.

The group $X^*(\widehat{Z}^\Gamma)$ acts transitively on $X^*(Z(\widehat{H})^\Gamma)_{\lambda'}$. Since $H^1(F, Z)$ is trivial, the group $\widehat{Z}^\Gamma$ is connected [K2, 1.2], whence

$$X^*(\widehat{Z}^\Gamma) \to X^*(Z(\widehat{H})^\Gamma)$$

is injective. Therefore $X^*(\widehat{Z}^\Gamma)$ acts simply transitively on $X^*(Z(\widehat{H})^\Gamma)_{\lambda'}$. Using the canonical isomorphism

$$\mathbf{B}(Z) \simeq X^*(\widehat{Z}^\Gamma)$$

of [K], we see that

$$\mathbf{B}(H)_{\mathbf{h}'} \simeq X^*(Z(\widehat{H})^\Gamma)_{\lambda'},$$

as desired.

LEMMA 4.16. *Let $f : H \to H'$ be an ad-isomorphism, and assume that H_{der}, H'_{der} are simply connected. Then f is good.*

Put $D = H/H_{\text{der}}$ and $D' = H'/H'_{\text{der}}$. Recall that $\widehat{D} = Z(\widehat{H})$ and $\widehat{D}' = Z(\widehat{H}')$. The map (4.9.1) can be thought of as the natural map

$$\mathbf{B}(H) \to \mathbf{B}(D),$$

using the identifications

$$\mathbf{B}(D) \simeq X^*(\widehat{D}^\Gamma) \simeq X^*(Z(\widehat{H})^\Gamma).$$

Thus our problem is to show that the square

$$\begin{array}{ccc} \mathbf{B}(H) & \longrightarrow & \mathbf{B}(H') \\ \downarrow & & \downarrow \\ \mathbf{B}(D) & \longrightarrow & \mathbf{B}(D') \end{array}$$

is cartesian. Let $\mathbf{d} \in \mathbf{B}(D)$ and let $\mathbf{d}'$ be its image in $\mathbf{B}(D')$. We must show that

$$\mathbf{B}(H)_{\mathbf{d}} \simeq \mathbf{B}(H')_{\mathbf{d}'}.$$

We claim that both of these fibers are in natural one-to-one correspondence with the set $\mathbf{B}(J^h_{\mathrm{sc}})$, where h is a basic 1-cocycle in $H(\overline{L})$ whose image in $\mathbf{B}(D)$ is $\mathbf{d}$, and where J^h is the inner form of H obtained from h as in 4.4. Of course this claim is a special case of Corollary 4.12, but in order to avoid circular reasoning, we must establish Corollary 4.12 directly in the case that H_{der} is simply connected. Looking back at the method used to derive Corollary 4.12 from Proposition 4.10, we see that it is enough to show that

$$\mathbf{B}(H_{\mathrm{sc}}) \simeq \ker[\mathbf{B}(H) \to \mathbf{B}(D)].$$

It follows from the exactness of

$$1 \to H_{\mathrm{sc}} \to H \to D \to 1$$

that $\mathbf{B}(H_{\mathrm{sc}})$ maps onto $\ker[\mathbf{B}(H) \to \mathbf{B}(D)]$. The fibers of the map

$$\mathbf{B}(H_{\mathrm{sc}}) \to \mathbf{B}(H)$$

coincide with the orbits of $D(F)$ on $\mathbf{B}(H_{\mathrm{sc}})$ (see 1.5). It follows from the triviality of $H^1(F, H_{\mathrm{sc}})$ (see [Kn]) that the map $H(F) \to D(F)$ is surjective. Therefore the orbits of $D(F)$ on $\mathbf{B}(H_{\mathrm{sc}})$ coincide with the orbits of $H(F)$ on $\mathbf{B}(H_{\mathrm{sc}})$. Looking back at 1.5, we see that the action of $H(F)$ on $\mathbf{B}(H_{\mathrm{sc}})$ is induced by the conjugation action of $H(F)$ on H_{sc}. It follows from 4.6 that this action is trivial. We conclude that $\mathbf{B}(H_{\mathrm{sc}}) \to \mathbf{B}(H)$ is injective, and our proof is complete.

4.17. Now we prove Proposition 4.10. Let $f\colon H \to H'$ be any ad-isomorphism. It is easy to construct (see [K4, 2.4.4]) a commutative diagram

$$\begin{array}{ccc} H_1 & \xrightarrow{f_1} & H'_1 \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ H & \xrightarrow{f} & H' \end{array}$$

in which the two vertical arrows are z-extensions. Clearly f_1 is an ad-isomorphism. By Lemma 4.16 f_1 is good. By Lemma 4.15 p' is good. It is clear that the composition of two good ad-isomorphisms is good. Therefore $p' \circ f_1 = f \circ p$ is good. By Lemma 4.15 p is good, and moreover

$$X^*(Z(\widehat{H}_1)^\Gamma) \to X^*(Z(\widehat{H})^\Gamma)$$

is surjective. Therefore the fact that $f \circ p$ is good implies that f is good (see 4.7). The proof of Proposition 4.10 is complete.

4.18. Let h_τ be a basic 1-cocycle of W in $H(\overline{L})$, let J be the F-group associated to h_τ in 3.3, and let $\nu \in \mathfrak{A}$ denote the Newton point of h_τ. As we noted in 4.4, J is an inner form of H (and G), so that the Newton maps for J and H both take values in $\mathfrak{A}$. It follows from (3.4.3) that the diagram

$$\begin{array}{ccc} \mathbf{B}(J) & \xrightarrow{\cdot h_\tau} & \mathbf{B}(H) \\ \downarrow & & \downarrow \\ \mathfrak{A} & \xrightarrow{\cdot \nu} & \mathfrak{A} \end{array} \tag{4.18.1}$$

commutes, where $\cdot\nu$ denotes translation by ν in the abelian group $\mathfrak{A}$ and the vertical arrows are Newton maps. Since h_τ is basic, it is evident that the map

$$\mathbf{B}(J) \xrightarrow{\cdot h_\tau} \mathbf{B}(H)$$

is bijective. Thus we conclude from (4.18.1) that the image of the Newton map for H is the translate by ν of the image of the Newton map for J. Note that if the center of H is connected, then the natural map

$$\mathbf{B}(H)_b \to \mathbf{B}(H_{\text{ad}})_b = H^1(F, H_{\text{ad}})$$

is surjective, so that every inner form of H is of the form J for a suitable 1-cocycle. In particular the Newton maps for inner forms of an adjoint group all have the same image, since the relevant Newton points ν are trivial in this case.

5. Simple description of $\mathbf{B}(H)$ for connected reductive H

Let G be a quasi-split connected reductive group over F. For such G a simple, concrete description of $\mathbf{B}(G)$ is given in [K]. Our goal here is to give an analogous description for all connected reductive groups over F. This is best accomplished by considering simultaneously all inner forms H of the given quasi-split group G.

5.1. We first need to recall from [K] the description of $\mathbf{B}(G)$ in the quasi-split case. By a parabolic subgroup of G we mean a parabolic subgroup of G defined over F. Fix a Borel subgroup B of G over F. As usual we refer to parabolic subgroups of G containing B as *standard* parabolic subgroups of G. We fix a maximal torus T in B over F, and for any standard parabolic subgroup P of G we write $P = MN$, where N is the unipotent radical of P and M is the unique Levi component of P containing T. We write A_P (or A_M) for the maximal split torus in the center of M. Let $\mathfrak{A}_P$ denote the $\mathbb{R}$-vector space $X_*(A_P) \otimes_{\mathbb{Z}} \mathbb{R}$. As usual P determines an open chamber $\mathfrak{A}_P^+$ in $\mathfrak{A}_P$, defined by

$$\mathfrak{A}_P^+ = \{x \in \mathfrak{A}_P \mid \langle \alpha, x \rangle > 0 \text{ for every root } \alpha \text{ of } A_P \text{ in Lie}(N)\}.$$

We also use the notation A, $\mathfrak{A}$, $\mathfrak{A}_{\mathbb{Q}}$, $\overline{C}$ and $\overline{C}_{\mathbb{Q}}$ from 4.1 (A is the maximal split torus in T).

Any element **g** in $\mathbf{B}(G)$ determines a standard parabolic subgroup $P_{\mathbf{g}}$, as follows. Let $\bar{\nu} \in \overline{C}$ be the image of **g** under the Newton map (see 4.2). The closed chamber $\overline{C}$ in $\mathfrak{A}$ is the disjoint union

$$\overline{C} = \coprod_P \mathfrak{A}_P^+,$$

where P runs over the standard parabolic subgroups of G (as usual we identify $\mathfrak{A}_P$ with a subspace of $\mathfrak{A}$). By definition $P_{\mathbf{g}}$ is the unique standard parabolic subgroup P for which $\bar{\nu} \in \mathfrak{A}_P^+$.

For any standard parabolic subgroup $P = MN$ of G, we denote by $\mathbf{B}(G)_P$ the subset of $\mathbf{B}(G)$ consisting of all elements **g** for which $P_{\mathbf{g}}$ is equal to P. Thus $\mathbf{B}(G)$ is the disjoint union

$$\mathbf{B}(G) = \coprod_P \mathbf{B}(G)_P, \tag{5.1.1}$$

where P runs through the set of standard parabolic subgroups in G. Of course $\mathbf{B}(G)_G$ is simply the set $\mathbf{B}(G)_b$ of basic elements in $\mathbf{B}(G)$ (see 4.4). For any basic element **m** in $\mathbf{B}(M)$, the image of **m** under the Newton map (for M) lies in $\mathfrak{A}_P$. We write $\mathbf{B}(M)_b^+$ for the subset of $\mathbf{B}(M)_b$ consisting of all **m** whose image under the Newton map lies in the subset $\mathfrak{A}_P^+$ of $\mathfrak{A}_P$.

It follows from [K, Sect. 6] that the canonical map

$$\mathbf{B}(M) \to \mathbf{B}(G)$$

induces a bijection

$$\mathbf{B}(M)_b^+ \simeq \mathbf{B}(G)_P. \tag{5.1.2}$$

There is a natural homomorphism

$$X^*(Z(\widehat{M})^\Gamma) \to \mathfrak{A}_P \tag{5.1.3}$$

obtained by composing the natural map

$$X^*(Z(\widehat{M})^\Gamma) \to X^*(Z(\widehat{M})^\Gamma)_{\mathbb{R}}$$

with the isomorphism (4.4.3)

$$\mathfrak{A}_P \simeq X^*(Z(\widehat{M})^\Gamma)_{\mathbb{R}}.$$

Let $X^*(Z(\widehat{M})^\Gamma)^+$ denote the subset of $X^*(Z(\widehat{M})^\Gamma)$ consisting of all elements whose image in $\mathfrak{A}_P$ lies in $\mathfrak{A}_P^+$. Combining (5.1.2) with the bijection (4.4.1)

$$\mathbf{B}(M)_b \simeq X^*(Z(\widehat{M})^\Gamma),$$

we get a bijection An *inner form* of G is a pair (H, Ψ) consisting of a connected reductive group H over F and a Γ-stable $G_{\mathrm{ad}}(\overline{F})$-orbit Ψ of $\overline{F}$-isomorphisms

$$\psi \colon G \to H.$$

Let (H_1, Ψ_1), (H_2, Ψ_2) be two inner forms of G. An isomorphism from (H_1, Ψ_1) to (H_2, Ψ_2) is an F-isomorphism $\alpha \colon H_1 \to H_2$ carrying Ψ_1 into Ψ_2. The group of automorphisms of (H, Ψ) is equal to $H_{\mathrm{ad}}(F)$. There is an obvious bijection from the set of isomorphism classes of inner forms of G to the set $H^1(F, G_{\mathrm{ad}})$, obtained by sending (H, Ψ) to the class of the 1-cocycle $\tau \mapsto \psi^{-1} \circ \tau(\psi)$, where ψ is any element in Ψ.

Consider triples $(H, \Psi, \mathbf{h})$ consisting of an inner form (H, Ψ) of G and an element $\mathbf{h} \in \mathbf{B}(H)$. An isomorphism from one triple $(H_1, \Psi_1, \mathbf{h}_1)$ to another $(H_2, \Psi_2, \mathbf{h}_2)$ is an F-isomorphism $\alpha \colon H_1 \to H_2$ carrying Ψ_1 into Ψ_2 and $\mathbf{h}_1$ into $\mathbf{h}_2$. Let $\mathbf{B}_s(G)$ denote the set of isomorphism classes of triples $(H, \Psi, \mathbf{h})$. Note that any ad-isomorphism $G \to G'$ (see 4.8) induces a natural map

$$\mathbf{B}_s(G) \to \mathbf{B}_s(G'). \tag{5.2.1}$$

There is an obvious map

$$\mathbf{B}_s(G) \to H^1(F, G_{\mathrm{ad}}), \tag{5.2.2}$$

sending $(H, \Psi, \mathbf{h})$ to the element in $H^1(F, G_{\mathrm{ad}})$ determined by the inner form (H, Ψ) of G. Let (H, Ψ) be an inner form of G, and let $\mathbf{x}$ denote the corresponding element in $H^1(F, G_{\mathrm{ad}})$. Then there is a canonical bijection from $\mathbf{B}(H)$ to the fiber of (5.2.2) over $\mathbf{x}$; to prove this use that $H_{\mathrm{ad}}(F)$ acts trivially on $\mathbf{B}(H)$ (see 4.6). Speaking loosely, $\mathbf{B}_s(G)$ is the disjoint union of the sets $\mathbf{B}(H)$ as H runs through the inner forms of G.

Let $(H, \Psi, \mathbf{h})$ be a triple as above. The map (4.9.1) for H produces from $\mathbf{h}$ an element λ in $X^*(Z(\widehat{H})^\Gamma)$, which we regard as an element of $X^*(Z(\widehat{G})^\Gamma)$. We define a map

$$\mathbf{B}_s(G) \to X^*(Z(\widehat{G})^\Gamma) \tag{5.2.3}$$

by sending $(H, \Psi, \mathbf{h})$ to λ.

5.3. We are going to give a simple, concrete description of $\mathbf{B}_s(G)$ that is quite analogous to the one we already have for $\mathbf{B}(G)$. Let $(H, \Psi, \mathbf{h})$ be a triple as above. The Newton map for H produces from $\mathbf{h}$ an element $\bar{\nu} \in \mathfrak{A}$. We define a map (still called the Newton map)

$$\mathbf{B}_s(G) \to \mathfrak{A} \tag{5.3.1}$$

by sending $(H, \Psi, \mathbf{h})$ to $\bar{\nu}$. Again the Newton map takes values in the subset $\overline{C}_{\mathbb{Q}}$ of $\mathfrak{A}$.

Just as in 5.1 we use the Newton map to associate a standard parabolic subgroup $P(H, \Psi, \mathbf{h})$ to $(H, \Psi, \mathbf{h})$, and for a given standard parabolic subgroup P of G we write $\mathbf{B}_s(G)_P$ for the subset of $\mathbf{B}_s(G)$ consisting of all (isomorphism classes of) triples $(H, \Psi, \mathbf{h})$ such that $P(H, \Psi, \mathbf{h})$ is equal to P.

Let $\widehat{Z}_{\text{sc}}$ denote the center of $(\widehat{G})_{\text{sc}}$, the simply connected cover of the derived group of $\widehat{G}$. Of course Γ operates on $\widehat{Z}_{\text{sc}}$, and as usual we denote by $\widehat{Z}_{\text{sc}}^{\Gamma}$ the group of fixed points. There is a canonical bijection [K2, 1.2]

$$H^1(F, G_{\text{ad}}) \to X^*(\widehat{Z}_{\text{sc}}^{\Gamma}). \tag{5.3.2}$$

Recall that we have chosen a maximal torus T of G contained in the Borel subgroup B. Let T_{sc} (respectively, T_{ad}) denote the inverse image (respectively, image) of T in G_{sc} (respectively, G_{ad}). There is a surjective homomorphism

$$T_{\text{sc}} \to T_{\text{ad}}.$$

Dual to this is the surjective homomorphism

$$(\widehat{T})_{\text{sc}} \to (\widehat{T})_{\text{ad}},$$

whose kernel is $\widehat{Z}_{\text{sc}}$. Thus $(\widehat{T})_{\text{sc}}$ is an extension of $(\widehat{T})_{\text{ad}}$ by the finite abelian group $\widehat{Z}_{\text{sc}}$. Since the group $X^*((\widehat{T})_{\text{ad}}) = X_*(T_{\text{sc}})$ has a basis that is permuted by Γ (for example the basis of simple coroots of T_{sc}), the group $(\widehat{T})_{\text{ad}}^{\Gamma}$ of Γ-invariants in $(\widehat{T})_{\text{ad}}$ is connected; hence the homomorphism

$$(\widehat{T})_{\text{sc}}^{\Gamma} \to (\widehat{T})_{\text{ad}}^{\Gamma}$$

is also surjective. Thus we get an extension

$$1 \to \widehat{Z}_{\text{sc}}^{\Gamma} \to (\widehat{T})_{\text{sc}}^{\Gamma} \to (\widehat{T})_{\text{ad}}^{\Gamma} \to 1 \tag{5.3.3}$$

of $(\widehat{T})_{\text{ad}}^{\Gamma}$ by $\widehat{Z}_{\text{sc}}^{\Gamma}$.

Dual to $T_{\text{sc}} \to T$ is a surjective homomorphism

$$\widehat{T} \to (\widehat{T})_{\text{ad}},$$

which induces a surjective homomorphism

$$\widehat{T}^{\Gamma} \to (\widehat{T})_{\text{ad}}^{\Gamma}. \tag{5.3.4}$$

Pulling back the extension (5.3.3) by means of the homomorphism (5.3.4), we obtain an extension

$$1 \to \widehat{Z}_{\text{sc}}^{\Gamma} \to \widehat{T}_s^{\Gamma} \to \widehat{T}^{\Gamma} \to 1 \tag{5.3.5}$$

of $\widehat{T}^{\Gamma}$ by $\widehat{Z}_{\text{sc}}^{\Gamma}$, where we have written $\widehat{T}_s$ for the fiber product of $\widehat{T}$ and $(\widehat{T})_{\text{sc}}$ over $(\widehat{T})_{\text{ad}}$.

For any standard parabolic subgroup $P = MN$ of G, the group T is a maximal torus in M, and therefore there is a canonical Γ-equivariant embedding

$$Z(\widehat{M}) \hookrightarrow \widehat{T},$$

which induces an embedding

$$Z(\widehat{M})^\Gamma \hookrightarrow \widehat{T}^\Gamma. \tag{5.3.6}$$

Pulling back the extension (5.3.5) by means of the homomorphism (5.3.6), we obtain an extension

$$1 \to \widehat{Z}_{\text{sc}}^\Gamma \to Z_s(\widehat{M})^\Gamma \to Z(\widehat{M})^\Gamma \to 1, \tag{5.3.7}$$

where we have written $Z_s(\widehat{M})$ for the inverse image under

$$\widehat{T}_s \to \widehat{T}$$

of the subgroup $Z(\widehat{M})$ of $\widehat{T}$.

Since $Z(\widehat{G})^\Gamma$ is the kernel of (5.3.4), there is a canonical isomorphism

$$Z_s(\widehat{G})^\Gamma = Z(\widehat{G})^\Gamma \times \widehat{Z}_{\text{sc}}^\Gamma,$$

and hence there is a canonical embedding

$$Z(\widehat{G})^\Gamma \hookrightarrow Z_s(\widehat{G})^\Gamma.$$

Combining this with the obvious embeddings

$$Z_s(\widehat{G})^\Gamma \hookrightarrow Z_s(\widehat{M})^\Gamma,$$

we obtain embeddings

$$Z(\widehat{G})^\Gamma \hookrightarrow Z_s(\widehat{M})^\Gamma. \tag{5.3.8}$$

Let $f: G \to G'$ be an ad-isomorphism. There is a unique Levi subgroup M' of G' such that $f^{-1}(M') = M$, and there is an obvious cartesian diagram

$$\begin{array}{ccc} Z_s(\widehat{M'})^\Gamma & \longrightarrow & Z_s(\widehat{M})^\Gamma \\ \downarrow & & \downarrow \\ Z(\widehat{M'})^\Gamma & \longrightarrow & Z(\widehat{M})^\Gamma. \end{array} \tag{5.3.9}$$

Since $\widehat{Z}_{\text{sc}}^\Gamma$ is a finite abelian group, we see from (5.3.7) that there is a canonical isomorphism

$$X^*(Z_s(\widehat{M})^\Gamma)_{\mathbb{R}} \simeq X^*(Z(\widehat{M})^\Gamma)_{\mathbb{R}},$$

which we compose with the isomorphism (4.4.3)

$$X^*(Z(\widehat{M})^\Gamma)_{\mathbb{R}} \simeq \mathfrak{A}_P,$$

obtaining an isomorphism

$$X^*(Z_s(\widehat{M})^\Gamma)_{\mathbb{R}} \simeq \mathfrak{A}_P. \tag{5.3.10}$$

Thus we have a canonical map

$$X^*(Z_s(\widehat{M})^\Gamma) \to \mathfrak{A}_P. \tag{5.3.11}$$

We denote by $X^*(Z_s(\widehat{M})^\Gamma)^+$ the subset of $X^*(Z_s(\widehat{M})^\Gamma)$ consisting of all elements whose image under (5.3.11) lies in the subset $\mathfrak{A}_P^+$ of $\mathfrak{A}_P$.

THEOREM 5.4. *There is a canonical bijection*

$$\mathbf{B}_s(G)_P \simeq X^*(Z_s(\widehat{M})^\Gamma)^+,$$

and this bijection is functorial with respect to ad-isomorphisms $f: G \to G'$. *The composition of this bijection with the map* (5.3.11) coincides with the restriction to $\mathbf{B}_s(G)_P$ of the Newton map. The composition of this bijection with the map

$$X^*(Z_s(\widehat{M})^\Gamma) \to X^*(Z(\widehat{G})^\Gamma)$$

dual to (5.3.8) coincides with the restriction to $\mathbf{B}_s(G)_P$ of the map (5.2.3). The composition of this bijection with the map

$$X^*(Z_s(\widehat{M})^\Gamma) \to X^*(\widehat{Z}_{\text{sc}}^\Gamma)$$

dual to the inclusion of $\widehat{Z}_{\text{sc}}^\Gamma$ in $Z_s(\widehat{M})^\Gamma$ coincides with the restriction to $\mathbf{B}_s(G)_P$ of the map

$$\mathbf{B}_s(G) \to H^1(F, G_{\text{ad}}) \simeq X^*(\widehat{Z}_{\text{sc}}^\Gamma)$$

obtained by composing (5.2.2) and (5.3.2). The bijection is characterized by the last three properties.

It follows from 5.2 and 4.13 that the obvious map

$$\mathbf{B}_s(G) \to \mathfrak{A} \times X^*(Z(\widehat{G})^\Gamma) \times X^*(\widehat{Z}_{\text{sc}}^\Gamma) \tag{5.4.1}$$

is injective. Therefore there can be at most one bijection satisfying the last three properties stated in the theorem.

We begin by constructing the desired bijection in the case that G is an adjoint group. Let (H, Ψ) be an inner form of G. Choose $\psi \in \Psi$ and let $g_\tau := \psi^{-1} \circ \tau(\psi)$ be the associated 1-cocycle of Γ in $G_{\text{ad}}(\overline{F}) = G(\overline{F})$. Of course we can restrict g_τ to W, obtaining a 1-cocycle of W in $G(\overline{L})$. As in (3.4.2) we have the map

$$\mathbf{B}(H) \to \mathbf{B}(G) \tag{5.4.2}$$

sending the class of a 1-cocycle h_τ of W in $H(\overline{L})$ to the class of the 1-cocycle $\psi^{-1}(h_\tau)g_\tau$. It is obvious that the map (5.4.2) is bijective and independent of the choice of ψ in Ψ. Looking back at (4.18.1), we see that the diagram

$$\begin{array}{ccc} \mathbf{B}(H) & \longrightarrow & \mathbf{B}(G) \\ \downarrow & & \downarrow \\ \mathfrak{A} & = & \mathfrak{A} \end{array} \tag{5.4.3}$$

commutes, where the two vertical arrows are Newton maps (from 3.2 we know that the Newton point of g_τ is trivial).

Let $\lambda_H \in X^*(\widehat{Z}_{\mathrm{sc}}^\Gamma)$ be the image under the map (5.3.2) of the class of g_τ. We have already noted (see the proof of Corollary 4.12) that the diagram

$$\begin{array}{ccc} \mathbf{B}(H) & \longrightarrow & \mathbf{B}(G) \\ \downarrow & & \downarrow \\ X^*(\widehat{Z}_{\mathrm{sc}}^\Gamma) & \xrightarrow{\cdot\lambda_H} & X^*(\widehat{Z}_{\mathrm{sc}}^\Gamma) \end{array} \tag{5.4.4}$$

commutes, where the vertical arrows are of type (4.9.1).

Let P be a standard parabolic subgroup of G. Viewing $\mathbf{B}(H)$ as a subset of $\mathbf{B}_s(G)$, we define $\mathbf{B}(H)_P$ to be the intersection of $\mathbf{B}(H)$ and $\mathbf{B}_s(G)_P$. It follows from the commutativity of (5.4.3) that the bijection (5.4.2) induces a bijection

$$\mathbf{B}(H)_P \to \mathbf{B}(G)_P. \tag{5.4.5}$$

Combining the bijections (5.4.5) for varying (H, Ψ), we get a bijection

$$\mathbf{B}_s(G)_P \to \mathbf{B}(G)_P \times X^*(\widehat{Z}_{\mathrm{sc}}^\Gamma), \tag{5.4.6}$$

the restriction to $\mathbf{B}(H)_P \subset \mathbf{B}_s(G)_P$ of (5.4.6) being given by

$$h_\tau \mapsto (\psi^{-1}(h_\tau)g_\tau, \lambda_H).$$

Combining (5.4.6) with the bijection (5.1.4), we get a bijection

$$\mathbf{B}_s(G)_P \to X^*(Z(\widehat{M})^\Gamma)^+ \times X^*(\widehat{Z}_{\mathrm{sc}}^\Gamma). \tag{5.4.7}$$

Since G is adjoint, there is a canonical splitting of the extension (5.3.5), obtained as follows: in this special case $\widehat{T}_s^\Gamma$ is the fiber product of $\widehat{T}^\Gamma$ with itself over $(\widehat{T})_{\mathrm{ad}}^\Gamma$, and therefore the diagonal map from $\widehat{T}^\Gamma$ to that fiber product provides the desired splitting. Since the extensions (5.3.7) are obtained as pull-backs from (5.3.5), they all have canonical splittings as well. Thus

$$Z_s(\widehat{M})^\Gamma = Z(\widehat{M})^\Gamma \times \widehat{Z}_{\mathrm{sc}}^\Gamma \tag{5.4.8}$$

in our special case, and we can view (5.4.7) as a bijection

$$\mathbf{B}_s(G)_P \to X^*(Z_s(\widehat{M})^\Gamma)^+. \tag{5.4.9}$$

It is easy to check that (5.4.9) has the three desired properties.

Now we consider the general case. There is a commutative square

$$\begin{array}{ccc} \mathbf{B}_s(G) & \longrightarrow & \mathbf{B}_s(G_{\text{ad}}) \\ \downarrow & & \downarrow \\ X^*(Z(\widehat{G})^\Gamma) & \longrightarrow & X^*(\widehat{Z}_{\text{sc}}^\Gamma), \end{array} \tag{5.4.10}$$

where the vertical maps are of type (5.2.3). It follows from Proposition 4.10 that the square (5.4.10) is cartesian. Let $P = MN$ be a standard parabolic subgroup of G and let P_1, M_1 denote the images in G_{ad} of P, M respectively. The inverse image of $\mathbf{B}_s(G_{\text{ad}})_{P_1}$ under

$$\mathbf{B}_s(G) \to \mathbf{B}_s(G_{\text{ad}})$$

is $\mathbf{B}_s(G)_P$; therefore the square

$$\begin{array}{ccc} \mathbf{B}_s(G)_P & \longrightarrow & \mathbf{B}_s(G_{\text{ad}})_{P_1} \\ \downarrow & & \downarrow \\ X^*(Z(\widehat{G})^\Gamma) & \longrightarrow & X^*(\widehat{Z}_{\text{sc}}^\Gamma) \end{array} \tag{5.4.11}$$

is cartesian as well. Using the bijection (5.4.9), we see that there is a canonical bijection

$$\mathbf{B}_s(G)_P \to X^*(Z_s(\widehat{M}_1)^\Gamma)^+ \times_Y X^*(Z(\widehat{G})^\Gamma), \tag{5.4.12}$$

where we have written Y for $X^*(\widehat{Z}_{\text{sc}}^\Gamma)$. The target of (5.4.12) is a subset of the abelian group

$$X^*(Z_s(\widehat{M}_1)^\Gamma) \times_Y X^*(Z(\widehat{G})^\Gamma),$$

and this abelian group can be identified with $X^*(A)$, where A is the group

$$A := (Z_s(\widehat{M}_1)^\Gamma \times Z(\widehat{G})^\Gamma)/\widehat{Z}_{\text{sc}}^\Gamma$$

($\widehat{Z}_{\text{sc}}^\Gamma$ is embedded in the product as follows: the first component of the embedding is the inverse of (5.3.8) for G_{ad} and the second component is the canonical map from $\widehat{Z}_{\text{sc}}^\Gamma$ to $Z(\widehat{G})^\Gamma$).

There are natural homomorphisms

$$Z_s(\widehat{M}_1)^\Gamma \to Z_s(\widehat{M})^\Gamma \tag{5.4.13}$$

and

$$Z(\widehat{G})^\Gamma \hookrightarrow Z_s(\widehat{M})^\Gamma, \tag{5.4.14}$$

the first coming from (5.3.9) and the second from (5.3.8). Together these yield a homomorphism

$$Z_s(\widehat{M}_1)^\Gamma \times Z(\widehat{G})^\Gamma \to Z_s(\widehat{M})^\Gamma$$

and this homomorphism yields an isomorphism

$$A \simeq Z_s(\widehat{M})^\Gamma. \tag{5.4.15}$$

In this way the target of (5.4.12) can be viewed as a subset of $X^*(Z_s(\widehat{M})^\Gamma)$ and this subset is easily seen to be $X^*(Z_s(\widehat{M})^\Gamma)^+$. Thus we get a canonical bijection

$$\mathbf{B}_s(G)_P \simeq X^*(Z_s(\widehat{M})^\Gamma)^+, \tag{5.4.16}$$

as desired. It is routine to check that this bijection satisfies all the properties stated in the theorem.

6. The Subset $\mathbf{B}(H,\mu)$ of $\mathbf{B}(H)$

In this section we define certain subsets $\mathbf{B}(H,\mu)$ of $\mathbf{B}(H)$. Motivation for introducing these subsets is given in the introduction.

6.1. Let G be a quasi-split connected reductive group over F, and let (H,Ψ) be an inner form of G. We use the same notation as in the last two sections. In particular B denotes a Borel subgroup of G over F, and T denotes a maximal F-torus in B. Let $\mu \in X_*(T)$ and suppose that μ lies in the closed Weyl chamber in $X_*(T)_\mathbb{R}$ determined by B. Of course we may also regard μ as a character on $\widehat{T}$, which we can restrict to the subgroup $Z(\widehat{G})^\Gamma$ of $\widehat{T}$, obtaining an element

$$\mu_1 \in X^*(Z(\widehat{G})^\Gamma) = X^*(Z(\widehat{H})^\Gamma).$$

We can also restrict μ to $\widehat{T}^\Gamma$; then, applying the homomorphism

$$X^*(\widehat{T}^\Gamma) \to \mathfrak{A}$$

(a special case of (5.1.3)) to this element of $X^*(\widehat{T}^\Gamma)$, we obtain an element

$$\mu_2 \in \mathfrak{A}.$$

Equivalently, viewing $\mathfrak{A}$ as the subspace of Γ-invariant elements in $X_*(T)_{\mathbb{R}}$, we have

$$\mu_2 = [\Gamma : \Gamma_\mu]^{-1} \sum_{\tau \in \Gamma/\Gamma_\mu} \tau(\mu), \tag{6.1.1}$$

where Γ_μ denotes the stabilizer of μ in Γ.

6.2. Let $\mathbf{B}(H,\mu)$ denote the subset of $\mathbf{B}(H)$ consisting of all $\mathbf{h} \in \mathbf{B}(H)$ such that the image of $\mathbf{h}$ under the map (4.9.1) is equal to μ_1 and such that the image $\bar{\nu} \in \mathfrak{A}$ of $\mathbf{h}$ under the Newton map (4.2.1) satisfies

$$\bar{\nu} \leqslant \mu_2. \tag{6.2.1}$$

Here $\leqslant$ is the usual order on $\mathfrak{A}$; thus $\bar{\nu} \leqslant \mu_2$ means that $\mu_2 - \bar{\nu}$ is a nonnegative linear combination of positive coroots in $X_*(T)_{\mathbb{R}}$, or, equivalently, a nonnegative linear combination of positive relative coroots in $\mathfrak{A}$. Let Ω_F be the relative Weyl group of the maximal split torus A in T; recall that Ω_F can be identified with the fixed points Ω^Γ of Γ in Ω, where Ω denotes the absolute Weyl group of T in G. It is known (see [A]) that (6.2.1) is equivalent to the the following condition:

$$\bar{\nu} \text{ lies in the convex hull of the orbit } \Omega_F \cdot \mu_2 \text{ of } \mu_2 \text{ under } \Omega_F. \tag{6.2.2}$$

6.3. Since μ_1, μ_2 depend only on the restriction of μ to $\widehat{T}^\Gamma$, the subset $\mathbf{B}(H,\mu)$ depends only on this restriction, or, equivalently, only on the image of μ in the group $X_*(T)_\Gamma$ of coinvariants of Γ in $X_*(T)$.

6.4. It follows easily from Theorem 5.4 that $\mathbf{B}(H,\mu)$ is a finite set. It is clear that $\mathbf{B}(H,\mu)$ contains the unique basic element in $\mathbf{B}(H)$ whose image under (4.4.1) is equal to μ_1, and it is clear that $\mathbf{B}(H,\mu)$ contains no other basic element. We say that the pair (H,μ) is *uniform* if $\mathbf{B}(H,\mu)$ has exactly one element, namely the basic element we just described. Again motivation for making this definition is given in the introduction.

6.5. Let T_{ad} denote the image of T in G_{ad}, and let μ_{ad} denote the image of μ under

$$X_*(T) \to X_*(T_{\text{ad}}).$$

Then the natural map $\mathbf{B}(H) \to \mathbf{B}(H_{\text{ad}})$ induces a bijection

$$\mathbf{B}(H,\mu) \simeq \mathbf{B}(H_{\text{ad}},\mu_{\text{ad}}). \tag{6.5.1}$$

Indeed, this follows immediately from Corollary 4.11. In particular (H,μ) is uniform if and only if $(H_{\text{ad}},\mu_{\text{ad}})$ is uniform.

Suppose that H is a product

$$H = H_1 \times H_2$$

and that μ_1, μ_2 are the two components of μ. Then there is an obvious bijection

$$\mathbf{B}(H,\mu) = \mathbf{B}(H_1,\mu_1) \times \mathbf{B}(H_2,\mu_2). \tag{6.5.2}$$

In particular (H,μ) is uniform if and only if (H_1,μ_1) and (H_2,μ_2) are both uniform.

Let E be a finite extension of F in $\overline{F}$, and let G_0 be a quasi-split connected reductive group over E. We use $R(G_0)$ to denote the F-group obtained from G_0 by Weil's restriction of scalars from E to F. Suppose that $G = R(G_0)$. By Shapiro's lemma every inner form (H,Ψ) is isomorphic to one of the form $(R(H_0), R(\Psi_0))$, where (H_0,Ψ_0) is an inner form of G_0. Of course T, B are of the form $R(T_0)$, $R(B_0)$ for a maximal torus T_0 and Borel subgroup B_0 in G_0 containing T_0. The dominant coweight μ lies in

$$X_*(T) = \mathrm{Ind}(X_*(T_0)),$$

where $\mathrm{Ind}(X_*(T_0))$ denotes the Γ-module induced by the Γ_E-module $X_*(T_0)$ (we denote by Γ_E the Galois group of $\overline{F}$ over E). Thus μ can be thought of as a function

$$\phi\colon \Gamma \to X_*(T_0)$$

satisfying

$$\phi(\rho\tau) = \rho\cdot\phi(\tau) \quad \text{for all } \rho\in\Gamma_E, \quad \tau\in\Gamma.$$

Pick a set Γ_0 of coset representatives for the cosets $\Gamma_E\backslash\Gamma$ and form the sum

$$\mu_0 = \sum_{\tau\in\Gamma_0} \phi(\tau) \in X_*(T_0),$$

noting that each $\phi(\tau)$ is dominant in $X_*(T_0)$, so that μ_0 is dominant as well. Of course the image of μ_0 in the group of coinvariants $X_*(T_0)_{\Gamma_E}$ is well-defined. It is easy to see that there is a canonical bijection

$$\mathbf{B}(H,\mu) = \mathbf{B}(H_0,\mu_0). \tag{6.5.3}$$

6.6. If $\mu = 0$, then (H,μ) is uniform. Indeed, in this case the Newton point $\bar\nu$ of any $\mathbf{h} \in \mathbf{B}(H,\mu)$ must be 0. Therefore $\mathbf{B}(H,\mu)$ consists of basic elements, and as we have seen, $\mathbf{B}(H,\mu)$ contains exactly one basic element.

LEMMA 6.7. *Suppose H is an F-simple adjoint group, and suppose that μ is nonzero. Suppose further that H is not anisotropic over F. Then (H,μ) is not uniform.*

By hypothesis H contains a proper parabolic subgroup Q. Choose a Levi subgroup L of Q. Let $P = MN$ be the unique standard parabolic subgroup of G such that $\psi(P)$ is conjugate to Q under $H(\overline{F})$ for all $\psi\in\Psi$. Let Ψ_M be the set of $\psi\in\Psi$ such that $\psi(P) = Q$ and $\psi(M) = L$; then Ψ_M is (the set of $\overline{F}$-points of) an F-torsor under M. In particular L is an inner form of M.

Now let w be an element of the relative Weyl group Ω_F of A in G. Restricting the character $w\mu$ on $\widehat{T}$ to the subgroup $Z(\widehat{M})^\Gamma$, we get an element of $X^*(Z(\widehat{M})^\Gamma)$, which by means of the canonical bijection

$$\mathbf{B}(L)_b \simeq X^*(Z(\widehat{M})^\Gamma)$$

determines a basic element $\mathbf{x}(w)$ in $\mathbf{B}(L)$. Let $\mathbf{h}(w) \in \mathbf{B}(H)$ denote the image of $\mathbf{x}(w)$ under the natural map

$$\mathbf{B}(L) \to \mathbf{B}(H).$$

The elements $\mathbf{h}(w)$ all belong to $\mathbf{B}(H,\mu)$. Since H is adjoint, the element $\mathbf{h}(w)$ is basic in $\mathbf{B}(H)$ if and only if its image under the Newton map is trivial, which happens if and only if the restriction of $w\mu$ to the identity component of $Z(\widehat{M})^\Gamma$ is trivial. Therefore $\mathbf{h}(w)$ is basic in $\mathbf{B}(H)$ if and only if $w\mu_2$ lies in the kernel K of the natural surjection $\mathfrak{A} \to \mathfrak{A}_M$ (dual to $Z(\widehat{M})^\Gamma \hookrightarrow \widehat{T}^\Gamma$).

Note that our hypothesis that μ is nonzero implies that μ_2 is nonzero (this is clear from (6.1.1) since μ is dominant and Γ preserves the cone of dominant coweights). Since G is F-simple and adjoint, the relative root system of A in G is irreducible (since G is quasi-split its relative Dynkin graph is the quotient by Γ of its absolute Dynkin graph [T, 2.5.3]). Therefore the representation of Ω_F on $\mathfrak{A}$ is irreducible, and hence $\Omega_F \cdot \mu_2$ spans $\mathfrak{A}$. We conclude that there exists $w \in \Omega_F$ such that $w\mu_2 \notin K$. The corresponding element $\mathbf{h}(w)$ in $\mathbf{B}(H,\mu)$ is not basic, and therefore (H,μ) is not uniform.

6.8. Using (6.5.1), (6.5.2), (6.5.3), we see that in order to classify all uniform pairs (H,μ) it suffices to classify the ones for which H is an absolutely simple adjoint group. By Lemma 6.7 we may further assume that H is anisotropic over F (otherwise (H,μ) is uniform only for $\mu = 0$). Any absolutely simple, adjoint, anisotropic group over F is an inner form of $P\mathrm{GL}_n$ [Kn]. Therefore we may assume that $G = P\mathrm{GL}_n$ for some $n \geqslant 2$ and that $H = D_{j/n}^\times/F^\times$, where $D_{j/n}$ denotes a central division algebra over F having dimension n^2 and Hasse invariant j/n (of course j must be relatively prime to n). We denote the algebraic group $D_{j/n}^\times/F^\times$ by $H_{j/n}$.

We make the usual choices for T, B (diagonal matrices and upper triangular matrices, taken modulo scalar matrices), and we represent coweights $\mu \in X_*(T)$ as n-tuples $(\mu_1,\dots,\mu_n)$ of integers, modulo constant n-tuples $(a,\dots,a)$. Of course μ is dominant if and only if

$$\mu_1 \geqslant \mu_2 \geqslant \dots \geqslant \mu_n. \tag{6.8.1}$$

For any integer k between 1 and n we write $\mu(k)$ for the n-tuple $(1,\dots,1,0,\dots,0)$ in which 1 is repeated k times and 0 is repeated $n-k$ times.

LEMMA 6.9. *The pairs $(H_{1/n},\mu(1))$ and $(H_{(n-1)/n},\mu(n-1))$ are uniform. There are no other uniform pairs of the form $(H_{j/n},\mu)$ except those for which μ is 0.*

It is more convenient to work with GL_n and $D_{j/n}^{\times}$ rather than their adjoint groups, and by (6.5.1) it is harmless to do so. We must show that the only uniform pairs $(D_{j/n}^{\times}, \mu)$ with μ nonconstant (in other words, not of the form $(a, \dots, a)$) are obtained by taking $j = 1$ and μ of the form

$$(1, 0, \dots, 0) + (a, \dots, a)$$

or by taking $j = n - 1$ and μ of the form

$$(1, \dots, 1, 0) + (a, \dots, a).$$

6.10. We begin the proof of the lemma by working out the image of the Newton map for $D_{j/n}^{\times}$. Of course $\overline{C}_{\mathbb{Q}}$ consists of all n-tuples $\nu = (\nu_1, \dots, \nu_n)$ of rational numbers satisfying

$$\nu_1 \geqslant \nu_2 \geqslant \dots \geqslant \nu_n. \tag{6.10.1}$$

For such an n-tuple and a rational number x we say that the *multiplicity* of x in ν is the number of indices i for which ν_i is equal to x, and we write $m_\nu(x)$ for this multiplicity.

It follows from 5.1 that the image of the Newton map for GL_n is the set E consisting of all elements $\nu \in \overline{C}_{\mathbb{Q}}$ such that

$$m_\nu(\nu_i) \cdot \nu_i \in \mathbb{Z} \quad \text{for } i = 1, \dots, n.$$

Let g_τ be a basic 1-cocycle of W in $\mathrm{GL}_n(\overline{L})$ whose image under the Newton map is

$$(-j/n, -j/n, \dots, -j/n).$$

Let J be the inner form of G associated to g_τ (see 4.4). Then J is isomorphic to $D_{j/n}^{\times}$ (see 2.1). It follows from 4.18 that the image E_j of the Newton map for $D_{j/n}^{\times}$ satisfies

$$(-j/n, -j/n, \dots, -j/n) + E_j = E.$$

Therefore E_j consists of all elements in $\overline{C}_{\mathbb{Q}}$ of the form

$$(j/n, j/n, \dots, j/n) + \nu$$

for some element ν in E.

Let $(\mu_1, \dots, \mu_n)$ be a dominant coweight for the diagonal torus in $G = \mathrm{GL}_n$. Thus μ satisfies (6.8.1). We assume that μ is nonconstant, so that $\mu_1 > \mu_n$. Since $Z(\widehat{G})^\Gamma = \mathbb{C}^\times$, the group $X^*(Z(\widehat{G})^\Gamma)$ is torsion-free. Therefore the Newton map

$$\mathbf{B}(D_{j/n}^{\times}) \to E_j \simeq E$$

is bijective, and under this bijection the subset $\mathbf{B}(D_{j/n}^{\times}, \mu)$ corresponds to the subset of E consisting of all elements ν such that

$$\nu + (j/n, \ldots, j/n) \leqslant \mu. \tag{6.10.2}$$

Recall the explicit form of the order $\leqslant$ on $\mathfrak{A} = \mathbb{R}^n$: $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ satisfy $x \leqslant y$ if and only if

$$\begin{aligned} x_1 &\leqslant y_1, \\ x_1 + x_2 &\leqslant y_1 + y_2, \\ &\vdots \\ x_1 + x_2 + \ldots + x_{n-1} &\leqslant y_1 + y_2 + \ldots + y_{n-1}, \\ x_1 + x_2 + \ldots + x_{n-1} + x_n &= y_1 + y_2 + \ldots + y_{n-1} + y_n. \end{aligned}$$

The unique basic element in $\mathbf{B}(D_{j/n}^{\times}, \mu)$ corresponds to the constant solution

$$\nu = (a, \ldots, a)$$

of (6.10.2), where a is determined by the condition

$$na + j = \mu_1 + \ldots + \mu_n.$$

In order to prove the lemma we must show that the inequality (6.10.2) has a nonconstant solution $\nu \in E$ except in the two special cases specified in the statement of the lemma. We can simplify this task considerably by means of the following remark. If (6.10.2) admits a nonconstant solution $\nu \in E$, then it admits a nonconstant solution in E of the special form

$$\nu' = (a/r, \ldots, a/r, b/s, \ldots, b/s), \tag{6.10.3}$$

where r, s are integers between 1 and $n - 1$ such that $r + s = n$, and where a/r is repeated r times and b/s is repeated s times. Indeed, if $\nu = (\nu_1, \ldots, \nu_n) \in E$ is nonconstant and satisfies (6.10.2), we let r be the multiplicity of ν_1 in ν and define $a, b \in \mathbb{Z}$ by

$$\begin{aligned} a &= r\nu_1 = \ldots = r\nu_r, \\ b &= \nu_{r+1} + \ldots + \nu_n. \end{aligned}$$

It is easy to see that $\nu' \in E$, and that $\nu' \leqslant \nu$; thus ν' is a nonconstant solution of (6.10.2) of the desired form.

An n-tuple ν' of the form (6.10.3) is nonconstant and lies in E if and only if

$$as > br, \tag{6.10.4}$$

and it satisfies (6.10.2) if and only if

$$a+b+j=\mu_1+\ldots+\mu_n \tag{6.10.5}$$

and

$$an+rj\leqslant n(\mu_1+\ldots+\mu_r). \tag{6.10.6}$$

Using (6.10.5) to eliminate b, we see that (6.10.2) has a nonconstant solution of the form (6.10.3) if and only if there exists $a\in\mathbb{Z}$ satisfying

$$r(\mu_1+\ldots+\mu_n)-rj<an\leqslant n(\mu_1+\ldots+\mu_r)-rj. \tag{6.10.7}$$

It is obvious that (6.10.7) has a solution whenever the difference between

$$n(\mu_1+\ldots+\mu_r)-rj$$

and

$$r(\mu_1+\ldots+\mu_n)-rj$$

is greater than or equal to n. Therefore, if (6.10.7) has no solutions we conclude that

$$n(\mu_1+\ldots+\mu_r)-r(\mu_1+\ldots+\mu_n)<n. \tag{6.10.8}$$

Now suppose that (6.10.2) has no nonconstant solutions in E. Then (6.10.8) holds for each r between 1 and $n-1$. Adding the inequalities (6.10.8) for $r=1$ and $r=n-1$, we find that

$$\mu_1-\mu_n<2.$$

Since μ_1, μ_2 are integers satisfying $\mu_1>\mu_n$, we conclude that $\mu_1-\mu_n=1$. Up to the addition of a constant vector (which is of no importance), μ must be equal to $\mu(k)$ for some k between 1 and $n-1$. Taking $r=k$ in (6.10.8), we find that

$$nk-k^2<n$$

which is equivalent to

$$(k-1)(k-n+1)\geqslant 0$$

and implies that $k=1$ or $k=n-1$. Thus μ is equal to $\mu(1)$ or $\mu(n-1)$ (up to constant vectors).

Suppose that $\mu=\mu(1)$. Then (6.10.7) reduces to

$$r-rj<an\leqslant n-rj. \tag{6.10.9}$$

If (6.10.2) has no nonconstant solutions in E, then (6.10.9) has no solution for $r=1$, which can happen only if j is congruent to 1 modulo n. Conversely, if $j=1$, then (6.10.9) reduces to

$$0<an\leqslant n-r,$$

which has no solutions (no matter what r is). Similarly, if $\mu = \mu(n-1)$, then (6.10.2) admits nonconstant solutions in E except when j is congruent to $n-1$ modulo n. The proof of the lemma is now complete.

6.11. Now we classify all uniform pairs (H, μ) with μ nonzero and H adjoint and F-simple. By Lemma 6.7 H is necessarily anisotropic over F, and hence we may assume that there exists a finite extension E of F and a central division algebra D over E such that $H = R(H_0)$ (as in 6.5 we use R to denote Weil's restriction of scalars from E to F), where H_0 is the E-group $D^\times/E^\times$. Write the Hasse invariant of D as j/n with $1 \leqslant j \leqslant n-1$ and $(j,n) = 1$. Of course $G_0 = P\mathrm{GL}_n(E)$ is a quasi-split inner form of H_0, and $R(G_0)$ is a quasi-split inner form of H.

Giving a dominant coweight μ for $R(G_0)$ is the same as giving a family of dominant coweights $\mu(\iota)$ for G_0, one for each embedding $\iota\colon E \to \overline{F}$ over F. We saw in 6.5 that (H, μ) is uniform if and only if (H_0, μ_0) is uniform, where

$$\mu_0 = \sum_\iota \mu(\iota).$$

Clearly μ_0 is nonzero if and only if μ is nonzero. Therefore, by Lemma 6.9 either $j = 1$ and $\mu_0 = \mu(1)$, or $j = n-1$ and $\mu_0 = \mu(n-1)$.

Suppose that $j = 1$ and $\mu_0 = \mu(1)$. Since there is no nontrivial way to decompose $\mu(1)$ as a sum of dominant coweights, the coweights $\mu(\iota)$ must be 0 except for one embedding ι_0, for which $\mu(\iota_0) = \mu(1)$. Similarly, if $j = n-1$, then the coweights $\mu(\iota)$ must be 0 except for one embedding ι_0, for which $\mu(\iota_0) = \mu(n-1)$.

7. The map $w_G\colon G(L) \to X^*(Z(\widehat{G})^I)$

7.1. Let H be a connected reductive group over L. Recall from 1.1 that we denote by I the group $\mathrm{Gal}(\overline{L}/L)$. In this section we are going to construct a natural surjective homomorphism

$$w_H\colon H(L) \to X^*(Z(\widehat{H})^I). \tag{7.1.1}$$

We will also see that when H is defined over F, the map w_G can be used to construct the map (4.9.1)

$$\mathbf{B}(H) \to X^*(Z(\widehat{H})^\Gamma).$$

7.2. We begin by constructing w_H in the case of tori. Let T be a torus over L. Then the natural map

$$X_*(T) = X^*(\widehat{T}) \xrightarrow{\text{res}} X^*(\widehat{T}^I)$$

identifies $X^*(\widehat{T}^I)$ with $X_*(T)_I$, the group of coinvariants of I in $X_*(T)$. Thus we seek to define a functorial surjection

$$w_T\colon T(L) \to X_*(T)_I. \tag{7.2.1}$$

Of course there is a natural surjection

$$q_T\colon X_*(T)_I \to \mathrm{Hom}(X^*(T)^I, \mathbb{Z}) \tag{7.2.2}$$

(an element μ in $X_*(T)$ determines a homomorphism $\lambda \mapsto \langle \lambda, \mu \rangle$ from $X^*(T)^I$ to $\mathbb{Z}$, where $\langle \cdot, \cdot \rangle$ denotes the canonical pairing between $X^*(T)$ and $X_*(T)$). There is an obvious functorial map

$$v_T : T(L) \to \mathrm{Hom}(X^*(T)^I, \mathbb{Z}), \tag{7.2.3}$$

sending $t \in T(L)$ to the homomorphism

$$\lambda \mapsto \mathrm{val}(\lambda(t))$$

from $X^*(T)^I$ to $\mathbb{Z}$. Here val denotes the usual valuation on L, normalized so that uniformizing elements have valuation 1. We are going to define w_T in such a way that

$$q_T \circ w_T = v_T. \tag{7.2.4}$$

Note that v_T is always surjective. Indeed, let T_a denote the maximal anisotropic subtorus of T and put $S = T/T_a$. Consider the commutative diagram

$$\begin{array}{ccc} T(L) & \longrightarrow & S(L) \\ \downarrow{\scriptstyle v_T} & & \downarrow{\scriptstyle v_S} \\ \mathrm{Hom}(X^*(T)^I, \mathbb{Z}) & \longrightarrow & \mathrm{Hom}(X^*(S)^I, \mathbb{Z}). \end{array}$$

The bottom arrow is an isomorphism, and the top arrow is surjective since $H^1(L, T_a)$ is trivial. Moreover v_S is obviously surjective, since S is split. Therefore v_T is surjective.

The map q_T is an isomorphism whenever $X_*(T)_I$ is torsion-free, and in this case we define w_T to be the unique map satisfying (7.2.4); since v_T is surjective, so is w_T. Of course w_T is functorial in T for such T. Recall that a torus T over L is said to be *induced* if $X_*(T)$ has a $\mathbb{Z}$-basis that is permuted by I. If T is induced, then $X_*(T)_I$ is torsion-free, so that w_T has been defined.

For any torus T there exists an induced torus R and a surjective map

$$X_*(R) \to X_*(T)$$

of I-modules. Moreover there exists another induced torus S and an I-module map

$$X_*(S) \to X_*(R)$$

such that

$$X_*(S) \to X_*(R) \to X_*(T) \to 0$$

is exact. In this way we get an exact sequence

$$S \xrightarrow{f} R \xrightarrow{g} T \to 1$$

in which the kernels of f and g are tori. The diagram

$$\begin{array}{ccccccc} S(L) & \longrightarrow & R(L) & \longrightarrow & T(L) & \longrightarrow & 1 \\ \downarrow{\scriptstyle w_S} & & \downarrow{\scriptstyle w_R} & & & & \\ X_*(S)_I & \longrightarrow & X_*(R)_I & \longrightarrow & X_*(T)_I & \longrightarrow & 0 \end{array} \tag{7.2.5}$$

is commutative and has exact rows (use that $H^1(L, \ker f)$ and $H^1(L, \ker g)$ are trivial). We define w_T^g to be the unique map from $T(L)$ to $X_*(T)_I$ making

$$\begin{array}{ccc} R(L) & \longrightarrow & T(L) \\ \downarrow{\scriptstyle w_R} & & \downarrow{\scriptstyle w_T^g} \\ X_*(R)_I & \longrightarrow & X_*(T)_I \end{array} \tag{7.2.6}$$

commute (the existence and uniqueness of w_T^g follow from (7.2.5)).

Let $T \to U$ be a map of tori. Choose an induced torus Q and a surjection

$$X_*(Q) \to X_*(U)$$

of I-modules, and let $h\colon Q \to U$ be the corresponding map of tori. We claim that

$$\begin{array}{ccc} T(L) & \longrightarrow & U(L) \\ \downarrow{\scriptstyle w_T^g} & & \downarrow{\scriptstyle w_U^h} \\ X_*(T)_I & \longrightarrow & X_*(U)_I \end{array} \tag{7.2.7}$$

commutes. Indeed, it is easy to construct an induced torus R', a surjective I-map

$$X_*(R') \to X_*(R)$$

and an I-map

$$X_*(R') \to X_*(U)$$

such that

$$\begin{array}{ccccc} R' & \longrightarrow & R & \longrightarrow & T \\ \| & & & & \downarrow \\ R' & \longrightarrow & Q & \longrightarrow & U \end{array}$$

commutes. The surjectivity of $R'(L) \to T(L)$ together with the functoriality of w for the maps $R' \to R$ and $R' \to Q$ of induced tori establishes the commutativity of (7.2.7).

It follows from the commutativity of (7.2.7) that w_T^g is independent of the choice of g. Thus we may define w_T to be any one of the maps w_T^g. The commutativity of (7.2.7) further implies that w_T is functorial in T. The map w_T is surjective and satisfies (7.2.4) (use (7.2.5) to deduce these statements from the corresponding ones for the induced torus R).

7.3. Let L' be a finite extension of L in $\overline{L}$, and let I' denote the subgroup $\mathrm{Gal}(\overline{L}/L')$ of I. Then the diagrams

$$\begin{array}{ccc} T(L') & \xrightarrow{w} & X_*(T)_{I'} \\ \Big\downarrow{\scriptstyle N} & & \Big\downarrow{\scriptstyle \alpha} \\ T(L) & \xrightarrow{w} & X_*(T)_I \end{array} \tag{7.3.1}$$

and

$$\begin{array}{ccc} T(L') & \xrightarrow{w} & X_*(T)_{I'} \\ \Big\uparrow{\scriptstyle \beta} & & \Big\uparrow{\scriptstyle N} \\ T(L) & \xrightarrow{w} & X_*(T)_I \end{array} \tag{7.3.2}$$

both commute. In (7.3.1) N is the norm map from $T(L')$ to $T(L)$ and α is the obvious surjection (induced by the identity map on $X_*(T)$). In (7.3.2) β is induced by the embedding $L \hookrightarrow L'$ and N is given by

$$N(\bar{\mu}) = \sum_{\tau \in I' \backslash I} \tau\mu$$

for an element $\bar{\mu} \in X_*(T)_I$ represented by an element $\mu \in X_*(T)$. It is easy to prove the commutativity of these diagrams by reducing to the case in which T is an induced torus and then using (7.2.4).

Diagram (7.3.1) suggests a shorter way to define the map w, as the referee pointed out. Choose a finite Galois extension L' of L in $\overline{L}$ that splits T. The norm map N identifies $T(L)$ with the coinvariants of $\mathrm{Gal}(L'/L)$ on $T(L')$ (see the appendix to Chapter 1 in [S1]). It is clear how to define w for $T(L')$, and we define w for $T(L)$ to be the unique map making the diagram (7.3.1) commute. It is easy to see that the resulting map is independent of the choice of L'.

7.4. Now we define the surjection w_H for any connected reductive group H. We begin with the case in which the derived group H_{der} of H is simply connected. Then we put

$$D = H/H_{\mathrm{der}}.$$

Recall that

$$\widehat{D} = Z(\widehat{H}).$$

We define w_H to be the unique map making the diagram

$$\begin{array}{ccc} H(L) & \xrightarrow{w_H} & X^*(Z(\widehat{H})^I) \\ \downarrow & & \| \\ D(L) & \xrightarrow{w_D} & X^*(\widehat{D}^I) \end{array} \tag{7.4.1}$$

commute. Note that w_H is surjective since w_D and the map

$$H(L) \to D(L)$$

are both surjective.

Now consider the general case. Pick a z-extension $H' \to H$ with kernel Z. The map $w_{H'}$ has already been defined. We define w_H to be the unique map making

$$\begin{array}{ccc} H'(L) & \xrightarrow{w_{H'}} & X^*(Z(\widehat{H}')^I) \\ \downarrow & & \downarrow \\ H(L) & \xrightarrow{w_H} & X^*(Z(\widehat{H})^I) \end{array} \tag{7.4.2}$$

commute. Of course uniqueness follows from the surjectivity of $H'(L) \to H(L)$ and existence follows from the commutativity of

$$\begin{array}{ccc} Z(L) & \longrightarrow & X^*(\widehat{Z}^I) \\ \downarrow & & \downarrow \\ H'(L) & \longrightarrow & X^*(Z(\widehat{H}')^I). \end{array}$$

The map w_H is surjective since the maps $w_{H'}$ and

$$X^*(Z(\widehat{H}')^I) \to X^*(Z(\widehat{H})^I)$$

are surjective. Using [K4, 2.4.4], one sees easily that w_H is independent of H' and that w_H is functorial in H.

There is an obvious homomorphism

$$v_H \colon H(L) \to \mathrm{Hom}(X_*(Z(\widehat{H}))^I, \mathbb{Z}), \tag{7.4.3}$$

sending $h \in H(L)$ to the homomorphism

$$\lambda \mapsto \mathrm{val}(\lambda(h))$$

from $X_*(Z(\widehat{H}))^I$ to $\mathbb{Z}$ (we view elements of $X_*(Z(\widehat{H}))^I$ as homomorphisms from H to $\mathbb{G}_m$ defined over L). There is an obvious surjective homomorphism

$$q_H \colon X^*(Z(\widehat{H})^I) \to \mathrm{Hom}(X_*(Z(\widehat{H}))^I, \mathbb{Z}), \tag{7.4.4}$$

whose kernel is the torsion subgroup of $X^*(Z(\widehat{H})^I)$. It is not hard to check that

$$q_H \circ w_H = v_H \tag{7.4.5}$$

(reduce first to the case in which H_{der} is simply connected and then to the case of tori).

7.5. Now suppose that H is a connected reductive group over F. Then the surjection w_H commutes with the action of the Frobenius element σ. Therefore w_H induces a map

$$B(H) \to X^*(Z(\widehat{H})^I)_{\langle\sigma\rangle}, \tag{7.5.1}$$

where the subscript $\langle\sigma\rangle$ indicates that we are taking coinvariants for the group $\langle\sigma\rangle$. Since $X^*(Z(\widehat{H})^I)$ can be identified with the group of coinvariants of I in $X^*(Z(\widehat{H}))$, we see that

$$\begin{aligned} X^*(Z(\widehat{H})^I)_{\langle\sigma\rangle} &= X^*(Z(\widehat{H}))_\Gamma \\ &= X^*(Z(\widehat{H})^\Gamma). \end{aligned}$$

Moreover $B(H)$ can be identified with $\mathbf{B}(H)$. Thus (7.5.1) can also be viewed as a map

$$\mathbf{B}(H) \to X^*(Z(\widehat{H})^\Gamma). \tag{7.5.2}$$

We claim that the map (7.5.2) coincides with the map (4.9.1). As usual one uses z-extensions to reduce to the case in which H is a torus. Then by [K, 2.2(b)] one reduces to the case $H = \mathbb{G}_m$, which is easy to treat directly.

7.6. Let T be a torus over F. We write $T(L)_0$ for the kernel of

$$v_T \colon T(L) \to \mathrm{Hom}(X^*(T)^I, \mathbb{Z})$$

and we write $T(L)_1$ for the kernel of

$$w_T \colon T(L) \to X_*(T)_I.$$

Obviously $T(L)_1$ is a subgroup of finite index in $T(L)_0$. Moreover $T(L)_1$ is equal to $T(L)_0$ if $X_*(T)_I$ is torsion-free, which happens whenever T is an induced torus.

We claim that

$$H^1(\langle\sigma\rangle, T(L)_1) = \{1\}. \tag{7.6.1}$$

Choose induced tori R, S over F and an exact sequence

$$X_*(S) \to X_*(R) \to X_*(T) \to 0$$

of Γ-modules. The diagram

$$\begin{array}{ccccccc} S(L) & \longrightarrow & R(L) & \longrightarrow & T(L) & \longrightarrow & 1 \\ \downarrow{\scriptstyle w_S} & & \downarrow{\scriptstyle w_R} & & \downarrow{\scriptstyle w_T} & & \\ X_*(S)_I & \longrightarrow & X_*(R)_I & \longrightarrow & X_*(T)_I & \longrightarrow & 0 \end{array}$$

is commutative with exact rows, and all the vertical arrows are surjective. Therefore the map

$$R(L)_1 \to T(L)_1$$

is surjective, and hence the map

$$H^1(\langle\sigma\rangle, R(L)_1) \to H^1(\langle\sigma\rangle, T(L)_1)$$

is surjective as well. Therefore it suffices to prove (7.6.1) for induced tori T.

We may assume that $T = R_{E/F}\mathbb{G}_m$ for a finite extension E of F in $\overline{F}$. Then

$$T(L) = M^\times \times \ldots \times M^\times,$$

where M is the compositum of E and L in $\overline{F}$, and

$$T(L)_1 = \mathfrak{o}_M^\times \times \ldots \times \mathfrak{o}_M^\times,$$

where $\mathfrak{o}_M$ denotes the valuation ring in M. By Shapiro's lemma we reduce to the case in which there is only one factor $M^\times$ (this occurs when E is totally ramified over F), and then by replacing L, F by M, E we reduce to the case in which T is $\mathbb{G}_m$. Thus we must show that the map

$$\sigma - 1 \colon \mathfrak{o}_L^\times \to \mathfrak{o}_L^\times$$

is surjective. This is an easy exercise (see the proof of Proposition 2.3 in [K]). We are done proving the claim.

Now consider the exact sequence

$$1 \to T(L)_1 \to T(L) \xrightarrow{w} X_*(T)_I \to 1.$$

We see from the associated long exact cohomology sequence of $\langle\sigma\rangle$-cohomology that

$$T(F) \to (X_*(T)_I)^{\langle\sigma\rangle} \tag{7.6.2}$$

is surjective, and that

$$B(T) \to X_*(T)_\Gamma$$

is an isomorphism. We already knew the second fact, but this alternative proof provides additional insight.

7.7. Again let H be a connected reductive group over F. The restriction of w_H to $H(F)$ provides a homomorphism

$$H(F) \to X^*(Z(\widehat{H})^I)^{\langle\sigma\rangle}. \tag{7.7.1}$$

We claim that (7.7.1) is surjective.

For tori (7.7.1) can be thought of as (7.6.2), which we already know is surjective. If the derived group H_{der} of H is simply connected, the surjectivity of (7.7.1) follows from the surjectivity of the map (7.6.2) for the torus H/H_{der}. For arbitrary H choose a z-extension

$$1 \to Z \to H' \to H \to 1.$$

Consider the commutative square

$$\begin{array}{ccc} H'(F) & \xrightarrow{(7.7.1)'} & X^*(Z(\widehat{H}')^I)^{\langle\sigma\rangle} \\ \downarrow & & \downarrow \\ H(F) & \xrightarrow{(7.7.1)} & X^*(Z(\widehat{H})^I)^{\langle\sigma\rangle}. \end{array}$$

We know that (7.7.1)$'$ is surjective. Moreover, since $\widehat{Z}^I$ is connected, the sequence

$$1 \to X^*(\widehat{Z}^I) \to X^*(Z(\widehat{H}')^I) \to X^*(Z(\widehat{H})^I) \to 1$$

is exact. Taking invariants under $\langle\sigma\rangle$, we find that

$$X^*(Z(\widehat{H}')^I)^{\langle\sigma\rangle} \to X^*(Z(\widehat{H})^I)^{\langle\sigma\rangle}$$

is surjective; here we used that the group

$$\begin{aligned} H^1(\langle\sigma\rangle, X^*(\widehat{Z}^I)) &= X^*(\widehat{Z}^I)_{\langle\sigma\rangle} \\ &= X^*(\widehat{Z})_\Gamma \end{aligned}$$

is torsion-free. We see from the commutative square above that (7.7.1) is surjective, as desired.

8. Algebraic 1-cocycles

Let T be a torus over F and let K be a finite Galois extension of F in $\overline{F}$ that splits T. Let $W_{K/F}$ be the Weil group associated to K/F (see B.3 for a review of $W_{K/F}$). In this section we will define a group

$$H^1_{\text{alg}}(W_{K/F}, T(K))$$

and a canonical isomorphism

$$H^1_{\text{alg}}(W_{K/F}, T(K)) \simeq \mathbf{B}(T).$$

8.1. Let $\tau \mapsto t_\tau$ be an abstract 1-cocycle of $W_{K/F}$ in $T(K)$ (of course $W_{K/F}$ acts on $T(K)$ in the obvious way, through its quotient Gal(K/F)). We say that t_τ is an *algebraic* 1-cocycle if there exists an element $\mu \in X_*(T)$ such that

$$t_x = \mu(x)$$

for all x in the subgroup $K^\times$ of $W_{K/F}$. The cocharacter μ is uniquely determined by the 1-cocycle and is fixed by Γ. We write $Z^1_{\text{alg}}(W_{K/F}, T(K))$ for the group of algebraic 1-cocycles of $W_{K/F}$ in $T(K)$. Any abstract 1-coboundary $\tau \mapsto t^{-1}\tau(t)$ is obviously algebraic (the associated μ is trivial). We define $H^1_{\text{alg}}(W_{K/F}, T(K))$ to be the quotient of $Z^1_{\text{alg}}(W_{K/F}, T(K))$ by the subgroup of 1-coboundaries.

Let $\mathcal{E}^1_{K/F}$ be the extension of Γ by $K^\times$ obtained from the extension $W_{K/F}$ of Gal(K/F) by $K^\times$ by pulling back along the surjection

$$\Gamma \to \text{Gal}(K/F);$$

thus $\mathcal{E}^1_{K/F}$ is the fiber product of $W_{K/F}$ and Γ over Gal(K/F). As in B.3, we let $\mathcal{E}_{K/F}$ denote the extension of Γ by $\overline{F}^\times$ obtained from $\mathcal{E}^1_{K/F}$ by pushing out along the injection

$$K^\times \hookrightarrow \overline{F}^\times.$$

Thus $\mathcal{E}^1_{K/F}$ and $\overline{F}^\times$ can be identified with subgroups of $\mathcal{E}_{K/F}$; the product of these two subgroups is $\mathcal{E}_{K/F}$ and their intersection is $K^\times$. Recall that $\mathcal{E}_{K/F}$ is a topological group (see B.3).

Let t_τ be an abstract 1-cocycle of $\mathcal{E}_{K/F}$ in $T(\overline{F})$. We say that t_τ is an *algebraic* 1-cocycle if the map $\tau \mapsto t_\tau$ is continuous for the discrete topology on $T(\overline{F})$ and there exists $\mu \in X_*(T)$ such that

$$t_x = \mu(x)$$

for all x in the subgroup $\overline{F}^\times$ of $\mathcal{E}_{K/F}$. Again μ is uniquely determined and invariant under Γ, and again 1-coboundaries are algebraic. We write

$$Z^1_{\text{alg}}(\mathcal{E}_{K/F}, T(\overline{F}))$$

for the group of algebraic 1-cocycles of $\mathcal{E}_{K/F}$ in $T(\overline{F})$, and

$$H^1_{\text{alg}}(\mathcal{E}_{K/F}, T(\overline{F}))$$

for its quotient by the subgroup of 1-coboundaries.

There is an obvious map

$$Z^1_{\text{alg}}(W_{K/F}, T(K)) \to Z^1_{\text{alg}}(\mathcal{E}_{K/F}, T(\overline{F})), \tag{8.1.1}$$

defined as follows. Let t_τ be an algebraic 1-cocycle of $W_{K/F}$ in $T(K)$, and let μ be the associated cocharacter. We inflate t_τ using the canonical surjection

$$\mathcal{E}^1_{K/F} \to W_{K/F},$$

obtaining a 1-cocycle t'_τ of $\mathcal{E}^1_{K/F}$ in $T(K)$ whose restriction to the subgroup $K^\times$ of $\mathcal{E}^1_{K/F}$ is given by μ. We let t''_τ be the unique 1-cocycle of $\mathcal{E}_{K/F}$ in $T(\overline{F})$ whose restriction to $\mathcal{E}^1_{K/F}$ is equal to t'_τ and whose restriction to $\overline{F}^\times$ is given by μ. Note that t''_τ is algebraic. The map $t_\tau \mapsto t''_\tau$ is the desired map (8.1.1).

Let Γ_K denote the subgroup $\text{Gal}(\overline{F}/K)$ of Γ. We use the canonical splitting of the extension

$$1 \to \overline{F}^\times \to \mathcal{E}_{K/F} \to \Gamma \to 1$$

over the subgroup Γ_K to identify Γ_K with an (open) subgroup of $\mathcal{E}_{K/F}$. Since T splits over K, the group $H^1(K, T)$ is trivial. Therefore the restriction to Γ_K of any algebraic 1-cocycle a_τ of $\mathcal{E}_{K/F}$ in $T(\overline{F})$ is a 1-coboundary. Therefore there exists a cohomologous 1-cocycle b_τ whose restriction to Γ_K is trivial. It then follows easily that (8.1.1) induces an isomorphism

$$H^1_{\text{alg}}(W_{K/F}, T(K)) \to H^1_{\text{alg}}(\mathcal{E}_{K/F}, T(\overline{F})). \tag{8.1.2}$$

8.2. Put $s = [K : F]$. Choose a uniformizing element π in F. Recall from B.2 that the choice of π determines an extension $\mathcal{D}_s$ of Γ by $\overline{F}^\times$. We define the notion of algebraic 1-cocycle of $\mathcal{D}_s$ in $T(\overline{F})$ in the same way we did for $\mathcal{E}_{K/F}$ (impose continuity and the existence of an appropriate cocharacter μ of T); in this way we get groups

$$Z^1_{\text{alg}}(\mathcal{D}_s, T(\overline{F})),$$

$$H^1_{\text{alg}}(\mathcal{D}_s, T(\overline{F})).$$

The isomorphisms (B.3.2) induce isomorphisms

$$Z^1_{\text{alg}}(\mathcal{D}_s, T(\overline{F})) \simeq Z^1_{\text{alg}}(\mathcal{E}_{K/F}, T(\overline{F})),$$

and the induced isomorphisms

$$H^1_{\text{alg}}(\mathcal{D}_s, T(\overline{F})) \simeq H^1_{\text{alg}}(\mathcal{E}_{K/F}, T(\overline{F}))$$

all coincide.

Let t_τ be an algebraic 1-cocycle of $\mathcal{D}_s$ in $T(\overline{F})$. For any representation $\rho\colon T \to \mathrm{GL}(V)$ of T on a finite dimensional vector space V over F we get a representation of $\mathcal{D}_s$ on $\overline{F} \otimes_F V$ by letting $\tau \in \mathcal{D}_s$ act on $\overline{F} \otimes_F V$ by the τ-linear automorphism

$$\rho(t_\tau) \circ (\tau \otimes \mathrm{id}_V)$$

(we are also denoting the image of τ in Γ by τ). Recall (see Appendix B) that giving a representation of $\mathcal{D}_s$ is the same as giving an object in $\mathcal{T}_s$, the category of σ-L-spaces whose slopes lie in the subgroup $\frac{1}{s}\mathbb{Z}$ of $\mathbb{Q}$. In this way t_τ determines a $\otimes$-functor β from $\mathbf{Rep}(T)$ to $\mathcal{T}_s$. Let ω_T denote the obvious fiber functor (over $\overline{F}$) $V \mapsto \overline{F} \otimes_F V$ on $\mathbf{Rep}(T)$, and let $\omega_\pi^{\overline{F}}$ be the fiber functor (over $\overline{F}$)) on $\mathcal{T}_s$ constructed in B.2. There is an obvious $\otimes$-isomorphism from $\omega_\pi^{\overline{F}} \circ \beta$ to ω_T.

As in 3.1 this construction yields a bijection from $Z^1_{\text{alg}}(\mathcal{D}_s, T(\overline{F}))$ to the set of $\otimes$-isomorphism classes of pairs (β, α), where β is an exact $\otimes$-functor from $\mathbf{Rep}(T)$ to $\mathcal{T}_s$, and α is a $\otimes$-isomorphism from $\omega_\pi^{\overline{F}} \circ \beta$ to ω_T. This in turn yields a bijection from $H^1_{\text{alg}}(\mathcal{D}_s, T(\overline{F}))$ to the set of $\otimes$-isomorphism classes of exact $\otimes$-functors β from $\mathbf{Rep}(T)$ to $\mathcal{T}_s$.

We claim that any exact $\otimes$-functor β from $\mathbf{Rep}(T)$ to *σ-L-spaces* factors through the full Tannakian subcategory $\mathcal{T}_s$. In other words we claim that the image of the Newton map

$$\mathbf{B}(T) \to X_*(T)^\Gamma \otimes \mathbb{Q}$$

is contained in the subgroup $X_*(T)^\Gamma \otimes (\frac{1}{s}\mathbb{Z})$. Since this image is the same (see [K]) as that of the map

$$X_*(T) \to X_*(T)^\Gamma \otimes \mathbb{Q}$$

sending μ to

$$\frac{1}{s} \sum_{\tau \in \mathrm{Gal}(K/F)} \tau(\mu),$$

we see that the claim is true. Therefore we get a bijection from $H^1_{\text{alg}}(\mathcal{D}_s, T(\overline{F}))$ to the set of $\otimes$-isomorphism classes of exact $\otimes$-functors β from **Rep**(T) to W_F-$\overline{L}$-*spaces*. Comparing this with what was proved in 3.1, we obtain a canonical isomorphism

$$H^1_{\text{alg}}(\mathcal{D}_s, T(\overline{F})) \simeq \mathbf{B}(T). \tag{8.2.2}$$

Let t_τ be an algebraic 1-cocycle of $\mathcal{D}_s$ in $T(\overline{F})$. Composing $\tau \mapsto t_\tau$ with the map (B.2.5) from W_F to $\mathcal{D}_s$, we get a (continuous) 1-cocycle of W_F in $T(\overline{F})$, which we can regard as a 1-cocycle of W_F in $T(\overline{L})$. This map on 1-cocycles induces the map (8.2.2).

Let t be a positive integer such that s divides t. Recall from B.2 that there is a canonical surjection

$$\mathcal{D}_t \to \mathcal{D}_s,$$

which gives rise to an inflation map

$$H^1_{\text{alg}}(\mathcal{D}_s, T(\overline{F})) \xrightarrow{i} H^1_{\text{alg}}(\mathcal{D}_t, T(\overline{F})). \tag{8.2.3}$$

It is easy to check the commutativity of the diagram

$$\begin{array}{ccc} H^1_{\text{alg}}(\mathcal{D}_s, T(\overline{F})) & \xrightarrow{i} & H^1_{\text{alg}}(\mathcal{D}_t, T(\overline{F})) \\ \downarrow & & \downarrow \\ \mathbf{B}(T) & = & \mathbf{B}(T), \end{array} \tag{8.2.4}$$

in which the vertical arrows are isomorphisms of type (8.2.2).

8.3. Combining (8.2.1) and (8.2.2), we get an isomorphism

$$H^1_{\text{alg}}(\mathcal{E}_{K/F}, T(\overline{F})) \simeq \mathbf{B}(T). \tag{8.3.1}$$

Let t_τ be an algebraic 1-cocycle of $\mathcal{E}_{K/F}$ in $T(\overline{F})$. Then by composing $\tau \mapsto t_\tau$ with the map (B.3.3) from W_F to $\mathcal{E}_{K/F}$, we get a (continuous) 1-cocycle of W_F in $T(\overline{F})$, which we view as a 1-cocycle of W_F in $T(\overline{L})$. This map on 1-cocycles induces the isomorphism (8.3.1). It follows from the discussion at the end of B.3 that the isomorphism (8.3.1) is independent of the choice of uniformizing element π.

8.4. Combining (8.3.1) and (8.1.2), we get a canonical isomorphism

$$H^1_{\text{alg}}(W_{K/F}, T(K)) \simeq \mathbf{B}(T). \tag{8.4.1}$$

It is easy to see that the diagram

$$\begin{array}{ccc} H^1(K/F, T(K)) & \longrightarrow & H^1_{\text{alg}}(W_{K/F}, T(K)) \\ \downarrow & & \downarrow{\scriptstyle (8.4.1)} \\ H^1(F, T) & \xrightarrow{(1.4.2)} & \mathbf{B}(T) \end{array} \tag{8.4.2}$$

commutes, where the top arrow is the inflation map for the surjection

$$W_{K/F} \to \text{Gal}(K/F)$$

and the left vertical arrow is the inflation map for the surjection

$$\Gamma \to \text{Gal}(K/F)$$

(the second inflation map is an isomorphism since $H^1(K, T)$ is trivial).

9. Hypercohomology

Let $f : T \to U$ be a map of F-tori. We regard $T \to U$ as a complex of length 2, concentrated in degrees 0 and 1. Let K be a finite Galois extension of F in $\overline{F}$ that splits T and U, and put $s = [K : F]$. In this section we will define hypercohomology groups $\mathbf{B}(T \to U)$ and $H^1_{\text{alg}}(W_{K/F}, T(K) \to U(K))$ and show that they are canonically isomorphic.

9.1. First we define $\mathbf{B}(T \to U)$. By a 1-hypercocycle of W_F in $T(\overline{L}) \to U(\overline{L})$ we mean a pair (t, u) consisting of a 1-cocycle t of W_F in $T(\overline{L})$ and an element $u \in U(\overline{L})$ such that $f(t) = \partial u$ (here ∂u denotes the coboundary of u, namely the 1-cocycle $\tau \mapsto u^{-1}\tau(u)$). By a 1-hypercoboundary we mean a pair of the form $(\partial t, f(t))$, where t is an element of $T(\overline{L})$. We let

$$\mathbf{B}(T \to U)$$

denote the group of 1-hypercocycles modulo 1-hypercoboundaries.

There is an exact sequence

$$\begin{aligned} 1 \to\ & \text{cok}[T(F) \to U(F)] \to \mathbf{B}(T \to U) \\ \to\ & \ker[\mathbf{B}(T) \to \mathbf{B}(U)] \to 1. \end{aligned} \tag{9.1.1}$$

Let C (respectively, W) denote the kernel (respectively, cokernel) of $f : T \to U$. Of course W is a torus, but C need not be. There is a second exact sequence

$$1 \to \mathbf{B}(C) \to \mathbf{B}(T \to U) \to W(F), \tag{9.1.2}$$

and if C is connected then the map

$$\mathbf{B}(T \to U) \to W(F)$$

is surjective.

9.2. Now we define $H^1_{\text{alg}}(W_{K/F}, T(K) \to U(K))$. By a 1-hypercocycle we now mean a pair (t, u) consisting of an algebraic 1-cocycle t of $W_{K/F}$ in $T(K)$ and an element $u \in U(K)$ such that $f(t) = \partial u$. By a 1-hypercoboundary we mean a pair of the form $(\partial t, f(t))$, where t is an element of $T(K)$. We let

$$H^1_{\text{alg}}(W_{K/F}, T(K) \to U(K))$$

denote the group of 1-hypercocycles modulo 1-hypercoboundaries.

9.3. There are also hypercohomology groups

$$H^1_{\text{alg}}(\mathcal{E}_{K/F}, T(\overline{F}) \to U(\overline{F})),$$
$$H^1_{\text{alg}}(\mathcal{D}_s, T(\overline{F}) \to U(\overline{F}))$$

(define these in the obvious way, using algebraic 1-cocycles). There are canonical isomorphisms

$$\begin{aligned} H^1_{\text{alg}}(W_{K/F}, T(K) \to U(K)) &\simeq H^1_{\text{alg}}(\mathcal{E}_{K/F}, T(\overline{F}) \to U(\overline{F})) \\ &\simeq H^1_{\text{alg}}(\mathcal{D}_s, T(\overline{F}) \to U(\overline{F})) \\ &\simeq \mathbf{B}(T \to U) \end{aligned} \tag{9.3.1}$$

analogous to (8.1.2), (8.2.1), (8.2.2). Indeed, the maps on 1-cocycles defining (8.1.2), (8.2.1), (8.2.2) can be used to define maps between the hypercohomology groups above, and these maps on hypercohomology are all isomorphisms since the maps (8.1.2), (8.2.1), (8.2.2) are isomorphisms (use the exact sequence (9.1.1) and its analogs for the other hypercohomology groups). The resulting isomorphism

$$H^1_{\text{alg}}(W_{K/F}, T(K) \to U(K)) \simeq \mathbf{B}(T \to U) \tag{9.3.2}$$

is independent of the choice of π.

9.4. The diagram analogous to (8.4.2)

$$\begin{array}{ccc} H^1(K/F, T(K) \to U(K)) & \longrightarrow & H^1_{\text{alg}}(W_{K/F}, T(K) \to U(K)) \\ \downarrow & & \downarrow {\scriptstyle (9.3.2)} \\ H^1(F, T \to U) & \longrightarrow & \mathbf{B}(T \to U) \end{array} \tag{9.4.1}$$

commutes, where we have written $H^1(K/F, T(K) \to U(K))$ for

$$H^1(\text{Gal}(K/F), T(K) \to U(K))$$

and $H^1(F, T \to U)$ for $H^1(\Gamma, T(\overline{F}) \to U(\overline{F}))$, as in [KS]. Note that the left vertical arrow in (9.4.1) is an isomorphism since T and U split over K. The bottom arrow is analogous to (1.4.2).

10. Hyperhomology

As in Section 9 let $f : T \to U$ be a map of F-tori and let K be a finite Galois extension of F in $\overline{F}$ that splits T and U. Let X, Y denote the cocharacter groups $X_*(T)$, $X_*(U)$ respectively. We regard

$$X \xrightarrow{f_*} Y$$

as a complex of length 2 placed in degrees 0 and 1. In this section we will define an isomorphism

$$H_0(W_{K/F}, X \to Y) \simeq H^1_{\mathrm{alg}}(W_{K/F}, T(K) \to U(K)).$$

10.1. The group $H_0(W_{K/F}, X \to Y)$ is the hyperhomology group studied in Section A.3 of [KS], and our discussion here closely parallels the one there. For $m \geqslant 0$ we write $C_m(X)$ for the group of (abstract) m-chains of $W_{K/F}$ in X, so that $H_m(W_{K/F}, X)$ is the m-th homology group of the complex

$$\ldots \to C_2(X) \xrightarrow{\partial} C_1(X) \xrightarrow{\partial} C_0(X).$$

We then get a double complex

$$\begin{array}{ccccccc} \cdots & \longrightarrow & C_2(X) & \xrightarrow{\partial} & C_1(X) & \xrightarrow{\partial} & C_0(X) \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & C_2(Y) & \xrightarrow{\partial} & C_1(Y) & \xrightarrow{\partial} & C_0(Y) \end{array}$$

with vertical maps induced by $f_* : X \to Y$, and from this double complex we get the complex

$$\ldots \to C_1(X) \oplus C_2(Y) \xrightarrow{\alpha} C_0(X) \oplus C_1(Y) \xrightarrow{\beta} C_0(Y),$$

with α given by

$$\alpha(x_1, y_2) = (\partial x_1, f_* x_1 - \partial y_2)$$

and β given by

$$\beta(x_0, y_1) = f_* x_0 - \partial y_1.$$

Then $H_0(W_{K/F}, X \to Y)$ is the quotient

$$\ker(\beta)/\mathrm{im}(\alpha)$$

and we refer to elements of $\ker(\beta)$ as 0-hypercycles.

We write $C^0(T)$ for the group of 0-cochains of $W_{K/F}$ in $T(K)$ and $Z^1_{\mathrm{alg}}(T)$ for the group $Z^1_{\mathrm{alg}}(W_{K/F}, T(K))$ of algebraic 1-cocycles of $W_{K/F}$ in $T(K)$; of course

$$C^0(T) = T(K).$$

We are going to define maps

$$\phi\colon C_1(X) \to C^0(T),$$
$$\psi\colon C_0(X) \to Z^1_{\mathrm{alg}}(T),$$

making the diagram

$$\begin{array}{ccccc} C_2(X) & \xrightarrow{\partial} & C_1(X) & \xrightarrow{\partial} & C_0(X) \\ \downarrow & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\psi} \\ 0 & \longrightarrow & C^0(T) & \xrightarrow{\partial} & Z^1_{\mathrm{alg}}(T) \end{array} \tag{10.1.1}$$

commute. Both ϕ and ψ will be functorial in T. Just as in [KS], we will use ϕ, ψ to define a homomorphism

$$H_0(W_{K/F}, X \to Y) \to H^1_{\mathrm{alg}}(W_{K/F}, T(K) \to U(K)) \tag{10.1.2}$$

sending the class of the 0-hypercycle (x_0, y_1) to the class of the 1-hypercocycle $(\psi(x_0), \phi(y_1))$.

It remains to define ϕ, ψ. We fix a (set-theoretic) section

$$s\colon \mathrm{Gal}(K/F) \to W_{K/F}$$

of the canonical surjection

$$W_{K/F} \to \mathrm{Gal}(K/F).$$

As usual this section gives us a 2-cocycle $a_{\rho,\tau}$ of $\mathrm{Gal}(K/F)$ in $K^\times$, defined by the equation

$$s(\rho)s(\tau) = a_{\rho,\tau}s(\rho\tau).$$

We now define ϕ exactly as in [KS, A.3]. It sends a 1-chain $w \mapsto x_w$ of $W_{K/F}$ in X to the element

$$\gamma = \prod_{\rho,\tau,a} \rho(x_{as(\tau)})(a^{-1}_{\rho,\tau}\rho(a)^{-1}), \tag{10.1.3}$$

of $T(K)$, where the product is taken over all

$$(\sigma, \tau, a) \in \mathrm{Gal}(K/F) \times \mathrm{Gal}(K/F) \times K^{\times}.$$

We define ψ as follows. Let

$$\mu \in C_0(X) = X$$

and put

$$\nu = \sum_{\tau \in \mathrm{Gal}(K/F)} \tau(\mu).$$

Define a map

$$t \colon W_{K/F} \to K^{\times}$$

by the equation

$$w = t(w)s(\rho),$$

where ρ denotes the image of $w \in W_{K/F}$ under

$$W_{K/F} \to \mathrm{Gal}(K/F).$$

Then ψ sends μ to the algebraic 1-cocycle

$$w \mapsto \nu(t(w)) \cdot \prod_{\tau \in \mathrm{Gal}(K/F)} \rho\tau(\mu)(a_{\rho,\tau}) \tag{10.1.4}$$

of $W_{K/F}$ in $T(K)$, where $w \in W_{K/F}$ and ρ denotes the image of w in $\mathrm{Gal}(K/F)$. A direct calculation [L, A.1] shows that the 1-cocycle condition is satisfied, and it is obvious that this 1-cocycle is algebraic. It is not hard to check that the cohomology class of the 1-cocycle is equal to the corestriction of the element of

$$H^1(K^{\times}, T(K)) = \mathrm{Hom}(K^{\times}, T(K))$$

determined by μ.

If $\nu = 0$, or, in other words, if μ lies in the subgroup $C_0(X)_0$ of $C_0(X)$ (the notation $C_0(X)_0$ comes from [KS]), then the first factor in (10.1.4) is 1, and the second factor coincides with the one used to define the map

$$\psi \colon C_0(X)_0 \to Z^1(T)$$

in [KS]. Thus the map ψ used in this paper extends the one in [KS]. In particular (10.1.1) commutes, since ∂ maps $C_1(X)$ into $C_0(X)_0$ and the analogous diagram in [KS] commutes.

10.2. The maps ϕ, ψ have all the desired properties, and thus the map (10.1.2) has now been defined. However we chose a section s of

$$W_{K/F} \to \mathrm{Gal}(K/F)$$

in order to define ϕ, ψ, and we need to check that the map (10.1.2) is independent of this choice.

Let s' be another section and let ϕ', ψ' be the corresponding maps. Let b_τ be the 1-cochain of $\mathrm{Gal}(K/F)$ in $K^\times$ defined by

$$s'(\tau) = b_\tau s(\tau)$$

for $\tau \in \mathrm{Gal}(K/F)$. Define a homomorphism

$$k\colon C_0(X) \to C^0(T)$$

by sending an element $\mu \in X$ to the element

$$k(\mu) := \prod_{\tau \in \mathrm{Gal}(K/F)} (\tau\mu)(b_\tau)$$

of $T(K)$ (this map is the obvious extension of the one in [KS]). Clearly k is functorial in T, and a routine calculation shows that

$$\phi' - \phi = k\partial$$

and

$$\psi' - \psi = \partial k.$$

It then follows easily that the homomorphism (10.1.2) does not change when s is replaced by s'.

10.3. We now show that the homomorphism (10.1.2) is an isomorphism. Using the 5-lemma as in [KS], we see it is enough to prove that (10.1.2) is an isomorphism in the special case in which either T or U is trivial. Thus we must show that the maps

$$H_1(W_{K/F}, X) \to T(F)$$

and

$$X_\Gamma \to H^1_{\mathrm{alg}}(W_{K/F}, T(K))$$

are isomorphisms. The first map is the usual Langlands isomorphism (see [KS] for a review). Composing the second map with the isomorphism (8.4.1), we get a functorial homomorphism

$$X_\Gamma \to \mathbf{B}(T), \tag{10.3.1}$$

which we must show is an isomorphism.

In fact (10.3.1) is equal to the isomorphism [K, 2.4.1]. By [K, 2.2(b)] it is enough to prove that (10.3.1) coincides with the map in [K] in the special case $T = \mathbb{G}_m$ (more precisely we use the obvious variant of [K, 2.2(b)] that applies to the category of tori over F that are split by K).

In order to prove that (10.3.1) coincides with the map in [K] in the special case $T = \mathbb{G}_m$, we need to introduce some homomorphisms taking values in $\mathbb{Q}$. Put $s = [K : F]$ and consider the extensions $\mathcal{D}_s$, $\mathcal{E}_{K/F}$ of Γ by $\overline{F}^\times$. Let $v_\mathcal{D}$ (respectively, $v_\mathcal{E}$) denote the unique continuous homomorphism from $\mathcal{D}_s$ (respectively, $\mathcal{E}_{K/F}$) to $\mathbb{Q}$ extending the valuation map

$$\mathrm{val}\colon \overline{F}^\times \to \mathbb{Q}$$

on the subgroup $\overline{F}^\times$ (we normalize the valuation on $\overline{F}$ so that it takes the value 1 on uniformizing elements for F). The existence and uniqueness of $v_\mathcal{D}$, $v_\mathcal{E}$ follow from the triviality of $H^i(\Gamma, \mathbb{Q})$ for $i \geqslant 1$. Clearly the isomorphisms (B.3.2)

$$\mathcal{D}_s \simeq \mathcal{E}_{K/F}$$

carry $v_\mathcal{D}$ into $v_\mathcal{E}$. Looking back at the definition of the homomorphism (B.2.5), we see that the composition

$$W_F \xrightarrow{(B.2.5)} \mathcal{D}_s \xrightarrow{v_\mathcal{D}} \mathbb{Q} \tag{10.3.2}$$

sends $\tau \in W_F$ to j/s, where j is the unique integer such that the restriction of τ to F^{un} is equal to σ^j; of course the composed map (10.3.2) is also equal to

$$W_F \xrightarrow{(B.3.3)} \mathcal{E}_{K/F} \xrightarrow{v_\mathcal{E}} \mathbb{Q}. \tag{10.3.3}$$

Now let $\mu \in X_*(\mathbb{G}_m)$ be the identity map on $\mathbb{G}_m$. Let c_w be the corresponding algebraic 1-cocycle of $W_{K/F}$ in $K^\times$ (defined by the formula (10.1.4)), and let c'_w be the algebraic 1-cocycle of $\mathcal{E}_{K/F}$ in $\overline{F}^\times$ obtained from c_w by means of (8.1.1). The map $w \mapsto \mathrm{val}(c'_w)$ is a continuous homomorphism from $\mathcal{E}_{K/F}$ to $\mathbb{Q}$ extending the homomorphism $s \cdot \mathrm{val}$ on the subgroup $\overline{F}^\times$ of $\mathcal{E}_{K/F}$, and therefore

$$\mathrm{val}(c'_w) = s \cdot v_\mathcal{E}(w).$$

Let c''_τ be the 1-cocycle of W_F in $\overline{F}^\times$ obtained from c'_w by means of the homomorphism (B.3.3). It follows from the discussion above that

$$\mathrm{val}(c''_\tau) = s \cdot (j/s) = j,$$

where j is the unique integer such that the restriction of $\tau \in W_F$ to F^{un} is equal to σ^j. Pick a cocycle c'''_τ of $\langle\sigma\rangle$ in $(F^{\mathrm{un}})^\times$ whose inflation to W_F is cohomologous to c''_τ. Then

$$\mathrm{val}(c'''_\sigma) = 1.$$

Of course c'''_σ is an element of $T(L)$ whose class in $B(T) = \mathbf{B}(T)$ is equal to the image of $\mu \in X_*(\mathbb{G}_m)$ under (10.3.1). Comparing with [K, 2.4], we see that this class in $B(T)$ is also the image of μ under the isomorphism

$$X_\Gamma \to B(T)$$

defined in [K]. This completes our proof.

10.4. Combining the isomorphisms (9.3.2) and (10.1.2), we obtain an isomorphism

$$H_0(W_{K/F}, X \to Y) \simeq \mathbf{B}(T \to U). \tag{10.4.1}$$

11. Duality for $\mathbf{B}(T \to U)$

We let $f : T \to U$, $f_* : X \to Y$ and K/F be as in Section 10. In this section we use (10.4.1) to prove a duality theorem for $\mathbf{B}(T \to U)$.

11.1. First we must topologize $\mathbf{B}(T \to U)$. Recall the exact sequence (9.1.1). We put the unique topology on $\mathbf{B}(T \to U)$ for which $\mathbf{B}(T \to U)$ is a topological group and the canonical map

$$U(F) \to \mathbf{B}(T \to U)$$

is open. We write

$$\mathrm{Hom}_{\mathrm{cont}}(\mathbf{B}(T \to U), \mathbb{C}^\times)$$

for the group of continuous homomorphisms from $\mathbf{B}(T \to U)$ to $\mathbb{C}^\times$.

11.2. Dual to $f : T \to U$ is a homomorphism

$$\widehat{f} : \widehat{U} \to \widehat{T}.$$

The hypercohomology groups $H^1(W_F, \widehat{U} \to \widehat{T})$ and $H^1(W_{K/F}, \widehat{U} \to \widehat{T})$ are defined in [KS, A.3], using continuous 1-cocycles of W_F and $W_{K/F}$ in $\widehat{U}$. The inflation map

$$H^1(W_{K/F}, \widehat{U} \to \widehat{T}) \to H^1(W_F, \widehat{U} \to \widehat{T}) \tag{11.2.1}$$

is an isomorphism. Recall from [KS, (A.3.8)] that there is a canonical isomorphism

$$\mathrm{Hom}\big(H_0(W_{K/F}, X \to Y), \mathbb{C}^\times\big) \simeq H^1_{\mathrm{abs}}(W_{K/F}, \widehat{U} \to \widehat{T}),$$

where the subscript abs indicates that we regard $W_{K/F}$ as an abstract group when forming the hypercohomology group. Combining this with the isomorphism (10.4.1), we get an isomorphism

$$\mathrm{Hom}\big(\mathbf{B}(T \to U), \mathbb{C}^\times\big) \simeq H^1_{\mathrm{abs}}(W_{K/F}, \widehat{U} \to \widehat{T}),$$

and it is clear that this isomorphism restricts to an isomorphism

$$\mathrm{Hom}_{\mathrm{cont}}\big(\mathbf{B}(T \to U), \mathbb{C}^\times\big) \simeq H^1(W_{K/F}, \widehat{U} \to \widehat{T}),$$

which we combine with (11.2.1) to get an isomorphism

$$\mathrm{Hom}_{\mathrm{cont}}(\mathbf{B}(T \to U), \mathbb{C}^\times) \simeq H^1(W_F, \widehat{U} \to \widehat{T}). \tag{11.2.2}$$

In 11.5 below we will prove that the isomorphism (11.2.2) is independent of the choice of K. Combining (11.2.2) with the canonical injection

$$H^1(F, T \to U) \to \mathbf{B}(T \to U)$$

(the bottom arrow in (9.4.1)), we recover the surjection

$$H^1(W_F, \widehat{U} \to \widehat{T}) \to \mathrm{Hom}_{\mathrm{cont}}(H^1(F, T \to U), \mathbb{C}^\times)$$

of [KS, Lemma A.3.B].

11.3. We are going to prove a rather technical lemma that will be used in 11.5 to prove that (11.2.2) is independent of the choice of K. The lemma will be used again in Section 12.

Let R denote the torus $R_{K/F}\mathbb{G}_m$ obtained from $\mathbb{G}_m$ by Weil's restriction of scalars from K to F. The group $G(K/F) := \mathrm{Gal}(K/F)$ acts on (the left of) R by F-automorphisms; for $\tau \in G(K/F)$ we write θ_τ for the corresponding F-automorphism of R. Put

$$R_1 := \prod_{\tau \in G(K/F)} R,$$

and consider the homomorphism

$$R \xrightarrow{\eta} R_1 \tag{11.3.1}$$

whose projection to the factor R indexed by $\tau \in G(K/F)$ is given by $\theta_\tau^{-1} - \mathrm{id}_R \in \mathrm{End}(R)$.

Of course $X_*(R)$ is the left regular representation of $G(K/F)$ on the group ring $\mathbb{Z}[G(K/F)]$. We write μ_K for the element of $X_*(R)$ corresponding to the unit element $1 \in \mathbb{Z}[G(K/F)]$. Then (R, μ_K) represents the functor $T \mapsto X_*(T)$ on the category of F-tori split by K. Note that

$$\tau(\mu_K) = \theta_\tau^{-1}(\mu_K)$$

for any $\tau \in G(K/F)$. It follows that the class of μ_K in $X_*(R)_\Gamma$ lies in the kernel of

$$X_*(R)_\Gamma \xrightarrow{\eta_*} X_*(R_1)_\Gamma.$$

Now let $\mathcal{C}_K$ denote the category whose objects are homomorphisms

$$f : T \to U$$

of F-tori split by K and whose morphisms are given by commutative diagrams

$$\begin{array}{ccc} T & \xrightarrow{f} & U \\ \downarrow & & \downarrow \\ T' & \xrightarrow{f'} & U'. \end{array}$$

Let $f: T \to U$ be an object in $\mathcal{C}_K$. Then giving a morphism from $\eta\colon R \to R_1$ to $f\colon T \to U$ is the same as giving an element $\mu \in X_*(T)$ and a family of elements $\mu_\tau \in X_*(U)$, one for each $\tau \in G(K/F)$, satisfying

$$f_*(\mu) = \sum_{\tau \in G(K/F)} (\tau - 1)(\mu_\tau). \tag{11.3.2}$$

The class of $\mu \in X_*(T)$ in $X_*(T)_\Gamma$ lies in the kernel of

$$X_*(T)_\Gamma \xrightarrow{f_*} X_*(U)_\Gamma.$$

Moreover, it is clear that for any element $\mu \in X_*(T)$ whose class in $X_*(T)_\Gamma$ lies in the kernel of

$$X_*(T)_\Gamma \xrightarrow{f_*} X_*(U)_\Gamma$$

there exists a morphism from $\eta\colon R \to R_1$ to $f\colon T \to U$ that carries $\mu_K \in X_*(R)$ into $\mu \in X_*(T)$.

We are almost ready to state the technical lemma. For any object $T \to U$ in $\mathcal{C}_K$ put

$$H(T \to U) := H_0(W_{K/F}, X \to Y)$$

(as usual $X = X_*(T)$, $Y = X_*(U)$). Of course H is an additive functor from the additive category $\mathcal{C}_K$ to the category of abelian groups. Suppose that we are given an additive functor I from $\mathcal{C}_K$ to the category of abelian groups, and that we are also given two natural transformations α, β from H to I.

LEMMA 11.4. *Suppose that the maps*

$$\alpha, \beta\colon H(T \to U) \to I(T \to U)$$

are equal whenever T is trivial or U is trivial. Suppose further that the obvious map

$$I(R \to R_1) \to I(R \to 1) \times I(1 \to R_1/\eta(R))$$

is injective. Then α is equal to β.

First we note that the maps

$$\alpha, \beta \colon H(R \to R_1) \to I(R \to R_1)$$

are equal. Indeed, this follows immediately from the hypotheses of the lemma (apply the first hypothesis to both $R \to 1$ and $1 \to R_1/\eta(R)$). It follows that the maps

$$\alpha, \beta : H(T \to U) \to I(T \to U) \tag{11.4.1}$$

are equal on all elements of $H(T \to U)$ that arise as the image of an element in $H(R \to R_1)$ for some morphism from $R \to R_1$ to $T \to U$.

There is an obvious exact sequence

$$\begin{aligned} \cdots \to\ & H(1 \to T) \to H(1 \to U) \to H(T \to U) \\ \to\ & X_\Gamma \to Y_\Gamma \to \cdots \end{aligned} \tag{11.4.2}$$

Let $x \in H(T \to U)$. We want to show that $\alpha(x) = \beta(x)$. It follows from the discussion in 11.3 that there is a morphism ξ from $R \to R_1$ to $T \to U$ and an element $y \in H(R \to R_1)$ such that x and $\xi(y)$ have the same image in $\ker[X_\Gamma \to Y_\Gamma]$. Since we have already seen that α, β have the same value on $\xi(y)$, we are reduced to the case in which x lies in the image of $H(1 \to U)$. Therefore the first hypothesis of the lemma, applied to $1 \to U$, implies that $\alpha(x) = \beta(x)$.

11.5. Now we use the lemma to prove that the isomorphism (11.2.2) is independent of the choice of K. As in [KS, A.3] the only nontrivial fact that we need is the commutativity of

$$\begin{array}{ccc} H_0(W_{K'/F}, X \to Y) & \longrightarrow & \mathbf{B}(T \to U) \\ \Big\downarrow{\scriptstyle p_*} & & \Big\| \\ H_0(W_{K/F}, X \to Y) & \longrightarrow & \mathbf{B}(T \to U). \end{array} \tag{11.5.1}$$

Here K' is a finite Galois extension of F in $\overline{F}$ containing K, and the map p_* is induced by the canonical surjection

$$p \colon W_{K'/F} \to W_{K/F}.$$

The horizontal maps are of type (10.4.1).

Note that the map p_* is an isomorphism (use the exact sequence (11.4.2)). Therefore (11.5.1) gives us two natural transformations α, β from H to I, where I

denotes the functor on $\mathcal{C}_K$ that sends $T \to U$ to $\mathbf{B}(T \to U)$. We claim that α, β, I satisfy the hypotheses of Lemma 11.4. The first point to check is that

$$\begin{array}{ccc} H_1(W_{K'/F}, X) & \longrightarrow & T(F) \\ \big\downarrow & & \big\| \\ H_1(W_{K/F}, X) & \longrightarrow & T(F) \end{array} \tag{11.5.2}$$

commutes. This is standard (and also follows from the commutativity of (A.3.11) in [KS]). The second point to check is that

$$\begin{array}{ccc} X_\Gamma & \longrightarrow & \mathbf{B}(T) \\ \downarrow\downarrow & & \big\| \\ X_\Gamma & \longrightarrow & \mathbf{B}(T) \end{array}$$

commutes. This follows from the fact, proved in 10.3, that both horizontal maps agree with the canonical map

$$X_\Gamma \to \mathbf{B}(T)$$

defined in [K]. The third point to check is that the natural map

$$\mathbf{B}(R \to R_1) \to \mathbf{B}(R \to 1) \times \mathbf{B}(1 \to R_1/\eta(R))$$

is injective.

More generally let us find a sufficient condition for the injectivity of

$$\mathbf{B}(T \xrightarrow{f} U) \to \mathbf{B}(T \to 1) \times \mathbf{B}(1 \to W), \tag{11.5.3}$$

where $W = U/f(T)$. It follows from (9.1.1) that the kernel of (11.5.3) is equal to the kernel of

$$\mathrm{cok}[T(F) \to U(F)] \to W(F),$$

which is equal to $V(F)/f(T(F))$, where V is the subtorus $f(T)$ of U. Let C denote the kernel of $T \to U$. Then $T(F) \to V(F)$ is surjective if $H^1(F, C)$ is trivial. Therefore we conclude that (11.5.3) is injective whenever $H^1(F, C)$ is trivial. This condition is satisfied by $\eta\colon R \to R_1$, since C is $\mathbb{G}_m$ in this case.

12. A valuation map on $\mathbf{B}(T \to U)$

We let $f : T \to U$, $f_* : X \to Y$ and K/F be as in Section 10. In this section we are going to define a surjection

$$\mathbf{B}(T \to U) \to H^1(\langle\sigma\rangle, X_I \to Y_I)$$

and study its properties.

12.1. We need to review group cohomology and homology for the infinite cyclic group $\langle\sigma\rangle$. Let $\mathbb{Z}[\langle\sigma\rangle]$ denote the integral group ring of $\langle\sigma\rangle$. There is an exact sequence

$$0 \to \mathbb{Z}[\langle\sigma\rangle] \xrightarrow{\sigma-1} \mathbb{Z}[\langle\sigma\rangle \xrightarrow{\alpha} \mathbb{Z} \to 0,$$

where $\sigma - 1$ denotes multiplication by $\sigma - 1$ and α is defined by

$$\alpha\Bigl(\sum_j m_j\sigma^j\Bigr) = \sum_j m_j.$$

Thus we get a projective resolution

$$\mathbb{Z}[\langle\sigma\rangle] \xrightarrow{\sigma-1} \mathbb{Z}[\langle\sigma\rangle]$$

of the trivial $\langle\sigma\rangle$-module $\mathbb{Z}$.

Let A be an abelian group on which $\langle\sigma\rangle$ acts. Then $H^\bullet(\langle\sigma\rangle, A)$ is the cohomology of the complex

$$A \xrightarrow{\sigma-1} A$$

and $H_\bullet(\langle\sigma\rangle, A)$ is the homology of the same complex. Therefore $H^m(\langle\sigma\rangle, A)$ and $H_m(\langle\sigma\rangle, A)$ vanish for $m \geqslant 2$ and

$$H^0(\langle\sigma\rangle, A) \;=\; A^{\langle\sigma\rangle} = H_1(\langle\sigma\rangle, A),$$
$$H^1(\langle\sigma\rangle, A) \;=\; A_{\langle\sigma\rangle} = H_0(\langle\sigma\rangle, A),$$

(as usual the superscript $\langle\sigma\rangle$ indicates invariants and the subscript $\langle\sigma\rangle$ indicates coinvariants).

Now let $\phi\colon A \to B$ be a map of $\langle\sigma\rangle$-modules. From ϕ we get a double complex

$$\begin{array}{ccc} A & \xrightarrow{\sigma-1} & A \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi} \\ B & \xrightarrow{\sigma-1} & B, \end{array}$$

which in turn gives rise to a complex

$$A \xrightarrow{(\sigma-1,\phi)} A \oplus B \xrightarrow{\phi-(\sigma-1)} B.$$

The cohomology (respectively, homology) of this complex is the hypercohomology (respectively, hyperhomology) of $A \to B$. Therefore

$$H_{1-m}(\langle\sigma\rangle, A \to B) \simeq H^m(\langle\sigma\rangle, A \to B) \qquad (12.1.1)$$

for all $m \in \mathbb{Z}$. Moreover $H^m(\langle\sigma\rangle, A \to B)$ vanishes unless $m = 0, 1, 2$, and

$$\begin{aligned} H^0(\langle\sigma\rangle, A \to B) &= \ker[A^{\langle\sigma\rangle} \to B^{\langle\sigma\rangle}], \\ H^1(\langle\sigma\rangle, A \to B) &= \ker(\phi - (\sigma - 1))/\mathrm{im}(\sigma - 1, \phi), \\ H^2(\langle\sigma\rangle, A \to B) &= \mathrm{cok}[A_{\langle\sigma\rangle} \to B_{\langle\sigma\rangle}]. \end{aligned} \tag{12.1.2}$$

We refer to elements of $\ker(\phi - (\sigma - 1))$ as *simplified* 1-hypercocycles (and also as simplified 0-hypercycles), and we refer to elements of $\mathrm{im}(\sigma - 1, \phi)$ as simplified 1-hypercoboundaries (and also as simplified 0-hyperboundaries).

There is an exact sequence

$$\begin{aligned} 1 \to \mathrm{cok}[A^{\langle\sigma\rangle} \to B^{\langle\sigma\rangle}] &\to H^1(\langle\sigma\rangle, A \to B) \\ &\to \ker[A_{\langle\sigma\rangle} \to B_{\langle\sigma\rangle}] \to 1 \end{aligned} \tag{12.1.3}$$

generalizing (9.1.1), and there is an exact sequence

$$\begin{aligned} 1 \to \big(\ker[A \to B]\big)_{\langle\sigma\rangle} &\to H^1(\langle\sigma\rangle, A \to B) \\ &\to \big(\mathrm{cok}[A \to B]\big)^{\langle\sigma\rangle} \to 1 \end{aligned} \tag{12.1.4}$$

analogous to (9.1.2) (it generalizes (9.1.2) in case C is connected).

12.2. We define $B(T \to U)$ to be the hypercohomology group

$$B(T \to U) := H^1(\langle\sigma\rangle, T(L) \to U(L)). \tag{12.2.1}$$

The inflation map for the surjection $W_F \to \langle\sigma\rangle$ yields an isomorphism

$$B(T \to U) \simeq \mathbf{B}(T \to U). \tag{12.2.2}$$

Recall the canonical surjection (7.2.1)

$$w_T \colon T(L) \to X_I.$$

Together the maps w_T and w_U induce a map of complexes from $[T(L) \to U(L)]$ to $[X_I \to Y_I]$, and this in turn induces a map

$$\mathbf{B}(T \to U) = B(T \to U) \to H^1(\langle\sigma\rangle, X_I \to Y_I). \tag{12.2.3}$$

We claim that the map (12.2.3) is surjective. Since w_T, w_U are surjective, it suffices to show that

$$H^2(\langle\sigma\rangle, T(L)_1 \to U(L)_1) \tag{12.2.4}$$

is trivial, where $T(L)_1$ denotes the kernel of w_T. But (12.2.4) is equal to

$$\mathrm{cok}\big[(T(L)_1)_{\langle\sigma\rangle} \to (U(L)_1)_{\langle\sigma\rangle}\big],$$

which is indeed trivial (see (7.6.1)).

12.3. Consider the canonical surjection $q\colon W_{K/F} \to \langle\sigma\rangle$. There is a natural map (analogous to inflation for hypercohomology)

$$H_0(W_{K/F}, X \to Y) \to H_0(\langle\sigma\rangle, X_I \to Y_I) \tag{12.3.1}$$

obtained as the composition of

$$H_0(W_{K/F}, X \to Y) \to H_0(W_{K/F}, X_I \to Y_I)$$

and

$$H_0(W_{K/F}, X_I \to Y_I) @ > q_* >> H_0(\langle\sigma\rangle, X_I \to Y_I).$$

12.4. Consider the diagram

$$\begin{array}{ccc} H_0(W_{K/F}, X \to Y) & \xrightarrow{(10.4.1)} & \mathbf{B}(T \to U) \\ \big\downarrow{\scriptstyle (12.3.1)} & & \big\downarrow{\scriptstyle (12.2.3)} \\ H_0(\langle\sigma\rangle, X_I \to Y_I) & \xrightarrow{(12.1.1)} & H^1(\langle\sigma\rangle, X_I \to Y_I). \end{array} \tag{12.4.1}$$

We are going to use Lemma 11.4 to prove that (12.4.1) commutes. We take I to be the functor sending $T \to U$ to $H^1(\langle\sigma\rangle, X_I \to Y_I)$, and we take α, β to be the two natural transformations from H to I given by the two paths in the diagram (12.4.1).

Let V denote the image $f(T)$ of T in U, let W denote the quotient torus U/V, and let C denote the kernel of f. We are interested in the kernel of the map

$$H^1(\langle\sigma\rangle, X_I \to Y_I) \to H^1(\langle\sigma\rangle, X_I) \times (X_*(W)_I)^{\langle\sigma\rangle}, \tag{12.4.2}$$

since we must check that (12.4.2) is injective for $\eta\colon R \to R_1$. It is easy to see that the kernel of (12.4.2) is equal to

$$\big(\mathrm{im}[X_*(V)_I \to Y_I]\big)^{\langle\sigma\rangle} / \mathrm{im}[X_I^{\langle\sigma\rangle} \to Y_I^{\langle\sigma\rangle}]. \tag{12.4.3}$$

Now suppose that C is connected. Then $X_*(T) \to X_*(V)$ is surjective, as is the induced map on I-coinvariants, so that in this case (12.4.3) is equal to

$$\big(\mathrm{im}[X_I \to Y_I]\big)^{\langle\sigma\rangle} / \mathrm{im}[X_I^{\langle\sigma\rangle} \to Y_I^{\langle\sigma\rangle}]. \tag{12.4.4}$$

The group (12.4.4) is trivial if

$$H^1(\langle\sigma\rangle, \ker[X_I \to Y_I]) \tag{12.4.5}$$

is torsion-free.

Now suppose that $f: T \to U$ is $\eta: R \to R_1$. Then C is $\mathbb{G}_m$, so that the kernel of (12.4.2) is equal to (12.4.4). It is easy to see that the kernel of $X_I \to Y_I$ is $\mathbb{Z}$ (with trivial action of $\langle\sigma\rangle$). Therefore the group (12.4.5) is torsion-free (isomorphic to $\mathbb{Z}$), and we conclude that (12.4.2) is injective for $\eta: R \to R_1$, as desired.

The next point to check is that

$$\begin{CD} X_\Gamma @>{(10.4.1)}>> \mathbf{B}(T) \\ @V{(12.3.1)}VV @VV{(12.2.3)}V \\ H_0(\langle\sigma\rangle, X_I) @>{(12.1.1)}>> H^1(\langle\sigma\rangle, X_I) \end{CD}$$

commutes. We identify $H^1(\langle\sigma\rangle, X_I)$ with X_Γ. Then we must show that the composed map

$$X_\Gamma \xrightarrow{(10.4.1)} \mathbf{B}(T) \xrightarrow{(12.2.3)} X_\Gamma$$

is the identity map on X_Γ. This follows from 7.5 and 10.3 (see the discussion of the map (10.3.1)).

The final point to check is that

$$\begin{CD} H_1(W_{K/F}, X) @>{(10.4.1)}>> T(F) \\ @V{(12.3.1)}VV @VV{(7.6.2)}V \\ H_1(\langle\sigma\rangle, X_I) @= (X_I)^{\langle\sigma\rangle} \end{CD} \tag{12.4.6}$$

commutes. Let

$$\xi: T(F) \to (X_I)^{\langle\sigma\rangle}$$

be the homomorphism obtained by going the long way around (12.4.6) (remember that (10.4.1) is an isomorphism). We want to show that ξ is equal to (7.6.2).

Observe that ξ is independent of the field K (use that the diagram (11.5.2) commutes). Of course ξ is functorial in T. Let E be a finite unramified extension of F in $\overline{F}$. Let R_E denote the F-torus $R_{E/F}(T_E)$, obtained by Weil's restriction of scalars from the torus T_E over E. The map ξ for the torus R_E can be thought of as a map

$$T(E) \to (X_I)^{\langle\sigma_E\rangle}, \tag{12.4.7}$$

where $\langle\sigma_E\rangle$ denotes the Frobenius automorphism of F^{un} over E (we used that $X_*(R_E)_I$ is the $\langle\sigma\rangle$-module induced by the $\langle\sigma_E\rangle$-module X_I).

Suppose that E' is a finite unramified extension of F in $\overline{F}$ containing E. Then there is a canonical embedding

$$R_E \hookrightarrow R_{E'}$$

and the functoriality of ξ implies that the diagram

$$\begin{array}{ccc} T(E) & \longrightarrow & (X_I)^{\langle \sigma_E \rangle} \\ \downarrow & & \downarrow \\ T(E') & \longrightarrow & (X_I)^{\langle \sigma_{E'} \rangle} \end{array}$$

commutes. Thus these maps fit together to give a functorial map

$$\widetilde{w}_T : T(F^{\text{un}}) \to X_I.$$

We must show that $\widetilde{w}_T$ is the restriction to $T(F^{\text{un}})$ of the map

$$w_T : T(L) \to X_I$$

defined in 7.2.

Choose an induced torus R over F and a surjection

$$X_*(R) \to X_*(T)$$

of Γ-modules. Then there is an exact sequence

$$1 \to C \to R \to T \to 1,$$

where C is a torus. Since $H^1(F^{\text{un}}, C)$ is trivial, the map

$$R(F^{\text{un}}) \to T(F^{\text{un}})$$

is surjective. Therefore it is enough to prove that w_T restricts to $\widetilde{w}_T$ in the case that T is an induced torus. Then X_I is torsion-free, and by using elements of X^Γ we reduce to the case in which T is $\mathbb{G}_m$.

Thus we must show that for $T = \mathbb{G}_m$ the map (12.4.7)

$$E^\times \to \mathbb{Z}$$

is the usual valuation map on E. Using the norm map $R_{E/F}\mathbb{G}_m \to \mathbb{G}_m$, we see that it is enough to show that the map

$$\xi : F^\times \to \mathbb{Z}$$

for $\mathbb{G}_m$ is the usual valuation map on F.

Thus it is enough to show that the diagram

$$\begin{array}{ccc} H_1(F^\times, \mathbb{Z}) & \longrightarrow & F^\times \\ {\scriptstyle q_*}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{val}} \\ H_1(\langle\sigma\rangle, \mathbb{Z}) & \longrightarrow & \mathbb{Z} \end{array} \tag{12.4.8}$$

commutes, where q is the canonical surjection $F^\times \to \langle\sigma\rangle$ (uniformizing elements in $F^\times$ map to σ). Let x be an element in $H_1(F^\times, \mathbb{Z})$. Choose a 1-cycle $a \mapsto x_a$ of $F^\times$ in $\mathbb{Z}$ representing x. Then (see (10.1.3)) the top horizontal arrow maps x to the element

$$\prod_{a \in F^\times} a^{-x_a} \in F^\times$$

and the valuation of this element is

$$\sum_{a \in F^\times} -\mathrm{val}(a) \cdot x_a. \tag{12.4.9}$$

The map q_* sends x to the class of the 1-cycle

$$\sigma^n \mapsto \sum_{\mathrm{val}(a)=n} x_a \tag{12.4.10}$$

of $\langle\sigma\rangle$ in $\mathbb{Z}$ (the sum is taken over all $a \in F^\times$ satisfying the stated condition).

Let A be any abelian group on which $\langle\sigma\rangle$ acts. Let $C_m(A)$ be the group of standard m-chains of $\langle\sigma\rangle$ in A, so that $H_\bullet(\langle\sigma\rangle, A)$ is the homology of the complex

$$\dots \to C_2(A) \xrightarrow{\partial} C_1(A) \xrightarrow{\partial} C_0(A).$$

The diagram

$$\begin{array}{ccccc} C_2(A) & \xrightarrow{\partial} & C_1(A) & \xrightarrow{\partial} & C_0(A) \\ \big\downarrow & & \big\downarrow & & \big\| \\ 0 & \longrightarrow & A & \xrightarrow{\sigma-1} & A \end{array}$$

commutes, where the vertical arrow $C_1(A) \to A$ sends a 1-chain $\sigma^n \mapsto a_n$ to the element

$$\sum_{n \in \mathbb{Z}} \gamma_n(a_n) \in A,$$

where γ_n denotes the unique element in the integral group ring of $\langle\sigma\rangle$ satisfying the equation

$$\gamma_n \cdot (\sigma - 1) = \sigma^{-n} - 1.$$

Therefore the bottom horizontal arrow in (12.4.8) maps the 1-cycle (12.4.10) to the integer

$$\sum_{n\in\mathbb{Z}} \gamma_n \sum_{\mathrm{val}(a)=n} x_a. \tag{12.4.11}$$

The element γ_n acts by multiplication by $-n$ on $\mathbb{Z}$. Therefore (12.4.11) is equal to (12.4.9), and we are done.

12.5. We return for a moment to the cohomology of the group $\langle\sigma\rangle$. Let X_1, X_2 be finitely generated abelian groups on which $\langle\sigma\rangle$ acts and let $h\colon X_1 \to X_2$ be a homomorphism. Dual to X_1, X_2 are diagonalizable $\mathbb{C}$-groups

$$D_i := \mathrm{Hom}(X_i, \mathbb{C}^\times) \quad (i = 1, 2)$$

on which $\langle\sigma\rangle$ acts. Of course X_i is equal to $X^*(D_i)$. There is a map $\widehat{h}\colon D_2 \to D_1$ dual to h. Since $\mathbb{C}^\times$ is an injective abelian group, there is a canonical isomorphism

$$\mathrm{Hom}\big(H_0(\langle\sigma\rangle, X_1 \to X_2), \mathbb{C}^\times\big) \simeq H^1(\langle\sigma\rangle, D_2 \to D_1), \tag{12.5.1}$$

analogous to [KS, (A.3.8)]. This gives us a $\mathbb{C}^\times$-valued pairing between

$$H_0(\langle\sigma\rangle, X_1 \to X_2)$$

and

$$H^1(\langle\sigma\rangle, D_2 \to D_1).$$

As in [KS] we have the following explicit formula for this pairing in terms of standard chains and cochains. Consider a 0-hypercycle $(x_1, x_2(w))$ and a 1-hypercocycle $(d_2(w), d_1)$. Thus $x_1 \in X_1$, and x_2 is a 1-chain of $\langle\sigma\rangle$ in X_2 such that

$$h(x_1) = \sum_{w\in\langle\sigma\rangle} \big(w^{-1}x_2(w) - x_2(w)\big);$$

similarly $d_1 \in D_1$, and d_2 is a 1-cocycle of $\langle\sigma\rangle$ in D_2 such that

$$\widehat{h}(d_2(w)) = d_1^{-1} w(d_1)$$

for all $w \in \langle\sigma\rangle$. Then the value of the pairing on the classes of these two elements is given by

$$\langle x_1, d_1\rangle \prod_{w\in\langle\sigma\rangle} \langle x_2(w), d_2(w)\rangle^{-1}. \tag{12.5.2}$$

In 12.4 we saw how to convert from standard 1-chains for $\langle\sigma\rangle$ to the simplified 1-chains we used in 12.1. Of course it is obvious how to convert a standard 1-cocycle of $\langle\sigma\rangle$ to a simplified 1-cocycle: take the value of the 1-cocycle at $\sigma \in \langle\sigma\rangle$. Converting (12.5.2) into the language of simplified chains and cochains, we find the following alternative description of our pairing. Consider a simplified 0-hypercycle (x_1, x_2) and a simplified 1-hypercocycle (d_2, d_1). Thus $x_1 \in X_1$ and $x_2 \in X_2$ satisfy

$$h(x_1) = (\sigma - 1)x_2;$$

similarly $d_1 \in D_1$ and $d_2 \in D_2$ satisfy

$$\widehat{h}(d_2) = (\sigma - 1)(d_1).$$

Then the value of the pairing on the classes of these two elements is given by

$$\langle x_1, d_1\rangle\langle\sigma(x_2), d_2\rangle. \tag{12.5.3}$$

Recall the canonical isomorphism (12.1.1)

$$H_0(\langle\sigma\rangle, X_1 \to X_2) \simeq H^1(\langle\sigma\rangle, X_1 \to X_2).$$

Using this isomorphism, we get a pairing between

$$H^1(\langle\sigma\rangle, X_1 \to X_2)$$

and

$$H^1(\langle\sigma\rangle, D_2 \to D_1).$$

It is also given by the formula (12.5.3) (recall that a simplified 0-hypercycle is the same as a simplified 1-hypercocycle).

12.6. We return to $f : T \to U$ and the canonical surjection (12.2.3)

$$\mathbf{B}(T \to U) \to H^1(\langle\sigma\rangle, X_I \to Y_I). \tag{12.6.1}$$

There is an injective inflation map

$$H^1(\langle\sigma\rangle, \widehat{U}^I \to \widehat{T}^I) \hookrightarrow H^1(W_F, \widehat{U} \to \widehat{T}). \tag{12.6.2}$$

There is a $\mathbb{C}^\times$-valued pairing (see (11.2.2)) between

$$\mathbf{B}(T \to U)$$

and

$$H^1(W_F, \widehat{U} \to \widehat{T}),$$

and by applying 12.5 to $X_I \to Y_I$ we get a $\mathbb{C}^\times$-valued pairing between

$$H^1(\langle\sigma\rangle, X_I \to Y_I)$$

and

$$H^1(\langle\sigma\rangle, \widehat{U}^I \to \widehat{T}^I).$$

We claim that these two pairings are compatible, in the sense that

$$\langle b, x'\rangle = \langle b', x\rangle$$

for any $b \in \mathbf{B}(T \to U)$ and $x \in H^1(\langle\sigma\rangle, \widehat{U}^I \to \widehat{T}^I)$, where b' denotes the image of b under (12.6.1) and x' denotes the image of x under (12.6.2). The only nontrivial fact needed to prove this claim is the commutativity of the diagram (12.4.1), which we have already established.

13. Canonical splittings

Let E be a finite unramified extension of F in $\overline{F}$ and put $r = [E : F]$. Thus σ^r is the Frobenius automorphism of F^{un} over E.

13.1. We return once again to the cohomology of the group $\langle\sigma\rangle$. Let A be an abelian group on which $\langle\sigma\rangle$ acts. Restricting A to the subgroup $\langle\sigma^r\rangle$ of $\langle\sigma\rangle$ and then inducing back up to $\langle\sigma\rangle$, we obtain a $\langle\sigma\rangle$-module

$$I(A) := \mathrm{Ind}_{\langle\sigma^r\rangle}^{\langle\sigma\rangle}(A).$$

We can identify $I(A)$ with the r-fold product $A \times \ldots \times A$ (as an abelian group). The action of σ on an r-tuple $(a_1, \ldots, a_r) \in I(A)$ is given by

$$\sigma(a_1, \ldots, a_r) = (\sigma(a_2), \ldots, \sigma(a_r), \sigma(a_1)). \tag{13.1.1}$$

There is a canonical automorphism θ of the $\langle\sigma\rangle$-module $I(A)$, given by

$$\theta(a_1, \ldots, a_r) = (a_r, a_1, \ldots, a_{r-1}). \tag{13.1.2}$$

There is an obvious injective $\langle\sigma\rangle$-map $i\colon A \to I(A)$, defined by

$$i(a) = (a, \ldots, a) \tag{13.1.3}$$

and an obvious surjective $\langle\sigma\rangle$-map $m\colon I(A) \to A$ defined by

$$m(a_1, \ldots, a_r) = a_1 \cdots a_r. \tag{13.1.4}$$

The sequence

$$1 \to A \xrightarrow{i} I(A) \xrightarrow{1-\theta} I(A) \xrightarrow{m} A \to 1$$

is exact, and therefore the exact sequence (12.1.4) for the complex $I(A) \xrightarrow{1-\theta} I(A)$ becomes

$$1 \to A_{\langle\sigma\rangle} \to H^1(\langle\sigma\rangle, I(A) \xrightarrow{1-\theta} I(A)) \to A^{\langle\sigma\rangle} \to 1. \qquad (13.1.5)$$

We claim that there is a canonical splitting of the exact sequence (13.1.5). We write $J(A)$ for the subgroup

$$A^{\langle\sigma\rangle} \times \dots \times A^{\langle\sigma\rangle}$$

of $I(A)$. On the subgroup $J(A)$ the automorphisms σ and θ of $I(A)$ are inverse to one another. Recall from 12.1 that a simplified 1-hypercocycle of $\langle\sigma\rangle$ in

$$I(A) \xrightarrow{1-\theta} I(A)$$

is a pair $(x, y) \in I(A) \times I(A)$ satisfying

$$(1-\theta)(x) = (\sigma - 1)(y).$$

For any $x \in J(A)$ the pair $(\sigma(x), x)$ is a simplified 1-hypercocycle of $\langle\sigma\rangle$ in

$$I(A) \xrightarrow{1-\theta} I(A).$$

We denote by

$$H^1(\langle\sigma\rangle, I(A) \xrightarrow{1-\theta} I(A))_J$$

the subgroup of $H^1(\langle\sigma\rangle, I(A) \xrightarrow{1-\theta} I(A))$ consisting of the classes of all simplified 1-hypercocycles of this special form. We claim that the surjection

$$H^1(\langle\sigma\rangle, I(A) \xrightarrow{1-\theta} I(A)) \to A^{\langle\sigma\rangle} \qquad (13.1.6)$$

induces an isomorphism

$$H^1(\langle\sigma\rangle, I(A) \xrightarrow{1-\theta} I(A))_J \to A^{\langle\sigma\rangle}. \qquad (13.1.7)$$

This will provide the desired splitting. Let $x \in J(A)$. The map (13.1.7) sends the class of $(\sigma(x), x)$ to $m(x)$. Since m maps $J(A)$ onto $A^{\langle\sigma\rangle}$, we see that (13.1.7) is surjective. Suppose that the class of $(\sigma(x), x)$ maps to the identity element of $A^{\langle\sigma\rangle}$. Then there exists $y \in J(A)$ such that $x = (1-\theta)(y)$; therefore $(\sigma(x), x)$ is equal to the simplified 1-hypercoboundary

$$\big((\sigma - 1)(y), (1-\theta)(y)\big),$$

and we see that (13.1.7) is injective as well.

We also need the following variant of the discussion above. Now we consider the complex

$$I(A) \xrightarrow{1-\theta^{-1}} I(A).$$

The sequence

$$1 \to A \xrightarrow{i} I(A) \xrightarrow{1-\theta^{-1}} I(A) \xrightarrow{m} A \to 1$$

is exact, so that we get an exact sequence

$$1 \to A_{\langle\sigma\rangle} \to H^1\big(\langle\sigma\rangle, I(A) \xrightarrow{1-\theta^{-1}} I(A)\big) \to A^{\langle\sigma\rangle} \to 1. \tag{13.1.8}$$

This exact sequence also has a canonical splitting. As the complementary subgroup

$$H^1\big(\langle\sigma\rangle, I(A) \xrightarrow{1-\theta^{-1}} I(A)\big)_J$$

we now take all classes that can be represented by simplified 1-hypercocycles of the form (x^{-1}, x) for some $x \in J(A)$.

13.2. We continue to use the notation of 13.1. We now let X be a finitely generated abelian group on which $\langle\sigma\rangle$ acts, and let $D_X = \mathrm{Hom}(X, \mathbb{C}^\times)$ be the diagonalizable $\mathbb{C}$-group dual to X. There is a canonical isomorphism of $\langle\sigma\rangle$-modules

$$D_{I(X)} \simeq I(D_X), \tag{13.2.1}$$

where I denotes the induction functor $\mathrm{Ind}_{\langle\sigma^r\rangle}^{\langle\sigma\rangle}$, as in 13.1. We denote the automorphism θ of 13.1 for the group $I(X)$ (respectively, $I(D_X)$) by θ_X (respectively, θ_D). Dual to

$$\theta_X : I(X) \to I(X)$$

is the automorphism

$$\widehat{\theta}_X : D_{I(X)} \to D_{I(X)}.$$

Since the functor $X \mapsto D_X$ is contravariant, the isomorphism (13.2.1) identifies $\widehat{\theta}_X$ with the *inverse* of θ_D.

From 13.1 we get an exact sequence

$$1 \to X_{\langle\sigma\rangle} \to H^1\big(\langle\sigma\rangle, I(X) \xrightarrow{1-\theta_X} I(X)\big) \to X^{\langle\sigma\rangle} \to 1 \tag{13.2.2}$$

and a subgroup

$$H^1\big(\langle\sigma\rangle, I(X) \xrightarrow{1-\theta_X} I(X)\big)_J \tag{13.2.3}$$

complementary to $X_{\langle\sigma\rangle}$. We also get an exact sequence

$$1 \to (D_X)_{\langle\sigma\rangle} \to H^1\big(\langle\sigma\rangle, D_{I(X)} \xrightarrow{1-\widehat{\theta}_X} D_{I(X)}\big) \to (D_X)^{\langle\sigma\rangle} \to 1 \tag{13.2.4}$$

and a subgroup

$$H^1\big(\langle\sigma\rangle, D_{I(X)} \xrightarrow{1-\widehat{\theta}_X} D_{I(X)}\big)_J \tag{13.2.5}$$

complementary to $(D_X)_{\langle\sigma\rangle}$ (since $\widehat{\theta}_X = \theta_D^{-1}$, we are using the variant discussed at the end of 13.1).

Recall from 12.5 the canonical pairing $\langle\cdot,\cdot\rangle$ between the groups

$$H^1(\langle\sigma\rangle, I(X) \xrightarrow{1-\theta_X} I(X))$$

and

$$H^1(\langle\sigma\rangle, D_{I(X)} \xrightarrow{1-\widehat{\theta}_X} D_{I(X)}).$$

The pairing is given by the formula (12.5.3). We claim that the subgroups (13.2.3) and (13.2.5) annihilate each other under this pairing. In other words we claim that

$$\langle(\sigma(x), x), (d^{-1}, d)\rangle = 1 \tag{13.2.6}$$

for any $x \in J(X)$ and any $d \in J(D_X)$. By (12.5.3) the left-hand side of (13.2.6) is equal to

$$\langle\sigma(x), d\rangle\langle\sigma(x), d^{-1}\rangle = 1,$$

which proves the claim.

13.3. Now let T be a torus over F, and put $X := X_*(T)$. Let R denote the F-torus $R_{E/F}(T_E)$ obtained from T_E by Weil's restriction of scalars. The Galois group $\mathrm{Gal}(E/F)$ acts naturally (on the left of) R by F-automorphisms, and we denote by θ the F-automorphism of R by which the Frobenius element $\sigma_{E/F}$ in $\mathrm{Gal}(E/F)$ acts. Under the canonical isomorphism

$$R(F) = T(E),$$

the action of θ on $R(F)$ goes over to the action of $\sigma_{E/F}$ on $T(E)$.

There is a canonical isomorphism of $\langle\sigma\rangle$-modules

$$R(L) = I(T(L)), \tag{13.3.1}$$

obtained as follows. We have

$$\begin{aligned} R(L) &= T(E \otimes_F L), \\ I(T(L)) &= T(L) \times \ldots \times T(L) \\ &= T(L \times \ldots \times L), \end{aligned}$$

and with these identifications (13.3.1) becomes the map

$$T(E \otimes_F L) \to T(L \times \ldots \times L)$$

induced by the L-algebra isomorphism

$$E \otimes_F L \to L \times \ldots \times L$$

sending $e \otimes l$ to the r-tuple

$$(\sigma^r(e)l, \dots, \sigma^2(e)l, \sigma(e)l).$$

Note that (13.3.1) carries the automorphism of $R(L)$ induced by $\theta \in \mathrm{Aut}_F(R)$ over to the automorphism of $I(T(L))$ denoted by θ in 13.1.

Consider the exact sequence

$$1 \to \mathbf{B}(T) \to \mathbf{B}(R \xrightarrow{1-\theta} R) \to T(F) \to 1 \tag{13.3.2}$$

(a special case of both (13.1.5) and (9.1.2)). From 13.1 we get a canonical subgroup

$$\mathbf{B}(R \xrightarrow{1-\theta} R)_J$$

of $\mathbf{B}(R \xrightarrow{1-\theta} R)$, complementary to the subgroup $\mathbf{B}(T)$.

Consider the Langlands dual complex

$$\widehat{R} \xrightarrow{1-\widehat{\theta}} \widehat{R}.$$

There is an obvious identification (of $\mathbb{C}$-groups) of $\widehat{R}$ with the r-fold product

$$\widehat{T} \times \dots \times \widehat{T}.$$

Let $\tau \in \Gamma$ and suppose that the restriction of τ to F^{un} is equal to σ. Then

$$\tau(\widehat{t}_1, \dots, \widehat{t}_r) = (\tau(\widehat{t}_2), \dots, \tau(\widehat{t}_r), \tau(\widehat{t}_1)).$$

Moreover the action of $\widehat{\theta}$ is given by

$$\widehat{\theta}(\widehat{t}_1, \dots, \widehat{t}_r) = (\widehat{t}_2, \dots, \widehat{t}_r, \widehat{t}_1).$$

The sequence

$$1 \to \widehat{T} \xrightarrow{i} \widehat{R} \xrightarrow{1-\widehat{\theta}} \widehat{R} \xrightarrow{m} \widehat{T} \to 1 \tag{13.3.3}$$

is exact, where i is defined by

$$\widehat{t} \mapsto (\widehat{t}, \dots, \widehat{t})$$

and m is defined by

$$(\widehat{t}_1, \dots, \widehat{t}_r) \mapsto \widehat{t}_1 \cdots \widehat{t}_r.$$

Let $f : T \to U$ be a map of F-tori, let C be the kernel of f, let W be the cokernel of f, and let V be the image of f. Assume that C is connected. There is an exact sequence

$$1 \to [T \to V] \to [T \to U] \to [1 \to W] \to 1 \tag{13.3.4}$$

and the obvious map from $[C \to 1]$ to $[T \to V]$ is a quasi-isomorphism. Applying **B** to (13.3.4) we get the exact sequence (9.1.2)

$$1 \to \mathbf{B}(C) \to \mathbf{B}(T \to U) \to W(F) \to 1. \tag{13.3.5}$$

Dual to (13.3.4) is the exact sequence

$$1 \to [\widehat{W} \to 1] \to [\widehat{U} \to \widehat{T}] \to [\widehat{V} \to \widehat{T}] \to 1. \tag{13.3.6}$$

Since C is connected, the map $\widehat{V} \to \widehat{T}$ is injective with cokernel $\widehat{C}$, and hence the obvious map from $[\widehat{V} \to \widehat{T}]$ to $[1 \to \widehat{C}]$ is a quasi-isomorphism. Applying the functor $H^\bullet(W_F, \cdot)$ of [KS, A.3] to (13.3.6), we get an exact sequence

$$1 \to H^1(W_F, \widehat{W}) \to H^1(W_F, \widehat{U} \to \widehat{T}) \to \widehat{C}^\Gamma \to 1 \tag{13.3.7}$$

(use that $H^2(W_F, \widehat{W})$ vanishes), and this exact sequence is obtained from (13.3.5) by applying the functor $\mathrm{Hom}_{\mathrm{cont}}(\cdot, \mathbb{C}^\times)$ (see Section 11).

Taking $T \xrightarrow{f} U$ to be $R \xrightarrow{1-\theta} R$, the exact sequence (13.3.7) becomes

$$1 \to H^1(W_F, \widehat{T}) \to H^1(W_F, \widehat{R} \xrightarrow{1-\widehat{\theta}} \widehat{R}) \to \widehat{T}^\Gamma \to 1, \tag{13.3.8}$$

and this sequence is obtained by applying $\mathrm{Hom}_{\mathrm{cont}}(\cdot, \mathbb{C}^\times)$ to (13.3.2).

There is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & H^1(\langle\sigma\rangle, \widehat{T}^I) & \longrightarrow & H^1(\langle\sigma\rangle, \widehat{R}^I \xrightarrow{1-\widehat{\theta}} \widehat{R}^I) & \longrightarrow & \widehat{T}^\Gamma & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & H^1(W_F, \widehat{T}) & \longrightarrow & H^1(W_F, \widehat{R} \xrightarrow{1-\widehat{\theta}} \widehat{R}) & \longrightarrow & \widehat{T}^\Gamma & \longrightarrow & 1 \end{array} \tag{13.3.9}$$

in which the vertical maps are inflation maps for the canonical surjection $W_F \to \langle\sigma\rangle$. Note that

$$\widehat{T}^I = \mathrm{Hom}(X_I, \mathbb{C}^\times),$$

and hence that the top row in (13.3.9) is the exact sequence (13.2.4) for the finitely generated abelian group X_I. Therefore 13.2 gives us a subgroup

$$H^1(\langle\sigma\rangle, \widehat{R}^I \xrightarrow{1-\widehat{\theta}} \widehat{R}^I)_J$$

of

$$H^1(\langle\sigma\rangle, \widehat{R}^I \xrightarrow{1-\widehat{\theta}} \widehat{R}^I)$$

complementary to the subgroup $H^1(\langle\sigma\rangle, \widehat{T}^I)$. By inflation we identify

$$H^1(\langle\sigma\rangle, \widehat{R}^I \xrightarrow{1-\widehat{\theta}} \widehat{R}^I)_J$$

with a subgroup of

$$H^1(W_F, \widehat{R} \xrightarrow{1-\widehat{\theta}} \widehat{R});$$

obviously this subgroup is complementary to $H^1(W_F, \widehat{T})$, so that we have produced a canonical splitting of the exact sequence occurring as the bottom row in (13.3.9).

PROPOSITION 13.4. *The subgroup* $\mathbf{B}(R \xrightarrow{1-\theta} R)_J$ *of* $\mathbf{B}(R \xrightarrow{1-\theta} R)$ *and the subgroup* $H^1(\langle\sigma\rangle, \widehat{R}^I \xrightarrow{1-\widehat{\theta}} \widehat{R}^I)_J$ *of* $H^1(W_F, \widehat{R} \xrightarrow{1-\widehat{\theta}} \widehat{R})$ *annihilate each other under the* $\mathbb{C}^\times$*-valued pairing between* $\mathbf{B}(R \xrightarrow{1-\theta} R)$ *and* $H^1(W_F, \widehat{R} \xrightarrow{1-\widehat{\theta}} \widehat{R})$ *obtained from* (11.2.2).

We have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbf{B}(T) & \longrightarrow & \mathbf{B}(R \xrightarrow{1-\theta} R) & \longrightarrow & T(F) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & (X_I)_{\langle\sigma\rangle} & \longrightarrow & H^1(\langle\sigma\rangle, Y_I \xrightarrow{1-\theta} Y_I) & \longrightarrow & (X_I)^{\langle\sigma\rangle} & \longrightarrow & 1,
\end{array} \tag{13.4.1}$$

where Y denotes the cocharacter group $X_*(R)$. The vertical maps are of type (12.2.3), and the bottom row is of type (13.2.2) (for the finitely generated abelian group X_I); of course we are using the obvious identification

$$Y_I = \mathrm{Ind}_{\langle\sigma^r\rangle}^{\langle\sigma\rangle}(X_I).$$

In 12.6 we proved the compatibility of two pairings. This compatibility implies that the diagram (13.3.9) is obtained by applying $\mathrm{Hom}_{\mathrm{cont}}(\cdot, \mathbb{C}^\times)$ to the diagram (13.4.1).

Since $H^1(\langle\sigma\rangle, \widehat{R}^I \xrightarrow{1-\widehat{\theta}} \widehat{R}^I)_J$ is a subgroup of $H^1(\langle\sigma\rangle, \widehat{R}^I \xrightarrow{1-\widehat{\theta}} \widehat{R}^I)$ it is enough to show that $H^1(\langle\sigma\rangle, \widehat{R}^I \xrightarrow{1-\widehat{\theta}} \widehat{R}^I)_J$ annihilates the image of $\mathbf{B}(R \xrightarrow{1-\theta} R)_J$ in

$$H^1(\langle\sigma\rangle, Y_I \xrightarrow{1-\theta} Y_I). \tag{13.4.2}$$

But this image is contained in the canonical subgroup of (13.4.2) complementary to $(X_I)_{\langle\sigma\rangle}$. Therefore the desired annihilation was proved in 13.2 (apply 13.2 to the finitely generated abelian group X_I).

Appendix

A. Automorphism groups of $\otimes$-functors

A.1. Let k be a commutative ring with 1. Let $G = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$ be affine schemes over k, and suppose that we are given a morphism

$$a\colon G \times X \to X$$

of schemes over k (the product is taken over $\mathrm{Spec}(k)$). We think of G, X as set-valued functors on the category of k-algebras and define a subfunctor X^G of X as follows: for any k-algebra R the set $X^G(R)$ consists of all elements $x \in X(R)$ such that

$$a(g, x_S) = x_S$$

for every R-algebra S and every $g \in G(S)$ (we use x_S to denote the image of x in $X(S)$). If G is a group scheme and a is an action of G on X, then we refer to points in $X^G(R)$ as G_R-fixed points in $X(R)$ (G_R denotes the group scheme over R obtained from G by extension of scalars).

Now assume that k is a field. Then we claim that X^G is represented by a closed subscheme of X. Let

$$a^*\colon B \to A \otimes_k B$$

be the k-algebra map induced by a. The set $X^G(R)$ can be identified with the set of k-algebra homomorphisms $f\colon B \to R$ such that the map

$$\mathrm{id}_A \otimes f\colon A \otimes_k B \to A \otimes_k R$$

vanishes on the subset M of $A \otimes_k B$ consisting of all elements of the form $a^*(b) - 1_A \otimes b$ for some $b \in B$. Pick a basis $\{a_i\}_{i \in I}$ for A as k-vector space. Any element $x \in A \otimes_k B$ can be written uniquely as $\sum_{i\in I} a_i \otimes b_i(x)$ and $\mathrm{id}_A \otimes f$ vanishes on x if and only if $f(b_i(x)) = 0$ for every $i \in I$. It follows that $\mathrm{id}_A \otimes f$ vanishes on M if and only if f vanishes on the set N of elements in B of the form $b_i(x)$ for some $x \in M$ and some $i \in I$. Therefore X^G is represented by the closed subscheme of X defined by the ideal in B generated by N.

A.2. Let k be a field, and let $\mathcal{T}, \mathcal{U}$ be Tannakian categories over k (see [D], [Sa]). Let $\beta\colon \mathcal{T} \to \mathcal{U}$ be an exact $\otimes$-functor. For any k-algebra R we define an R-linear $\otimes$-category $\mathcal{U}^R$ as in 3.3. Recall that $\mathcal{U}^R$ has the same objects as $\mathcal{U}$, and that for objects X, Y in $\mathcal{U}$ one has

$$\mathrm{Hom}_{\mathcal{U}^R}(X, Y) = \mathrm{Hom}_{\mathcal{U}}(X, Y) \otimes_k R.$$

As in 3.3 there is an obvious $\otimes$-functor

$$\mathcal{U} \to \mathcal{U}^R.$$

Composing this functor with β, we get a $\otimes$-functor

$$\beta^R : \mathcal{T} \to \mathcal{U}^R.$$

We then let $J_\beta(R)$ denote the group of $\otimes$-automorphisms of β^R.

We claim that the functor J_β is representable by an affine group scheme over k. Suppose that $\mathcal{U}$ has a fiber functor $\omega_\mathcal{U}$ over a nonzero k-algebra S. We define a fiber functor $\omega_\mathcal{T}$ on $\mathcal{T}$ by

$$\omega_\mathcal{T} := \omega_\mathcal{U} \circ \beta.$$

Then $\omega_\mathcal{T}, \omega_\mathcal{U}$ determine k-groupoids $\mathcal{G}, \mathcal{H}$ acting transitively on $\mathrm{Spec}(S)$ (see [D]), and the pullbacks of $\mathcal{G}$, $\mathcal{H}$ along the diagonal map

$$\mathrm{Spec}(S) \to \mathrm{Spec}(S) \times_{\mathrm{Spec}(k)} \mathrm{Spec}(S)$$

are affine group schemes G, H over S. The $\otimes$-functor β induces a homomorphism

$$\nu : H \to G$$

over S, and we denote by G_ν the centralizer of ν in G, by which we mean the subfunctor of H-fixed points in G (see A.1) for the conjugation action of H on G. We claim further that there is a canonical isomorphism

$$(J_\beta)_S \simeq G_\nu,$$

where $(J_\beta)_S$ is the group scheme over S obtained from J_β by extension of scalars.

In fact the first claim follows from the second. Indeed, $\mathcal{U}$ has a fiber functor over some field S containing k. It is easy to see that J_β is a sheaf for the faithfully flat topology (on the category of affine schemes over k). Therefore it is enough to prove that $(J_\beta)_S$ is representable by an affine group scheme over S, and this follows from A.1 (assuming the truth of the second claim).

Now we prove the second claim. Let R be any S-algebra. From $\omega_\mathcal{T}, \omega_\mathcal{U}$ we get fiber functors $\omega^R_\mathcal{T}$, $\omega^R_\mathcal{U}$ on $\mathcal{T}, \mathcal{U}$ over R, and the corresponding groupoids $\mathcal{G}_R$, $\mathcal{H}_R$ (respectively, group schemes) G_R, H_R are obtained from $\mathcal{G}$, $\mathcal{H}$ (respectively, G, H) by extending scalars from $S \otimes_k S$ to $R \otimes_k R$ (respectively, from S to R).

Giving an element $a \in J_\beta(R)$ is the same as giving a compatible family of elements

$$a_X \in \left(\mathrm{End}_{\mathcal{H}_R}(\omega^R_\mathcal{U} \beta X) \otimes_k R\right)^\times,$$

one for each object X in $\mathcal{T}$ (compatible means functorial and compatible with all finite tensor products). But $\omega^R_\mathcal{U} \beta X = \omega^R_\mathcal{T} X$ and

$$\mathrm{End}_{\mathcal{H}_R}(\omega^R_\mathcal{T} X) \otimes_k R = \mathrm{End}_{H_R}(\omega^R_\mathcal{T} X),$$

since the action of the groupoid $\mathcal{H}_R$ on $\omega^R_{\mathcal{T}} X$ determines descent data (from R to k) on $\mathrm{End}_{H_R}(\omega^R_{\mathcal{T}} X)$, and $\mathrm{End}_{\mathcal{H}_R}(\omega^R_{\mathcal{T}} X)$ is equal to the k-vector space obtained from $\mathrm{End}_{H_R}(\omega^R_{\mathcal{T}} X)$ by descent. Moreover $\mathrm{End}_{H_R}(\omega^R_{\mathcal{T}} X)$ can be identified with the fixed points of the action of H_R on the R-module $\mathrm{End}_R(\omega^R_{\mathcal{T}} X)$. Therefore giving $a \in J_\beta(R)$ is the same as giving an H_R-fixed point in the set of compatible families of elements

$$a'_X \in \mathrm{End}_R(\omega^R_{\mathcal{T}} X)^\times$$

and this in turn is the same as giving an H_R-fixed point in $G(R)$. Therefore $J_\beta(R)$ is equal to $G^H(R)$, where H acts on G by conjugation, which proves the second claim.

B. The Galois gerbs $\mathcal{D}_s$

B.1. Let $\mathcal{T}$ be a Tannakian category over F (see [D], [Sa]). We suppose that $\mathcal{T}$ admits a fiber functor over $\overline{F}$, and we fix such a fiber functor ω. Then in the usual way ω determines an affine group scheme G over $\overline{F}$. We assume further that G is of finite type over $\overline{F}$, so that G is a linear algebraic group over $\overline{F}$. Of course $G(\overline{F})$ is equal to the group of $\otimes$-automorphisms of the fiber functor ω.

Let $\tau \in \Gamma$. By a τ-linear $\otimes$-automorphism of ω we mean a family of τ-linear isomorphisms

$$g_X : \omega(X) \to \omega(X),$$

one for each object X in $\mathcal{T}$, functorial in X and compatible with finite tensor products. Let $\mathcal{G}_\tau$ be the set of all τ-linear $\otimes$-automorphisms of ω, and let $\mathcal{G}$ be the disjoint union

$$\mathcal{G} := \coprod_{\tau \in \Gamma} \mathcal{G}_\tau.$$

Then $\mathcal{G}$ is a group (under composition) and there is an exact sequence

$$1 \to G(\overline{F}) \to \mathcal{G} @>q>> \Gamma \to 1, \tag{B.1.1}$$

the fiber of q over $\tau \in \Gamma$ being $\mathcal{G}_\tau$ (to prove that the map $\mathcal{G} \to \Gamma$ is surjective use that any two fiber functors for $\mathcal{T}$ over $\overline{F}$ are isomorphic).

The extension $\mathcal{G}$ of Γ by $G(\overline{F})$ is called the *Galois gerb* associated to $\mathcal{T}$ and ω (see [LR]). There is a natural topology on $\mathcal{G}$ making $\mathcal{G}$ into a topological group. The induced topology on the subgroup $G(\overline{F})$ is discrete, and the induced topology on the quotient group Γ is the usual Krull topology. The topology is defined as follows. There exists a finite Galois extension K of F in $\overline{F}$ and a fiber functor ω_0 on $\mathcal{T}$ over K. Choose a $\otimes$-isomorphism between ω and the fiber functor $\omega_0^{\overline{F}}$ obtained from ω_0 by extension of scalars from K to $\overline{F}$. Let Γ_K denote the subgroup $\mathrm{Gal}(\overline{F}/K)$ of Γ. Our choices determine a section of $\mathcal{G} \to \Gamma$ over the subgroup Γ_K

of Γ (since each $\omega(X)$ has acquired a K-structure and hence a canonical τ-linear automorphism for each $\tau \in \Gamma_K$). Any two sections of this type become conjugate under $G(\overline{F})$ after restricting to a suitably small open subgroup of Γ. Using our chosen section we express $q^{-1}(\Gamma_K)$ as the semidirect product

$$q^{-1}(\Gamma_K) = G(\overline{F}) \rtimes \Gamma_K.$$

We put the discrete topology on $G(\overline{F})$, the Krull topology on Γ_K, and the product topology on $q^{-1}(\Gamma_K)$. We give $\mathcal{G}$ the unique topology for which it is a topological group and the inclusion

$$q^{-1}(\Gamma_K) \hookrightarrow \mathcal{G}$$

is an open mapping. It is easy to see that this topology is independent of the choices we made.

By a representation ρ of $\mathcal{G}$ we mean a discrete, semilinear, algebraic action of $\mathcal{G}$ on a finite dimensional $\overline{F}$-vector space V (discrete means that the stabilizer in $\mathcal{G}$ of any vector in V is an open subgroup of $\mathcal{G}$, semilinear means that elements in $\mathcal{G}_\tau$ act by τ-linear automorphisms of V, and algebraic means that the restriction of ρ to $G(\overline{F})$ is a representation of the algebraic group G). For any object X in $\mathcal{T}$ there is an obvious representation of $\mathcal{G}$ on $\omega(X)$, and the resulting $\otimes$-functor $X \mapsto \omega(X)$ from $\mathcal{T}$ to the $\otimes$-category of representations of $\mathcal{G}$ is a $\otimes$-equivalence of $\otimes$-categories.

B.2. Let $\mathcal{T}$ be the Tannakian category *σ-L-spaces* (see Section 2). Let s be a positive integer. We denote by $\mathcal{T}_s$ the full Tannakian subcategory of $\mathcal{T}$ consisting of all σ-L-spaces (V, Φ) whose slopes lie in the subgroup $\frac{1}{s}\mathbb{Z}$ of $\mathbb{Q}$.

Let F_s denote the fixed field of σ^s on F^{un}; of course F_s is the unique unramified extension of F in $\overline{F}$ having degree s. The Tannakian category $\mathcal{T}_s$ has fiber functors over F_s, and any two such fiber functors are isomorphic. We can single out one such fiber functor by choosing a uniformizing element π for F. Then the desired fiber functor ω_π is given by

$$\omega_\pi(V, \Phi) := \bigoplus_{n \in \mathbb{Z}} V^{\pi^{-n}\Phi^s} \tag{B.2.1}$$

together with the obvious isomorphism

$$\omega_\pi\left(\bigotimes_{i \in I} V_i\right) = \bigotimes_{i \in I} \omega_\pi(V_i).$$

The group of automorphisms of ω_π is $\mathbb{G}_m(F_s)$ (an element $x \in \mathbb{G}_m(F_s)$ acts on $V^{\pi^{-n}\Phi^s}$ by x^n).

By considering semilinear $\otimes$-automorphisms of ω_π as well, we get an extension

$$1 \to \mathbb{G}_m(F_s) \to \mathcal{D}_s^0 \to \mathrm{Gal}(F_s/F) \to 1. \tag{B.2.2}$$

Extending scalars from F_s to $\overline{F}$ we get a fiber functor $\omega_\pi^{\overline{F}}$ on $\mathcal{T}_s$ over $\overline{F}$, and thus we also have the extension

$$1 \to \mathbb{G}_m(\overline{F}) \to \mathcal{D}_s \to \Gamma \to 1, \tag{B.2.3}$$

where $\mathcal{D}_s$ denotes the Galois gerb associated to $\mathcal{T}$ and $\omega_\pi^{\overline{F}}$. Of course the extension (B.2.3) is obtained from the extension (B.2.2) by pulling back along the canonical surjection

$$\Gamma \to \mathrm{Gal}(F_s/F)$$

and then pushing out along the canonical injection

$$\mathbb{G}_m(F_s) \hookrightarrow \mathbb{G}_m(\overline{F}).$$

For any σ-L-space (V, Φ) the σ-linear automorphism Φ preserves the subspace $\omega_\pi(V, \Phi)$ of V. Since the resulting σ-linear automorphism of $\omega_\pi(V, \Phi)$ is functorial and compatible with tensor products, there is a canonical element $\varphi_s \in \mathcal{D}_s^0$ lying over the Frobenius element in $\mathrm{Gal}(F_s/F)$, namely the unique element that acts by Φ on $\omega_\pi(V, \Phi)$ for all (V, Φ). Note that the s-th power of φ_s is equal to $\pi \in \mathbb{G}_m(F_s)$. The element $\varphi_s \in \mathcal{D}_s^0$ determines a homomorphism

$$\langle\sigma\rangle \to \mathcal{D}_s^0, \tag{B.2.4}$$

namely the unique one that sends the generator σ of the infinite cyclic group $\langle\sigma\rangle$ to the element φ_s in $\mathcal{D}_s^0$. We now define a continuous homomorphism

$$W_F \to \mathcal{D}_s \tag{B.2.5}$$

as follows. Let Γ_s denote the group $\mathrm{Gal}(F_s/F)$. Then the fiber product

$$\mathcal{D}_s^0 \times_{\Gamma_s} \Gamma$$

is a subgroup of $\mathcal{D}_s$, and the homomorphism (B.2.5) factors through this subgroup, its first component being the map

$$W_F \to \langle\sigma\rangle \to \mathcal{D}_s^0$$

obtained by composing the canonical surjection $W_F \to \langle\sigma\rangle$ with the map (B.2.4) from $\langle\sigma\rangle$ to $\mathcal{D}_s^0$, and its second component being the canonical injection

$$W_F \hookrightarrow \Gamma.$$

It is clear that the map (B.2.5) is a section of

$$\mathcal{D}_s \to \Gamma$$

over the subgroup W_F of Γ. The pair consisting of the extension $\mathcal{D}_s$ of Γ by $\mathbb{G}_m(\overline{F})$ and the section (B.2.5) over W_F has no nontrivial automorphisms.

Now suppose that t is a positive integer such that s divides t, say $t = su$. For any object (V, Φ) in $\mathcal{T}_s$ (which also can be regarded as an object in $\mathcal{T}_t$) there is a canonical isomorphism

$$F_t \otimes_{F_s} \left(\bigoplus_{n \in \mathbb{Z}} V^{\pi^{-n}\Phi^s} \right) \to \bigoplus_{m \in \mathbb{Z}} V^{\pi^{-m}\Phi^t} \tag{B.2.6}$$

(to prove this use descent theory for F_t/F_s). The isomorphism (B.2.6) determines a map of extensions

$$\begin{CD} 1 @>>> \mathbb{G}_m(\overline{F}) @>>> \mathcal{D}_t @>>> \Gamma @>>> 1 \\ @. @VuVV @VVV @| @. \\ 1 @>>> \mathbb{G}_m(\overline{F}) @>>> \mathcal{D}_s @>>> \Gamma @>>> 1, \end{CD} \tag{B.2.7}$$

where the left vertical arrow is the map $x \mapsto x^u$. It is easy to see that the diagram

$$\begin{CD} W_F @>>> \mathcal{D}_t \\ @| @VVV \\ W_F @>>> \mathcal{D}_s \end{CD}$$

commutes, where the horizontal maps are of type (B.2.5).

B.3. Now let K be any finite Galois extension of F in $\overline{F}$. Put $s = [K : F]$. Let $W_{K/F}$ denote the Weil group of K/F. Recall that $W_{K/F}$ is the subgroup of $\mathrm{Gal}(K^{\mathrm{ab}}/F)$ consisting of elements that induce on F^{un} an integral power of σ (here K^{ab} denotes the maximal abelian extension of K in $\overline{F}$). Obviously $W_{K/F}$ is a quotient of W_F, and there is an exact sequence

$$1 \to K^\times \to W_{K/F} \to \mathrm{Gal}(K/F) \to 1, \tag{B.3.1}$$

in which we use the reciprocity isomorphism for K to identify $K^\times$ with a subgroup of $\mathrm{Gal}(K^{\mathrm{ab}}/K)$. We normalize the reciprocity isomorphism in the same way Serre does [S2], so that Frobenius elements in $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ correspond to uniformizing elements in $K^\times$.

Pulling back the extension (B.3.1) along the canonical surjection

$$\Gamma \to \mathrm{Gal}(K/F)$$

and then pushing it out along the canonical injection

$$\mathbb{G}_m(K) \hookrightarrow \mathbb{G}_m(\overline{F}),$$

we get an extension $\mathcal{E}_{K/F}$ of Γ by $\mathbb{G}_m(\overline{F})$. The surjection $\mathcal{E}_{K/F} \to \Gamma$ has a canonical section over the subgroup $\Gamma_K := \mathrm{Gal}(\overline{F}/K)$ of Γ. We use this section to topologize $\mathcal{E}_{K/F}$ in the same way that we topologized $\mathcal{G}$ in B.1. The induced topology on the subgroup $\mathbb{G}_m(\overline{F})$ of $\mathcal{E}_{K/F}$ is discrete, and the induced topology on the quotient group Γ is the Krull topology.

The extensions $\mathcal{E}_{K/F}$ and $\mathcal{D}_s$ are isomorphic (both correspond to $\frac{1}{s} \in \mathbb{Q}/\mathbb{Z}$ under the canonical isomorphism from $H^2(F, \mathbb{G}_m)$ to $\mathbb{Q}/\mathbb{Z}$), and the isomorphism between them is unique up to an inner automorphism of $\mathcal{E}_{K/F}$ coming from an element in $\mathbb{G}_m(\overline{F})$ (since $H^1(F, \mathbb{G}_m)$ is trivial). Using one of these isomorphisms

$$\mathcal{D}_s \simeq \mathcal{E}_{K/F}, \tag{B.3.2}$$

the map (B.2.5) gives us a section

$$W_F \to \mathcal{E}_{K/F} \tag{B.3.3}$$

of

$$\mathcal{E}_{K/F} \to \Gamma$$

over the subgroup W_F of Γ, and if we make a different choice of isomorphism (B.3.2) the section (B.3.3) is replaced by a conjugate under some element of $\mathbb{G}_m(\overline{F})$. Suppose that we make a different choice of uniformizing element π. Then the section (B.3.3) is multiplied by a 1-cocycle of W_F in $\mathbb{G}_m(\overline{F})$ that is cohomologous to one obtained by inflation from a 1-cocycle of $\langle\sigma\rangle$ in the group of units in $F_s^\times$. Note that the isomorphism (B.3.2) is an isomorphism of topological groups and hence that the map (B.3.3) is continuous.

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