LEZIONE 22 NOVEMBRE

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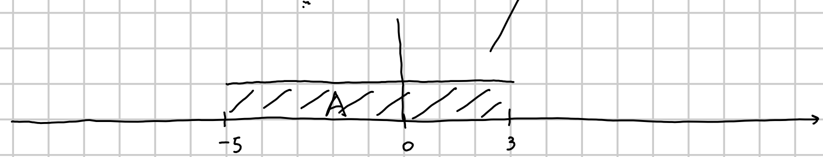
$$\int_{-5}^{3} (3x+1)\,dx = \int_{-5}^{3} 3x\,dx + \int_{-5}^{3} 1\,dx =$$

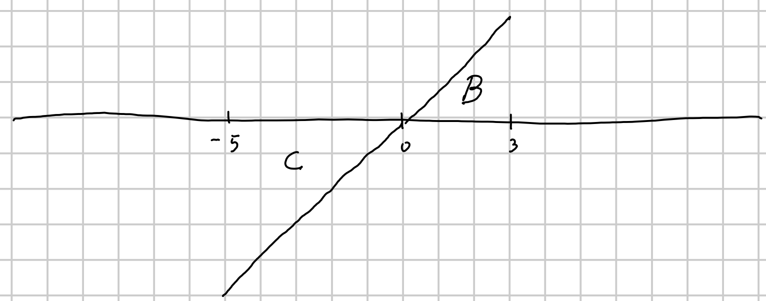
$$= 3\underbrace{\int_{-5}^{3} x\,dx} + \underbrace{\int_{-5}^{3} 1\cdot dx}$$

A = 8



$$\int_{-5}^{3} 1\,dx = \text{Area } A = B\cdot h = 8\cdot 1 = 8$$

π



$$\int_{-5}^{3} x\,dx = \text{Area } B - \text{Area } C = \frac{3\cdot 3}{2} - \frac{5\cdot 5}{2} = \frac{9}{2} - \frac{25}{2}$$

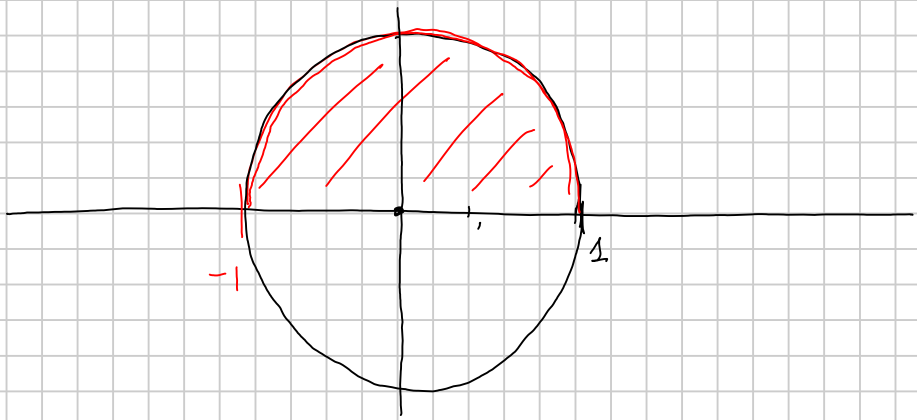
$$= -\frac{16}{2} = -8$$

$$\int_{-5}^{3} 3x+1\,dx = 3\underbrace{\int_{-5}^{3} x\,dx} + \int_{-5}^{3} 1\,dx = 3\cdot(-8)+8 = -16$$

53

$$\int_{-1}^{1} \sqrt{1-x^2}\, dx$$

$y = +\sqrt{1-x^2}$ $\qquad$ $y^2 = 1-x^2$ $\qquad$ $x^2+y^2=1$



$$\int_{-1}^{1} \sqrt{1-x^2}\, dx = \text{Area tratteggiata di Rosso}$$

$$= \frac{1}{2}\ \text{Area cerchio di raggio } 1$$

$$= \frac{1}{2}\pi = \frac{\pi}{2}$$

TEOREMA 1 Se $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ (o definite in un intervallo) e $f'(x) = g'(x)$ per ogni x, allora esiste una costante c tale che

$$f(x) = g(x) + c \quad \text{per ogni } x$$

[IL CASO FONDAMENTALE È SE $f'(x) = 0$ per ogni ALLORA $f(x) = c$ per ogni x]

TEOREMA FONDAMENTALE DEL CALCOLO INTEGRALE

$f : \mathbb{R} \longrightarrow \mathbb{R}$ continua (tutte le funzioni che noi useremo sono continue e la funzione può essere definita anche solo su ~~tutto~~ un intervallo)

Fissiamo $a \in \mathbb{R}$

Definiamo

$$F(x) = \int_a^x f(t)\, dt$$

Allora $F'(x) = f(x)$

CONSEGUENZA

VOGLIAMO CALCOLARE $\int_a^b f(x)\, dx$ E SUPPONIAMO DI SAPERE CHE $f(x) = g'(x)$ per ogni x.

Allora $\int_a^b f(x)\, dx = g(b) - g(a)$

DIMOSTRAZIONE

$F(x) = \int_a^x f(t)\, dt$ NOI VOGLIAMO CALCOLARE $\underline{F(b)}$

SAPPIAMO ANCHE $F(a) = \int_a^a f(t)\, dt = 0$

SAPPIAMO ANCHE CHE $F'(x) = f(x) = g'(x)$

ALLORA SAPPIAMO ANCHE $F(x) = g(x) + c$

$$\int_a^b f(t)\, dt = F(b) = F(b) - \underbrace{F(a)}_{=0} = g(b) + c - (g(a) + c) = g(b) - g(a)$$

#

Se $g'(x) = f(x)$ $\quad \int_a^b f(x)\, dx = g(b) - g(a) = [g]_a^b = g\Big|_a^b$

DEFINIZIONE Se $f: \mathbb{R} \to \mathbb{R}$ (o f definita su un intervallo)

e g è una funzione $g: \mathbb{R} \to \mathbb{R}$ tale che $g'(x) = f(x)$

g si dice una primitiva di f.

ESEMPI

$$\int_a^b g'(x)\, dx = g(b) - g(a)$$

PRENDIAMO $g(x) = x^{\alpha}$ $g'(x) = \alpha x^{\alpha-1}$

$$\int_a^b \underline{\alpha x^{\alpha-1}}\, dx = b^{\alpha} - a^{\alpha} = \left[x^{\alpha} \right]_a^b$$

QUINDI SE $\alpha \neq 0$ DIVIDO PER α

$$\int_a^b x^{\alpha-1}\, dx = \left[\frac{1}{\alpha} x^{\alpha} \right]_a^b = \frac{b^{\alpha}}{\alpha} - \frac{a^{\alpha}}{\alpha}$$

Se pongo $\beta = \alpha - 1$ E $\beta \neq -1$ $\alpha = \beta + 1$

$$\boxed{\int_a^b x^{\beta}\, dx} = \left[\frac{x^{\beta+1}}{\beta+1} \right]_a^b = \frac{b^{\beta+1}}{\beta+1} - \frac{a^{\beta+1}}{\beta+1}$$

$\beta = 7$ $$\int_3^5 x^7\, dx = \left[\frac{x^8}{8} \right]_3^5 = \frac{5^8}{8} - \frac{3^8}{8}$$

$$\int_1^2 \sqrt{x}\, dx = \int_1^2 x^{1/2}\, dx = \left[\frac{x^{3/2}}{3/2} \right]_1^2 = \frac{2^{3/2}}{3/2} - \frac{1^{3/2}}{3/2}$$

$$\int x^{\beta} \qquad \beta \neq -1$$

$$\int_1^3 \frac{1}{x}\, dx = \log(3) - \log(1) = \log(3)$$

$$f = g' \qquad g(x) = \log x \qquad g'(x) = \frac{1}{x}$$

ESERCIZIO

a) $\displaystyle\int_1^3 e^x \, dx$ b) $\displaystyle\int_0^\pi \sin(x)\, dx$

a) $$\int_1^3 e^x \, dx = \left[g \right]_1^3 = g(3) - g(1) = \left[e^x \right]_1^3 = e^3 - e$$

Se $f(x) = e^x$ voglio trovare $g(x)$ tale che $g'(x) = f(x)$

Nel nostro caso possiamo prendere $g(x) = e^x$

b) $$\int_0^\pi \sin(x)\, dx = \left[-\cos(x) \right]_0^\pi = -\cos\pi + \cos(0) = 1 + 1 = 2$$

$$D(\cos(x)) = -\sin(x)$$

$$D(-\cos(x)) = \sin(x)$$

$$\int f + g = \int f + \int g \qquad \int \lambda f = \lambda \int f.$$

$\lambda \in \mathbb{R}$

PER LE DERIVATE C'È ANCHE UNA FORMULA SEMPLICE PER IL CALCOLO DELLA DERIVATA DEL PRODOTTO E DELLA COMPOSIZIONE

INTEGRAZIONE PER PARTI

$$(F\,G)' = F'G + F\cdot G'$$

$$\left[F\cdot G\right]_a^b = \int_a^b (F\,G)'(x)\,dx = \int_a^b F'(x)\,G(x)\,dx + \int_a^b F(x)\,G'(x)\,dx$$

$$\left[F\cdot G\right]_a^b = \int_a^b F'(x)\,G(x)\,dx + \int_a^b F(x)\,G'(x)\,dx$$

CHIAMO $F'(x) = f(x)$

$$\left[F\cdot G\right]_a^b = \int_a^b f(x)\,G(x)\,dx + \int_a^b F(x)\,G'(x)\,dx$$

$$\boxed{\int_a^b f(x)\,G(x)\,dx = \left[F\,G\right]_a^b - \int_a^b F(x)\,G'(x)\,dx}$$

DOVE $F'(x) = f(x)$

ESEMPIO

$$\int_1^4 x\cdot e^x\,dx = \left[F\,G\right]_1^4 - \int_1^4 G'(x)\,F(x)\,dx$$

G f

$G(x)=x$ $\quad f(x)=e^x$ $\quad F(x)=e^x$

$$= \left[e^x\,x\right]_1^4 - \int_1^4 1\,e^x\,dx =$$

$$= 4e^4 - e^1\cdot 1 - \int_1^4 e^x\,dx =$$

$$= 4e^4 - e - \left[e^x \right]_1^4 = 4e^4 \cancel{- e} - e^4 \cancel{+ e} = 3e^4$$

$$\int_2^5 \underset{\substack{\| \\ f}}{1} \cdot \underset{\substack{\| \\ G}}{\log(x)} \, dx = \left[F \cdot G \right]_2^5 - \int_2^5 F \, G' \, dx =$$

$f(x) = 1 \qquad G(x) = \log x$

$F(x) = x \qquad G'(x) = \frac{1}{x}$

$$= \left[x \log x \right]_2^5 - \int_2^5 \cancel{x} \cdot \frac{1}{\cancel{x}} \, dx$$

$$= \left[x \log x \right]_2^5 - \underline{\int_2^5 1 \, dx}$$

$$= \left[x \log x \right]_2^5 - \left[x \right]_2^5$$

ESERCIZIO

$$\int_0^\pi \underset{G}{x} \; \underset{f}{\sin(x)} \, dx \qquad\qquad \int_0^2 \underset{G}{x^2} \; \underset{f}{e^x} \, dx$$

$$\int_0^\pi \underset{G}{x} \; \underset{f}{\sin(x)} \, dx = \left[F \cdot G \right]_0^\pi - \int_0^\pi G'(x) \, F(x) \, dx$$

$G(x) = x \qquad G'(x) = 1$

$f(x) = \sin(x) \qquad F(x) = -\cos x$

$$= \left[(-\cos x) \cdot x \right]_0^\pi - \int_0^\pi 1 \, \underline{(-\cos x)} \, dx$$

$$= -\underline{\cos \pi} \cdot \pi \underline{- (-\cos 0) \cdot 0} + \int_0^\pi \cos x \, dx$$

$$= \pi + \int_0^\pi \cos(x) \, dx = \pi + \left[\sin(x) \right]_0^\pi =$$

$$= \pi + \sin \pi - \sin 0 = \pi$$

INTEGRAZIONE PER SOSTITUZIONE

$y = g(x)$

Se $h(x) = f(g(x))$ $\quad \underline{h'(x)} = f'(y)\ g'(x)$

$$= \underline{f'(g(x))\ g'(x)}$$

$$h(b) - h(a) = \int_a^b h'(x)\,dx = \underline{\int_a^b f'(g(x))\ g'(x)\,dx}$$

$$h(b) - h(a) = f(g(b)) - f(g(a)) = \int_{\underline{g(a)}}^{\overline{g(b)}} f'(x)\,dx$$

$$\int_\alpha^\beta f'(x)\,dx = f(\underline{\underline{\beta}}) - f(\underline{\underline{\alpha}})$$

QUINDI

$$\int_a^b f'(g(x))\ g'(x)\,dx = \int_{g(a)}^{g(b)} f'(x)\,dx$$

Se AL POSTO DI f metto F

$$\int_a^b F'(g(x))\ g'(x)\,dx = \int_{g(a)}^{g(b)} F'(x)\,dx$$

E SE F' LO CHIAMO ℓ

$$\boxed{\int_a^b \ell(g(x)) \cdot g'(x)\,dx = \int_{g(a)}^{g(b)} \ell(x)\,dx}$$

ESEMPIO

$$\int_0^2 2x\, e^{x^2}\, dx = \int_0^2 \underline{e^{g(x)}\, g'(x)}\, dx$$

$$\boxed{g(x) = x^2}$$

$g'(x) = 2x$

$e^{g(x)} \quad g'(x)$

$\ell(x) = e^x$

$$= \int_{g(0)}^{g(2)} e^x\, dx =$$

$$= \int_0^4 e^x\, dx = \left[e^x\right]_0^4 = e^4 - 1$$

ESEMPIO

$$\int_0^3 x\, \underline{e^{x^2}}\, dx = \frac{1}{2}\int_0^3 2x\, e^{x^2}\, dx$$

$g'(x) \quad e^{g(x)}$

Se $g(x) \overset{\shortparallel}{=} x^2 \qquad g'(x) = 2x$

$$= \frac{1}{2}\int_0^9 e^x\, dx = \frac{1}{2}\left(e^9 - 1\right)$$

$$= \frac{1}{2}\left[e^x\right]_0^9 = \frac{1}{2}\left(e^9 - 1\right)$$

ESEMPIO

$$\int_0^\pi \sin\left(\frac{1}{2}x + \pi\right) dx = \int_0^\pi \sin\left(g(x)\right) dx$$

con $g(x) = \frac{1}{2}x + \pi$

e $g'(x) = \frac{1}{2}$

$$= 2\int_0^\pi \sin\left(g(x)\right) \frac{1}{2}\, dx = 2\int_0^\pi \sin\left(g(x)\right) g'(x)\, dx =$$

$$= 2 \int_{g(0)}^{g(\pi)} \sin(x)\, dx = 2 \int_{\pi}^{\frac{3}{2}\pi} \sin(x)\, dx$$

$$g(x) = \frac{x}{2} + \pi$$

ESR.

$$\int_0^1 e^{x+1}\, dx \qquad \boxed{\int_0^1 e^{-2x+1}\, dx}$$