Local triviality

23/03/2023
Andrea Gallese

Setup

$$\begin{tikzcd}
 & & 0 \arrow[r] \arrow[d] & \left(H^1(K_n/K, E(K_n))[p]\right) \ni \tilde{d}(n) \arrow[d] & \\
0 \arrow[r] & \dfrac{E(K)}{pE(K)} \arrow[r, "\delta"] \arrow[d] & H^1(K, E[p]) \arrow[r] \arrow[d] & H^1(K,E)[p] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \left(\dfrac{E(K_n)}{pE(K_n)}\right)^{\mathcal{G}_n} \arrow[r, "\delta_n"'] & H^1(K_n, E[p])^{\mathcal{G}_n} \arrow[r] & \left(H^1(K,E)[p]\right)^{\mathcal{G}_n} &
\end{tikzcd}$$

$c(n) \longmapsto d(n)$, $\quad \tilde{d}(n) \longmapsto d(n)$

$P_n \longmapsto \delta_n(P_n)$

- $K/\mathbb{Q}$ campo quadr, disc $K \neq -3, -4$ $\qquad D := \operatorname{disc} K$
- $E/\mathbb{Q}$ curv. ell. conduttore N, $\quad X_0(N) \xrightarrow{\varphi} E$
- Heegner condition $N = \mathcal{N} \cdot \bar{\mathcal{N}}$
- p s.t. $GL_2(\mathbb{F}_p) \simeq \operatorname{Gal}(\mathbb{Q}(E[p])/\mathbb{Q})$
- ℓ s.t. $*$ Frob_ℓ is the class of cpx conj. in $\operatorname{Gal}(\mathbb{Q}(E[p])/\mathbb{Q})$
  $*$ $\ell + 1 \equiv 0 \equiv a_\ell \ (p)$ [but not
- $n = \prod \ell$ as above, n sqrfree, $(n, D \cdot p \cdot N) = 1$
- $P_n = \sum_{\sigma \in \mathcal{G}_n/G_n} \sigma D_n y_n \in E(K_n)$ $\qquad (\sigma_\ell - 1) D_\ell = \ell + 1 - \operatorname{Tr}_\ell$

$$\begin{tikzcd}[row sep=small]
K_n \arrow[d, dash] \arrow[dd, bend left=40, dash, "G_n"'] \arrow[ddd, bend left=60, dash, "\mathcal{G}_n"] \\
K_1 \arrow[d, dash] \\
K \arrow[d, dash] \\
\mathbb{Q}
\end{tikzcd}$$

Thm v finite place of K, $v \nmid n$

$$\begin{array}{ccc} H^1(K, E)[p] & \xrightarrow{res} & H^1(K_v, E)[p] \\ d(n) & \longmapsto & 0 \end{array}$$

Rmk

$$H^1(K_v^{nr}/K_v, E(K_v^{nr}))[p] \hookrightarrow H^1(\mathbb{F}_v, \tilde{E}(\mathbb{F}_v$$

$$\tilde{d(n)} \in H^1(K_n/K_v, E(K_n)) \subseteq \underset{\substack{L/K_v \\ L \text{ unram.}}}{\bigcup} H^1(L/K_v, E(L))$$

(injection ~~since~~ from

$$\begin{array}{ccccccccc} & & 0 & & & & & & \\ & & \downarrow & & & & & & \\ 0 & \to & \hat{E} & \to & E(K_v^{nr}) & \to & \tilde{E} & \to & 0 \\ & & \wr\downarrow [p] & & \downarrow & & \downarrow & & \\ 0 & \to & \hat{E} & \to & E(K_v^{nr}) & \to & \tilde{E} & \to & 0 \end{array}$$)

$\tilde{E}$ ↳ special fiber Néron model? No: good reduction case, for now

$\boxed{\text{Case } v \nmid N}$

To prove the theorem, it thus suffices to prove that $[\tilde{d(n)}_v] = 0$ in $H^1(G_{\mathbb{F}_v}, \tilde{E}_v)$

Thm Let E be an elliptic curve over a finite field $\mathbb{F}_q$ the first cohomology group $H^1(\mathbb{F}_q, E)$ vanishes

Thm (Lang's isogeny) Let $\mathbb{F}_q$ be a finite field, F be the Frob. Let $G/\mathbb{F}_q$ be a smooth connected alg. grp. The map

$$\begin{array}{ccc} G & \longrightarrow & G \\ x & \longmapsto & x^{-1}F(x) \end{array}$$

is surjective

Proof ($G = E$ an elliptic curve) We have to prove that Id-Frob is surjective, which is obvious since it's a non-zero isogeny □

Proof of $H^1(\mathbb{F}_q, G) = (0)$

$$H^1(\mathbb{F}_q, G) = \frac{\{\alpha : G_{\mathbb{F}_q} \to G(\overline{\mathbb{F}_q}) \text{ (continuous)} \mid \alpha_{\sigma\tau} = \alpha_\sigma \cdot {}^\sigma\alpha_\tau\}}{\{\sigma \mapsto \sigma(x) - x \mid x \in G(\overline{\mathbb{F}_q})\}}$$

Let α be a cocycle and let $y = \alpha(F)$. By Lang's isogeny thm, $y = F(x) - x$ for some $x \in G(\overline{\mathbb{F}_q})$. Hence, α and the coboundary $\sigma \mapsto \sigma x - x$ agree on F, hence (by continuity) they agree everywhere. $\square$

2nd approach: PHS

A <u>PRINCIPAL HOMOGENEOUS SPACE</u> under E is a curve $C/\mathbb{F}_q$ connected, smooth, with a simply transitive action $\mu : E \times C \to C$

$$\begin{array}{ccc} H^1(G_F, E) & \xleftrightarrow{\sim} & \{\text{PHS under } E\} \\ [\sigma \mapsto \sigma x - x] & \longleftarrow & C \end{array}$$

Fix $x \in C(\bar{k})$ $\qquad k = \mathbb{F}_q$

(Here $a-b$, for $a, b \in C$, is the unique $c \in E$ s.t. $\mu(c, b) = a$) $\boxed{\text{NOTE}}$ If $x \in C(k)$, the cocycle $\sigma \mapsto \sigma x - x$ is trivial!

In particular, a PHS is trivial iff $C(k) \neq \emptyset$

Thus, it suffices to show that a (smooth) curve of genus 1 over $\mathbb{F}_q$ has a rat'l pt. This is true by the Lang-Weil bound, or by noticing that

$C(\bar{k}) \simeq E(\bar{k})$ "is" a smooth connected alg gp

and (w/ some care) we can apply Lang to obtain a rat'l pt on C

$\boxed{\text{Case } v \mid N}$

E has bad red. [or at least can have bad red.]
we are over K and not $\mathbb{Q}$

Let $\mathcal{E}$ be a Néron model over $\mathcal{O}_{K_v}$

Note that the special fiber $\tilde{\mathcal{E}}$ is disconnected, and
$H^1(G_{\mathbb{F}_v}, \tilde{\mathcal{E}}^0) = (0)$ [Lang's isogeny again, here $\tilde{\mathcal{E}}_0$ is even simpler]

Our cohomology class $d(n)$ corresponds to the cocycle

$$\gamma \mapsto \frac{(\gamma - 1)}{p} P_n$$

Similarly to last time, one can write

$$\frac{\gamma - 1}{p} \tilde{P}_n = \sum_{d \mid n} c_d y_d$$

<u>Claim</u> ~~y_d~~ the image of y_d in $\mathcal{E}/\mathcal{E}^0$ has order prime to p.

This suffices: from $0 \to \mathcal{E}^0 \to \mathcal{E} \to \mathcal{E}/\mathcal{E}^0 \to 0$

one gets $H^1(\mathbb{F}_v, \mathcal{E}) \hookrightarrow H^1(\mathbb{F}_v, \mathcal{E}/\mathcal{E}^0)$, and since it ~~the level of~~ is p-torsion, $d(n)$ goes to 0

Spreading out $X_0(N)$ over $\mathcal{O}_K$

We look for a (non-complete) curve $Z_0 / \operatorname{Spec} \mathcal{O}_K$ that satisfies:

- $Z_0(N)_K$ is smooth and irreducible
- $Z_0(N)_v$ " " " "
- $x_n \in Z_0(N)(\mathcal{O}_K)$

We use the functorial interpretation. Define the functor

$$\mathcal{Z}_0(N) : \mathrm{Sch} \longrightarrow \mathrm{Set}$$

$$S \longmapsto \{ (A \xrightarrow{\tau} A')$$

where * $A_s \xrightarrow{\tau_s} A'_s$ is an isogeny of ell curves with kernel $(\tau_s) \simeq \mathbb{Z}/N\mathbb{Z}$, or

* $A_s \simeq \mathbb{P}^1 \times \mathbb{Z}/N\mathbb{Z}$ / (gluing) ~~incollamento~~

$\downarrow$

$A_{s'} \simeq \mathbb{P}^1$ (projection) ~~proiezione~~

for all geometric pts $s \in S$. $\}/\sim$

(I have some doubts about the definition of this functor)

Thm (Katz-Mazur)

$\mathcal{Z}_0(N)$ is representable by $Z_0(N)$, a smooth, geom. irred. scheme over $\mathbb{Z}$.

Facts

*
$$\begin{tikzcd} X_0(N)(k_v) \arrow[r, "\varphi_{k_v}"] & E(k_v) \\ Z_0(N)(\mathcal{O}_v) \arrow[u, "\text{take generic fibre}"] & \mathcal{E}_{\mathcal{O}_v}(\mathcal{O}_v) \arrow[u] \end{tikzcd}$$

The cusp "0" is in the image of the map on the left (and no other cusp is). The reason is that the cusps of $X_0(N)$ are cusp(d) for $d | N$, and the "special fibre" at cusp(d) is a d-gon.

* At least one of x_n, $w_N x_n$ ~~belongs~~ comes from an $\mathcal{O}_v$-pt of $Z_0(N)$

* they are dual to each other (& $\mathbb{C}/\mathcal{O}_n$ is not supersingular: ℓ splits in K)

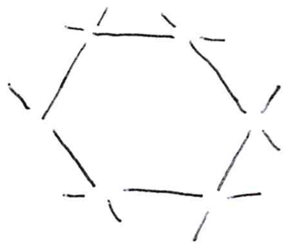
Indeed, x_n "is" the isogeny $\mathbb{C}/\mathcal{O}_n \longrightarrow \mathbb{C}/\mathcal{N}^{-1}$

$w_N(x_n)$ " " " $\mathbb{C}/\mathcal{N}\mathcal{O}_n \longrightarrow \mathbb{C}/\mathcal{O}_n$

One of the two is separable* (hence has ker $\simeq \mathbb{Z}/N\mathbb{Z}$) mod v!

Going back to our diagram of schemes,

$$0,\ x_n \in \underbrace{\mathcal{X}_0(N)_{\mathbb{F}_v}}_{\text{irreducible!}} \longrightarrow \mathcal{E}_{\mathbb{F}_v}$$



Hence, 0 and x_n land into THE SAME component!

We still need to translate by $\underbrace{(0)-(\infty)}_{\text{rat'l pt!}}$, which is torsion, and $E_{tors}[p](\mathbb{Q}) = \{0\}$.

more precisely: ~~our~~ the pt we care about is $\varphi(x_n)-\varphi(\infty)$, which differs from $\varphi(x_n)-\varphi(0)$ by a non-p-torsion pt