On Singularity Formation Under Mean Curvature Flow

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Mean Curvature Flow

The mean curvature flow is a family of hypersurfaces $M_t \subset \mathbb{R}^{d+1}$ whose smooth immersions $\psi(\cdot, t) : N \to M_t \subset \mathbb{R}^{d+1}$ satisfy the partial differential equation

$$(\partial_t \psi)^N = -H(\psi)$$

where $(\partial_t \psi)^N$ is the normal component of $\partial_t \psi$ and $H(x)$ is the mean curvature of M_t at a point $x \in M_t$.

Applications and Connections

- Material Science (interface motion between different materials or different phases).
- Image recognition.
- Connection to the Ricci flow.
- Topological classification of surfaces and submanifolds.

Some Key Works: Existence

- First mathematical treatment (using geometric measure theory): Brakke [1978];
- Short time existence: Brakke, Huisken, Evans and Spruck, Ilmanen, Ecker and Huisken [1991];
- Weak solutions: Evans and Spruck, Chen, Giga and Goto [1991];

Some Key Works: Singularities

The most interesting problem here is formation of singularities.

- Collapse of convex hypersurfaces: Huisken [1984], extensions: White [2000, 2003], Huisken and Sinestrari [2007-2009];
- Neckpinching for rotationally symmetric hypersurfaces: Grayson, Ecker, Huisken, M. Simon, Dziuk and Kawohl, Smoczyk, Altschuler, Angenent and Giga, Soner and Souganidis [1990-1995];
- MCF with surgery and topological classification of surfaces and submanifolds: Huisken and Sinestrari [2007-2009];
- Nature of the singular set: White [2000, 2003], Colding and Minicozzi [2012].

Symmetries and Solitons

T_λ is a (generalized) symmetry group of the MCF, if $\{T_\lambda\}$ is a one-parameter group, i.e. $T_0 = \mathbf{1}$, $T_t \circ T_s = T_{t+s}$, and

$$H(T_\lambda \psi) = b(\lambda) H(\psi) \qquad (\Rightarrow b(st) = b(s)b(t)).$$

Given a (generalized) symmetry group, T_λ of the MCF, the soliton is defined as

$$\psi(t) = T_{\lambda(t)} \varphi.$$

MCF is invariant under

- *Translations*: $\psi \to \psi + h, \ \forall h \in \mathbb{R}^{d+1}$;
- *Rotations*: $\psi \to R\psi, \ \forall R \in O(d+1)$;
- *Scaling*: $\psi \to \quad \lambda\psi, \quad t \to \lambda^{-2} t, \quad \lambda > 0.$

Related to these symmetries are three types of solitons: translational, rotational and scaling solitons.

Scaling solitons

The solitons corresponding to the scaling symmetry are of the form

$M(t) \equiv M^{\lambda(t)} := \lambda(t)M$, or $\psi(u,t) = \lambda(t)\varphi(u)$, where $\lambda(t) > 0$.

Plugging this into MCF and using $H(\lambda\varphi) = \lambda^{-1}H(\varphi)$ gives

$$H(\varphi) = a\langle \nu, \varphi \rangle, \quad \text{and} \quad \lambda\dot{\lambda} = -a. \tag{1}$$

Since $H(\varphi)$ is independent of t, then so should be $\lambda\dot{\lambda} = -a$.
Solving the last equation, we find $\lambda = \sqrt{\lambda_0^2 - 2at}$.

i) $a > 0 \Rightarrow \lambda \to 0$ as $t \to T := \frac{\lambda_0^2}{2a} \Rightarrow M^\lambda$ is a shrinker.
ii) $a < 0 \Rightarrow \lambda \to \infty$ as $t \to \infty \Rightarrow M^\lambda$ is an expander.

For φ solving (1), M is called the self-similar surface.
$a = 0 \;\Rightarrow M$ is a minimal surface.

Rescaled MCF

To understand dynamics near scaling soliton, we rescale the MCF:

$$\varphi(u,\tau) := \lambda^{-1}(t)\psi(u,t), \qquad \tau := \int_0^t \frac{dt'}{\lambda(t')^2} \ .$$

Important point: we do not fix $\lambda(t)$ but consider it as free parameter to be found from MCF. The rescaled surface satisfies

$$(\partial_\tau \varphi)^N = -H(\varphi) + a\langle \varphi, \nu(\varphi)\rangle, \qquad a = -\dot{\lambda}\lambda \ .$$

- The rescaled MCF is a gradient flow for the Huisken functional

  $$V_a(\varphi) := \int_{M^\lambda} e^{-\frac{a}{2}|x|^2},$$

  where $M^\lambda = \lambda^{-1}(t)M$ is the rescaled surface M.

(MCF is a gradient flow for the area functional $V(\psi) = V_{a=0}(\psi)$.)

Self-similar Surfaces

Static solutions of the rescaled MCF

$$(\partial_\tau \varphi)^N = -H(\varphi) + a\langle \varphi, \nu(\varphi)\rangle, \qquad a = -\dot{\lambda}\lambda.$$

- are self-similar surfaces,

$$H(\varphi) - a\langle \nu(\varphi), \varphi\rangle = 0, \qquad a \in \mathbb{R}.$$

We expect that as $\tau \to \infty$, solutions to the rescaled MCF converge to self-similar surfaces.

Hence one wants to classify self-similar surfaces and determine which ones of them are stable.

Theorem. (Huisken, Colding-Minicozzi) Under certain conditions, the only self-similar surfaces are planes, spheres and cylinders.

For $a = 0$, φ is a minimal surface $\Rightarrow$ cf. Bernstein conjecture.

Linearized Stability

$\varphi =$ a self-similar surface $\Longrightarrow$

$$\varphi_{\lambda,z,g} := T_g^{\mathrm{rot}} T_z^{\mathrm{transl}} T_\lambda^{\mathrm{scal}} \varphi$$

is also a self-similar surface. Consider the manifold

$$\mathcal{M}_{\mathrm{self-sim}} := \{\varphi_{\lambda,z,g} : (\lambda, z, g) \in \mathbb{R}_+ \times \mathbb{R}^{d+1} \times SO(d+1)\}.$$

Definition (Linearized stability of self-similar surfaces)

We say that a self-similar surface φ, with $a > 0$, is *linearly stable* iff

$\mathrm{Hess}^N V_a(\varphi) > 0 \quad$ on $\quad \{\text{scaling, transl., rot. modes}\}^\perp$.

(I.e. the only unstable motions allowed are scaling, transl., rot..)

Symmetries and Spectrum of Hessian

Theorem. The hessian $\text{Hess}^N V_a(\varphi)$ of $V_a(\varphi)$ in the normal direction at a self-similar $d-$dimensional surface φ has

1. (Colding-Minicozzi) the simple eigenvalue $-2a$,
2. (Colding-Minicozzi) the eigenvalue $-a$ of multiplicity $d+1$,
3. the eigenvalue 0 of multiplicity $\frac{1}{2}(d-1)d$ (unless φ is a sphere).

These eigenvalues are due to *rescaling, translations and rotations* of the surface. The eigenvalue 0 distinguishes between a *sphere, a cylinder and a generic surface*.

Proof. Let $H_a(\varphi) := H(\varphi) - a\varphi \cdot \nu(\varphi)$. To prove say the first statement, we observe that, since $H_{\lambda^{-2}a}(\lambda\varphi) = \lambda^{-1} H_a(\varphi)$,

$$H_{\lambda^{-2}a}(\lambda\varphi) = 0, \ \forall \lambda \in \mathbb{R}_+.$$

Differentiating this equation w.r.to λ at $\lambda = 1$, and reparametrizing the result, we arrive at the desired eigenvalue equation. $\square$

Spectrum and Stability

The spectral theorem above gives unstable and central manifolds corresponding to the eigenvalues $-2a$, $-a$ and 0.

Hence, if these are the only non-positive eigenvalues, then we expect the stability in the transverse direction to $\mathcal{M}_{\mathrm{self-sim}}$. Otherwise, we expect instability.

Spectral Picture of Collapse: Sphere and Cylinder

For the $d-$sphere of the radius $\sqrt{\frac{a}{d}}$, the normal hessian > 0 on (scaling and translational modes)$^{\perp}$ $\Rightarrow$ by the definition above, it is linearly stable.

For the $(d+1)-$cylinder of the radius $\sqrt{\frac{a}{d}}$, the normal hessian has, in addition to the eigenvalues above,

1. the eigenvalue $-a$ of multiplicity 1, due to translations along the axis of the cylinder,
2. the eigenvalue 0 of multiplicity $d+1$, which originates in a "shape instability".

Hence the $(d+1)-$cylinder is linearly unstable.

Using the eigenfunction corresponding to the shape instability eigenvalue, we find the approximate neck profile

$$\nu_{ab} := \sqrt{\frac{d + by^2}{a}}, \; b > 0.$$

Spectrum and Mean convexity

The spectral information tells us about the geometry of φ. In particular, we have the following result

Theorem

Let φ be a self-similar surface. Then:
(a) (Colding-Minicozzi) For $a > 0$ (shrinker),

$$\mathrm{Hess}^N V_a(\varphi) \geq -2a \text{ iff } H(\varphi) > 0.$$

(b) For $a < 0$ (expander), $H(\varphi)$ changes the sign.

Proof.

First, one shows that the normal hessian, $\mathrm{Hess}^N V_a(\varphi)$, has a positivity improving property. Therefore the Perron-Frobenius theory applies and gives the result. □

On Singularity Formation Under Mean Curvature Flow II

We continue with the mean curvature flow, which is defined by the initial value problem

$$(\partial_t \psi)^N = -H(\psi)$$

for the family of hypersurfaces $M_t \subset \mathbb{R}^{d+1}$ defined by smooth immersions

$$\psi(\cdot, t) : N \to M_t \subset \mathbb{R}^{d+1}.$$

Here $(\partial_t \psi)^N$ is the normal component of $\partial_t \psi$ and $H(x)$ is the mean curvature of M_t at a point $x \in M_t$.

We are interested in understanding how the singularities form under this flow.

Huisken's Conjecture

Under MCF, the $\mathrm{vol}(M_t) \to 0$ as $t \to t_* \implies$ closed surfaces collapse. How this collapse takes place?
There are three explicit solutions of MCF:

- Collapsing Euclidean spheres with radii decreasing as $\sqrt{2d(t_* - t)}$;
- Collapsing Euclidean cylinders with radii decreasing as $\sqrt{2(d-1)(t_* - t)}$;

Conjecture [Huisken]: Generic singularities are spheres and cylinders.
Partial results: Huisken, White, Colding and Minicozzi

Results:

- The spherical collapse is asymptotically stable.
- The cylindrical collapse is unstable.

Theorem. (W. Kong-I.M.S.) Let a surface M_0 be close to S^d in H^s, $s > \frac{d}{2} + 1$. Then $\exists t_* < \infty$, s.t. MCF has a solution M_t for $0 \leq t < t_*$ and

- $M_t \to z_*$, for some z_*, as $t \to t_*$;
- M_t are defined by immersions of S^d,

$$\psi(\omega, t) = z(t) + u(\omega, t)\omega,$$

$$\rho(t) = \sqrt{\tau}\left(1 + O_{H^s}(\tau^\beta)\right),$$

with $\tau := 2d(t_* - t)$, $\alpha := \frac{1}{2}(d + \frac{1}{2} - \frac{1}{2d})$ and $\beta := \frac{1}{2}(1 - \frac{1}{2d})$.

Graphs over Cylinders

Our next result deals with initial conditions M_0, which are graphs over $(d+1)-$dimensional cylinders C^{d+1} along the $x_{d+2}-$axis in $\mathbb{R}^{d+2}$,

$$\psi_0(\omega, x) = (u_0(\omega, x)\omega, x).$$

It combines two results, one with Zhou Gang on equivariant graphs (surfaces of revolution), i.e.

$$u_0(\omega, x) \text{ is independent of } \omega,$$

and one in general case with Zhou Gang and Dan Knopf.

Neckpinching

Theorem. (Zhou Gang-S, Zhou Gang-Knopf-S) Let $d \geq 1$ and (informally for brevity)

M_0 be a surface close to a cylinder, $\mathcal{C}^{d+1}$,

M_0 has an arbitrary shallow waist and is even w.r.to the waist.

Then M_t is defined by an immersion

$$\psi(\omega, x, t) = (u(\omega, x, t)\omega, x)$$

of C^{d+1}, where $(\omega, x) \in \mathcal{C}^{d+1}$ and $u(\omega, x, t)$ satisfies

(i) There exists a finite time t^* such that $u(\cdot, t) > 0$ for $t < t^*$ and $\lim_{t \to t^*} \inf u(\cdot, t) \to 0$;

(ii) If $u_0 \partial_x^2 u_0 \geq -1$ then there exists a function $u_*(\omega, x) > 0$ such that $u(\omega, x, t) \geq u_*(\omega, x)$ for $\mathbb{R} \backslash \{0\}$ and $t \leq t^*$.

Dynamics of Scaling Parameter

Theorem. (Zhou Gang-S, Zhou Gang-Knopf-S)

(iii) There exist C^1 functions $\zeta(\omega, x, t)$, $\lambda(t)$ and $b(t)$ such that

$$u(\omega, x, t) = \lambda(t)[\sqrt{\frac{d + b(t)y^2}{a(t)}} + \zeta(\omega, y, t)]$$

with $y := x/\lambda(t)$, $a(t) = -\lambda(t)\partial_t\lambda(t)$ and

$$\|\langle y\rangle^{-m}\partial_y^n\zeta(\omega, y, t)\|_\infty \leq cb^2(t), \; m + n = 3.$$

(iv) The parameters $\lambda(t)$ and $b(t)$ satisfy (with $\tau := 2d(t^* - t)$)

$$\lambda(t) = \tau^{\frac{1}{2}}(1 + o(1)) \qquad \text{(scaling eigenvalue)}$$

$$b(t) = -\frac{d}{\ln\tau}(1 + O(\frac{1}{|\ln\tau|^{3/4}})) \qquad \text{(shape eigenvalue).}$$

Comparison with Previous Results

A result similar to (ii) (axi-symmetric surfaces) but for a different set of initial conditions was proven by H.M.Soner and P.E.Souganidis.

The previous result closest to ours is that by S. Angenent and D. Knopf on the axi-symmetric neckpinching for the Ricci flow.

Some ideas of the proof are close to those of Bricmont and Kupiainen on NLH.

All works mentioned above deal with *surfaces of revolution* of barbell shapes (*far from cylinders*) which are either compact (Dirichlet b.c.) or periodic (Neumann b.c.).

These works rely on parabolic maximum principle going back to Hamilton and monotonicity formulae for an entropy functional $\int_{M_t}$ backward heat kernel$(x, t)d\mu_t$, due to Huisken and Giga and Kohn.

Key Steps in Proof

Rescaling

Spectrum

Collar lemma

Estimates of the linear evolution

Bootstrap

Rescaled MCF

At the first step, we rescale the MCF:

$$\varphi(u,\tau) := \lambda^{-1}(t)\psi(u,t), \quad \tau := \int_0^t \frac{dt'}{\lambda(t')^2} \ .$$

Important point: we do not fix $\lambda(t)$ but consider it as free parameter to be found from MCF. The rescaled surface satisfies

$$(\partial_\tau \varphi)^N = -H(\varphi) + a\langle \varphi, \nu(\varphi)\rangle, \quad a = -\dot{\lambda}\lambda \ .$$

Next, we look for solutions which are graphs over the cylinder (C^{d+1}),

$$\varphi(\omega, y, \tau) = (v(\omega, y, \tau)\omega, y)$$

where $(\omega, y) \in \mathcal{C}^{d+1}$.

Collar Lemma (Fixed Cylinder)

Let $\varphi = \text{graph}_{\mathcal{C}^{d+1}}\, \rho$ and introduce the manifold of necks

$$M_{neck} := \{\nu_{ab} : a, b \in \mathbb{R}^+, b \leq \epsilon\}.$$

Lemma

There exist a small neigbourhood $\mathcal{U}_{path}$ of M_{neck} in $C^1([0,T], \langle y \rangle^3 L^\infty)$, such that

$$v(y,\omega,\tau) = \nu_{a(\tau),b(\tau)}(y) + \phi(y,\omega,\tau),$$

with

$$\phi(\cdot,\tau) \perp 1,\ \ a(\tau)y^2 - 1 \ in\ L^2(\mathbb{R} \times \mathbb{S}^d, e^{-\frac{a(\tau)}{2}y^2} dyd\omega).$$

The vectors 1 and $(1 - ay^2)$ which are almost tangent vectors to the manifold, M_{neck}, provided b is sufficiently small.

Effective Equations

Substitute $\varphi(\omega, y, \tau) = (v(\omega, y, \tau)\omega, y)$, where v is given by $v(y, \tau) = \nu_{a(\tau),b(\tau)}(y) + \phi(y, \tau)$, into the rescaled MCF to obtain

$$\partial_\tau \phi = -L_{ab}\phi + F_{ab} + N_{ab}(\phi)$$

where L_{ab} is the Hessian of the Huisken entropy on the neck ν_{ab},

$$L_{ab} := -\partial_y^2 + ay\partial_y - 2a - \frac{a}{d}\Delta_{\mathbb{S}^d} + V_{ab}(y),$$

$F_{ab} \approx$ a sum of generators of broken symmetries (the source term) and $N_{ab}(\phi)$ is a nonlinearity. Remember that

$$\phi(\cdot, \tau) \perp 1,\ a(\tau)y^2 - 1 \text{ in } L^2(\mathbb{R} \times \mathbb{S}^d, e^{-\frac{a(\tau)}{2}y^2} dy d\omega).$$

Project the above equation on $1,\ a(\tau)y^2 - 1 \implies$ the *equations for the parameters* $a,\ b \implies$ *need to estimate* ϕ.

Key Propagation Estimate

Key propagation estimate: The propagator $U(\tau,\sigma)$ generated by $-L_{ab}$ satisfies $(\tau \geq \sigma \geq 0)$

$$\|\langle z\rangle^{-3}U(\tau,\sigma)g\|_\infty \lesssim e^{-c(\tau-\sigma)}\|\langle z\rangle^{-3}g\|_\infty,$$

where $g \perp 1,\ a(\tau)y^2 - 1$ in $L^2(\mathbb{R}, e^{-\frac{a(\tau)}{2}y^2}dy)$.
By Duhamel principle we rewrite the differential equation for $\phi(y,\tau)$ as

$$\phi(\tau) = U(\tau,0)\phi(0) + \int_0^\tau U(\tau,\sigma)(F+N)(\sigma)d\sigma.$$

Using this and the *key propagation estimate*, we estimate the functions

$$M_{m,n}(\tau) := \max_{\sigma\leq\tau} b^{-\frac{m+n+1}{2}}(\sigma)\|\langle y\rangle^{-m}\partial_y^n\phi(\cdot,\sigma)\|_\infty,$$

where $b(t) \approx -\frac{d}{\ln\tau}$ and $(m,n) = (3,0),\ (\frac{11}{10},0),\ (2,1),\ (1,2)$.

Bootstrap

For the estimating functions $M_{m,n}(\tau)$, $(m,n) = (3,0)$, $(\frac{11}{10}, 0)$, $(2,1)$, $(1,2)$, we let

$$M := (M_{i,j}) \quad \text{and} \quad |M| := \sum_{i,j} M_{i,j}.$$

Lemma. Assume that for $\tau \in [0, T]$ and

$$|M(\tau)| \leq b^{-\frac{1}{4}}(\tau), \ v(y,\tau) \geq \frac{1}{4}\sqrt{2(d-1)}, \text{ and } \partial_y^n v(\cdot,\tau) \in L^\infty,$$

for $n = 0, 1, 2$. Then there exists a nondecreasing polynomial $P(M)$ s.t. on the same time interval,

$$M_{m,n}(\tau) \leq M_{m,n}(0) + b^{\frac{1}{2}}(0)P(M(\tau)),$$

Corollary. Assume $|M(0)| \ll 1$. On any interval $[0, T]$,

$$|M(\tau)| \leq b^{-\frac{1}{4}}(\tau) \Longrightarrow |M(\tau)| \lesssim 1.$$

Hessian on the Neck

Consider the Hessian of the Huisken entropy on the neck $\varphi_{ab} = \text{graph}_{\mathcal{C}^{d+1}} \nu_{ab}$:

$$L_{ab} := \underbrace{-\partial_y^2 + ay\partial_y - 2a - \frac{a}{d}\Delta_{\mathbb{S}^d}}_{\text{normal hess on cyl}} + V_{ab}(y,\omega).$$

By the collar lemma it acts on functions in

$$X^{\perp} := \{\phi(\cdot,\tau) \in L^2(\mathbb{R} \times \mathbb{S}^d, e^{-\frac{a(\tau)}{2}y^2} dyd\omega) : \\ \phi(\cdot,\tau) \perp 1,\ a(\tau)y^2 - 1\}.$$

Let $U(\tau,\sigma)$, $\tau \geq \sigma \geq 0$, be the propagator generated by $-L_{ab}$. The main step in the proof involves showing the *key propagation estimate*: $\forall g \in X^{\perp}$,

$$\|\langle z\rangle^{-3} U(\tau,\sigma) g\|_\infty \lesssim e^{-c(\tau-\sigma)} \|\langle z\rangle^{-3} g\|_\infty.$$

Estimating the Linear Propagator. I

Write $L_{ab} = L_{a0} + V$, with $L_{a0} := -\partial_y^2 + ay\partial_y - 2a$ (the normal hessian at the cylinder), and use that V is slowly varying in y to do a multiplicativ perturbation (adiabatic) theory.
For the integral kernel $K(x, y)$ of $U(\tau, \sigma)$ (for simplicity, we do not display the variables of $\mathbb{S}^d$), we have the representation

$$K(x, y) = K_0(x, y)\langle e^V \rangle(x, y),$$

where $K_0(x, y)$ is the integral kernel of the operator $e^{-(\tau-\sigma)L_{a0}}$ and

$$\langle e^V \rangle(x, y) = \int e^{\int_\sigma^\tau V(\omega(s)+\omega_0(s),s)ds} d\mu(\omega).$$

Here $d\mu(\omega)$ is a harmonic oscillator (Ornstein-Uhlenbeck) probability measure on the continuous paths $\omega : [\sigma, \tau] \to \mathbb{R}$ with the boundary condition $\omega(\sigma) = \omega(\tau) = 0$ and

$$(-\partial_s^2 + a^2)\omega_0 = 0 \text{ with } \omega(\sigma) = y \text{ and } \omega(\tau) = x.$$

Estimating the Linear Propagator. II

To estimate $U(x,y)$ for $e^{a(\tau-\sigma)} \leq b^{-1/32}(\tau)$ we use the explicit formula

$$K_0(x,y) = 4\pi(1-e^{-2ar})^{-\frac{1}{2}}\sqrt{a}e^{2ar}e^{-a\frac{(x-e^{-ar}y)^2}{2(1-e^{-2ar})}},$$

where $r := \tau - \sigma$, and the bound

$$|\partial_y \langle e^V\rangle(x,y)| \leq b^{\frac{1}{2}} r,$$

which follows from the definition of $\langle e^V \rangle$ and the properties

$$V(y,\tau) \geq 0 \text{ and } |\partial_y V(y,\tau)| \lesssim b^{\frac{1}{2}}(\tau).$$

Then we iterate using the semi-group property $\Rightarrow$ estimate of the remainder ϕ. $\square$

Extensions

We do not fix the cylinder and look for surfaces of the form

$$\psi(x, \omega, t) = \lambda(t) g(t) \varphi(y, \omega, \tau) + z(t),$$

where $(\lambda,\ z,\ g) : [0, T) \to \mathbb{R} \times \mathbb{R}^{d+2} \times SO(d+2)$,
to be determined later,

$$y = \lambda^{-1}(t)(x - x_0(t)), \quad \tau = \tau(t) := \int_0^t \lambda^{-2}(s) ds,$$

and $\varphi(\cdot, \tau) : \mathcal{C}^{d+1} \to \mathbb{R}^{d+2}$ is a normal graph over the fixed cylinder.

The time dependent parameters $\lambda(t),\ z(t),\ g(t)$ are chosen so that $\varphi(\cdot, \tau)$ is orthogonal to the non-positive (scaling, translation and rotation) modes of the normal hessian on the cylinder.

Then we proceed as before.

Thank-you for your attention.