Lectures on Mean Curvature Flow (MAT 1063 HS)

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Abstract

The mean curvature flow arises material science and condensed matter physics and has been recently successfully applied by Huisken and Sinestrari to topological classification of surfaces and submanifolds. It is closely related to the Ricci and inverse mean curvature flow.

The most interesting aspect of the mean curvature flow is formation of singularities, which is the main theme of these lectures.

Background on geometry of surfaces and some technical statements are given in appendices.

(we often use different notation for the same objects, as we did not decide on the notation.)

Contents

1 General properties of the mean curvature flow

The mean curvature flow, starting with a hypersurface S_0 in $\mathbb{R}^{n+1}$, is the family of hypersurfaces $S(t)$ given by *immersions* $x(\cdot, t)$ which satisfy the evolution equation

$$\begin{cases} \frac{\partial x}{\partial t} & = & -H(x)\nu(x) \\ x|_{t=0} & = & x_0 \end{cases} \tag{1.1}$$

where x_0 is an immersion of S_0, $H(x)$ and $\nu(x)$ are mean curvature and the outward unit normal vector at $x \in S(t)$, respectively. The terms used above are explained in Appendix A.

In this lecture we describe some general properties of the mean curvature flow, (1.1). We begin with writing out (1.1) for various explicit representations for surfaces S_t.

1.1 Mean curvature flow for level sets and graphs

We rewrite out (1.1) for the level set and graph representation of S. Below, all differential operations, e.g. ∇, Δ, are defined in the corresponding Euclidian space (either $\mathbb{R}^{n+1}$ or $\mathbb{R}^n$).

1) Level set representation $S = \{\varphi(x, t) = 0\}$. Then, by (A.2) of Appendix A, we have

$$\nu(x) = \frac{\nabla\varphi}{|\nabla\varphi|}, \qquad H(x) = \operatorname{div}\left(\frac{\nabla\varphi}{|\nabla\varphi|}\right). \tag{1.2}$$

We compute $0 = \frac{d\varphi}{dt} = \nabla_x \varphi \cdot \frac{\partial x}{\partial t} + \frac{\partial \varphi}{\partial t}$ and therefore $\frac{\partial \varphi}{\partial t} = \nabla\varphi \cdot \frac{\nabla\varphi}{|\nabla\varphi|} \operatorname{div}\left(\frac{\nabla\varphi}{|\nabla\varphi|}\right)$, which gives

$$\frac{\partial \varphi}{\partial t} = |\nabla\varphi| \operatorname{div}\left(\frac{\nabla\varphi}{|\nabla\varphi|}\right). \tag{1.3}$$

2) Graph representation: S = graph of f. In this case S is the zero level set of the function $\varphi(x) = x_{n+1} - f(u)$, where $u = (x_1, \ldots, x_n)$ and $x = (u, x_{n+1})$, and using (1.3) with this function, we obtain

$$\frac{\partial f}{\partial t} = \sqrt{|\nabla f|^2 + 1} \operatorname{div}\left(\frac{\nabla f}{\sqrt{|\nabla f|^2 + 1}}\right). \tag{1.4}$$

Denote by $\operatorname{Hess} f$ the standard euclidean hessian, $\operatorname{Hess} f := \left(\frac{\partial^2 f}{\partial u^i \partial u^j}\right)$. Then we can rewrite (1.4) as

$$\frac{\partial f}{\partial t} = \Delta f - \frac{\nabla f \operatorname{Hess} f \nabla f}{|\nabla f|^2 + 1}. \tag{1.5}$$

1.2 Different form of the mean curvature flow

Multiplying the equation (1.1) in $\mathbb{R}^{n+1}$ by $\nu(x)$, we obtain the equation

$$\nu(x) \cdot \frac{\partial x}{\partial t} = -H(x). \tag{1.6}$$

In opposite direction we have

Proposition 1. *If x satisfies* (1.6)*, then there is a (time-dependent) reparametrization φ of S, s.t. $x \circ \varphi$ satisfies* (1.1)*.*

Proof. Denote $(\frac{\partial x}{\partial t})^T := \frac{\partial x}{\partial t} - (\nu \cdot \frac{\partial x}{\partial t})\nu$ (the projection of $\frac{\partial x}{\partial t}$ onto $T_x S$) and let φ satisfy the ODE $\dot{\varphi} = -(dx)^{-1}(\frac{\partial x}{\partial t} \circ \varphi)^T$ (parametrized by $u \in U$). Then $\frac{\partial}{\partial t}(x \circ \varphi) = (\frac{\partial}{\partial t} x) \circ \varphi + dx\ \dot{\varphi}$. Substituting $\dot{\varphi} = -(dx)^{-1}(\frac{\partial x}{\partial t} \circ \varphi)^T$ into this, we obtain $\frac{\partial}{\partial t}(x \circ \varphi) = (\nu \cdot (\frac{\partial}{\partial t} x) \circ \varphi)\nu = -H$. □

Thus the MCF in the form (1.6) is invariant under reparametrization, while the form (1.1) is obtained by fixing a specific parametrization (fixing the gauge).

1.3 Mean curvature flow and the volume functional

We show that the mean curvature arises from the first variation of the surface area functional. We do this for surfaces which are graphs of some function, which is locally the case for any surface.

Recall that locally the surface volume functional can be written as

$$V(S \cap U) = \int_U \sqrt{g} d^n u,$$

where $g := \det(g_{ij})$. We show that the MCF is a gradient flow of the surface area functional. We do this for surfaces which are graphs of some function, which is locally the case for any surface. (We do an immersion case later.)

Recall that, if $E : M \to \mathbb{R}$ is a functional on an infinite dimensional manifold M, then the Gâteaux derivative $dE(u)$, $u \in M$, of $E(u)$, is a linear functional on X defined as

$$dE(u)\xi = \frac{\partial}{\partial\lambda} E(u_s)|_{s=0}, \tag{1.7}$$

where u_s is a path on M satisfying $u_s|_{s=0} = u$ and $\partial_s u_s|_{s=0} = \xi$, with $\xi \in T_u M$, if the latter derivative exists.

If M is a Riemannian manifold, with an inner product (Riemann metric) $h(\xi, \eta)$, then there exists a vector field, $\text{grad}_g E(u)$, such that

$$\langle \text{grad}_h E(u), \xi \rangle = dE(u)\xi. \tag{1.8}$$

This vector is called the gradient of $E(u)$ at u w.r.t. the metric h.

We want to compute the Gâteaux derivative of the area functional V. Because of reparametrization (see Proposition 1), it suffices to look only at normal variations, ψ_s, of the immersion ψ, i.e. generated by vector fields η, directed along the normal ν: $\eta = f\nu$. We begin with

Proposition 2. *For a surface S given locally by an immersion ψ and normal variations, $\eta = f\nu$, we have*

$$dV(\psi)\eta = \int_U H\nu \cdot \eta \, \sqrt{g} d^n u. \tag{1.9}$$

Proof. We want to show for $g \equiv g(\psi) := \det(g_{ij})$ that

$$d\sqrt{g}\eta = H\sqrt{g}\nu \cdot \eta. \tag{1.10}$$

To this end we use the representation $\det(g_{ij}) = e^{\text{Tr} \ln(g_{ij})}$, which gives $dg\eta = g \, \text{Tr}[(g_{ij})^{-1} d(g_{ij})\eta]$ and therefore

$$dg\eta = gg^{ji} dg_{ij}\eta. \tag{1.11}$$

Next, we compute $dg_{ij}\eta = 2\left\langle \frac{\partial\psi}{\partial u^i}, \frac{\partial\eta}{\partial u^j} \right\rangle = 2\frac{\partial}{\partial u^j}\left\langle \frac{\partial\psi}{\partial u^i}, \eta \right\rangle - 2\left\langle \frac{\partial}{\partial u^j}\frac{\partial\psi}{\partial u^i}, \eta \right\rangle$. For normal variations, $\eta = f\nu$, the last relation and Lemma 41 of Appendix A give

$$dg_{ij}\eta = 2b_{ij}(\nu \cdot \eta), \tag{1.12}$$

where b_{ij} are the matrix elements of the second fundamental form. This relation, together with the relation $dg(\psi)\eta = g(\psi)g^{ji} dg_{ij}\eta$, proven above, and, implies $dg\eta = 2(\nu \cdot \eta)gg^{ji}b_{ij}$, which, due to Lemma 41, gives

$$dg\eta = 2gH(\nu \cdot \eta).$$

This, together with $d\sqrt{g}\eta = \frac{1}{2}g^{-1/2}dg\eta$, gives in turn (1.10). □

The equation (1.9) and the definition (1.8) imply

Theorem 3. *The mean curvature flow is a gradient flow with the functional V and the metric $\int_S \xi\eta$:*

$$\partial_t \psi = -\operatorname{grad} V(\psi).$$

Corollary 4. *If S evolves according to* (1.1)*, then $V(S)$ decreases. In fact, $\partial_t V(S) = -\int_S H^2(x) < 0$.*

Proof. The assertion on $\psi(U)$ follows from (1.9) as

$$\partial_t V(\psi) = \int_U H\nu \cdot \partial_t \psi = -\int_\psi H^2(x),$$

which after using a partition of unity proves the statement. □

As a result, the area of a closed surface shrink under the mean curvature flow. Note also that the equation (1.12) implies $\partial_t g_{ij} = 2b_{ij}(\nu \cdot \partial_t \psi)$, which, together with (1.6), gives the following equation for the evolution of the metric

$$\partial_t g_{ij} = -2b_{ij} H. \tag{1.13}$$

Graph representation. If S is locally a graph, $S = \operatorname{graph} f,\ f : U \to \mathbb{R}$, i.e. $\psi(u) = (u, f(u))$, then the surface area functional is given by

$$V(f) = \int_U \sqrt{1 + |\nabla f|^2} d^n u.$$

Then the mean curvature flow

$$\partial_t f = -\operatorname{grad} V(f),$$

where grad is defined in the metric $h_f(\xi, \eta) := \int_U \xi\eta \frac{d^n u}{\sqrt{1+|\nabla f|^2}}$. Indeed, if $S = \operatorname{graph} f,\ f : U \to \mathbb{R}$, i.e. $\psi(u) = (u, f(u))$, then the proof of (1.9) is simplified. We compute, using that $\xi|_{\partial U} = 0$,

$$dV(f)\xi = \partial_s|_{s=0} V(f + s\xi) = \int_U \frac{\nabla f}{\sqrt{1 + |\nabla f|^2}} \cdot \nabla \xi d^n u,$$

which, after integrating by parts and using the equation (A.3) of Appendix A, gives

$$dV(f)\xi = \int_U H\xi d^n u. \tag{1.14}$$

Using (1.14) we find

$$dV(f)\xi = -\int_U H\xi d^n u = -\int_U \sqrt{1 + |\nabla f|^2} H\xi \frac{d^n u}{\sqrt{1 + |\nabla f|^2}}.$$

So we have $\operatorname{grad} V(f) = -\sqrt{1 + |\nabla f|^2} H(x)$ which together with (1.4) gives the result.

1.4 Symmetries

Mean curvature is invariant under translations, rotations and scaling. The latter is defined as

$$S(t) \to \lambda S(\lambda^{-2}t) \quad \Leftrightarrow \quad x \to \quad \lambda x, \quad t \to \lambda^{-2}t.$$

The invariance under translations and rotations is obvious. To prove the invariance under scaling, we first show that

$$H(\lambda x) = \lambda^{-1} H(x). \tag{1.15}$$

Indeed, looking at the definitions $B = \left(b_{ij} = \frac{\partial^2 \psi}{\partial u_i \partial u_j} \cdot \nu\right)$ and $G = \left(g_{ij} := \left\langle \frac{\partial \psi}{\partial u^i}, \frac{\partial \psi}{\partial u^j} \right\rangle\right)$, we see that

$$B \ \to \ \lambda B \quad \text{ and } \quad G \to \ \lambda^2 G \quad \text{if} \quad \psi \to \lambda\psi.$$

Hence $G^{-1}B \to \lambda^{-1} G^{-1} B$, which due to the equation (**??**) of Appendix A, implies (1.15).

Now, let $\tau := \lambda^{-2}t$, and $x^\lambda(t) := \lambda x(\tau)$. Then

$$\partial_t x^\lambda = \lambda\lambda^{-2}(\partial_\tau x)(\tau) = -\lambda^{-1} H(x(\tau))\nu(x(\tau)).$$

On the other hand, $\nu^\lambda \equiv \nu(x^\lambda(\tau)) = \nu \equiv \nu(x(\tau))$ and $H^\lambda \equiv H(x^\lambda(\tau)) = \lambda^{-1} H(x(\tau)) \equiv \lambda^{-1} H$ (by (1.15)), which gives

$$\partial_t x^\lambda = -H^\lambda \nu^\lambda. \tag{1.16}$$

We can also use the scaling $x(u,t) \to \lambda x(\lambda^{-1}u, \lambda^{-2}t)$, together with the fact that $H \to H$, under $u \to \lambda^{-1}u$. Indeed, $B \ \to \ \lambda^{-2}B$ and $G \to \ \lambda^{-2}G$ if $\psi(u) \to \psi(\lambda^{-1}u)$, and therefore $G^{-1}B \to G^{-1}B$.

For $S =$ graph f, $f(u) \to \lambda f(\lambda^{-1}u) \Leftrightarrow \psi(u) \to \lambda\psi\left(\lambda^{-1}u\right)$, where $\psi(u) := (u, f(u))$. Indeed,

$$\psi(u) := (u, f(u)) \to \left(u, \lambda f(\lambda^{-1}u)\right) = \lambda\left(\lambda^{-1}u, f(\lambda^{-1}u)\right) =: \lambda\psi\left(\lambda^{-1}u\right).$$

1.5 Special solutions

Static solutions: Static solutions satisfy the equation $H(x) = 0$ on S, which, by Proposition 2, is the equation for critical points of the volume functional $V(\psi)$ (the Euler - Lagrange equation for $V(\psi)$). Thus, static solutions of the MCF (1.1) are *minimal surfaces*.

Spherically symmetric (equivariant) solutions:

a) Sphere $x(t) = R(t)\hat{x}$, where $\hat{x} = x/\|x\|$, or $\{\varphi(x,t) = 0\}$, where $\varphi(x,t) := |x|^2 - R(t)^2$, or graph f, where $f(u,t) = \sqrt{R(t)^2 - |u|^2}$. ($S_R = RS^n$, where S^n is the unit n-sphere.) Then we have

$$H(x) = \text{div}\left(\frac{\nabla\varphi}{|\nabla\varphi|}\right) = \text{div}(\hat{x}) = \frac{n}{R}$$

and therefore we get $\dot{R} = -\frac{n}{R}$ which implies $R = \sqrt{R_0^2 - 2nt}$. So this solution shrinks to a point.

b) Cylinder $x(t) = (R(t)\hat{x'}, x'')$, where $x = (x', x'') \in \mathbb{R}^{k+1} \times \mathbb{R}^m$. Then $H(x) = \frac{n-1}{R}$ and $\dot{R} = -\frac{n-1}{R}$ which implies $R = \sqrt{R_0^2 - 2(n-1)t}$. (In the implicit function representation the cyliner is given by $\{\varphi = 0\}$, where $\varphi := r - R$, with $r = \left(\sum_{i=1}^{n-1} x_i^2\right)^{\frac{1}{2}}$.)

Motion of torus (H. M. Soner and P. E. Souganidis).

Solitons - Self-similar surfaces Consider solutions of the MCF of the form $S(t) \equiv S^{\lambda(t)} := \lambda(t)S$ (standing waves), or $x(u,t) = \lambda(t)y(u)$, where $\lambda(t) > 0$. Plugging this into (1.1) and using $H(\lambda y) = \lambda^{-1}H(y)$, gives $\dot{\lambda}y = -\lambda^{-1}H(y)\nu(y)$, or $\lambda\dot{\lambda}y = -H(y)\nu(y)$. Multiplying this by $\nu(y)$, we obtain

$$H(y) = a\langle \nu, y\rangle, \qquad \text{and} \qquad \lambda\dot{\lambda} = -a. \tag{1.17}$$

Since $H(y)$ is independent of t, then so should be $\lambda\dot{\lambda} = -a$. Solving the last equation, we find $\lambda = \sqrt{\lambda_0^2 - 2at}$.

i) $a > 0 \Rightarrow \lambda \to 0$ as $t \to T := \frac{\lambda_0^2}{2a} \Rightarrow S^\lambda$ is a shrinker.

ii) $a < 0 \Rightarrow \lambda \to \infty$ as $t \to \infty \Rightarrow S^\lambda$ is an expander.

The equation (1.17) has the solutions: a is time-independent and x is one of the following

a) Sphere $x = R\hat{x}$, where $R = \sqrt{\frac{n}{a}}$.

b) Cylinder $x = (R\hat{x'}, x'')$, where $x = (x', x'') \in \mathbb{R}^{k+1} \times \mathbb{R}^m$, where $R = \sqrt{\frac{k}{a}}$.

As stated in the following theorem, these solutions are robust.

Theorem 5 (Huisken). *Let S satisfy $H = ax \cdot \nu$ and $H \geq 0$. We have*

(i) If $n \geq 2$, and S is compact, then S is a sphere of radius $\sqrt{\frac{n}{a}}$.

(ii) If $n = 2$ and S is a surface of revolution, then S is the cylinder of radius $\sqrt{\frac{n-1}{a}}$.

Now we consider self-similar surfaces in the graph representation. Let

$$f(u,t) = \lambda\chi(\lambda^{-1}u), \qquad \lambda \quad \text{depends on} \quad t. \tag{1.18}$$

Substituting this into (1.4) and setting $y = \lambda^{-1}u$ and $a = -\dot{\lambda}\lambda$, we find

$$\sqrt{1 + |\nabla_y\chi|^2}H(\chi) = a(y\partial_y - 1)\chi. \tag{1.19}$$

Translation solitons. These are solutions of the MCF of the form $S(t) \equiv S + h(t)$ (traveling waves), or $x(u,t) = y(u) + h(t)$, where $h(t) \in \mathbb{R}^{n+1}$. Plugging this into (1.1) and using $H(y + h) = H(y)$, gives $\dot{h} = -H(y)\nu(y)$, or

$$H(y) = v \cdot \nu(y), \qquad \text{and} \qquad \dot{h} = v. \tag{1.20}$$

Since $H(y)$ is independent of t, then so should be $\dot{h} = v$. **(more to come)**

Breathers. MCF periodic in t.

2 Self-similar surfaces and rescaled MCF

Recall, that static solutions of the MCF (1.1) satisfy the equation $H(x) = 0$ on S. Thus, static solutions of the MCF are *minimal surfaces*. On the other hand, *self-similar surfaces* (or *scaling solitons*) satisfy the equation (1.17), i.e.

$$H(y) = a\langle \nu, y\rangle, \tag{2.1}$$

where a is a constant. In view of this equation, we see that the minimal surfaces are special cases of self-similar ones corresponding to $a = 0$. In fact, $a = 0$ separates two types of evolution: contracting $a > 0$ (λ decreasing) and expanding $a < 0$ (λ increasing). (Remember that $a = -\lambda\partial_t\lambda$ is the negative of the speed of scaling λ.) (We see that *scaling solitons generalize the notion of the minimal surface.*)

2.1 Rescaled MCF.

First we note that the self-similar surfaces are static solutions of the rescaled MCF,

$$\partial_\tau \varphi = -(H(\varphi) - a\varphi \cdot \nu(\varphi))\nu(\varphi), \tag{2.2}$$

where $a = -\dot{\lambda}\lambda$, which is obtained by rescaling the surface ψ and time t as

$$\varphi(u, \tau) := \lambda^{-1}(t)\psi(u, t), \quad \tau = \int_0^t \lambda^{-2}(s)ds, \tag{2.3}$$

and then reparametrizing the obtained surface $S^{\text{resc}}(\tau) := \lambda(t)S(t)$ as in the proof of Proposition 1. Indeed, since $\lambda H = H^\lambda$ (or $\lambda H(\psi) = H(\lambda^{-1}\psi)$), we find

$$\begin{aligned}\partial_\tau \varphi &= \lambda^2\partial_t\varphi = \lambda^2(-\dot{\lambda}\lambda^{-2}\psi + \lambda^{-1}\partial_t\psi)\\ &= -\dot{\lambda}\lambda\varphi - \lambda H(\psi)\nu(\psi) = a\varphi - H(\varphi)\nu(\varphi).\end{aligned}$$

Then the mean curvature flow equation (1.1) implies

$$\partial_\tau \varphi = -H(\varphi)\nu(\varphi) + a\varphi, \tag{2.4}$$

which after the reparametrization gives (2.2). This is another analogy with minimal surfaces.

For static solutions, a =const. Then solving the equation $\dot{\lambda}\lambda = -a$, we obtain the parabolic scaling:

$$\lambda = \sqrt{2a(T - t)} \quad \text{and} \quad \tau(t) = -\frac{1}{2a}\ln(T - t), \tag{2.5}$$

where $T := \lambda_0^2/2T$, which was already discussed in connection with the scaling solitons.

Now, we know that minimal surfaces are critical points of the volume functional $V(\psi)$ (by Proposition 2, the equation $H(x) = 0$ on S is the Euler - Lagrange equation for $V(\psi)$). Are self-similar surfaces critical points of some modification of the volume functional? (Recall that, because of reparametrization (see Proposition 1), it suffices to look only at normal variations, ψ_s, of the immersion ψ, i.e. generated by vector fields η, directed along the normal ν: $\eta = f\nu$.) The answer to this question is yes and is given in the following

Proposition 6. *Let $\rho(x) = e^{-\frac{a}{2}|x|^2}$ and $V_a(\varphi) := \int_{S^\lambda} \rho$. For a surface S given locally by an immersion ψ and normal variations, $\eta = f\nu$, we have*

$$dV_a(\varphi)\eta = \int_U (H - a\varphi \cdot \nu)\nu \cdot \eta \ \rho d^n u, \tag{2.6}$$

Proof. The definition of $V_a(\varphi)$ gives $V_a(\varphi) = \int_{S^\lambda} \rho(\varphi)\sqrt{g(\varphi)}$, where $\rho(\varphi) = e^{-\frac{a}{2}|\varphi|^2}$. We have $d\rho(\varphi)\eta = -a\rho(\varphi)\eta$. Using this formula and the equation $d\sqrt{g}\eta = H\sqrt{g}\nu \cdot \eta$ proven above (see (1.10)) and the fact that we are dealing with normal variations, $\eta = f\nu$, we obtain (2.6). □

By the definition (1.8) and the formula (2.6), we have $\mathrm{grad}_h V_a(\varphi) = (H - a\varphi \cdot \nu)\nu$ in the Riemann metric $h(\xi, \eta) := \int_{S^\lambda} \xi\eta\rho$. This and the equation (2.2) and Proposition 6 imply

Corollary 7. *Assume a in* (2.2) *is constant (i.e. the rescaling* (2.3) *is parabolic,* (2.5)*). Then*

a) The modified area functional $V_a(\varphi)$ is momotonically decaying under the rescaled flow (2.2)*, more precisely,*

$$\partial_\tau V_a(\varphi) = -\int_{S^\lambda} \rho|H - a\nu \cdot \varphi|^2. \tag{2.7}$$

b) The renormalized flow (2.2) *is a gradient flow for the modified area functional $V_a(\varphi)$ and the Riemann metric $h(\xi, \eta) := \int_{S^\lambda} \xi\eta\rho$:*

$$\dot{\varphi} = -\,\mathrm{grad}_h V_a(\varphi).$$

The relation (2.7) is the Huisken monotonicity formula (earlier results of this type were obtained by Giga and Kohn and by Struwe).

Most interesting minimal surfaces are not just critical points of the volume functional $V(\psi)$ but are minimizers for it. What about self-similar surfaces? We see that

(i) $\inf V_a(\varphi) = 0$ and, for compact minimal surfaces, $V_a(\varphi)$ is minimized by any sequence shrinking to a point.

(ii) $V_a(\varphi)$ is unbounded from above. This is clear for $a < 0$. To see this for $a < 0$, we construct a sequence of surfaces lying inside a fixed ball in $\mathbb{R}^{n+1}$ and folding tighter and tighter.

Thus self-similar surfaces are neither minimizers nor maximizers of $V_a(\varphi)$. We conjecture that they are saddle points satisfying min-max principle: $\sup_V \inf_{\varphi: V(\varphi)=V} V_a(\varphi)$. One can try use this principle (say in the form of the mountain pass lemma) to find solutions of (2.1).

2.2 Normal hessians.

Recall that the hessian of a functional $E(\varphi)$ is defined as $\mathrm{Hess}\, E(\varphi) := d\, \mathrm{grad}\, V_a(\varphi)$. Note that unlike the Gâteaux derivative, d, the gradient grad and therefore the hessian, Hess, depends on the Riemann metric on the space on which $E(\varphi)$ is defined.

Since the tangential variations lead to reparametrization of the surface, in what follows we are dealing with normal variations, $\eta = f\nu$. (In physics terms, specifying normal variations is called fixing the gauge.) We use the following notation for a linear operator, A, on normal vector fields on S: $A^N f = A(f\nu)$. For instance, $\mathrm{Hess}^N E(\varphi) f = \mathrm{Hess}\, E(\varphi)(f\nu)$ and

$$d^N F(\varphi) f = dF(\varphi)(f\nu). \tag{2.8}$$

We consider the hessian of the modified volume functional $V_a(\varphi)$, at a self-similar φ (i.e. $H(\varphi) = a\varphi \cdot \nu$) and in the normal direction (i.e. for normal variations, $\eta = f\nu$) in the Riemann metric $h(\xi, \eta) := \int_{S^\lambda} \xi \eta \rho$. In what follows, we call this hessian the normal hessian and denote it by $\mathrm{Hess}^N V_a(\varphi)$.

Before we proceed, we mention the following important property of $V_a(\varphi)$: the equation $H(\varphi) - a\varphi \cdot \nu(\varphi) = 0$ breaks the scaling and translational symmetry. Indeed, using the relations

$$H(\lambda\varphi) = \lambda^{-1} H(\varphi),\ \nu(\lambda\varphi) = \nu(\varphi),\ \forall \lambda \in \mathbb{R}_+, \tag{2.9}$$

$$H(\varphi + h) = H(\varphi),\ \nu(\varphi + h) = \nu(\varphi),\ \forall h \in \mathbb{R}^{n+1}, \tag{2.10}$$

$$H(g\varphi) = H(\varphi),\ \nu(g\varphi) = g\nu(\varphi),\ \forall g \in O(n+1), \tag{2.11}$$

using that $g \in O(n+1)$ are isometries in $\mathbb{R}^{n+1}$and using the notation $H_a(\varphi) := H(\varphi) - a\varphi \cdot \nu(\varphi)$, we obtain

$$H_{\lambda^{-2}a}(\lambda\varphi) = \lambda^{-1} H_a(\varphi),\ \forall \lambda \in \mathbb{R}_+, \tag{2.12}$$

$$H_a(\varphi + h) + ah \cdot \nu(\varphi) = H_a(\varphi),\ \forall h \in \mathbb{R}^{n+1}, \tag{2.13}$$

$$H_a(g\varphi) = H_a(\varphi),\ \forall g \in O(n+1). \tag{2.14}$$

We want to address the spectrum of the normal hessian, $\mathrm{Hess}^\perp V_a(\varphi)$. First, we note that the tangential variations lead to zero modes of the full hessian, $\mathrm{Hess}\, V_a(\varphi)$. Indeed, we have

Proposition 8. *The full hessian,* $\mathrm{Hess}\, V_a(\varphi)$*, of the modified volume functional* $V_a(\varphi)$*, has the eigenvalue* 0 *with the eigenfunctions which are tangential vector fields on* S

Proof. We consider a family α_s of diffeormorphisms of U, with $\alpha_0 = \mathbf{1}$ and $\partial_s \varphi \circ \alpha_s|_{s=0} = \xi$, a tangential vector field, reparametrizing the immersion φ, and define the family $\varphi \circ \alpha_s$ of variations of φ. Then $\varphi \circ \alpha_s$ satisfies again the soliton equation, $H_a(\varphi \circ \alpha_s) = 0$. Differentiating the latter equation w.r.to s at $s = 0$ and using that $\partial_s \varphi \circ \alpha_s|_{s=0} = \xi$, is a tangential vector field, we obtain

$$dH_a(\varphi)\xi = 0, \tag{2.15}$$

which proves the proposition. □

Theorem 9. *The hessian,* $\mathrm{Hess}^N V_a(\varphi)$*, of the modified volume functional* $V_a(\varphi)$*, at a self-similar* φ *(i.e.* $H(\varphi) = a\varphi \cdot \nu$*) and in the normal direction (i.e. for normal variations,* $\eta = f\nu$*), has*

- *the eigenvalue* $-2a$ *with the eigenfunction* $\varphi\cdot\nu(\varphi)$,
- *the eigenvalue* $-a$ *with the eigenfunctions* $\nu^j(\varphi),\ j=1,\dots,n+1$, *and,*
- *the eigenvalue* 0 *with the eigenfunctions* $\sigma_j\varphi\cdot\nu(\varphi),\ j=1,\dots,\frac{1}{2}n(n-1)$, *where* σ_j *are generators of the Lie algebra of* $SO(n+1)$, *unless* φ *is a sphere.*

Proof. If an immersion φ satisfies the soliton equation $H(\varphi)=a\varphi\cdot\nu(\varphi)$, then by (2.12), we have $H_{\lambda^{-2}a}(\lambda\varphi)=0$ for any $\lambda>0$. Differentiating this equation w.r.to λ at $\lambda=1$, we obtain $dH_a(\varphi)\varphi=-2a\varphi\cdot\nu(\varphi)$.

Now, choosing ξ to be equal to the tangential projection, φ^T, of φ, and subtracting the equation (2.15) from the last equation, we find $dH_a(\varphi)(\varphi\cdot\nu(\varphi))\nu(\varphi)=-a\varphi\cdot\nu(\varphi)$. Since by the definition (2.17), $dH_a(\varphi)f\nu(\varphi)=d^NH_a(\varphi)f$, this proves the first statement.

To prove the second statement, we observe that the soliton equation implies, by (2.13), that $H_a(\varphi+sh)+ash\cdot\nu(\varphi)=0$ and any constant vector field h. Differentiating this equation w.r.to s at $s=0$, we obtain $dH_a(\varphi)h=-ah\cdot\nu(\varphi)$. Now, choosing ξ to be equal to the tangential projection, h^T, of h, and subtracting the equation (2.15) from the last equation, we find $dH_a(\varphi)(h\cdot\nu(\varphi))\nu(\varphi)=-ah\cdot\nu(\varphi)$, which together with (2.17) gives the second statement.

Finally, to prove the third statement, we differentiate the equation $H_a(g(s)\varphi)=0$, where $g(s)$ is a one-parameter subgroup of $O(n+1)$, w.r.to s at $s=0$, to obtain $dH_a(\varphi)\sigma\varphi=0$, where σ denotes the generator of $g(s)$. Now, choosing ξ in (2.15) to be equal to the tangential projection, $(\sigma\varphi)^T$, of $\sigma\varphi$, and subtracting the equation (2.15) from the last equation, we find $dH_a(\varphi)(\sigma\varphi\cdot\nu(\varphi))\nu(\varphi)=0$, which together with (2.17) gives the third statement. □

Remark 1. *a) For* $a\neq 0$, *the soliton equation,* $\varphi\cdot\nu(\varphi)=a^{-1}H(\varphi)$, *and Proposition 3 imply that the mean curvature* H *is an eigenfunction of* $\mathrm{Hess}^N V_a(\varphi)$ *with the eigenvalue* $-2a$.

b) Strictly speaking, if the self-similar surface is not compact, then $\varphi\cdot\nu(\varphi)$ *and* $\nu^j(\varphi),\ j=1,\dots,n+1$, *generalized eigenfunctions of* $\mathrm{Hess}^N V_a(\varphi)$. *In the second case, the Schnol-Simon theorem (see Appendix C.1 or [?]) implies that the points* $-2a$ *and* $-a$ *belong to the essential spectrum of* $\mathrm{Hess}^N V_a(\varphi)$.

c) We show below that the normal hessian, $\mathrm{Hess}^N_{\mathrm{sph}} V_a(\varphi)$, *on the sphere of the radius* $\sqrt{\frac{a}{n}}$, *given in* (2.23), *has no other eigenvalues below* $\frac{2a}{n}$, *besides* $-2a$ *and* $-a$. *A similar statement, but with* n *replaced by* $n-1$, *we have for the cylinder.*

We call the eigenfunction $\varphi\cdot\nu(\varphi)$, $\nu^j(\varphi)$ and $\sigma_j\varphi\cdot\nu(\varphi),\ j=1,\dots,n+1$, the *scaling, translational and rotational modes.* They originate from the normal projections, $(\lambda\varphi)^N$ and $(sh)^N$, of scaling, translation and rotation variations.

2.3 Self-similar surfaces.

Linearized Stability. Given a self-similar surface φ, we consider for example the manifold of surfaces obtained from φ by symmetry transformations,

$$\mathcal{M}_\varphi := \{\lambda g\varphi + z : (\lambda, z,\ g)\in\mathbb{R}_+\times\mathbb{R}^{n+1}\times SO(n+1)\}.$$

By the spectral theorem above, it has unstable and central manifolds corresponding to the eigenvalues $-2a$, $-a$ and 0. Hence, we can expect only the dynamical stability in the transverse direction.

Definition 1 (Linearized stability of self-similar surfaces)**.** *We say that a self-similar surface ϕ, with $a > 0$, is linearly stable (for the lack of a better term) iff the normal hessian satisfies $\mathrm{Hess}^N V_a(\varphi) > 0$ on the subspace $\big(\mathrm{span}\{\varphi \cdot \nu(\varphi),\ \nu^i(\varphi),\ i = 1, \ldots, n+1,\ \sigma_j \varphi \cdot \nu(\varphi),\ j = 1, \ldots, \frac{1}{2}d(d-1)\}\big)^\perp$ (i.e. on $\big(\mathrm{span}\{$scaling, translational, rotational modes$\}\big)^\perp$).*

(I.e. the only unstable motions allowed are scaling, translations and rotations.)

Another notion of stability was introduced by analogy with minimal surfaces in [3]:

Definition 2 ($F-$stability of self-similar surfaces, [3])**.** *We say that a self-similar surface ϕ, with $a > 0$, is $F-$stable iff the normal hessian satisfies $\mathrm{Hess}^N V_a(\varphi) \geq 0$ on the subspace $\mathrm{span}\{\varphi \cdot \nu(\varphi),\ \nu^j(\varphi),\ j = 1, \ldots, n+1\}^\perp$.*

Remark 2. *1) The $F-$stability, at least in the compact case, says that the $\mathrm{Hess}^N V_a(\varphi)$ has the smallest possible negative subspace, i.e φ has the smallest possible Morse index.*

2) The reason the $F-$stability works in the non-compact case is that, due to the separation of variables for the cylinder = (compact surface) $\times \mathbb{R}^k$, the orthogonality to the negative eigenfunctions of the compact factor removes the entire branches of the essential spectrum. This might not work for warped cylinders.

3) For minimal surfaces (strict) stability implies the the surface is a (strict) minimizer of the volume functional $V(\psi)$. As is already suggested by the discussion above, this is not so for self-similar surfaces, they are saddle points possibly satisfying some min-max principle.

4) Remark 1(c) shows that the spheres and cylinders are $F-$stable. However, we show in Section 5 that cylinders are dynamically unstable.

In what follows $a > 0$. Theorem 9 implies that if ϕ is not spherically symmetric, then 0 is an eigenvalue of $\mathrm{Hess}^N V_a(\varphi)$ of multiplicity at least $n + 1$. This gives the first statement in the following corollary, while the second one follows from Remark 1(c) and the definition of the $F-$stability (see also the first part of Remark 2(5)).

Corollary 10. *(a) If a self-similar surface with $a > 0$ satisfies $\mathrm{Hess}^N V_a(\varphi) > 0$ on the subspace $\mathrm{span}\{\varphi \cdot \nu(\varphi),\ \nu^j(\varphi),\ j = 1, \ldots, n+1\}^\perp$, then it a sphere or a cylinder.*

(b) There are no smooth, embedded self-similar ($a > 0$), $F-$stable surfaces in $\mathbb{R}^{n+1}$ close to $S^k \times \mathbb{R}^{n-k}$, where S^k is the round $k-$sphere of radius $\sqrt{\frac{k}{a}}$.

A slight but a key improvement of this result is a hard theorem:

Theorem 11 (Colding- Minicozzi)**.** *The only smooth, complete, embedded self-similar ($a > 0$), $F-$stable surfaces in $\mathbb{R}^{n+1}$ of* **polynomial growth** *are $S^k \times \mathbb{R}^{n-k}$, where S^k is the round $k-$sphere of radius $\sqrt{\frac{k}{a}}$.*

Remark 3. *The difference between this theorem and (trivial) Corollary 10 is that the latter requires a slightly stronger condition* $\mathrm{Hess}^N V_a(\varphi) > 0$ *on the subspace* $\mathrm{span}\{\varphi \cdot \nu(\varphi),\ \nu^j(\varphi),\ j = 1, \ldots, n+1\}^\perp$*, then the* $F-$*stability.*

To prove Theorem 11, we begin with the following result

Theorem 12. *For a self-similar surface with* $a > 0$, $\mathrm{Hess}^N V_a(\varphi) \geq -2a$ *iff* $H(\varphi) > 0$. *Hence if it is* $F-$*stable, then* $H(\varphi) > 0$.

To prove this theorem we will use the Perron-Frobenius theory (see Appendix C.2) and its extension as given in [26]. We begin with

Definition 3. *We say that a linear operator on* $L^2(S)$ *has a positivity improving property iff either* A*, or* e^{-A}*, or* $(A+\mu)^{-1}$*, for some* $\mu \in \mathbb{R}$*, takes non-negative functions into positive ones.*

Proposition 13. *The normal hessian,* $\mathrm{Hess}^N V_a(\varphi)$*, has a positivity improving property.*

Proof. By standard elliptic/parabolic theory, e^Δ and $(-\Delta+\mu)^{-1}$, for any $\mu > 0$, has strictly positive integra kernel and therefore is positivity improving. To lift this result to $\mathrm{Hess}^N V_a(\varphi) := -\Delta - |W|^2 - a\mathbf{1} + \varphi \cdot \nabla$ we proceed exactly as in [26]. □

Since, the operator $\mathrm{Hess}^N V_a(\varphi)$ is bounded from below and has a positivity improving property, it satisfies the assumptions of the Perron-Frobenius theory (see Appendix C.2) and its extension in [26] to the case when the positive solution in question is not an eigenfunction. The latter theory, Proposition 3 and Remark 1 imply the statement of Theorem 12.

Corollary 14 (Dan Ginsberg)**.** *Let* φ *be a self-similar surface. We have*

(a) For $a < 0$ *(*φ *is an expander),* $H(\varphi)$ *changes the sign.*

(b) For $a = 0$, $\inf \mathrm{Hess}^N V_a(\varphi) < 0$.

(c) For $a > 0$*, if* φ *is an entire graph over* $\mathbb{R}^n$ *and is weakly stable, then* φ *is a hyperplane.*

Indeed, (a) follows directly from Theorems 9 and 12, while (b) follows from the fact that for $a = 0$, 0 is an eigenvalue of the multiplicity $n+2$ and therefore, by the Perron-Frobenius theory, it is not the lowest eigenvalue of $\mathrm{Hess}^N V_a(\varphi)$. (c) is based on the fact that entire graphs over $\mathbb{R}^n$ cannot have strictly postive mean curvature. (check, references)

Theorem 15 (Colding- Minicozzi, Huisken)**.** *The only smooth, complete, embedded self-similar surfaces in* $\mathbb{R}^{n+1}$*, with* $a > 0$*, polynomial growth and* $H(\varphi) > 0$*, are* $S^k \times \mathbb{R}^{n-k}$*, where* S^k *is the round* $k-$*sphere of radius* $\sqrt{\frac{k}{a}}$.

Proof. Denote $L = \mathrm{Hess}^N V_a(\varphi)$. We extend the operator L to tensors by using the connection ∇ and the Laplce- Beltrami operator $\Delta = g^{ij}\nabla_i\nabla_j$ on tensors. The proof of this theorem is based on the following

Lemma 16. *We have the following relations*

(a) [Colding- Minicozzi, Huisken] $|\nabla|W||^2 \leq |\nabla W|^2$.

(b) $(1+\frac{2}{n+1})|\nabla|W||^2 \le |\nabla W|^2 + \frac{2n}{n+1}|\nabla H|^2$.

(c) [Colding- Minicozzi, Simons] $LW = -2aW$

(d) [Colding- Minicozzi, Simons] If $|W|$ *does not vanish, then* $L|W| \ge |W|$.

(e) more bounds.

Proof. A long string of elementary inequalities, see [3], Lemmas 10.2, 10.8, Proposition 10. 14. □

This lemma implies the relations

(a) [Colding- Minicozzi, Huisken] $|\nabla|W||^2 = |\nabla W|^2$.

(b) [Huisken] If $H > 0$, then $|W| = \beta H$, for some $\beta > 0$.

The last two relations and somewhat lengthy deliberations imply Theorem 15. □

(Compare with surfaces of constant mean curvature)

Theorems 12 and 15 imply Theorem 11.

There is a considerable literature on stable minimal surfaces. Much of it related to the Bernstein conjecture:

The only entire minimal graphs are linear functions.

It was shown it is true for $n \le 7$:

(a) Bernstein for $n = 2$;

(b) De Georgi for $n = 3$;

(c) Almgren for $n = 4$; .

(d) Simons for $n = 5, 6, 7$.

(b) and (d) used in part results of Fleming on minimal cones. These results were extended by Schoen, Simon and Yau.

Bombieri, De Georgi and Giusti constructed a contra example for $n > 7$.

Now, we compute explicit expression for the normal hessian, $\mathrm{Hess}^N V_a(\varphi)$.

Theorem 17. *The normal hessian,* $\mathrm{Hess}^N V_a(\varphi)$ *at a self-similar* φ *(i.e.* $H(\varphi) = a\varphi \cdot \nu$*) is given by*

$$\mathrm{Hess}^N V_a(\varphi) = -\Delta - |W|^2 - a(\mathbf{1} - \varphi \cdot \nabla). \tag{2.16}$$

Proof. Recall that by the definition (1.8) and the formula (2.6), we have $\mathrm{grad}_h V_a(\varphi) = (H - a\varphi\cdot\nu)\nu$ in the Riemann metric $h(\xi,\eta) := \int_{S^\lambda} \xi\eta\rho$. For normal variations, $\eta = f\nu$, this gives $\mathrm{grad}_h V_a^{\mathrm{norm}}(\varphi) = (H(\varphi) - a\varphi\cdot\nu)$. We have to compute $d\,\mathrm{grad}_h V_a^{\mathrm{norm}}(\varphi) = d(H - a\varphi\cdot\nu)\nu$ w.r.to normal variations. We claim that

$$d^N Hf = (-\Delta - |W|^2)f, \tag{2.17}$$

$$d^N\varphi\cdot\nu f = f - \varphi\cdot\nabla f. \tag{2.18}$$

We begin with some simple calculations. Denote $\varphi_j := \partial_{u^j}\varphi$. We have

Lemma 18 (Simons, Schoen-Simon-Yau, Hamilton, Huisken, Sesum)**.** *We have the following identities*

(a) $d\nu(\varphi)\eta = -g^{ij}\langle\nu,\partial_i\eta\rangle\varphi_j = -\langle\nu,\nabla\eta\rangle$.

(b) $db_{ij}(\varphi)\eta = -\langle\nu,(\partial_i\partial_j + \Gamma_{ij}^k\partial_k)\eta\rangle$.

(c) $dg^{ij}\eta = g^{im}g^{jn}dg_{ij}\eta,\ \ dg_{ij}\eta = \langle\varphi_m,\partial_i\eta\rangle + \langle\varphi_m,\partial_i\eta\rangle$.

Proof. For (a), use that $\langle\nu(\varphi_s),\partial_i\varphi_{si}\rangle = 0$ to obtain $\langle\partial_s\nu(\varphi_s),\partial_i\varphi_{si}\rangle = -\langle\nu(\varphi_s),\partial_i\partial_s\varphi_{si}\rangle$, which implies the desired statement.

To prove (b), we use Lemma 41, which says $b_{ij} = -\langle\frac{\partial^2\varphi}{\partial u_i\partial u_j},\nu\rangle$, the first relation, the equation $d\varphi\eta = \eta$ and the Gauss formula

$$\frac{\partial^2\psi}{\partial u_i\partial u_j} = b_{ij}\nu + \Gamma_{ij}^k\varphi_k \tag{2.19}$$

(see e.g. [24]), to find $db_{ij}(\varphi)\eta = -\langle\frac{\partial^2}{\partial u_i\partial u_j}d\varphi\eta,\nu\rangle - \langle\frac{\partial^2\varphi}{\partial u_i\partial u_j},d\nu\eta\rangle = -\langle\frac{\partial^2}{\partial u_i\partial u_j}\eta,\nu\rangle - g^{mn}\langle(b_{ij}\nu + \Gamma_{ij}^k\varphi_k,\varphi_n\rangle\langle\nu,\partial_m\eta\rangle$. Next, using $\langle\nu,\varphi_n\rangle = 0$ and $\langle(\varphi_k,\varphi_n\rangle = g_{kn}$ gives $db_{ij}(\varphi)\eta = -\langle\frac{\partial^2}{\partial u_i\partial u_j}\eta,\nu\rangle - g^{mn}g_{kn}\Gamma_{ij}^k\langle\nu,\partial_m\eta\rangle$, which implies the statement (c).

The first statement in (c) follows by differentiating the relation $gg^{-1} = \mathbf{1}$, where $g = (g_{ij})$, which gives $d(g^{-1}) = g^{-1}dgg^{-1}$, and the second, from the definition $g_{ij} = \langle\varphi_i,\varphi_j\rangle$ and the relation $d\varphi\eta = \eta$. □

Now, notice that the definition $\Delta f = \mathrm{div}\,\mathrm{grad}\, f = \nabla_i\nabla^i f$ and relations to

$$\mathrm{div}\, V = \nabla_i V^i = \frac{\partial V^i}{\partial u^i} + \Gamma_{mi}^m V^i,\ \ (\mathrm{grad}(f))^i = g^{ij}\frac{\partial f}{\partial u^j}, \tag{2.20}$$

imply $\Delta f = (\frac{\partial}{\partial u^i} + \Gamma_{mi}^m)g^{ij}\frac{\partial f}{\partial u^j}$ which gives the relation $g^{ij}(\partial_i\partial_j + \Gamma_{ij}^k\partial_k)\eta = \Delta\eta$. Using this relation, Lemma 18 and the definition $H = g^{ij}b_{ij}$, we find

$$dH\eta = g^{im}g^{jn}b_{ij}dg_{mn}\eta - \langle\nu,\Delta\eta\rangle. \tag{2.21}$$

Next, using Lemma 18 and the equation $d\varphi\eta = \eta$, we find

$$d\langle\varphi,\nu\rangle\eta = \langle\eta,\nu\rangle - g^{ij}\langle\nu,\partial_i\eta\rangle\langle\varphi,\varphi_j\rangle. \tag{2.22}$$

Now, taking $\eta = f\nu$, we arrive at (2.17). □

2.4 Hessians for spheres and cylinders

We finish this section with the discussion of the normal hessians on the $n-$sphere and $(n,k)-$cylinder.

Explicit expressions. The examples of Subsection A.2 show that

1) For the $n-$sphere S^n_R of radius $R=\sqrt{\frac{n}{a}}$ in $\mathbb{R}^{n+1}$, we have $b_{ij}=R^{-1}g_{ij}=Rg^{\text{stand}}_{ij}$, where g_{ij} and g^{stand}_{ij} are the metrics on S^n_R and the standard $n-$sphere $\mathbb{S}^n=S^n_1$, respectively, which gives $|W|^2=nR^{-2}=a$. Moreover, since $\nu(\varphi)=\varphi$, we have that $\varphi\cdot\nabla=0$ and therefore

$$\mathrm{Hess}^N_{\mathrm{sph}}\,V_a(\varphi)=-\frac{a}{n}\Delta_{\mathbb{S}^n}-2a, \tag{2.23}$$

on $L^2(\mathbb{S}^n)$, where $\Delta_{\mathbb{S}^k}$ is the Laplace-Beltrami operator on the standard $n-$sphere $\mathbb{S}^k$.

2) For the $n-$cylinder $\mathcal{C}^n_R=S^{n-k}_R\times\mathbb{R}^k$ of radius $R=\sqrt{\frac{n-k}{a}}$ in $\mathbb{R}^{n+1}$, we have $b_{ij}=R^{-1}g_{ij}=Rg^{\text{stand}}_{ij}$, for $i,j=1,\dots,n-k$, and $b_{\alpha\beta}=\delta_{\alpha\beta},\ \alpha,\beta=n-k+1,\dots,n$. Here g_{ij} and g^{stand}_{ij} are the metrics on S^{n-1}_R and $\mathbb{S}^{n-1}=S^{n-1}_1$, respectively. This gives

$$|W|^2=(n-k)R^{-2}=a. \tag{2.24}$$

Moreover, letting the standard round cylinder $\mathcal{C}^n=S^{n-k}\times\mathbb{R}^k$ be naturally embedded as $\mathcal{C}^n=\{(\omega,x):|\omega|^2=1\}\subset\mathbb{R}^{n+1}$, we see that the immersion φ is given by $\varphi(\omega,y)=(\chi(\omega,y),y)$ for some $\chi(\omega,y)$, where $y=y(x):=\lambda^{-1}(t)x$. This gives $\varphi\cdot\nabla=\omega\cdot\nabla_{\mathbb{S}^{n-k}}+y\cdot\nabla_y=y\cdot\nabla_y$, which together with the previous relation, implies

$$\mathrm{Hess}^N_{\mathrm{cyl}}\,V_a(\varphi)=-\Delta_y-ay\cdot\nabla_y-\frac{a}{n-k}\Delta_{\mathbb{S}^{n-k}}-2a, \tag{2.25}$$

acting on $L^2(\mathcal{C}^n)$.

Spectra of $L^{\mathrm{sph}}_a:=\mathrm{Hess}^N_{\mathrm{sph}}\,V_a(\varphi)$ and $L^{\mathrm{cyl}}_a:=\mathrm{Hess}^N_{\mathrm{cyl}}\,V_a(\varphi)$. We describe the spectra of the normal hessians on the $n-$sphere and $(n,k)-$cylinder, of the radii $\sqrt{\frac{a}{n}}$ and $\sqrt{\frac{a}{n-1}}$, respectively.

It is a standard fact that the operator $-\Delta=-\Delta_{\mathbb{S}^n}$ is a self-adjoint operator on $L^2(\mathbb{S}^n)$. Its spectrum is well known (see [29]): it consists of the eigenvalues $l(l+n-1),\ l=0,1,\dots,$ of the multiplicities $m_\ell=\begin{pmatrix} n+l\\ n\end{pmatrix}-\begin{pmatrix} n+l-2\\ n\end{pmatrix}$. Moreover, the eigenfunctions corresponding to the eigenvalue $l(l+n-1)$ are the restrictions to the sphere $\mathbb{S}^n$ of harmonic polynomials on $\mathbb{R}^{n+1}$ of degree l and denoted by Y_{lm} (the spherical harmonics),

$$-\Delta Y_{lm}=l(l+n-1)Y_{lm},\quad l=0,1,2,3,\dots,\ m=1,2,\dots,m_l. \tag{2.26}$$

In particular, the first eigenvalue 0 has the only eigenfunction 1 and the second eigenvalue n has the eigenfunctions $\omega^1,\cdots,\omega^{n+1}$.

Consequently, the operator $L^{\mathrm{sph}}_a:=\mathrm{Hess}^N_{\mathrm{sph}}\,V_a(\varphi)=-\frac{a}{n}(\Delta_{\mathbb{S}^n}+2n)$ is self-adjoint and its spectrum consists of the eigenvalues $\frac{a}{n}(l(l+n-1)-2n)=a(l-2)+\frac{a}{n}l(l-1),\ l=0,1,\dots,$ of the multiplicities

m_ℓ. In particular, the first $n+2$ eigenvectors of $L_a^{\rm sph}$ (those with $l=0,1$) correspond to the non-positive eigenvalues,

$$L_a^{\rm sph}\omega^0=-2a\omega^0, \quad L_a^{\rm sph}\omega^j=-a\omega^j,\ j=1,\dots,n+1, \tag{2.27}$$

and are due to the scaling ($l=0$) and translation ($l=1$) symmetries.

We proceed to the cylindrical hessian $L_a^{\rm cyl}:={\rm Hess}^N_{\rm cyl}V_a(\varphi)$, given in (2.25). The variables in this operator separate and we can analyze the operators $-\Delta_y-ay\cdot\nabla$ and $-\frac{a}{n-k}(\Delta_{\mathbb{S}^{n-k}}+2(n-k))$ separately. The operator $-\frac{a}{n-k}(\Delta_{\mathbb{S}^{n-k}}+2n)$ was already analyzed above. The operator $-\Delta_y-ay\cdot\nabla$ is the Ornstein - Uhlenbeck generator, which can be unitarily mapped by the gauge transformation

$$v(y,w)\to v(y,w)e^{-\frac{a}{4}|y|^2}$$

into the the harmonic oscillator Hamiltonian $H_{harm}:=-\Delta_y+\frac14 a^2|y|^2-ka$. Hence the linear operator $-\Delta_y-ay\cdot\nabla$ is self-adjoint on the Hilbert space $L^2(\mathbb{R},e^{-\frac{a}{2}|y|^2}dy)$. Since, as was already mentioned, the operator $-\frac{a}{n-k}(\Delta_{\mathbb{S}^{n-k}}+2(n-k))$ is self-adjoint on the Hilbert space $L^2(\mathcal{C}^n)$, we conclude that the linear operator $L_a^{\rm cyl}$ is self-adjoint on the Hilbert space $L^2(\mathbb{R}^k\times\mathbb{S}^{n-k},e^{-\frac{a}{2}|y|^2}dydw)$. Moreover, the spectrum of $-\Delta_y-ay\cdot\nabla$ is $\left\{a\sum_1^k s_i : s_i=0,1,2,3,\dots\right\}$, with the normalized eigenvectors denoted by $\phi_{s,a}(y)$, $s=(s_1,\dots,s_k)$,

$$(-\Delta_y-ay\cdot\nabla)\phi_{s,a}=a\sum_1^k s_i\phi_{s,a}, \quad s_i=0,1,2,3,\dots. \tag{2.28}$$

Using that we have shown that the spectrum of $-\frac{a}{n-k}(\Delta_{\mathbb{S}^{n-k}}+2(n-k))$ is $\frac{a}{n-k}(l(l+n-k-1)-2(n-k))=\frac{a}{n-k}l(l+n-k-1)-2a$, $l=0,1,\dots$, and denoting $r=\sum_1^k s_i$, $s_i=0,1,2,3,\dots$, we conclude the spectrum of the linear operator $L_a^{\rm cyl}$, for $k=1$, is

$${\rm spec}(L_a^{\rm cyl})=\left\{(r-2)a+\frac{a}{n-1}\ell(\ell+n-2) : r=0,1,2,3,\dots;\ \ell=0,1,2,\dots\right\}, \tag{2.29}$$

with the normalized eigenvectors given by $\phi_{r,l,m,a}(y,w):=\phi_{r,a}(y)Y_{lm}(w)$. This equation shows that the non-positive eigenvalues of the operator $L_a^{\rm cyl}$, for $k=1$, are

- the eigenvalue $-2a$ of the multiplicity 1 with the eigenfunction $\phi_{0,0,0,a}(y)=(\frac{a}{2\pi})^{\frac14}$ $((r,l)=(0,0))$, due to scaling of the transverse sphere;
- the eigenvalue $-a$ of the multiplicity n with the eigenfunctions $\phi_{0,1,m,a}(y)=(\frac{a}{2\pi})^{\frac14}w^m$, $m=1,\dots,n$ $((r,l)=(1,1))$, due to transverse translations;
- the eigenvalue 0 of the multiplicity n with the eigenfunctions $\phi_{1,1,m,a}(y)=(\frac{a}{2\pi})^{\frac14}\sqrt{a}yw^m$, $m=1,\dots,n$ $((r,l)=(0,1))$, due to rotation of the cylinder;
- the eigenvalue $-a$ of the multiplicity 1 with the eigenfunction $\phi_{1,0,0,a}(y)=(\frac{a}{2\pi})^{\frac14}\sqrt{a}y$ $((r,l)=(1,0))$;
- the eigenvalue 0 of the multiplicity 1 with the eigenfunction $\phi_{2,0,0,a}(y)=(\frac{a}{2\pi})^{\frac14}(1-ay^2)$ $((k,l)=(2,0))$.

The last two eigenvalues are not of the broken symmetry origin and are not covered by Theorem 9. They indicate instability of the cylindrical collapse

By the description of the spectra of the normal hessians of $L_a^{\rm sph} := \mathrm{Hess}^N_{\rm sph} V_a(\varphi)$ and $L_a^{\rm cyl} := \mathrm{Hess}^N_{\rm cyl} V_a(\varphi)$, *we conclude that the spherical collapse is linearly stable while the cylindrical one is not.*

We will show in Sections 4 and 5 that indeed the spherical collapse is (nonlinearly) stable while the cylindrical one is not. We will also show that the last two eigenvalues of $L_a^{\rm cyl}$ in the list above are due to translations of the point of the neckpinch on the axis of the cylinder and due to shape instability, respectively.

2.5 Minimal and self-similar submanifolds

For a self-similar hypersurface S in a manifold M, with immersion φ (satisfying $H(\varphi) = a\varphi \cdot \nu$), the normal hessian, $\mathrm{Hess}^N V_a(\varphi)$ is given by

$$\mathrm{Hess}^N V_a(\varphi) = -\Delta - |W|^2 - a(\mathbf{1} - \varphi \cdot \nabla) - Ric^M(\nu,\nu), \tag{2.30}$$

where $Ric^M(\nu,\nu)$ is the Ricci curvature of M.

Appendix: Graph representation. As an exercise, we prove (2.6) for a local patch, $S^\lambda \cap W$ given by a graph of a function $f^\lambda : U^\lambda \to \mathbb{R}$, $S^\lambda \cap W = \text{graph } f^\lambda$ which is the rescaling

$$f^\lambda(v,\tau) = \lambda^{-1} f(u,t), \quad v = \lambda^{-1}u,\ \tau = \int_0^t \lambda^{-2}(s)ds, \tag{2.31}$$

of S =graph f. In this case, using that $\sqrt{1+|\nabla f^\lambda|^2}\lambda^{-1}H^\lambda = \sqrt{1+|\nabla f|^2}H$ and $\partial_t f = \dot\lambda f^\lambda - \dot\lambda y \cdot \nabla_v f^\lambda f^\lambda + \lambda\lambda^{-2}\partial_\tau f^\lambda$, we find that $f^\lambda(v,\tau)$ satisfies the equation

$$\partial_\tau f^\lambda = \sqrt{1+|\nabla f^\lambda|^2}\mathrm{div}\left(\frac{\nabla f^\lambda}{\sqrt{1+|\nabla f^\lambda|^2}}\right) - a(v\partial_v - 1)f^\lambda. \tag{2.32}$$

In the rest of the derivation we omit the superindex λ. Let $x = (v, f(v))$. Since $|x|^2 = |v|^2 + |f(v)|^2$, we can write in local coordinates

$$V_a(f) := \int_\psi \rho = \int_U e^{-\frac{a}{2}(v^2+f^2)^2}\sqrt{1+|\nabla f|^2}d^n v.$$

We compute $dV_a(f)\xi = \int_U \rho\left(-af\xi + \frac{\nabla f \cdot \nabla \xi}{\sqrt{1+|\nabla f|^2}}\right)$. Integration by parts of the second term gives

$$dV_a(f)\xi = \int_{U^\lambda} \rho\left[-af\sqrt{1+|\nabla f|^2} + a(v + f\nabla f)\frac{\nabla f}{\sqrt{1+|\nabla f|^2}} - \mathrm{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right)\right]\xi d^n v.$$

The first two terms on the r.h.s. come from differentiating ρ. Reducing them to the common denominator and using that $H = \mathrm{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right)$, the mean curvature, we find

$$dV_a(f)\xi = -\int_U \rho\left(H - a\frac{(v\cdot\nabla - 1)f}{\sqrt{1+|\nabla f|^2}}\right)\xi d^n v. \tag{2.33}$$

Now $\nu = \frac{(-\nabla f,1)}{\sqrt{1+|\nabla f|^2}}$ and $x = (v, f(v))$. Hence we have $\nu \cdot x = \frac{-v\cdot\nabla f+f}{\sqrt{1+|\nabla f|^2}}$, which gives (2.6).

As before, (2.33) implies that (2.32) is the gradient flow for the functional $V_a := \int_{U^\lambda} e^{-\frac{a}{2}(v^2+f^2)^2}\sqrt{1+|\nabla f|^2}d^n v$ and the metric $h(\xi,\eta) := \int_{U^\lambda}\xi\eta\frac{\rho}{\sqrt{1+|\nabla f^\lambda|^2}}$:

$$\dot{f^\lambda} = -\operatorname{grad}_h J_a(f^\lambda).$$

Indeed, he equation (2.33) implies

$$\operatorname{grad}_h V_a(f^\lambda) = -\left(H - \frac{(y\cdot\nabla - 1)f^\lambda}{\sqrt{1+|\nabla f^\lambda|^2}}\right)\sqrt{1+|\nabla f^\lambda|^2}.$$

which gives the result.

3 Global Existence vs. Singularity Formation

The next two results indicate some possible scenarios for the long-time behavior of surface under mean curvature flow. Definition: S is uniformly convex if all curvatures of $S \geq \delta$ for some $\delta > 0$.

Theorem 19. *(Ecker-Huisken) If S_0 is an entire graph over a hyperplane in $\mathbb{R}^{n+1}$, satisfying certain growth conditions, then the MCF S_t, starting at S_0, exists $\forall t$ and converges to a a hyperplane.*

Theorem 20. *(Huisken) Let S_0 be a compact, uniformly convex hypersurface, then there is $T \in (0,\infty)$ s.t. $\lambda^{-1}S \to S_\infty$, for $\lambda = \sqrt{2n(T-t)}$.*

In the both cases, the rescaled flow converges to a surface S_∞, which satisfies $H = -\nu\cdot x$ in the first case and $H = \nu \cdot x$ in the second. We sketch the proof of Theorem 20

Idea of proof of the Huisken theorem. We conduct the proof in four steps.

Step #1: Rescaling. To understand the asymptotic behavior of the surface, we rescale it so that the rescaled surface converges to a limit (similarly as in traveling wave and blowup problems). Consider the new surface S^λ given by (blowup variables) (2.3) Its immersion φ satisfies the equation

$$\partial_\tau\varphi = -H(\varphi)\nu(\varphi) + a\varphi, \tag{3.1}$$

where $a = -\dot{\lambda}\lambda$.

Step #2: Lyapunov functional. On this step one proves Huisken monotonicity formula(2.7), which we recall here

$$\partial_\tau \int_{S^\lambda}\rho = -\int_{S^\lambda}\rho|H - a\nu\cdot\varphi|^2. \tag{3.2}$$

Step #3: Compactness. We prove the global existence of the rescaled flow $S^\lambda(\tau)$. By the local existence theorem there $T > 0$, s.t. the flow exists on the interval $[0,T)$. Assume that T is the maximal existence time. Let $T < \infty$. Let W be the Weingarten map of the surface and let $|W|^2 := Tr(W^2) = g^{ij}g^{kl}b_{ik}b_{jl}$, where recall b_{ik} are the matrix elements of W in the natural basis, i.e. (b_{ik}) is the second fundamental form. (Thus $|W|^2$ is the square root of the sum of squares of principal curvatures.) We want to use the following key result:

Theorem 21 (Compactness theorem: Langer, $n = 3$)**.** *Given constants $A > 0, E > 0, p > 2$, the set Ω of immersed surfaces $\psi : S \to \mathbb{R}^n$ satisfying*

- $V(\psi) = \int_S d\mathrm{vol} < A$*;*
- $E_p(\psi) := \int_S \mathrm{Tr}\, |W|^p dA < E$ *(p???);*
- $\int_S \psi d\mathrm{vol} = 0$ *(center of gravity at 0);*

is compact in the sense that $\forall\ \{\psi^n\} \subset \Omega$, $\exists$ surface diffeomorphisms ϕ^n such that a subsequence of $\{\psi^n \circ \phi^n\}$ converges in the C^1 topology to an immersion ψ in Ω. Here S is compact and without boundary.

It is shown in [13, 15] that for compact, uniformly convex hypersurfaces, S_0, there is $T \in (0, \infty)$ s.t. the Weingarten map W^λ of the rescaled surface S^λ satisfies

$$\sup_S |W^\lambda| \leq c. \tag{3.3}$$

Next, we have the following estimate which follows from (2.7): for any x_0,

$$V(S^\lambda \cap B_R(x_0)) \leq e^{aR^2/2} \int \rho_{x_0} \leq e^{aR^2/2} \int \rho_{x_0}|_{t=0}.$$

Then by Langer's theorem, the family $S^\lambda(\tau) \cap B_R(x_0),\ 0 \leq \tau < T$, is compact (up to the closure) for each $R > 0$ in the C^1 topology. Hence $S^\lambda(T) := \lim_{\tau' \to T} S^\lambda(\tau')$ exists on each ball $B_R(x_0)$, for a subsequence $\{\tau'\},\ \tau' \to T$, so we can take $S^\lambda(T)$ for the new initial condition and continue the flow beyond T. This leads to the contradiction. Hence $T = \infty$.

Since the family $S^\lambda(\tau),\ 0 \leq \tau < \infty$, is still compact by Langer's theorem, there exists a subsequence $\{\tau'\},\ \tau' \to \infty$, s.t. $S_\infty := \lim_{\tau' \to \infty} S^\lambda(\tau')$ exists on each ball $B_R(x_0)$.

Step #4: Identification of S_∞. To show that if $S^\lambda(\tau) \to S_\infty$, then S_∞ satisfies

$$H(y) = a\nu \cdot y, \tag{3.4}$$

one uses the monotonicity formula. Indeed, integrating (3.2), we obtain

$$\int_0^\infty d\tau \int_{S^\lambda} \rho |H - a\nu \cdot x|^2 = \int_{S^\lambda} \rho|_{\tau=0} - \int_{S^\lambda} \rho|_{\tau=\infty} \leq \int_{S^\lambda} \rho|_{\tau=0}.$$

This gives $\int_{S^\lambda} \rho |H^\lambda - a\nu^\lambda \cdot x^\lambda|^2 \to 0$ as $\tau \to \infty$. So we get (3.4).

By the Huisken classification theorem, stating that if $n \geq 2,\ H \geq 0$, and if S compact and satisfies (3.4), then S is the sphere of radius $\sqrt{\frac{n}{a}}$, we conclude that S is a sphere in the second case. □

In the original (non-rescaled) variables the estimate (3.3) becomes $\sup_S |W| \leq \frac{c}{T-t}$. A collapsing solution is called of type I if $|W|$ is bounded as $|W| \leq C(t^* - t)^{-\frac{1}{2}}$. Otherwise, it is called of type II (see Huisken). It was conjectured that the generic collapse is of type I. Indeed, all collapses investigated in the papers above are of type I.

MCF of submanifolds. For hypersurfaces immersed in manifolds, the theory of collapse and neckpinching are expected to go through with minimal modification. The reason for this is that they are local theories. The volume preserving flow is affected by the geometry of ambient space in a more substantial way, see Section **??**.

4 Stability of elementary solutions I. Spherical collapse

The next two results indicate some possible scenarios for the long-time behavior of surface under mean curvature flow.

Theorem 22.
- *(Ecker-Huisken) The hyperplanes are asymptotically stable.*
- *(Kong-Sigal) The spheres are asymptotically stable.*
- *(Gang - Sigal) The cylinders are unstable.*

We prove the second statement which we formulate precisely. Recall that we are considering the mean curvature flow, starting with a hypersurface S_0 in $\mathbb{R}^{n+1}$, which is given by an immersion, $x_0 : \hat{S} \to \mathbb{R}^{n+1}$), of some fixed $n-$dimensional hypersurface $\hat{S} \subset \mathbb{R}^{n+1}$ (i.e. $x(\cdot,t) : \hat{S} \to \mathbb{R}^{n+1}$). We look for solutions of the mean curvature flow,

$$\begin{cases} \frac{\partial x}{\partial t} & = & -H(x)\nu(x) \\ x|_{t=0} & = & x_0, \end{cases} \tag{4.1}$$

as immersions, $x(\cdot,t) : \hat{S} \to S$, of $\hat{S}$ giving the hypersurfaces $S(t)$. (Recall that $H(x)$ and $\nu(x)$ are mean curvature and the outward unit normal vector, at $x \in S(t)$, respectively.)

Theorem 23. *Let a surface M_0, defined by an immersion $x_0 \in H^s(\mathbb{S}^n)$, for some $s > \frac{n}{2} + 1$, be close to $\mathbb{S}^n$, in the sense that $\|x_0 - \mathbf{1}\|_{H^s} \ll 1$. Then there exist $t_* < \infty$ and $z_* \in \mathbb{R}^{n+1}$, s.t. (4.1) has the unique solution, $M_t,\ t < t_*$, and this solution contracts to the point z_*, as $t_* \to \infty$. Moreover, M_t is defined,f up to a (time-dependent) reparametrization, by an immersion $x(\cdot,t) \in H^s(\mathbb{S}^n)$, with the same s, of the form*

$$x(\omega,t) = z(t) + \lambda(t)\rho(\omega,t)\omega,$$

where ρ in turn can be written as

$$\rho(\omega,t) = \sqrt{\frac{n}{a(t)}} + \phi(\omega,t), \tag{4.2}$$

with $\lambda(t),\ z(t),\ a(t)$ and $\phi(\cdot,t)$ having the following asymptotic behaviour

$$\lambda(t) = \sqrt{2a_* s} + O(s^{\kappa_1}), \tag{4.3}$$

$$a(t) = -\lambda(t)\dot{\lambda}(t) = a_* + O(s^{\kappa_2}), \tag{4.4}$$

$$z(t) = z_* + O(s^{\kappa_3}) \tag{4.5}$$

$$\|\phi(\cdot,t)\|_{H^s} \lesssim s^{\frac{1}{2n}}, \tag{4.6}$$

where $s := t_* - t$, $\kappa_1 := \frac{1}{2} + \frac{1}{2a_*}(1 - \frac{1}{2n}),\ \kappa_2 := \frac{1}{2a_*}(1 - \frac{1}{2n})$ and $\kappa_3 := \frac{1}{2a_*}(n + \frac{1}{2} - \frac{1}{2n})$. Moreover, $|z_*| \ll 1$.

Note that our condition on the initial surface does not impose any restrictions of the mean curvature of this surface.

The form of expression (4.2) above is a reflection of a large class of symmetries of the mean curvature flow:

- (4.1) is invariant under rigid motions of the surface, i.e. $x \mapsto Rx + a$, where $R \in O(n+1)$, $a \in \mathbb{R}^{n+1}$ and $x = x(u,t)$ is a parametrization of S_t, is a symmetry of (4.1).
- (4.1) is invariant under the scaling $x \mapsto \lambda x$ and $t \mapsto \lambda^{-2} t$ for any $\lambda > 0$.

We utilize these symmetries in an essential way by defining the centres, $z(t)$, of the closed surfaces M_t and using the rescaling of the equation (4.1) by a parameter $\lambda(t)$. This leads to a transformed MCF depending explicitly on these parameters and determining their behaviour. Then we define Lyapunov-type functionals and derive a series of differential inequalities for them from which we obtain our main results. We omit some technical details for which we refer to (appendices of) [23, 1].

Notation. The relation $f \lesssim g$ for positive functions f and g signifies that there is a numerical constant C, s.t. $f \leq Cg$.

4.1 Collapse center

In this section we introduce a notion of the 'center' of a surface, close to a unit sphere, $\mathbb{S}^n$, in $\mathbb{R}^{n+1}$, and show that such a center exists. We will show in Section 4.8 that the centers $z(t)$ of the solutions M_t to (4.1) converge to the collapse point, z_*, of Theorem 23. Let $\omega = (\omega_1, \dots, \omega_{n+1}) \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$. For a closed surface M, given by an immersion $x : \mathbb{S}^n \to \mathbb{R}^{n+1}$, we define the center, z, by the relations $\int_{\mathbb{S}^n} ((x-z)\cdot\omega)\omega^j = 0,\ j = 1, \dots, n+1$. The reason for this definition will become clear in Section 4.6. We have

Proposition 24. *Assume a surface M is given by an immersion $x : \mathbb{S}^n \to \mathbb{R}^{n+1}$, with $y := \lambda^{-1}(x - \bar{z}) \in H^1(\mathbb{S}^n, \mathbb{R}^{n+1})$ close, in the $H^1(\mathbb{S}^n, \mathbb{R}^{n+1})$-norm, to the identity $\mathbf{1}$, for some $\lambda \in \mathbb{R}^+$ and $\bar{z} \in \mathbb{R}^{n+1}$. Then there exists $z \in \mathbb{R}^{n+1}$ such that $\int_{\mathbb{S}^n} ((x-z)\cdot\omega)\omega^j = 0,\ j = 1, \dots, n+1$.*

Proof. By replacing x by x^{new}, if necessary, we may assume that $\bar{z} = 0$ and $\lambda = 1$. Let $x \in H^1(\mathbb{S}^n, \mathbb{R}^{n+1})$. The relations $\int_{\mathbb{S}^n} ((x-z)\cdot\omega)\omega^j = 0\ \forall j$ are equivalent to the equation $F(x,z) = 0$, where $F(x,z) = (F_1(x,z), \dots, F_{n+1}(x,z))$, with

$$F_j(x,z) = \int_{\mathbb{S}^n} ((x-z)\cdot\omega)\omega^j,\ j = 1, \dots, n+1.$$

Clearly F is a C^1 map from $H^1(\mathbb{S}^n, \mathbb{R}^{n+1}) \times \mathbb{R}^{n+1}$ to $\mathbb{R}^{n+1}$. We notice that $F(\mathbf{1}, 0) = 0$. We solve the equation $F(x,z) = 0$ near $(\mathbf{1}, 0)$, using the implicit function theorem. To this end we calculate the derivatives $\partial_{z^i} F_j = -\int_{\mathbb{S}^n} \omega^i \omega^j = -\frac{1}{n+1}\delta_{ij}|\mathbb{S}^n|$ for $j = 1, \cdots, n+1$. The above relations allow us to apply implicit function theorem to show that for any x close to $\mathbf{1}$, there exists z, close to 0, such that $F(x,z) = 0$. □

Assume we have a family, $x(\cdot,t) : \mathbb{S}^n \to \mathbb{R}^{n+1},\ t \in [0,T]$, of immersions and functions $\bar{z}(t) \in \mathbb{R}^{n+1}$ and $\lambda(t) \in \mathbb{R}^+$, s.t. $\lambda^{-1}(t)(x(\omega,t) - \bar{z}(t))$, in the $H^1(\mathbb{S}^n, \mathbb{R}^{n+1})$-norm, to the identity $\mathbf{1}$ (i.e. a unit sphere). Then

Proposition 24 implies that there exists $z(t) \in \mathbb{R}^{n+1}$, s.t.

$$\int_{\mathbb{S}^n} ((x(\omega,t) - z(t)) \cdot \omega)\omega^j = 0, \ j = 1, \dots, n+1. \tag{4.7}$$

To apply the above result to the immersion $x(\cdot,t) : \mathbb{S}^n \to \mathbb{R}^{n+1}$, solving (4.1), we pick $\bar{z}(t)$ to be a piecewise constant function constructed iteratively, starting with $\bar{z}(t) = 0$ for $0 \le t \le \delta$ for δ sufficiently small (this works due to our assumption on the initial conditions), and $\bar{z}(t) = z(\delta)$ for $\delta \le t \le \delta + \delta'$ and so forth (see Subsection 4.8). This gives $z(t) \in \mathbb{R}^{n+1}$, s.t. (4.7) holds.

4.2 Rescaled equation

Instead of the surface M_t, it is convenient to consider the new, rescaled surface $\tilde{M}_\tau = \lambda^{-1}(t)(M_t - z(t))$, where $\lambda(t)$ and $z(t)$ are some differentiable functions to be determined later, and $\tau = \int_0^t \lambda^{-2}(s)ds$. The new surface is described by y, which is, say, an immersion of some fixed $n-$dimensional hypersurface $\Omega \subset \mathbb{R}^{n+1}$, i.e. $y(\cdot,\tau) : \Omega \to \mathbb{R}^{n+1}$, (or a local parametrization of $\tilde{M}_\tau$, i.e. $y(\cdot,\tau) : U \to M_t$). Thus the new collapse variables are given by

$$y(\omega,\tau) = \lambda^{-1}(t)(x(\omega,t) - z(t)) \text{ and } \tau = \int_0^t \lambda^{-2}(s)ds. \tag{4.8}$$

Let $\dot{\lambda} = \frac{\partial \lambda}{\partial t}$ and $\frac{\partial z}{\partial \tau}$ be the τ-derivative of $z(t(\tau))$, where $t(\tau)$ is the inverse function of $\tau(t) = \int_0^t \lambda^{-2}(s)ds$. Using that $\frac{\partial x}{\partial t} = \frac{\partial z}{\partial t} + \dot{\lambda}y + \lambda \frac{\partial y}{\partial \tau}\frac{\partial \tau}{\partial t} = \lambda^{-2}\frac{\partial z}{\partial \tau} + \dot{\lambda}y + \lambda^{-1}\frac{\partial y}{\partial \tau}$ and $H(\lambda y) = \lambda^{-1}H(y)$, we obtain from (4.1) the equation for y, λ and z:

$$\frac{\partial y}{\partial \tau} = -H(y)\nu(y) + ay - \lambda^{-1}\frac{\partial z}{\partial \tau} \text{ and } a = -\lambda\dot{\lambda}. \tag{4.9}$$

Thus M_t is given by the datum (y, λ, z), satisfying Eq. (4.9). Note that the equation (4.9) has static solutions ($a =$ a positive constant, $z = 0$, $y(\omega) = \sqrt{\frac{n}{a}}\omega, \ \omega \in \mathbb{S}^n$).

One can reformulate (and extend if necessary) standard results on the local well-posedness for the mean curvature flow (see e.g. [18]) to show that for an initial condition $y_0 \in H^s(\mathbb{S}^n)$, with $s > \frac{n}{2} + 1$, and given functions $a(\tau), \ z(\tau) \in C^1 \cap L^\infty(\mathbb{R})$, there is $T > 0$, s.t. (4.9) has a unique solution, $y \in H^s(\mathbb{S}^n)$, with $s > \frac{n}{2} + 1$, on the time interval $[0, T)$ and either $T = \infty$ or $T < \infty$ and $\|y\|_{C^\alpha} \to \infty$ and $\tau \to T$.

Our goal is to prove the following result.

Theorem 25. *Let $\rho_0 \in H^s(\mathbb{S}^n)$ satisfy $\|\rho_0 - \mathbf{1}\|_{H^s} \ll 1$ for some $s > \frac{n}{2} + 1$, and let $\lambda_0 > 0$ and $|z_0| \ll 1$. Then (4.9) with initial data $(y_0 = \rho_0(\omega)\omega, \lambda_0, z_0)$ has a unique solution (y, λ, z) for $\forall \tau$, where $y(\omega,\tau)$ is, up to a (time-dependent) reparametrization, of the form*

$$y(\omega,\tau) = \rho(\omega,\tau)\omega, \tag{4.10}$$

with $\rho(\cdot,\tau) \in H^s(\mathbb{S}^n)$, $\rho(\omega,\tau) = \sqrt{\frac{n}{a(\tau)}} + \phi(\omega,\tau)$, and $\lambda(\tau)$ and $z(\tau)$ satisfying $\lambda(\tau) = \lambda_0 e^{-\int_0^\tau a(s)ds}$ and $|z(\tau) - z_*| \lesssim e^{-(n+\frac{1}{2}-\frac{1}{2n})\tau}$, for some $a(\tau) = a_* + O(e^{-(1-\frac{1}{2n})\tau})$, with $|a_* - n| \le \frac{1}{2}$ and $|z_*| \ll 1$.

This theorem together with (4.8) implies Theorem 23 (see Section 4.8).

4.3 Differential equation for the graph function ρ

In the next proposition Δ is the Laplace-Beltrami operator in is the standard metric on $\mathbb{S}^n$, $\nabla_i \rho = \frac{\partial \rho}{\partial u^i}$ (in a local parametrization $x = x(u)$) and $(\text{Hess}\, \rho)_{ij} = \frac{\partial^2 \rho}{\partial u^i u^j} - \Gamma^k_{ij} \frac{\partial \rho}{\partial u^k}$, where $\Gamma^k_{ij} = \frac{1}{2} g^{kn} (\frac{\partial g_{in}}{\partial u^j} + \frac{\partial g_{jn}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^n})$. Here and in what follows the summation over the repeated indices is assumed. (Hess is the hessian on $\mathbb{S}^n$ and, if ∇ the Riemann connection on $\mathbb{S}^n$, which acts on vector fields as $(\nabla_i v)_j = \frac{\partial v_j}{\partial u^i} - \Gamma^k_{ij} v_k$, then $(\text{Hess})_{ij} = \nabla_i \nabla_j$.) For more details see Appendix A. In this section we prove the following

Proposition 26. *Let $\tilde{M}_\tau = \lambda^{-1}(t)(M_t - z(t))$ be defined by an immersion $y(\omega, \tau) = \rho(\omega, \tau)\omega$ of $\mathbb{S}^n$ for some functions $\rho(\cdot, \tau) : \mathbb{S}^n \to \mathbb{R}^+$, differentiable in their arguments and let $z(\tau) \in C^1(\mathbb{R}^+, \mathbb{R}^{n+1})$. Then $\tilde{M}_\tau$ satisfies* (4.9) *if and only if ρ and z satisfy the equation*

$$\frac{\partial \rho}{\partial \tau} = G(\rho) + a\rho - \lambda^{-1} z_\tau \cdot \omega + \lambda^{-1} \tilde{z}_\tau \cdot \rho^{-1} \nabla \rho, \tag{4.11}$$

where $\tilde{z}_{\tau k} = \frac{\partial x^i}{\partial u^k} z^i_\tau$, $z_\tau := \frac{\partial z}{\partial \tau}$ and **(check the sign of the last term on the r.h.s.)**

$$G(\rho) = \frac{1}{\rho^2} \Delta \rho - \frac{n}{\rho} - \frac{\nabla \rho \cdot \text{Hess}(\rho) \nabla \rho - \rho |\nabla \rho|^2}{\rho^2 (\rho^2 + |\nabla \rho|^2)}. \tag{4.12}$$

Proof. By a reparametrization (see Proposition 1), the equation (4.9) is equivalent to the equation

$$\frac{\partial y}{\partial \tau} \cdot \nu(y) = -H(y) + (ay - \lambda^{-1} \frac{\partial z}{\partial \tau}) \cdot \nu(y). \tag{4.13}$$

For the graph $y(\omega, \tau) = \rho(\omega, \tau)\omega$, we have $\frac{\partial y}{\partial \tau} = \frac{\partial \rho}{\partial \tau} \omega$. By results of Appendix A.8, we have $\nu \cdot \omega = p^{-1/2} \rho$ and $\nu \cdot y = p^{-1/2} \rho^2$, where $p := \rho^2 + |\nabla \rho|^2$. These relations, together with the equation (4.13), the expression (A.33) for the mean curvature $H(y)$ and a similar computation $\frac{\partial z}{\partial \tau} \cdot \nu(y) = z_\tau \cdot \omega - + \tilde{z}_\tau \cdot \rho^{-1} \nabla \rho$, give the equation (4.11). □

4.4 Reparametrization of solutions

Our goal is to decompose a solution, $\rho(\omega, \tau)$, of the equation (4.11) into a leading part which is a sphere of some radius τ-dependent and small fluctuation around this part. This would give a convenient reparametrization of our solution. We begin with an easy task of decomposing the initial condition for (4.11). Given ρ_0 we define a_0 by $\frac{n}{a_0} = \langle \rho_0 \rangle_{\mathbb{S}^n}$, where by $\langle f \rangle_{\mathbb{S}^n}$ we denote the average, $\langle f \rangle_{\mathbb{S}^n} := \frac{1}{|\mathbb{S}^n|} \int_{\mathbb{S}^n} f$, of f over $\mathbb{S}^n$, so that

$$\rho_0(\omega) = \sqrt{\frac{n}{a_0}} + \phi_0(\omega), \quad \text{with } \phi_0 \perp 1 \text{ in } L^2(\mathbb{S}^n). \tag{4.14}$$

(Here and in what follows, all the norms and inner products are in the sense of $L^2(\mathbb{S}^n)$).)

A similar decomposition of a solution, $\rho(\omega, \tau)$ is more subtle. We decompose $\rho(\omega, \tau)$ into the sphere, which (a) is closest to it and (b) is a stationary solution to (4.11), and a remainder. By (a) the remainder (fluctuation) is orthogonal to this sphere and by (b), the radius of the sphere is equal to $\sqrt{\frac{n}{a(\tau)}}$, where $a(\tau)$ is the parameter-function appearing in (4.11). In other words, we would like to find $a(\tau)$, s.t.

$$\rho(\cdot, \tau) = \sqrt{\frac{n}{a(\tau)}} + \phi(\omega, \tau), \quad \text{with } \phi \perp 1 \text{ in } L^2(\mathbb{S}^n). \tag{4.15}$$

A subtle point here is that the solution, $\rho(\omega,\tau)$, of the equation (4.11) depends on $a(\tau(t))$. To overcome this problem, we recall that the $a(\tau)$ entering (4.11) is related to the scaling parameter $\lambda(t)$ entering the definition $\rho(\omega,\tau) := \lambda(t)^{-1}R(\omega,t)$, with $R(\omega,t) := (x(\omega,t) - z(t))\cdot\omega$ (see Eqs (4.8) and (4.10)) and $\tau = \tau(t)$ given by (4.8), as $\lambda(t)\partial_t\lambda(t) = a(\tau(t))$ and $\lambda(t_0) = \lambda_0$. Solving the latter equation, we obtain $\lambda(t) = \lambda(a,\lambda_0)(t)$, where $\lambda(a,\lambda_0)(t)$ is the positive function given by

$$\lambda(a,\lambda_0)(t) := (\lambda_0^2 - 2\int_{t_0}^t a(\tau(s))ds)^{1/2}. \tag{4.16}$$

Since $0 = \langle\phi(\cdot,\tau),1\rangle = \langle\rho(\cdot,\tau) - \sqrt{\frac{n}{a(\tau)}}\rangle$, (4.15) is equivalent to the equation

$$\langle R(\cdot,t)\rangle_{\mathbb{S}^n} = \lambda(a,\lambda_0)(t)\sqrt{\frac{n}{a(\tau(t))}}. \tag{4.17}$$

Hence we would like to define $a(\tau)$ implicitly by the latter equation. To do this, we need some definitions. For any time t_0 and constant $\delta > 0$, we define $I_{t_0,\delta} := [t_0, t_0+\delta]$ and

$$\mathcal{A}_{t_0,\delta} := C^1(I_{t_0,\delta}, [n - \frac{1}{2}, n + \frac{1}{2}]). \tag{4.18}$$

Suppose R is such that, for some $\bar a \in \mathcal{A}_{t_0,\delta}$,

$$\sup_{t\in I_{t_0,\delta}} |\lambda(\bar a)(t)^{-1}\langle R(\cdot,t)\rangle_{\mathbb{S}^n} - \sqrt{\frac{n}{\bar a(t)}}| \le \nu. \tag{4.19}$$

We define the set

$$\mathcal{U}_{t_0,\delta,\lambda_0,\nu} := \{R \in C^1(I_{t_0,\delta}, L^2(\mathbb{S}^n)) \mid (4.19) \text{ holds for some } \bar a(t) \in \mathcal{A}_{t_0,\delta}\}. \tag{4.20}$$

Proposition 27. *Suppose $\delta \le \min(\frac{\lambda_0^2}{n-\frac12}, \frac{\sqrt{n-\frac12}}{4(n+\frac12)^{3/2}})$ and $\nu \le \frac{1}{8\lambda_0} n^{\frac14}(n-\frac12)^{5/4}$. Then there exists a unique C^1 map $g: \mathcal{U}_{t_0,\delta,\lambda_0,\nu} \to \mathcal{A}_{t_0,\delta}$, such that for any $R \in \mathcal{U}_{t_0,\delta,\lambda_0,\nu}$ and for $a(\tau(t)) = g(R)(t)$, we have (4.15), for $\rho(\omega,\tau) := \lambda(a,\lambda_0)(t)^{-1}R(\omega,t)$ and for any $t = t(\tau) \in I_{t_0,\delta}$ and for $|a(t) - \bar a(t)| < [8(n+\frac12)^{3/2}M_1]^{-1}$ for some $\bar a(t) \in \mathcal{A}_{t_0,\delta}$ and for $M_1 := (n-\frac12)^{-1/2}((n-\frac12)^{-1} + 2\delta)$.*

Proof. In this proof we write $\lambda(a)$ instead of $\lambda(a,\lambda_0)$ and $a(t)$ instead of $a(\tau(t))$. Define the C^1 function $a(t)$ implicitely by the equation (4.17), which as was mentioned above is equivalent to (4.15). We solve this equation using the inverse function theorem. Define the C^1 function $f : \mathcal{A}_{t_0,\delta} \to \mathcal{A}_{t_0,\delta}$, by $f(a) := \lambda(a)(t)\sqrt{\frac{n}{a(t)}}$. Then (4.17) can be rewritten as $f(a) = \langle R(\cdot,t)\rangle_{\mathbb{S}^n}$. We compute

$$d_a f(a)\alpha = -\sqrt{\frac{n}{a}}(\lambda(a)\frac{1}{2a}\alpha + \frac{1}{\lambda(a)}\int_{t_0}^t \alpha(s)ds). \tag{4.21}$$

By the assumption $a \in \mathcal{A}_{t_0,\delta}$ and the definition of $\mathcal{A}_{t_0,\delta}$, we have $n - \frac12 \le a(t) \le n + \frac12$. Using this, we estimate

$$\|d_a f(a)\alpha\|_\infty \ge \|\frac{\sqrt{n}}{2a^{3/2}}\alpha\|_\infty - \|\sqrt{\frac{n}{a}}\int_{t_0}^t \alpha(s)ds\|_\infty \ge (\frac{\sqrt{n}}{2(n+\frac12)^{3/2}} - \sqrt{\frac{n}{n-\frac12}}\delta)\|\alpha\|_\infty. \tag{4.22}$$

Hence $d_a f(a)$ is invertible, provided that $\delta < \frac{\sqrt{n-\frac{1}{2}}}{2(n+\frac{1}{2})^{3/2}}$. Hence the inverse function theorem implies that for any $a_\# \in \mathcal{A}_{t_0,\delta}$ there exists a neighborhood $\mathcal{U}_{a_\#}$ of $\sqrt{\frac{n}{a_\#}}$ in $C^1(I_{t_0,\delta}, L^2(\mathbb{S}^n))$ and a unique C^1 map $g : \mathcal{V}_{a_\#} := \{R \in C^1(I_{t_0,\delta}, L^2(\mathbb{S}^n)) \,|\lambda(a_\#)^{-1}R \in \mathcal{U}_{a_\#}\} \to \mathcal{A}_{t_0,\delta}$, such that $g(R)$ solves (4.17) for all $R \in \mathcal{V}_{a_\#}$.

With little more work, we can obtain a quantitate description of the neighbourhood $\mathcal{U}_{a_\#}$ (see Appendix 4.9), which completes the proof of Proposition 27. □

For the $a(\tau)$ found in the proposition, we have that $\rho(\omega,\tau) = \lambda(a,\lambda_0)(t)^{-1}R(\omega,t)$ satisfies

$$\rho(\cdot,\tau) - \sqrt{\frac{n}{a(\tau)}} \perp 1 \text{ in } L^2(\mathbb{S}^n). \tag{4.23}$$

Furthermore, if $y(\omega,\tau) := \lambda^{-1}(t)(x(\omega,t) - z(t)) = \rho(\omega,\tau)\omega$, where $\tau = \tau(t)$ is given in (4.8) and the family $z(t)$ is the one obtained in the paragraph after (4.7) and therefore (4.7) holds, then applying (4.7) to $y(\omega,\tau)$, we conclude that

$$\int_{\mathbb{S}^n} \rho(\omega,\tau)\omega^j = 0,\ j = 1,\dots,n+1. \tag{4.24}$$

The last two relations give

$$\rho(\cdot,\tau) - \sqrt{\frac{n}{a(\tau)}} \perp \omega^j,\ j = 0,\dots,n+1, \text{ in } L^2(\mathbb{S}^n), \tag{4.25}$$

where we used the notation $\omega^0 = 1$. If x satisfies (4.1) and $\rho(\omega,\tau) := \lambda(t)^{-1}R(\omega,t)$, with $R(\omega,t) := (x(\omega,t) - z(t)) \cdot \omega$ (see Eqs (4.8) and (4.10)) and $\tau = \tau(t)$ given by (4.8), then ρ satisfies (4.11), with a the same as in (4.15).

4.5 Lyapunov-Schmidt decomposition

(changed ϕ to ξ) Let ρ solve (4.11) and assume it can be written as $\rho(\omega,\tau) = \rho_{a(\tau)} + \xi(\omega,\tau)$, with $\rho_a = \sqrt{\frac{n}{a}}$ and $\xi \perp \omega^j,\ j = 0,\dots,n+1$. Plugging this into equation (4.11), we obtain the equation

$$\frac{\partial \xi}{\partial \tau} = -L_a\xi + N(\xi) + \lambda^{-1}\tilde{z}_\tau \cdot \rho^{-1}\nabla\xi + F, \tag{4.26}$$

where $L_a = -dG_a(\rho_a)$, with $G_a(\rho) := G(\rho) + a\rho$, $\rho = \rho_a + \xi$, $N(\xi) = G(\rho_a + \xi) - G(\rho_a) - dG(\rho_a)\xi$, $F = -\partial_\tau\rho_a - \lambda^{-1}z_\tau \cdot \omega$ and, recall, $z_\tau = \frac{\partial z}{\partial \tau}$ and $\tilde{z}_{\tau k} = \frac{\partial x^i}{\partial u^k} z^i_\tau$. Let $a_\tau = \frac{\partial a}{\partial \tau}$. We compute

$$\begin{aligned} L_a &= \tfrac{a}{n}(-\Delta - 2n),\ F = \tfrac{\sqrt{n}}{2}a^{-3/2}a_\tau - \lambda^{-1}z_\tau \cdot \omega, \\ N(\xi) &= -\tfrac{(\rho_a+\rho)\xi\Delta\xi}{\rho^2\rho_a^2} - \tfrac{n\xi^2}{\rho\rho_a^2} - \tfrac{\nabla\xi\cdot \mathrm{Hess}(\xi)\nabla\xi - \rho|\nabla\xi|^2}{\rho^2(\rho^2+|\nabla\xi|^2)}. \end{aligned} \tag{4.27}$$

Now, we project (4.26) onto $\mathrm{span}\{\omega^j,\ j = 0,\dots,n+1\}$. By $\xi \perp \omega^j,\ j = 0,\dots,n+1$, we have

$$\frac{\sqrt{n}}{2}a^{-3/2}a_\tau|\mathbb{S}^n| = \left\langle N(\xi) + \lambda^{-1}\tilde{z}_\tau \cdot \rho^{-1}\nabla\xi, 1\right\rangle, \tag{4.28}$$

$$-c\lambda^{-1}z^j_\tau = \left\langle N(\xi) + \lambda^{-1}\tilde{z}_\tau \cdot \rho^{-1}\nabla\xi, \omega^j\right\rangle, \tag{4.29}$$

where $j = 1,\dots,n+1$, and $c := \int_{\mathbb{S}^n}(\omega^j)^2 = \frac{1}{n+1}|\mathbb{S}^n|$. Indeed, this equation follows from

(i) $\left\langle \partial_\tau \xi, \omega^j \right\rangle = -\left\langle \xi, \partial_\tau \omega^j \right\rangle = 0$, $\left\langle L_a \xi, \omega^j \right\rangle = \left\langle \xi, L_a \omega^j \right\rangle = 0$, $j = 0, \ldots, n+1$;

(ii) $\langle F, 1\rangle = \left\langle \frac{\sqrt{n}}{2} a^{-3/2} a_\tau, 1 \right\rangle = \frac{\sqrt{n}}{2} a^{-3/2} a_\tau |\mathbb{S}^n|$; $\left\langle F, \omega^j \right\rangle = -\lambda^{-1} \left\langle z_\tau \cdot \omega, \omega^j \right\rangle = -c\lambda^{-1} z_\tau^j$, $j = 1, \ldots, n+1$.

Remark 4. *The operator* $L_a = -dG_a(\rho_a)$ *is the hessian* $\operatorname{Hess} V_a(\rho_a) = d \operatorname{grad} V_a(\rho_a)$ *of the modified volume functional* $V_a(\rho)$ *at the sphere* ρ_a.

4.6 Linearized operator

In this section we discuss the operator $L_a := -\partial G_a(\sqrt{\frac{n}{a}}) = \frac{a}{n}(-\Delta - 2n)$ acting on $L^2(\mathbb{S}^n)$, which is the linearization of the map $-G_a(\rho) = -G(\rho) - a\rho$ at $\rho_a = \sqrt{\frac{n}{a}}$. We have already encountered this operator in Section 2, as the normal hessian of modified volume functional around a sphere. Its spectrum was described in that section. Here we mention only the following relevant for us fact: The only non-positive eigenvalues of L_a are

$$L_a \omega^0 = -2a\omega^0, \qquad L_a \omega^j = -a\omega^j, \ j = 1, \ldots, n+1 \tag{4.30}$$

and the next eigenvalue is $\frac{2a}{n}$ and consequently, on their orthogonal complement we have

$$\langle \xi, L_a \xi \rangle \geq \frac{2a}{n} \|\xi\|^2 \text{ if } \xi \perp \omega^j, \ j = 0, \ldots, n+1. \tag{4.31}$$

This coercivity estimate will play an important role in our analysis. This is the reason why we need the conditions (4.25).

The $n+2$ non-positive eigenvectors of L_a correspond to the change in the size, a, of the sphere (the $j=0$ eigenvalue) and its translations (the $j = 1, \ldots, n+1$ eigenvalues). Indeed, let $\alpha = (r, z)$, where $r := \frac{n}{a}$, and ρ_α be the real function on $\mathbb{S}^n$, whose graph is the sphere S_α in $\mathbb{R}^{n+1}$ of radius R, centered at $z \in \mathbb{R}^{n+1}$, i.e. $\operatorname{graph}(\rho) := \{\rho(\omega)\omega : \omega \in S^n\} = S_\alpha$. The function ρ_α satisfies the equation $|\rho_\alpha(x)\hat{x} - z| = r$, or $\rho_\alpha(x)^2 + |z|^2 - 2\rho_\alpha(x) z \cdot \hat{x} = r^2$, and therefore it is given by

$$\rho_\alpha(\hat{x}) = z \cdot \hat{x} + \sqrt{r^2 - |z|^2 + (z \cdot \hat{x})^2},$$

where, recall, $\hat{x} = \frac{x}{|x|}$. Differentiating this relation with respect to R and z^j and expanding in z, we obtain that

$$\partial_r \rho_\alpha(x) = 1 + O(|z|), \ \partial_{z^j} \rho_\alpha(x) = \hat{x}^j + O(|z|), \tag{4.32}$$

which gives - in the leading order - the $n+2$ non-positive eigenvectors of L_a.

The fact that the $n+2$ non-positive eigenvectors of L_a are related to change of the radius and position of the euclidean sphere comes from the scaling and translational symmetries of (4.1). Indeed, by these symmetries, the equation $G_a(\rho_\alpha) = 0$ holds for any α. Differentiating it w.r.to α, we find $\partial G_a(\rho_\alpha)\partial_\alpha \rho_\alpha + \partial_\alpha G_a(\rho_\alpha) = 0$. If we introduce the operator $L_\alpha := -\partial G_a(\rho_\alpha)$, then the latter equations become $L_\alpha \partial_r \rho_\alpha = -2a\partial_r \rho_\alpha$, $L_\alpha \partial_z \rho_\alpha = 0$, i. e. $\partial_\alpha \rho_\alpha$ are eigenfunctions of the operator L_α. Since $L_\alpha = L_a + O(|z|)$, these equations imply (4.30).

The $n+2$ non-positive eigenvectors of L_a span the unstable - central subspace of the tangent space of the fixed point manifold, $\{S_{r,z} \ |r \in \mathbb{R}^+, \ z \in \mathbb{R}^{n+1}\}$ (manifold of spheres), for the rescaled MCF. We use the parameters of size, a, of the sphere (1 parameter) and its translations z ($n+1$ parameters) to make the fluctuation $\xi(\omega, \tau) := \rho(\omega, \tau) - \sqrt{\frac{n}{a}}$ orthogonal to the unstable-central modes, so that to be able to control it.

4.7 Lyapunov functional

We introduce the operator $L = -\Delta - 2n$, acting on $L^2(\mathbb{S}^n)$, which is related to the linearized operator $L_a := -\partial G_a(\sqrt{\frac{n}{a}}) = \frac{a}{n}(-\Delta - 2n)$, introduced above, as $L_a := \frac{a}{n}L$. Using the spectral information about obtained above we see that on functions ξ obeying (4.26) and $\xi \perp \omega^j,\ j = 0, \dots, n+1$, tit is bounded below as $\langle L_0\xi, \xi\rangle \geq \|\xi\|^2$, we derive in this section some differential inequalities for certain Sobolev norms of such a ξ. These inequalities allow us to prove a priori estimates for these Sobolev norms. For $k \geq 1$, we define the functional $\Lambda_k(\xi) = \frac{1}{2}\left\langle \xi, L^k\xi\right\rangle$.

Using the coercivity estimate (4.31) and standard elliptic estimates, we obtain (see Proposition 8 of [1], with $R^2 = n/a$)

Proposition 28. *There exist constants $c > 0$ and $C > 0$ such that*

$$c\|\xi\|^2_{H^k} \leq \Lambda_k(\xi) \leq C\|\xi\|^2_{H^k}.$$

Proposition 29. *Let $k > \frac{n}{2} + 1$. If ϕ satisfies (4.26), then there exists a constant $C > 0$ such that*

$$\partial_\tau\Lambda_k(\xi) \leq -\frac{2a}{n}\Lambda_k(\xi) - [\frac{a}{2n} - C(\Lambda_k(\xi)^{1/2} + \Lambda_k(\xi)^k)]\|L^{\frac{k+1}{2}}\xi\|^2. \tag{4.33}$$

Proof. We have, using (4.26),

$$\begin{aligned}\partial_\tau\frac{1}{2}\left\langle \xi, L^k\xi\right\rangle = \left\langle \partial_\tau\xi, L^k\xi\right\rangle = -\frac{a}{n}\left\langle L\xi, L^k\xi\right\rangle + \left\langle N(\xi), L^k\xi\right\rangle \\ + \left\langle \lambda^{-1}\tilde{z}_\tau \cdot \rho^{-1}\nabla\xi, L^k\xi\right\rangle + \left\langle F, L^k\xi\right\rangle,\end{aligned} \tag{4.34}$$

where, recall, we use the notation $\rho = \rho_a + \xi$. We consider each term on the right hand side. We have by the coercivity estimate (4.31)

$$\begin{aligned}\left\langle L\xi, L^k\xi\right\rangle &= \tfrac{1}{2}\|L^{\frac{k+1}{2}}\xi\|^2 + \tfrac{1}{2}\left\langle L^{\frac{k}{2}}\xi, L_0L_0^{\frac{k}{2}}\xi\right\rangle \\ &\geq \tfrac{1}{2}\|L^{\frac{k+1}{2}}\xi\|^2 + \left\langle L^{\frac{k}{2}}\xi, L^{\frac{k}{2}}\xi\right\rangle \\ &= \tfrac{1}{2}\|L^{\frac{k+1}{2}}\xi\|^2 + 2\Lambda_k(\xi).\end{aligned} \tag{4.35}$$

To estimate the next term we need the following inequality which is a special case of Lemma 12 of Appendix D of [1]

$$\|L^{\frac{k-1}{2}}N(\xi)\| \lesssim (\Lambda_k^{1/2}(\xi) + \Lambda_k^k(\xi))\|L_a^{\frac{k+1}{2}}\xi\|. \tag{4.36}$$

(The operator L_a of this paper is identified with the first (main) part of the operator $L_{R0} := -\frac{1}{R^2}(\Delta + n) + \frac{n}{R^2}\mathrm{Av}$, where $\mathrm{Av}\,\xi := \frac{1}{|\Gamma|}\int_\Gamma \xi$, of [1] , once we set $R^2 = n/a$.) This estimate implies that

$$\begin{aligned}|\left\langle N(\xi), L^k\xi\right\rangle| &= |\left\langle L_a^{\frac{k-1}{2}}N(\xi), L^{\frac{k+1}{2}}\xi\right\rangle| \\ &\leq \|L^{\frac{k-1}{2}}N(\xi)\|\|L^{\frac{k+1}{2}}\xi\| \\ &\leq C(\Lambda_k^{1/2}(\xi) + \Lambda_k^k(\xi))\|L^{\frac{k+1}{2}}\xi\|^2.\end{aligned} \tag{4.37}$$

To estimate the third term in (4.34), we use Eq. (4.29) to obtain

$$\lambda^{-1}|z_\tau| \lesssim \left| \langle N(\xi) + \lambda^{-1}\tilde{z}_\tau \cdot \rho^{-1}\nabla\xi, \omega\rangle \right| \\ \lesssim \|N(\xi)\|_{L^1} + \lambda^{-1}|z_\tau| \|\nabla\xi\|_{L^1}. \tag{4.38}$$

Next, we estimate $N(\xi)$. Using (4.27), where, recall, $\rho = \rho_a + \xi$, and assuming that $|\xi| \leq \frac{1}{2}\rho_a$, we have that

$$\|N(\xi)\|_{L^1} \lesssim (\|\nabla\xi\|^2_{L^4} + \|\xi\|_{H^1})\|\xi\|_{H^2}. \tag{4.39}$$

This together with (4.38) gives, provided that $\|\xi\|_{H^1} \ll 1$,

$$|z_\tau| \lesssim \lambda(\|\nabla\xi\|^2_{L^4} + \|\xi\|_{H^1})\|\xi\|_{H^2}. \tag{4.40}$$

From (4.40) and Proposition 28 we obtain that

$$\left\langle \lambda^{-1}\tilde{z}_\tau \cdot \rho^{-1}\nabla\xi, L^k\xi \right\rangle = \left\langle L^{\frac{k-1}{2}}(\lambda^{-1}\tilde{z}_\tau \cdot \rho^{-1}\nabla\xi), L^{\frac{k+1}{2}}\xi \right\rangle \leq C(\Lambda_k^{1/2}(\xi) + \Lambda_k^k(\xi))\|L^{\frac{k+1}{2}}\xi\|^2. \tag{4.41}$$

We have, by (4.30), (4.25) (i.e. $L\omega^j = -2n\delta_{j0}$, $\langle \omega^j, \xi\rangle = 0$) and the self-adjointness of L_a, that $\langle \omega^j, L_a^k\xi\rangle = 0,\ j = 0, \ldots, n+1$, and therefore

$$\langle F, L_a^k\xi\rangle = 0. \tag{4.42}$$

Relations (4.34)-(4.42) yield (4.33). □

4.8 Proof of Theorem 25

We begin with reparametrizing the initial condition. Applying Proposition 24, to the immersion $x_0(\omega)$ and the number $\lambda_0 = 1$, we find $z_0 \in \mathbb{R}^{n+1}$, s.t. $\int_{\mathbb{S}^n} \rho_0(\omega)\omega^j = 0,\ j = 1, \ldots, n+1$, where $\rho_0(\omega) = (x_0(\omega) - z_0)\cdot\omega$. Then we use (4.14) for $\rho_0(\omega)$ to obtain a_0 and $\xi_0(\omega)$, s.t. $\rho_0 = \rho_{a_0} + \xi_0$, with $\xi_0 \perp 1$. Here, recall, $\rho_a = \sqrt{\frac{n}{a}}$. The last two statements imply that $\xi_0 \perp \omega^j,\ j = 0, \ldots, n+1$, where, recall, $\omega^0 = 1$. If the initial condition, $x_0(\omega)$, is sufficiently close to the identity, then a_0 and $\xi_0(\omega)$ satisfy $\Lambda_k(\xi_0)^{\frac{1}{2}} + \Lambda_k(\xi_0)^k \leq \frac{1}{10C}$, $\Lambda_k(\xi_0) \ll 1$ and $|a_0 - n| \leq \frac{1}{10}$ (see (4.14)), where the constant C is the same as in Proposition 29.

Now we use the local existence result for the mean curvature flow. For $\delta > 0$ sufficiently small, the solution, $x(\omega, t)$, in the interval $[0, \delta]$, stays sufficiently close to the standard sphere $\mathbb{S}^n$. Hence we can apply Proposition 24, with $\bar{z}(t) = 0$, to this solution in order to find $z(t)$, s.t.

$$\int_{\mathbb{S}^n} ((x(\omega, t) - z(t))\cdot\omega)\omega^j = 0,\ j = 1, \ldots, n+1, \text{ and } z(0) = z_0.$$

By Proposition 26, $y(\omega, \tau) := \lambda(t)^{-1}(x(\omega, t) - z(t)) = \rho(\omega, \tau)\omega$, with $\rho(\omega, \tau) = (x(\omega, t) - z(t))\cdot\omega$ and $\lambda(t)$ satisfying (4.11). Finally we apply Proposition 27 to $R(\omega, t) := (x(\omega, t) - z(t))\cdot\omega = \lambda(t)\rho(\omega, \tau)$ to obtain $a(\tau)$ and $\xi(\omega, \tau)$ s.t. $\rho(\omega, \tau) = \rho_{a(\tau)} + \xi(\omega, \tau)$, with $\xi \perp \omega^j,\ j = 0, \ldots, n+1$. We repeat this procedure on the interval $[\delta, \delta + \delta']$ with $\bar{z}(t) := z(\delta)$ and so forth. This gives $T_1 > 0$, $z(t(\tau))$, $\rho(\omega, \tau)$, $a(\tau)$ and $\xi(\omega, \tau)$, $\tau \leq T_1$, s.t. $x(\omega, t) = z(t) + \lambda(t)\rho(\omega, \tau(t))$ and $\rho(\omega, \tau) = \rho_{a(\tau)} + \xi(\omega, \tau)$, with ρ and λ satisfying (4.11) and $\xi \perp \omega^j$, $j = 0, \cdots, n+1$.

Now, let
$$T = \sup\{\tau > 0 : \Lambda_k(\xi(\tau))^{\frac{1}{2}} + \Lambda_k(\xi(\tau))^k \le \frac{a}{4nC}\}.$$
By continuity, $T > 0$. Assume $T < \infty$. Then $\forall \tau \le T$ we have by Proposition 29 that $\partial_\tau \Lambda_k(\xi) \le -\frac{2a}{n}\Lambda_k(\xi)$. We integrate this equation to obtain $\Lambda_k(\xi) \le \Lambda_k(\xi_0)e^{-\frac{1}{2}\tau} \le \Lambda_k(\xi_0)$. This implies
$$\Lambda_k(\xi(T))^{\frac{1}{2}} + \Lambda_k(\xi(T))^k \le \Lambda_k(\xi_0)^{\frac{1}{2}} + \Lambda_k(\xi_0)^k \le \frac{a}{8nC}.$$
Therefore proceeding as above we see that there exists $\delta > 0$ such that $\Lambda_k(\xi(\tau))^{\frac{1}{2}} + \Lambda_k(\xi(\tau))^k \le \frac{a}{4nC}$, for $\tau \le T + \delta$, which contradicts the assumption that $T < \infty$ is the maximal existence time. So $T = \infty$ and $\Lambda_k(\xi) \le \Lambda_k(\xi_0)e^{-\frac{2}{n}\int_0^\tau a(s)ds}$. By Proposition 28 we know that $\|\xi\|^2_{H^k} \lesssim \Lambda_k(\xi_0)e^{-\frac{2}{n}\int_0^\tau a(s)ds}$.

Now, we use Eq. (4.28) to obtain
$$\begin{aligned}\left|a^{-3/2}a_\tau\right| &\lesssim \left|\left\langle N(\xi) + \lambda^{-1}\tilde{z}_\tau \cdot \rho^{-1}\nabla\xi, 1\right\rangle\right| \\ &\lesssim \|N(\xi)\|_{L^1} + \lambda^{-1}|z_\tau|\|\nabla\xi\|_{L^1}\end{aligned} \tag{4.43}$$
This together with (4.39) gives
$$|a^{-3/2}a_\tau| \lesssim (\|\nabla\xi\|^2_{L^4} + \|\xi\|_{H^1})\|\xi\|_{H^2}. \tag{4.44}$$
This implies $|a(\tau) - n| \le \frac{1}{4}$, and
$$\begin{aligned}|a(\tau)^{-\frac{1}{2}} - a(0)^{-\frac{1}{2}}| &\le \frac{1}{2}\int_0^\tau |a(s)^{-\frac{3}{2}}a_\tau(s)|ds \lesssim \int_0^\tau \Lambda_k(\xi)ds \\ &\le \Lambda_k(\xi_0)\int_0^\tau e^{-\frac{1}{2}s} \ll 1.\end{aligned} \tag{4.45}$$
Next, by (4.40)
$$\begin{aligned}|z(\tau) - z(0)| &\le \int_0^\tau |z_\tau(s)|ds \lesssim \int_0^\tau \lambda(s)\Lambda_k(\xi)ds \\ &\le \Lambda_k(\xi_0)\int_0^\tau e^{-(n+\frac{1}{2})s} \ll 1.\end{aligned} \tag{4.46}$$
Observe that $\lambda_0^2 - \lambda(t)^2 = 2\int_0^t a(\tau(s))ds$. Let t_* be the zero of the function $\lambda_0^2 - 2\int_0^t a(\tau(s))ds$. Since $|a(\tau) - n| \le \frac{1}{2}$, we have $t_* < \infty$ and $\lambda(t) \to 0$ as $t \to t_*$. Similarly to (4.45), we know that
$$|a(\tau_2)^{-1/2} - a(\tau_1)^{-1/2}| \lesssim \int_{\tau_1}^{\tau_2} e^{-\frac{1}{2}s}ds \to 0,$$
as $\tau_1, \tau_2 \to \infty$. Hence there exists $a_* > 0$, such that $|a(\tau) - a_*| \lesssim e^{-(1-\frac{1}{2n})\tau}$. Similar arguments show that there exists $z_* \in \mathbb{R}^{n+1}$ such that $|z(\tau) - z_*| \lesssim e^{-(n+\frac{1}{2}-\frac{1}{2n})\tau}$. Then $\lambda^2 = \lambda_0^2 - 2\int_0^t a(\tau(s))ds = 2\int_t^{t_*} a(\tau(s))ds = 2a_*(t_* - t) + o(t_* - t)$. The latter relation implies that $\tau = \int_0^t \frac{ds}{\lambda(s)^2} = \frac{1}{2a_*}\int_0^t \frac{ds}{(t_*-s)(1+o(1))}$ and therefore $e^{-(1-\frac{1}{2n})\tau} = O((t_*-t)^{\frac{1}{2a_*}(1-\frac{1}{2n})})$. So $\lambda(t) = \sqrt{2a_*(t_*-t)} + O((t_*-t)^{\frac{1}{2}+\frac{1}{2a_*}(1-\frac{1}{2n})})$, $\rho(\omega, \tau(t)) = \sqrt{\frac{n}{a(\tau(t))}} + \xi(\omega, \tau(t))$, and $\|\xi(\omega, \tau(t))\|_{H^k} \lesssim (t_* - t)^{\frac{1}{2n}}$. The latter inequality with $k = s$, together with estimates on a, z and λ obtained above and the relation $R(\omega, t) = \lambda(t)\rho(\omega, t)$, proves Theorem 25.

4.9 Appendix. Decomposition technicalities

In this appendix we complete the proof of Proposition 27 by giving a precise estimate of the neighbourhood in the inverse function theorem used there. Recall, that in this proof we write $\lambda(a)$ instead of $\lambda(a, \lambda_0)$ and $a(t)$ instead of $a(\tau(t))$. We follow the proof of the inverse function theorem. Since $R \in \mathcal{U}_{t_0,\delta,\lambda_0,\nu}$, there is $\bar{a}(t)$ s.t. (4.19) hold. We expand the function $f(a)$ in a around this $\bar{a} \in \mathcal{A}_{t_0,\delta}$:

$$f(a) = f(\bar{a}) + d_a f(\bar{a})\alpha + r(\alpha), \tag{4.47}$$

where $\alpha := a - \bar{a}$ and $r(\alpha)$ is defined by this equation. Now we rewrite the equation $f(a) = \langle R(\cdot, t)\rangle_{\mathbb{S}^n}$ as a fixed point problem $\alpha = \Phi(\alpha)$, where $\alpha := a - \bar{a}$ and $\Phi(\alpha) = d_a f(\bar{a})^{-1}[b(t) - f(\bar{a}) - r(a)]$, with the notation $b(t) := \langle R(\cdot, t)\rangle_{\mathbb{S}^n}$.

Now, we estimate $\Phi(\alpha)$. First, we estimate the remainder $r(\alpha)$. To this end, we use the definitions (4.18) and (4.16) to estimate $|a - n| \le \frac{1}{2}$ and $\lambda_1 := \sqrt{\lambda_0^2 - 2\delta(n + \frac{1}{2})} \le \lambda(a) \le \lambda_0$. Using the latter estimates, the relation

$$d_a^2 f(a)(\alpha, \alpha) = \sqrt{\frac{n}{a}}\Big(\lambda(a)\frac{3}{4a^2}\alpha^2 + \frac{\alpha}{a\lambda(a)}\int_{t_0}^t \alpha(s)ds - \lambda(a)^{-3}(\int_{t_0}^t \alpha(s)ds)^2\Big),$$

and the definition $\|\alpha\|_\infty := \sup_{s\in I_{t_0,\delta}} |\alpha(s)|$, we obtain

$$\|d_a^2 f(a)(\alpha, \alpha)\|_\infty \le M\|\alpha\|_\infty^2,$$

where $M := \sqrt{\frac{n}{n-\frac{1}{2}}}\Big(\lambda_0 \frac{3}{4(n-\frac{1}{2})^2} + \frac{1}{(n-\frac{1}{2})\lambda_1}\delta - \delta^2\lambda_1^{-3}\Big)$, which together with the standard remainder formula, shows that $r(\alpha)$ satisfies

$$\|r(\alpha)\|_\infty \le \sup_a \|d_a^2 f(a)(\alpha, \alpha)\|_\infty \le M\|\alpha\|_\infty^2. \tag{4.48}$$

Next, by the equation (4.47), we have

$$\begin{aligned} r(\alpha') - r(\alpha) = f(a') - f(a) - d_a f(\bar{a})(a' - a) &= \int_0^1 \partial_s f(sa' + (1 - s)a)ds - d_a f(a_\sharp)(a' - a) \\ &= \int_0^1 ds[d_a f(sa' + (1 - s)a) - d_a f(\bar{a})](a' - a). \end{aligned}$$

Next, by the expression (4.21) and the definition (4.18), which implies the inequality $n - \frac{1}{2} \le \bar{a}(t) \le n + \frac{1}{2}$, we have

$$\|d_a f(a)\|_\infty \le \|\frac{\sqrt{n}}{2a^{3/2}}\alpha\|_\infty - \|\sqrt{\frac{n}{a}}\int_{t_0}^t \alpha(s)ds\|_\infty \le \frac{\sqrt{n}}{2(n - \frac{1}{2})^{3/2}} + \sqrt{\frac{n}{n - \frac{1}{2}}}\delta, \tag{4.49}$$

which, together with the previous expression, gives, for $\|\alpha\|_\infty,\ \|\alpha'\|_\infty \le r$,

$$\|r(\alpha') - r(\alpha)\|_\infty \le \sup_a \|d_a f(a)\|2r\|a' - a\|_\infty \le \sqrt{n}M_1 r\|\alpha' - \alpha\|_\infty, \tag{4.50}$$

where, recall, $M_1 := (n - \frac{1}{2})^{-1/2}((n - \frac{1}{2})^{-1} + 2\delta)$.

Now, to estimate $\Phi(\alpha) = d_a f(\bar{a})^{-1}[b(t) - f(\bar{a}) - r(a)]$, we first use the expression (4.22), to obtain $\|(d_a f(a))^{-1}\|_\infty \le (\frac{\sqrt{n}}{2(n+\frac{1}{2})^{3/2}} - \sqrt{\frac{n}{n-\frac{1}{2}}}\delta)^{-1}$. Since by our assumptions, $\delta \le \frac{\sqrt{n-1}}{4(n+\frac{1}{2})^{3/2}}$, we have

$$\|d_a f(\bar{a})^{-1}\| \le \frac{4(n+\frac{1}{2})^{3/2}}{\sqrt{n}}. \tag{4.51}$$

By the definition (4.20) of $\mathcal{U}_{t_0,\delta,\lambda_0,\nu}$, we have $|b(t) - f(\bar{a})(t)| = |\lambda(\bar{a})(t)(\lambda(\bar{a})(t)^{-1}b(t) - \sqrt{\frac{n}{\bar{a}(t)}}| \le \lambda_0\nu$. Collecting the estimates above, we obtain, for $\|\alpha\|_\infty,\ \|\alpha'\|_\infty \le r$,

$$\|\Phi(\alpha)\|_\infty \le \frac{4(n+\frac{1}{2})^{3/2}}{\sqrt{n}}(\lambda_0\nu + Mr^2),$$
$$\|\Phi(\alpha') - \Phi(\alpha)\|_\infty \le 8(n+\frac{1}{2})^{3/2}M_1 r\|\alpha' - \alpha\|_\infty.$$

Now, choosing r and δ satisfying $\frac{4(n+\frac{1}{2})^{3/2}}{\sqrt{n}}(\lambda_0\nu+Mr^2) \le r$ and $8(n+\frac{1}{2})^{3/2}M_1 r < 1$, we see that that the map $\Phi(\alpha)$ is a contraction on the ball $\|\alpha\|_\infty \le r$ and therefore has a unique fixed point there. The inequalities on r and ν are satisfied for $\nu \le \frac{1}{16}\frac{n}{\lambda_0 M}(n+\frac{1}{2})^{-3/2}$ and $r < [8(n+\frac{1}{2})^{3/2}M_1]^{-1}$. Since $\delta \le \min(\frac{\lambda_0^2}{n-\frac{1}{2}}, \frac{\sqrt{n-\frac{1}{2}}}{4(n+\frac{1}{2})^{3/2}})$, we have $M \ge 3\lambda_0\frac{\sqrt{n}}{(n-\frac{1}{2})^{5/2}}$ and therefore $\nu \le \frac{1}{8\lambda_0}n^{\frac{1}{4}}(n-\frac{1}{2})^{5/4}$. $\square$

5 Stability of elementary solutions II. Neckpinching

We consider boundariless hypersurfaces, $\{\mathcal{M}_t | \ t \ge 0\}$, in $\mathbb{R}^{n+2}$, given by immersions $X(\cdot,t) : \mathbb{S}^n \times \mathbb{R} \to \mathbb{R}^{n+1}$, evolving by the mean curvature flow Then X satisfy the evolution equation:

$$\frac{\partial X}{\partial t} = -H(X)\nu(X), \tag{5.1}$$

where $\nu(X)$ and $H(X)$ are the outward unit normal vector and mean curvature at $X \in \mathcal{M}_t$, respectively.

We are interested in evolution of surfaces starting with those, close to cylinders. Our goal is to show that

(a) for initial surfaces arbitrary close to the cylinder, the MCF becomes singular in a finite time by collapsing the radius at one point along the axis - the neckpinch;

(b) while the point and time of the neckpinch depend on the initial conditions, the profile of the neckpinch is universal;

(c) describe the asymptotics of the parameters involved.

The initial surface could be arbitrary close to a cylinder, which shows that the cylindrical collapse to the axis is unstable.

There are two new elements in the analysis of the neckpinch as compared to the collapse to a point, which make the problem much more difficult:

(i) cylinders have a larger group of transformations compared to spheres: besides of shifts there are rotations of the axis (tilting),

(ii) the linearized map on the cylinder acts on functions on the cylinder - a non-compact domain - which make it much more difficult to analyse.

In the rest of this section we present the latest result on the neckpinch for a fixed cylinder axis, explain its proof and discuss possible extension to the case when the axis is allowed to move (tilt).

5.1 Existing Results

Write points in $\mathbb{R}^{n+2}$ as $(x, w) = (x, \omega^1, \ldots, \omega^{n+1})$. (We label components of points of $\mathbb{R}^{n+2}$ as $(x^0, x^1, \ldots, x^{n+1})$.) The round cylinder $\mathbb{R} \times \mathbb{S}^n$ is naturally embedded as $\mathcal{C}^{n+1} = \{(x, w) : |w|^2 = 1\} \subset \mathbb{R}^{n+2}$. We say a surface given by immersion X is a *(normal) graph over the cylinder* $\mathcal{C}^{n+1}$, if there a function $u : \mathcal{C}^{n+1} \to \mathbb{R}_+$ (called the graph function) s.t. $X = X_u$, where

$$X_u : (x, w) \mapsto (x, u(x, w)w). \tag{5.2}$$

Assume the solution, $S(t)$, of the MCF with an initial condition $\mathcal{M}_0$, given by a graph $X_0 = X_{u_0}$ over the cylinder $\mathcal{C}^{n+1}$ with a graph function $u_0(x, \omega)$, is given by a graph $X_0 = X_{u_0}$ over the cylinder $\mathcal{C}^{n+1}$ with a graph function $u(x, \omega, t)$. We say $S(t)$ undergoes a neckpinching at time t^* and at a point x_* if $\inf u(\cdot, t) > 0$ for $t < t^*$ and $\inf u(\cdot, t) \to 0$ as $t \to t^*$ and $\inf u(\cdot, t) \to 0$ at the single point x_*.

All existing results on the neckpinch, except for [8, 9], consider axi-symmetric, compact or periodic surfaces, i.e. initial graph functions $u_0(x, \omega)$, independent of ω and defined on a finite interval, say $[a, b]$, for some $-\infty < a < b < \infty$, with $u_0(x) > 0$ for $a < x < b$ and either $u_0(a) = u_0(b) = 0$ or $\partial_x u_0(a) = \partial_x u_0(b) = 0$ (the Dirichlet or Neumann boundary conditions). [8] considers a much more difficult case of axi-symmetric surfaces, with are initial conditions, $u_0(x)$, defined on the entire axis $\mathbb{R}$ and satisfying $u_0(x) > 0\ \forall x \in \mathbb{R}$ and $\liminf_{|x|\to\infty} u_0(x, \omega) > 0$, while [9] treats non-axi-symmetric surfaces along the entire axis $\mathbb{R}$, but with some symmetries fixing the cylinder axis.

For simplicity, we present the result of [9] and then discuss a general (unpublished) approach to general, non-axi-symmetric surfaces, without fixing the cylinder axis. In what follows, we use the notation $\langle x \rangle := (1 + x^2)^{\frac{1}{2}}$.

Initial conditions. We consider initial surfaces, S_0, given by immersions X_{u_0}, which are graphs over the cylinder $\mathcal{C}^{n+1}$, with the graph functions u_0, which are positive, have, modulo small perturbations, global minima at the origin, are slowly varying near the origin and are even w.r. to the origin.

More precisely, we assume that (a) $u_0(x, \omega)$ satisfies for $(k, m) = (3, 0)$, $(\frac{11}{10}, 0)$, $(1, 2)$ and $(2, 1)$ the estimates

$$\begin{gathered} \|u_0 - (\tfrac{2n+\varepsilon_0 x^2}{2\varsigma_0})^{\frac{1}{2}}\|_{k,m} \leq C\varepsilon_0^{\frac{k+m+1}{2}}, \\ u_0(x, \omega) \geq \sqrt{n}, \end{gathered} \tag{5.3}$$

weighted bounds on derivatives to the order 5.

symmetries: $u_0(x,\omega) = u_0(-x,\omega)$, either $u_0(x,\omega) = u_0(x)$, or $n = 2$ and $u_0(x,\theta) = u_0(x,\theta+\pi)$.

The last condition fixes the the point of the neckpinch and the axis of the cylinder. Here we used the norms

$$\|\phi\|_{k,m} := \left\|\langle x\rangle^{-k}\partial_x^m\phi\right\|_{L^\infty}.$$

Neckpinching profile.

Theorem 30 ([8, 9]). *Under initial conditions described above and for ε_0 sufficiently small, we have*

(i) there exists a finite time t_* such that $\inf u(\cdot,t) > 0$ for $t < t_*$ and $\lim_{t\to t_*} u(\cdot,t) > 0$, for $x \neq 0$ and $= 0$, for $x = 0$;

(ii) there exist C^1 functions $\zeta(y,\omega,t)$, $\lambda(t)$, $c(t)$ and $b(t)$ such that

$$u(x,\omega,t) = \lambda(t)[\sqrt{\frac{2n+b(t)y^2}{c(t)}} + \zeta(y,\omega,t)] \tag{5.4}$$

with $y := \lambda^{-1}(t)x$ and the remainder $\zeta(y,\omega,t)$ satisfying, for some constant c,

$$\sum_{m+n=3,n\le 2} \|\langle y\rangle^{-m}\partial_y^n\zeta(y,\omega,t)\|_\infty \le cb^2(t). \tag{5.5}$$

Dynamics of scaling parameter

Theorem 31 ([8, 9]). *(iii) the parameters $\lambda(t)$, $b(t)$ and $c(t)$ satisfy the estimates*

$$\begin{aligned} \lambda(t) &= s^{\frac{1}{2}}(1+o(1)); \\ b(t) &= -\tfrac{n}{\ln|s|}(1+O(\tfrac{1}{|\ln|s|}|^{3/4})); \\ c(t) &= 1+\tfrac{1}{\ln|s|}(1+O(\tfrac{1}{\ln|s|})). \end{aligned} \tag{5.6}$$

Here $s := t_* - t$ and $\lambda_0 = \frac{1}{\sqrt{2\varsigma_0+\frac{\varepsilon_0}{n}}}$, with ς_0, $\varepsilon_0 > 0$ depending on the initial datum and $o(1)$ is in $s := t_* - t$.

(iv) if $u_0\partial_x^2 u_0 \ge -1$ then there exists a function $u_*(x) > 0$ such that $u(x,t) \ge u_*(x)$ for $\mathbb{R}\backslash\{0\}$ and $t \le t_*$.

5.2 Rescaled surface

Let $\mathcal{M}_t$ denote a smooth family of smooth surfaces, given by immersions $X(\cdot,t) : \mathbb{R}\times\mathbb{S}^n \to \mathbb{R}^{n+1}$ and evolving by the mean curvature flow, (5.1). It is convenient to rescale it as follows

$$Y(y,w,\tau) = \lambda^{-1}(t)X(x,w,t), \tag{5.7}$$

where $y = y(x,t)$ and $\tau = \tau(t)$, with

$$y(x,t) := \lambda^{-1}(t)[x - x_0(t)] \quad \text{and} \quad \tau(t) := \int_0^t \lambda^{-2}(s)\,\mathrm{d}s, \tag{5.8}$$

respectively, where $x_0(t)$ marks the center of the neck. If the point of the neckpinch is fixed by a symmetry of the initial condition, then we set $x_0(t) = 0$. We derive the equation for Y. For most of the derivation, we suppress time dependence.

Let $\dot\lambda = \frac{\partial \lambda}{\partial t}$. Using $X = \lambda Y$ and the relations $H(X) = \lambda^{-1} H(Y)$ and $\nu(X) = \nu(Y)$ (see (2.9) - (2.11)), $\frac{\partial X}{\partial t} = \dot\lambda Y + \lambda \frac{\partial Y}{\partial t}$, $\frac{\partial Y}{\partial t} = \lambda^{-2}\frac{\partial Y}{\partial \tau} + \frac{\partial y}{\partial t}\frac{\partial Y}{\partial y}$ and $\frac{\partial y}{\partial t} = -\dot\lambda\lambda^{-1} y$, and the definition $a := -\dot\lambda \lambda^{-1}$, we obtain from (5.1) the equation for Y, λ, z and g:

$$\frac{\partial Y}{\partial \tau} = -H(Y)\nu(Y) + aY - ay\partial_y Y. \tag{5.9}$$

The equation (5.9) has the static solutions: a = a positive constant, $Y(x,w) = (x, \sqrt{\frac{n}{a}}w)$.

5.3 Linearized map

Consider the equation (5.9). This equation has the static solutions $Y(y,w) = X_\rho(y,w) = (y, \rho w)$, with $\rho = \sqrt{\frac{n}{a}}$. They describe the cylinders of the radii ρ, shifted by z and rotated by g. We would lie to investigate the linearized stability of these solutions. To this end, we linearize the r.h.s. of (5.9) around these static solutions, considering only variations in the normal direction, to obtain (see (2.25))

$$L_a := -\partial_y^2 - ay\partial_y + \frac{a}{n}(\Delta_{\mathbb{S}^n} + 2n)$$

where $\Delta_{\mathbb{S}^n}$ is the Laplace-Beltrami operator on $\mathbb{S}^n$. We have already encountered this operator in Section 2, as the normal hessian of modified volume functional around a cylinder. (Recall that the operator $-\frac{a}{n}(\Delta_{\mathbb{S}^n} + 2n)$, as the normal hessian of modified volume functional around a sphere. It played an important role in our study of spherical collapse.)

For a fixed a, properties of the operator L_a were described in Section 2: it is self-adjoint on the Hilbert space $L^2(\mathbb{R}\times\mathbb{S}^n, e^{-\frac{a}{2}y^2}dydw)$ and its spectrum consists of the eigenvalues $[k - 2 + \frac{1}{n}l(l+n-1)^2]a,\ k = 0,1,2,3,\dots;\ \ell = 0,1,2,\dots,$ of the multiplicities m_l, with the normalized eigenvectors given by

$$\phi_{k,l,m,a}(y,w) := \phi_{k,a}(y) Y_{lm}(w), \tag{5.10}$$

where, recall, $\phi_{k,a}(y)$ are the normalized eigenvectors of the linear operator $-\partial_y^2 - ay\partial_y$ corresponding to the eigenvalues $ka : k = 0,1,2,3,\dots,$ and Y_{lm} are the eigenfunctions of $-\Delta_{\mathbb{S}^n}$ corresponding to the eigenvalue $l(l+n-1)$ (the spherical harmonics). This implies that the only non-positive eigenvalues of L_a are $-2a,\ -a,\ 0$, corresponding to the eigenvectors, labeled by $(k,l) = (0,0),(0,1),(1,0),(1,1),(2,0)$, with multiplicities $1, n+2; n+2$ (so that the total multiplicity of the non-positive eigenvalues is $2n+5$). The first three eigenfunctions of the operator $-\partial_y^2 - ay\partial_y$, corresponding to the eigenvalues, $-2a,\ -a,\ 0$, are

$$\phi_{0,a}(y) = (\frac{a}{2\pi})^{\frac14},\ \phi_{1,a}(y) = (\frac{a}{2\pi})^{\frac14}\sqrt{a}y,\ \phi_{2,a}(y) = (\frac{a}{2\pi})^{\frac14}(1 - ay^2), \tag{5.11}$$

and first $n+2$ eigenfunctions of the operator $-\Delta_{\mathbb{S}^n}$, corresponding to the first eigenvalue 0, and the second eigenvalue n are 1 and $w^1, \cdots, w^{n+1}$, respectively. Hence, L_a has

- the eigenvalue $-2a$ of the multiplicity 1 with the eigenfunction $\phi_{0,0,0,a}(y) = (\frac{a}{2\pi})^{\frac{1}{4}}$ $((k,l) = (0,0))$, due to scaling of the transverse sphere;
- the eigenvalue $-a$ of the multiplicity $n+1$ with the eigenfunctions $\phi_{0,1,m,a}(y) = (\frac{a}{2\pi})^{\frac{1}{4}} w^m,\ m = 1,\dots,n+1$ $((k,l) = (1,1))$, due to transverse translations;
- the eigenvalue 0 of the multiplicity $n+1$ with the eigenfunctions $\phi_{1,1,m,a}(y) = (\frac{a}{2\pi})^{\frac{1}{4}} \sqrt{a} y w^m,\ m = 1,\dots,n+1$ $((k,l) = (0,1))$, due to rotation of the cylinder;
- the eigenvalue $-a$ of the multiplicity 1 with the eigenfunction $\phi_{1,0,0,a}(y) = (\frac{a}{2\pi})^{\frac{1}{4}} \sqrt{a} y$ $((k,l) = (1,0))$, due to translations of the point of the neckpinch on the axis of the cylinder;
- the eigenvalue 0 of the multiplicity 1 with the eigenfunction $\phi_{2,0,0,a}(y) = (\frac{a}{2\pi})^{\frac{1}{4}} (1 - ay^2)$ $((k,l) = (2,0))$, due to shape instability.

The modes $\phi_{1,a}$ and $\phi_{2,a}$ are due to the location of the neckpinch (the shift in the x direction) and its shape (see [8] and below). To see how they originate, we replace the cylindrical, static (i.e. y and τ-independent) solution $Y(y,w) = X_{V_a}(y,w) = (y, \rho_a w)$, with $V_a = \sqrt{\frac{n}{a}}$, to (5.9) with a constant, by a modulated cylinder solution

$$Y_{ab}(y,w) = X_{V_{ab}}(y,w) = (y, V_{ab}(y)w), \quad \text{with} \quad V_{ab}(y) = \sqrt{\frac{2n + by^2}{2a}}, \tag{5.12}$$

which is derived later and which incorporates some essential features of the neckpinch. They are approximate static solutions to (5.9) with a constant, $-H(Y_{ab}) + aY_{ab} \cdot \nu(Y_{ab}) \approx 0$. Differentiating it w.r.to a, x_0 (see (5.8)) and b and using that b is sufficiently small and therefore by^2 can be neglected in a bounded domain, we arrive at vectors proportional to $\phi_{0,a},\ \phi_{1,a}$ and $\phi_{2,a}$.

Recall from Section 5.8 that the modes $\phi_{k,l,m,a},\ (k,l,m) = (0,1,1),\dots,(0,1,n+1)$, are coming from the transverse shifts, and $\phi_{k,l,m,a},\ (k,l,m) = (1,1,1),\dots,(1,1,n+1)$, (i.e. $\phi_{1,a}(y)w^m \simeq yw^m,\ m = 1,\dots,n+1$), from the rotations of the cylinder. Hence fixing the cylinder axis, eliminates these modes.

5.4 Equations for the graph function

Assume that for each t in the the existence interval, the transformed immersion $Y(y,w,\tau)$, given in (5.7), is a graph over the cylinder $Y(y,w,\tau) = (y, v(y,w,\tau)w)$. where y and τ are blowup variables defined by (5.8), with $x_0 = 0$. Then, **similarly to the derivation of the equation** (4.11) and using $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial \tau} \partial_t \tau$, for any function f of the variables τ, $\partial_t \tau = \lambda^{-2}$ and $\partial_t y = -\lambda^{-2} \dot{\lambda} x x = \lambda^{-2} a y x$, where, recall, $a = -\lambda \dot{\lambda}$, we see that (5.9) implies

$$\partial_\tau v = G(v) - ay\partial_y v + av, \tag{5.13}$$

where the map $G(v)$ given by

$$\begin{aligned} G(v) := v^{-2}\Delta v + \partial_y^2 v - nv^{-1} &- \frac{v^{-4}\langle \nabla v, \mathrm{Hess}\, v\ \nabla v\rangle - v^{-2}|\nabla v|^2}{1 + (\partial_y v)^2 + v^{-2}|\nabla v|^2} \\ &- \frac{2v^{-2}\partial_y v \langle \nabla v, \nabla \partial_y v\rangle + (\partial_y v)^2 \partial_y^2 v}{1 + (\partial_y v)^2 + v^{-2}|\nabla v|^2}, \end{aligned} \tag{5.14}$$

with ∇ and Hess denoting the Levi-Civita covariant differentiation and hessian on a round $\mathbb{S}^n$ of radius one, with the components ∇_i and $\nabla_i \nabla_j$ in some local coordinates, respectively.

Initial conditions for v. If $\lambda_0 = 1$, then the initial conditions for u given in Theorem 30 implies that there exists a constant δ such that the initial condition $v_0(y)$ is even and satisfy for $(k, m) = (3, 0)$, $(\frac{11}{10}, 0)$, $(1, 2)$ and $(2, 1)$ the estimates

$$\begin{gathered} \|v_0(y,\omega) - (\tfrac{2n+\varepsilon_0 y^2}{1-\frac{1}{n}\varepsilon_0})^{\frac{1}{2}}\|_{k,m} \le C\varepsilon_0^{\frac{k+m+1}{2}}, \\ v_0(y,\omega) \ge \sqrt{n}, \end{gathered} \tag{5.15}$$

weighted bounds on derivatives to the order 5,

symmetries: $v_0(y,\omega) = v_0(-y,\omega),\ v_0(y,\theta) = v_0(y,\theta+\pi)$.

5.5 Reparametrization of the solution space

Adiabatic solutions. The equation (5.13) has the following cylindrical, static (i.e. y and τ-independent) solution

$$V_a := \sqrt{\frac{n}{a}} \text{ and } a \text{ is constant } \iff u_{cyl}(t). \tag{5.16}$$

In the original variables t and x, this family of solutions corresponds to the cylinder (homogeneous) solution $X(t) = (x, R(t)\omega)$, with $R(t) = \sqrt{R_0^2 - 2nt}$.

A larger family of approximate solutions is obtained by assuming that v is slowly varying and ignoring $\partial_\tau v$ and $\partial_y^2 v$ in the equation for v to obtain the equation

$$ayv_y - av + \frac{n}{v} = 0$$

(adiabatic or slowly varying approximation). This equation has the general solution

$$V_{ab}(y) := \sqrt{\frac{2n + by^2}{2a}}, \tag{5.17}$$

with $b \in \mathbb{R}$. Note also that for by^2 small, $V_{ab}(y) \approx \sqrt{\frac{n}{a}}(1 + \frac{by^2}{4n}) = \sqrt{\frac{n}{a}} + \frac{b}{4a\sqrt{an}}(\frac{2\pi}{a})^{\frac{1}{4}}[\phi_{0,a}(y) + \phi_{2,a}(y)]$, i.e. the static solution $\sqrt{\frac{n}{a}}$ is modified by vectors along its unstable and central subspaces.

In what follows we take $b \ge 0$ so that V_{ab} is smooth. Note that $V_{a0} = V_a$.

The rescaled equation (5.13) has two unknowns v and a. Hence we need an extra restriction specifying a or λ.

Reparametrization of solutions. We want to parametrize a solution to (5.13) by a point on the manifold M_{as}, formed by the family of almost solutions (5.17) of equation (5.13),

$$M_{as} := \{V_{ab} \,|\, a, b \in \mathbb{R}_+, b \leq \epsilon\},$$

equippped with the Riemannian metric

$$\langle \eta, \eta^{'} \rangle := \int \eta\eta^{'} e^{-\frac{ay^2}{2}} dydw, \tag{5.18}$$

and the fluctuation almost orthogonal to this manifold (large slow moving and small fast moving parts of the solution):

$$v(w, y, \tau) = V_{a(\tau)b(\tau)}(y) + \phi(w, y, \tau), \tag{5.19}$$

with

$$\phi(\cdot, \tau) \perp \phi_{ka(\tau)} \text{ in } L^2(\mathbb{R} \times \mathbb{S}^n, e^{-\frac{a(\tau)}{2}y^2} dydw),\ k = 0, 1, 2. \tag{5.20}$$

(Provided b is sufficiently small, the vectors $\phi_{0a} = 1$ and $\phi_{2a} = (1 - ay^2)$ are almost tangent vectors to the manifold, M_{as}. Note that ϕ is already orthogonal to $\phi_{1a} = \sqrt{a}y$, since by our initial conditions the solutions are even in y. For technical reasons, it is more convenient to require the fluctuation to be almost orthogonal to the manifold M_{as}.)

As in the spherical collapse case, one can show (see [8] and Appendix 5.11) that the scaling $\lambda(t)$ can be chosen in such a way that the representation (5.19) – (5.20) is satisfied.

5.6 Lyapunov-Schmidt splitting (effective equations)

In what follows we consider only surfaces of revolution. The decomposition (5.19) becomes

$$v(y, \tau) = V_{a(\tau),b(\tau)}(y) + \phi(y, \tau), \tag{5.21}$$

Substitute (5.21) into (MCF) to obtain

$$\partial_\tau \phi = -L_{ab}\phi + F_{ab} + N_{ab}(\phi) \tag{5.22}$$

where L_{ab} is the linear operator given by

$$L_{ab} := -\partial_y^2 + ay\partial_y - 2a + \frac{aby^2}{2 + \frac{by^2}{n}}$$

and the functions $F(a, b)$ and $N(a, b, \phi)$ are defined as

$$F_{ab} := \frac{1}{2}(\frac{2n + by^2}{2a})^{\frac{1}{2}}[\Gamma_1 + \Gamma_2 \frac{y^2}{2n + by^2} + -\frac{1}{n}\frac{b^3y^4}{(2n + by^2)^2}], \tag{5.23}$$

with

$$\begin{aligned} \Gamma_1 &:= \tfrac{\partial_\tau a}{2a} + 2a - 1 + \tfrac{b}{n}, \\ \Gamma_2 &:= -\partial_\tau b - b(2a - 1 + \tfrac{b}{n}) - \tfrac{b^2}{n}, \\ N_{ab}(\xi) &:= -\tfrac{n}{v}\tfrac{2a}{2n+by^2}\xi^2 - \tfrac{(\partial_y v)^2\partial_y^2 v}{1+(\partial_y v)^2}. \end{aligned} \tag{5.24}$$

Here we ordered the terms in F_{ab} according to the leading power in y^2.

Remember that

$$\phi(\cdot,\tau)\perp 1,\ a(\tau)y^2-1 \text{ in } L^2(\mathbb{R}, e^{-\frac{a(\tau)}{2}y^2}dy). \tag{5.25}$$

Project the above equation on $1,\ a(\tau)y^2-1 \implies$ the *equations for the parameters* $a,\ b$.

Estimating ϕ Let $U(\tau,\sigma)$ be the propagator generated by $-L_{ab}$. By Duhamel principle we rewrite the differential equation for $\phi(y,\tau)$ as

$$\phi(\tau)=U(\tau,0)\phi(0)+\int_0^\tau U(\tau,\sigma)(F+N)(\sigma)d\sigma. \tag{5.26}$$

The key problem in estimating $\phi(y,\tau)$ is to estimate the propagator $U(\tau,\sigma)$. We have to use (5.25).

Proposition 32. *There exist constants $c,\delta>0$ such that if $b(0)\le\delta$, then for any $g\perp 1,\ a(\tau)y^2-1$ in $L^2(\mathbb{R}, e^{-\frac{a(\tau)}{2}y^2}dy)$ and $\tau\ge\sigma\ge 0$, we have*

$$\|\langle z\rangle^{-3}U(\tau,\sigma)g\|_\infty \lesssim e^{-c(\tau-\sigma)}\|\langle z\rangle^{-3}g\|_\infty.$$

Estimating the linear propagator. I. Recall $U(\tau,\sigma)$ is the propagator generated by $-L_{ab}$. We write $L_{ab}=L_0+V,$ where

$$L_0:=-\partial_y^2+ay\partial_y-2a,\ V(y,\tau)\ge 0 \text{ and } |\partial_y V(y,\tau)|\lesssim b^{\frac12}(\tau). \tag{5.27}$$

(L_0 is the Ornstein-Uhlenbeck generator related to the harmonic oscillator Hamiltonian.)

Denote the integral kernel of $U(\tau,\sigma)$ by $K(x,y)$. We have the representation (see Appendix D)

$$K(x,y)=K_0(x,y)\langle e^V\rangle(x,y), \tag{5.28}$$

where $K_0(x,y)$ is the integral kernel of the operator $e^{-(\tau-\sigma)L_0}$ and

$$\langle e^V\rangle(x,y)=\int e^{\int_\sigma^\tau V(\omega(s)+\omega_0(s),s)ds}d\mu(\omega). \tag{5.29}$$

Here $d\mu(\omega)$ is a harmonic oscillator (Ornstein-Uhlenbeck) probability measure on the continuous paths $\omega:[\sigma,\tau]\to\mathbb{R}$ with the boundary condition $\omega(\sigma)=\omega(\tau)=0$ and

$$(-\partial_s^2+a^2)\omega_0=0 \text{ with } \omega(\sigma)=y \text{ and } \omega(\tau)=x. \tag{5.30}$$

Estimating the linear propagator. II. By a standard formula we have

$$K_0(x,y)=4\pi(1-e^{-2ar})^{-\frac12}\sqrt{a}e^{2ar}e^{-a\frac{(x-e^{-ar}y)^2}{2(1-e^{-2ar})}},$$

where $r:=\tau-\sigma$. To estimate $U(x,y)$ we use that, by the explicit formula for $K_0(x,y)$ given above,

$$|\partial_y^k K_0(x,y)|\lesssim \frac{e^{-akr}}{(1-e^{-2ar})^k}(|x|+|y|+1)^k K_0(x,y),$$

and by an elementary estimate

$$|\partial_y \langle e^V \rangle(x,y)| \leq b^{\frac{1}{2}} r. \tag{5.31}$$

Estimate for $e^{ar} \leq \beta^{-1/32}(\tau)$ $(r := \tau - \sigma)$ and then iterate using the semi-group property.

Now, (5.28) implies Equation (5.31) by the following lemma.

Lemma 33. *Assume in addition that the function $V(y,\tau)$ satisfies the estimates*

$$V \leq 0 \text{ and } |\partial_y V(y,\tau)| \lesssim \beta^{-\frac{1}{2}}(\tau) \tag{5.32}$$

where $\beta(\tau)$ is a positive function. Then

$$|\partial_y \int e^{\int_\sigma^\tau V(\omega_0(s)+\omega(s),s)ds} d\mu(\omega)| \lesssim |\tau - \sigma| \sup_{\sigma \leq s \leq \tau} \beta^{\frac{1}{2}}(\tau)$$

Proof. By Fubini's theorem

$$\partial_y \int e^{\int_\sigma^\tau V(\omega_0(s)+\omega(s),s)ds} d\mu(\omega) = \int \partial_y [\int_0^\tau V(\omega_0(s)+\omega(s),s)ds] e^{\int_\sigma^\tau V(\omega_0(s)+\omega(s),s)ds} d\mu(\omega)$$

Equation (5.32) implies

$$|\partial_y \int_\sigma^\tau V(\omega_0(s)+\omega(s),s)ds| \leq |\tau-\sigma| \sup_{\sigma \leq s \leq \tau} \beta^{\frac{1}{2}}(\tau)|, \text{ and } e^{\int_\sigma^\tau V(\omega_0(s)+\omega(s),s)ds} \leq 1.$$

Thus

$$|\partial_y \int e^{\int_\sigma^\tau V(\omega_0(s)+\omega(s),s)ds} d\mu(\omega)| \lesssim |\tau-\sigma| \sup_{\sigma \leq s \leq \tau} \beta^{\frac{1}{2}}(\tau)| \int d\mu(\omega) = |\tau-\sigma| \sup_{\sigma \leq s \leq \tau} \beta^{\frac{1}{2}}(\tau)|$$

to complete the proof. □

Derivation of Proposition 32 from (5.28) **and** (5.31)**.**

Bootstrap Let (a, b, ϕ) be the neck parametrization of the rescaled solution $v(y,\tau)$. To control the function $\phi(y,\tau)$, we use the estimating functions

$$M_{k,m}(T) := \max_{\tau \leq T} b^{-\frac{k+m+1}{2}}(\tau) \|\langle y \rangle^{-k} \partial_y^m \phi(\cdot,\tau)\|_\infty, \tag{5.33}$$

with $(k,m) = (3,0),\ (\frac{11}{10},0),\ (2,1),\ (1,2)$. Let $|M| := \sum_{i,j} M_{i,j}$ and

$$M := (M_{i,j}),\ (i,j) = (3,0),\ (\frac{11}{10},0),\ (1,2),\ (2,1). \tag{5.34}$$

Using Proposition 32 and a priori estimates obtained with help of a maximum principle (see Appendix **??**), we find

Proposition 34. *Assume that for* $\tau \in [0,T]$ *and*

$$|M(\tau)| \le b^{-\frac{1}{4}}(\tau),\ v(y,\tau) \ge \frac{1}{4}\sqrt{2n},\ \text{and}\ \partial_y^m v(\cdot,\tau) \in L^\infty,\ m=0,1,2.$$

Then there exists a nondecreasing polynomial $P(M)$ *s.t. on the same time interval,*

$$M_{k,m}(\tau) \le M_{k,m}(0) + b^{\frac{1}{2}}(0)P(M(\tau)), \tag{5.35}$$

Corollary 35. *Assume* $|M(0)| \ll 1$. *On any interval* $[0,T]$,

$$|M(\tau)| \le b^{-\frac{1}{4}}(\tau) \Longrightarrow |M(\tau)| \lesssim 1.$$

The analysis presented above goes through also in the non-radially symmetric case, with the cylinder axis fixed, but with one caveat. The feeder bounds have to be extended to the non-radially symmetric case and new bounds on θ- and mixed derivatives, $\partial_y^n\partial_\theta^m v(y,\theta,\tau)$, have to be obtained. This requires additional tools (differential inequalities for Lyapunov-type functionals.

5.7 General case. Transformed immersions

Let $\mathcal{M}_t$ denote a smooth family of smooth hypersurfaces, given by immersions $X(\cdot,t)$ and evolving by the mean curvature flow, (5.1). Instead of the surface $\mathcal{M}_t$, it is convenient to consider the new, rescaled surface $\tilde{\mathcal{M}}_\tau = \lambda^{-1}(t)g(t)^{-1}(\mathcal{M}_t - z(t))$, where $\lambda(t)$ and $z(t)$ are some differentiable functions and $g(t) \in SO(n+2)$, to be determined later, and $\tau = \tau(t) := \int_0^t \lambda^{-2}(s)ds$. If we look for X as a graph over a cylinder, then we have to rescale also the variable x along the cylinder axis, $y = \lambda^{-1}(t)(x - x_0(t))$ (see (5.8)). The new surface is described by Y, which is an immersion of the fixed cylinder $\mathcal{C}^{n+1}$, i.e. $Y(\cdot,\tau) : \mathcal{C}^{n+1} \to \mathbb{R}^{n+2}$, given by

$$Y(y,w,\tau) = \lambda^{-1}(t)g^{-1}(t)(X(x,w,t) - z(t)). \tag{5.36}$$

where $y = y(x,t)$ and $\tau = \tau(t)$ are given by $y = \lambda^{-1}(t)(x - x_0(t))$ and $\tau = \int_0^t \lambda^{-2}(s)ds$.

We derive the equation for Y. For most of the derivation, we suppress the time dependence. Let $\dot\lambda = \frac{\partial\lambda}{\partial t}$. Using that $X = z + \lambda g Y$ and the relations (2.9) - (2.11) , which imply $H(X) = \lambda^{-1}H(Y)$ and $\nu(X) = g\nu(Y)$, and using

$$\frac{\partial X}{\partial t} = \dot z + \dot\lambda g Y + \lambda \dot g Y + \lambda g \dot Y,$$

we obtain from (5.1) the equation for Y, λ, z and g:

$$\frac{\partial Y}{\partial \tau} = -H(Y)\nu(Y) + (a - y\partial_y)Y - g^{-1}\dot g Y - \lambda^{-1}g^{-1}\dot z. \tag{5.37}$$

The equation (5.37) has the static solutions ($a =$ a positive constant, $z =$constant, $g =$constant, $x_0 =$constant, $Y(x,w) = (x, \sqrt{\frac{n}{a}}w)$).

Note that we do not fix $\lambda(t)$ and $x_0(t)$, $z(t)$ and $g(t)$ but instead consider them as free parameters to be determined from the evolution of v in equation (5.37).

Remark 5. *Another way to write* (5.36) *is* $Y = (T_\lambda T_z T_g X) \circ S_\lambda$, *where* $T_\lambda,\ T_z$ *and* T_g *are the scaling, translation and rotation maps defined by* $T_\lambda X := \lambda^{-1}X,\ T_z X := X - z,\ z \in \mathbb{R}^{n+2}$, *and* $T_g X := g^{-1}X$, $g \in SO(n+2)$, *and acting pointwise and* $S_\lambda^{-1} : (x,\omega,t) \to (y = \lambda^{-1}(t)x, w, \tau = \int_0^t \lambda^{-2}(s)ds)$.

5.8 Translational and rotational zero modes

(do not probably need) Recall that the maps $X = X_{\rho z g}$, with z, g and ρ constant, describe the cylinders of the radii ρ, shifted by z and rotated by g. They are static solutions of (5.37) with a constants, i.e. solve

$$-H(X_{zg\rho}) + aX_{zg\rho} \cdot \nu(X_{\rho zg}) = 0, \tag{5.38}$$

provided $\rho = \sqrt{\frac{n}{a}}$. Differentiating this equation w.r. to a, z^i and g^{ib}, $b = n+2$, we arrive at the scaling, translational and rotational modes. We expect that these are exactly the modes mentioned in Section 2 and Subsection 5.3.

5.9 Collapse center and axis

In this section, extending [23], we introduce a notion of the 'centre' and 'axis' of a surface, close to the round cylinder $\mathcal{C}^{n+1}$, and show that such centre and axis exist. We define the cylinder axis, A, by a unit vector $\alpha \in \mathbb{S}^{n+1}$ so that $A := \mathbb{R}\alpha$. Moreover, we identify the axis vector α with an element $g \in SO(n+2)/SO(n+1)$ s.t. $ge_{n+2} = \alpha$.

Our eventual goal will be to show that the centers $z(t)$ and axis vectors $\alpha(t)$ of the solutions M_t to the MCF converge to the collapse point, z_*, and axis vector, α_*. For a boundaryless surface S, given by an immersion $X : \mathcal{C}^{n+1} \to \mathbb{R}^{n+2}$, we define the center, z, and the axis vector, α (or g s.t. $ge_{n+2} = \alpha$), by the relations

$$\int_{\mathcal{C}^{n+1}} (P^{\perp} g^{-1}(X(x,w) - z) \cdot w) x^k w^i = 0, \ i = 1, \dots, n+1, \ k = 0, 1, \tag{5.39}$$

where $P^{\perp}$ is the orthogonal projection to the subspace $\{(0, w)\} \subset \mathbb{R}^{n+2}$. Here and in what follows, the integrals over the set $\mathcal{C}^{n+1}$ (or $\mathbb{R} \times \mathbb{S}^n$), which do not specify the measure, are taken w.r.to the measure $e^{-\frac{a}{2}x^2} dwdx$ or $e^{-\frac{a}{2}y^2} dwdy$. These are $2n+2$ equations for the $2n+2$ unknowns $z \in \operatorname{Ran} P_g^{\perp}$ and $g \in SO(n+2)/SO(n+1)$, where $SO(n+1)$ is the group of rotations of the subspace $\{(w, 0)\} \subset \mathbb{R}^{n+2}$. Note that the rotated, shifted and dilated (transformed) cylinders, $\mathcal{C}_{\lambda zg}$, defined by the immersions $X_{\lambda zg} : \mathcal{C}^{n+1} \to \mathbb{R}^{n+2}$, given by

$$X_{\lambda zg}(x, w) := z + g(x, \lambda w), \tag{5.40}$$

satisfy (5.39), which justifies our interpretation of its solutions as the centre and axis vector of a surface.

Denote by $H^1(\mathcal{C}^{n+1}, \mathbb{R}^{n+2})$ the Sobolev space with the measure $e^{-\frac{a}{2}x^2} dwdx$. We introduce the following

Definition 4. *A surface $\mathcal{M}$, given by an immersion $X : \mathcal{C}^{n+1} \to \mathbb{R}^{n+2}$, is said to be H^1-close to a transformed cylinder $\mathcal{C}_{\lambda zg}$, iff $Y := \lambda^{-1} g^{-1}(X - z) \in H^1(\mathcal{C}^{n+1}, \mathbb{R}^{n+2})$ close, in the $H^1(\mathcal{C}^{n+1}, \mathbb{R}^{n+2})$-norm, to the identity $\mathbf{1} : \mathcal{C}^{n+1} \to \mathcal{C}^{n+1}$.*

Proposition 36. *Assume a surface $\mathcal{M}$, given by an immersion $X : \mathcal{C}^{n+1} \to \mathbb{R}^{n+2}$, is H^1-close to a transformed cylinder $\mathcal{C}_{\bar{\lambda}\bar{z}\bar{g}}$, for some $\bar{\lambda} \in \mathbb{R}^+$, $\bar{z} \in \mathbb{R}^{n+1}$ and $\bar{g} \in SO(n+2)$. Then there exists $g \in SO(n+2)/SO(n+1)$ and $z \in \operatorname{Ran} P_g^{\perp}$ such that (5.39) holds.*

Proof. By replacing X by X^{new}, if necessary, we may assume that $\bar{z}=0$, $\bar{g}=\mathbf{1}$ and $\lambda=1$. The relations (5.39) are equivalent to the equation $F(X,z,g)=0$, where

$$F(X,z,g)=(F_{ik}(X,z,g),\ i=1,\ldots,n+1,\ k=0,1),$$

with $F_{ik}(X,z,g)$ equal to the l.h.s. of (5.39). Clearly F is a C^1 map from $H^1(\mathcal{C}^{n+1},\mathbb{R}^{n+2})\times\mathbb{R}^{n+1}\times SO(n+2)/SO(n+1)$ to $\mathbb{R}^{2n+2}$. We notice that $F(\mathbf{1},0,\mathbf{1})=0$. We solve the equation $F(X,z,g)=0$ near $(\mathbf{1},0,\mathbf{1})$, using the implicit function theorem. To this end we have to calculate the derivatives of F w.r.to z and g at $X=\mathbf{1},\ z=0,\ g=\mathbf{1}$. The derivatives with respect to z^i are easy $\partial_{z^i}F_{jk}|_{z=0,g=\mathbf{1}}=-\int_{\mathcal{C}^{n+1}}w^iw^jx^k$, which gives

$$\partial_{z^i}F_{jk}|_{z=0,g=\mathbf{1}}=-\frac{1}{n+1}|\mathbb{S}^n|\delta_{ij}\delta_{k,0}, \tag{5.41}$$

for $i,j=1,\ldots,n+1,\ k=0,1$. To calculate the derivatives of F w.r.to g, we write $g\in SO(n+2)/SO(n+1)$ as a product of rotations g^{ab} in two-dimensional planes, i.e. involving only variables x^a and x^b, with $a=0$ and $b=1,\ldots,n+1$, and denote by $\partial_{g^{ab}}$ the derivative w.r.to the angle of the corresponding rotation.Using that $\partial_{g^{ab}}g^{-1}|_{z=0,g=\mathbf{1}}=-\ell_{ab}$, the generator of the rotation in the $ab-$plane and using that $\mathbf{1}(x,w)=(x,w)$, we calculate

$$\partial_{g^{ab}}F_{jk}|_{X=\mathbf{1},z=0,g=\mathbf{1}}=-\int_{\mathcal{C}^{n+1}}(\ell_{ab}(w,x)\cdot(0,w))\omega^jx^k. \tag{5.42}$$

We see that, if $a,b\in\{1,\ldots,n+1\}$, then $\ell_{ab}(x,w)\cdot(0,w)=-w^bw^a+w^aw^b+\sum_{i\neq a,b}(w^i)^2$ and therefore $\int_{\mathbb{S}^n}(\ell_{ab}(x,w)\cdot(0,w))\omega^j=0$. Furthermore, if $a=0$ and $b=1,\ldots,n+1$, then $\ell_{ab}(x,w)\cdot(0,w)=xw^b+\sum_{i\neq a,b}(w^i)^2$ and therefore $\int_{\mathbb{S}^n}(\ell_{ab}(x,w)\cdot(0,w))\omega^j=\delta_{b,j}x$. This gives, for $a=0$,

$$\partial_{g^{ai}}F_{jk}|_{X=\mathbf{1},z=0,g=\mathbf{1}}=c_n\int_{\mathbb{R}}x^2e^{-\frac{a}{2}x^2}dx\delta_{ij}\delta_{k,1}, \tag{5.43}$$

for $i,j=1,\ldots,n+1,\ k=0,1$. Hence dF is invertible and we can apply implicit function theorem to show that for any X close to $\mathbf{1}$, there exists z and g, close to 0 and $\mathbf{1}$, respectively, such that $F(X,z,g)=0$. $\square$

Assume that the transformed surfaces, $\tilde{\mathcal{M}}_\tau$, i.e. the immersions (5.36), are graphs over the round cylinder $\mathcal{C}^{n+1}$ determined by the functions $v:\mathcal{C}^{n+1}\times[0,T)\to\mathbb{R}_+,\ z(t)\in\mathbb{R}^{n+2},\ g(t)\in SO(n+2)$, as $Y(y,\omega,\tau)=(y,v(y,w,\tau)w)$, or

$$X(x,w,t)=z(t)+\lambda(t)g(t)(y,v(y,w,\tau)w). \tag{5.44}$$

Assume also there are functions $\bar{z}(t)\in\mathbb{R}^{n+2}$, $\bar{g}(t)\in SO(n+2)/SO(n+1)$ and $\bar{\lambda}(t)\in\mathbb{R}^+$, s.t. $X(\cdot,t)$ is H^1-close to a transformed cylinder $\mathcal{C}_{\bar{\lambda}\bar{z}\bar{g}}$ in the sense of the definition (4). Then Proposition 36 implies that there exists $z(t)\in\mathbb{R}^{n+2}$ and $g(t)\in SO(n+2)/SO(n+1)$, s.t. (5.39) holds. If X is a $(\lambda,z,g)-$graph over $\mathcal{C}^{n+1}$, i.e. it is of the form (5.44), then v satisfy

$$\int_{\mathcal{C}^{n+1}}v(y,w,\tau)y^kw^j=0,\ j=1,\ldots,n+2,\ k=0,1. \tag{5.45}$$

To apply Proposition 36 to the immersion $X(\cdot,t):\mathcal{C}^{n+1}\to\mathbb{R}^{n+2}$, solving (5.1), we pick $\bar{z}(t)$ to be a piecewise constant function constructed iteratively, starting with $\bar{z}(t)=0$ and $\bar{\alpha}(t)=e_1$ for $0\le t\le\delta$ for δ sufficiently small (this works due to our assumption on the initial conditions), and $\bar{z}(t)=z(\delta)$ and $\bar{\alpha}(t)=\bar{\alpha}(\delta)$ for $\delta\le t\le\delta+\delta'$ and so forth. This gives $z(t)\in\mathbb{R}^{n+1}$ and $g(t)\in O(n+2)$, s.t. (5.45) holds.

5.10 Equations for the graph function

Assume that for each t in the the existence interval, the transformed immersion $Y(y, w, \tau)$, given in (5.36), is a graph over the cylinder

$$Y(y, w, \tau) := \lambda^{-1}(t)g(t)^{-1}(X(x, w, t) - z(t)) = (y, v(y, w, \tau)w), \tag{5.46}$$

where y and τ are blowup variables defined by (5.8). Then, similarly to the derivation of the equation (4.11) - (4.12) or (5.13), and using $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial \tau}\partial_t\tau$, for any function f of the variables τ, $\partial_t\tau = \lambda^{-2}$ and

$$\partial_t y = -\lambda^{-2}\dot{\lambda}(x - x_0) - \lambda^{-1}\dot{x}_0 = \lambda^{-2}(ay - \lambda^{-1}\frac{\partial x_0}{\partial \tau}),$$

where, recall, $a = -\lambda\dot{\lambda}$, we see that (5.37) implies (cf. (5.13))

$$\partial_\tau v = G(v) - a(y\partial_y - 1)v - \lambda^{-1}\frac{\partial x_0}{\partial \tau}\partial_y v - g^{-1}\frac{\partial g}{\partial \tau}v - \lambda^{-1}g^{-1}\frac{\partial z}{\partial \tau} \tag{5.47}$$

where $\frac{\partial z}{\partial \tau}$ be the τ-derivative of $z(t(\tau))$, etc, with $t(\tau)$ the inverse function of $\tau(t) = \int_0^t \lambda^{-2}(s)ds$, and the map $G(v)$ given by (4.12). For v have the orthogonal decomposition (5.19) - (5.20). This together with (5.45) implies that $\phi(\cdot, \tau)$ satisfy (again in $L^2(\mathbb{R} \times \mathbb{S}^n, e^{-\frac{a(\tau)}{2}y^2}dydw)$)

$$\phi(\cdot, \tau) \perp 1, y, y^2,\ y^k w^j,\ j = 1, \ldots, n + 1,\ k = 0, 1. \tag{5.48}$$

Remark 6. *Other approaches: 1) Write a point, p, on the hypersurface $\mathcal{M}_t$ as say $p = q + x\omega + u(x, \nu)\nu$, where q gives the shift of the cylinder axis, ω is the unit vector in the direction of the cylinder axis, $\omega \cdot q = 0$, ν is the outward unit normal to the cylinder surface. To find the equation for u, we compute $\dot{p} \cdot \nu_{\mathcal{M}_t} = \sum_{i=1}^4 \gamma_i\varphi_i + \gamma_0\dot{u}$, for some γ_i and for $\varphi_1 = \cos\theta$, $\varphi_1 = \cos\theta, \varphi_2 = \sin\theta$, $\varphi_3 = x\cos\theta$, $\varphi_4 = x\sin\theta$, while $v(y, \theta, \tau)$ is now decomposed as $v(y, \theta, \tau) = V_{a(\tau),b(\tau)} + \phi(y, \theta, \tau)$, with $\phi \perp \{1, y, y^2\}$. One can also consider a moving frame (e_1, e_2, e_3), with $e_3 = \omega$, so that $\nu = e_1\cos\theta + e_2\sin\theta$.*

2) Write $v(y, \theta, \tau) = \tilde{V}_{a(\tau),b(\tau)} + \phi(y, \theta, \tau)$, where $\tilde{V}_{a(\tau),b(\tau)} = V_{a(\tau),b(\tau)} + \beta_0(\tau)y + \beta_1(\tau)\cos\theta + \beta_2(\tau)\sin\theta + \beta_3(\tau)y\cos\theta + \beta_4(\tau)y\sin\theta$ and $\phi \perp \{1, y, y^2, \cos\theta, \sin\theta, y\cos\theta, y\sin\theta\}$. In other words, one "dresses" up the main, finite dimensional term absorbing into it all the neutral-unstable modes (singular, secular behaviour).

5.11 Appendix. Decomposition technicalities

To formulate the exact statement we need some definitions. First, we note that the representation (5.19) – (5.20) is equivalent to the conditions

$$\int (v(w, y, \tau) - V_{ab}(y))\phi_{ka}(y)e^{-\frac{a(\tau)}{2}y^2}dydw = 0,\ k = 0, 1, 2, \tag{5.49}$$

with $\phi_{ka}(y)$ appear in (5.10) and $V_{ab}(y)$, the approximate (adiabatic) solution ($n = 1$) of (5.13) defined in (5.17).

Next, for any time t_0, we denote $I_{t_0,\delta} := [t_0, t_0 + \delta]$. We say that $\lambda(t)$ is *admissible* on $I_{t_0,\delta}$ if $\lambda \in C^2(I_{t_0,\delta}, \mathbb{R}^+)$ and $-\lambda\partial_t\lambda \in [1/4, 1]$.

Definition 5. *Let $t_* > 0$, $\epsilon_0 > 0$ and $\delta > 0$ and fix $t_0 \in [0, t_*)$ and $\lambda_0 > 0$. We say that a function $u \in C^1([0, t_*), \langle x\rangle^3 L^\infty)$ admits an orthogonal decomposition iff there are*

- *$\lambda(t)$, admissible on $I_{t_0,\delta}$, $a(\tau(t)) \in C^1(I_{t_0,\delta}, [\frac{1}{4}, 1])$ and $b(\tau(t)) \in C^1(I_{t_0,\delta}, (0, \epsilon_0])$,*

s.t. for $t \in I_{t_0,\delta}$, the functions

- *$v(w, y, \tau) := \lambda^{-1}(t)u(w, \lambda(t)y, t)$, with τ and t related by (5.8), and $a(\tau)$*

satisfy the conditions (5.49) and

$$-\lambda(t)\partial_t\lambda(t) = a(\tau(t)), \qquad \lambda(t_0) = \lambda_0. \tag{5.50}$$

Proposition 5.3 and Lemma 5.4 of [8] imply

Proposition 37. *Let $t_* > 0$ and fix $t_0 \in [0, t_*)$ and $\lambda_0 > 0$. There are $\epsilon_0 > 0$ and $\delta = \delta(\lambda_0, u) > 0$ and $\lambda(t)$, admissible on $I_{t_0,\delta}$, s.t. if*

(i) $u \in C^1([0, t_*), \langle x\rangle^3 L^\infty)$,

(ii) $\inf_{x\in\mathbb{R}} u(w, x, t) > 0$,

(iii) $\|v(\cdot, \tau_0) - V_{a_0 b_0}(\cdot)\|_{3,0} \ll b_0$ *for some* $a_0 \in [1/4, 1]$, $b_0 \in (0, \epsilon_0]$,

where $v(w, y, \tau) := \lambda^{-1}(t)u(w, \lambda(t)y, t)$, with τ and t related by (5.8), then the function u admits an orthogonal decomposition

6 Sections to do

- Ricci flow
- Volume-preserving mean curvature flow (after Antonopoulou, Karali, Sigal)
- Submanifolds
- Dissipation: evolution of graphs over a plane (after Ecker and Huisken)
- Local existence (after Huisken and Polden)

A Elements of Theory of Surfaces

In this appendix we sketch some results from the theory of surfaces in $\mathbb{R}^{n+1}$. For detailed expositions see [24, 2].

A.1 Mean curvature

By a *hypersurface* S in $\mathbb{R}^{n+1}$, we often mean an n dimensional surface element given by a map $\psi : U \to S$, where $U \subset \mathbb{R}^n$ is open, s.t. $d\psi(u)$ is one-to-one (maximal rank) for every $u \in U$. Such maps are called (local) *immersions*. Here $d\psi(u)$ is the linear map from $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$, defined as $d\psi(u)\xi = \partial_s|_{s=0}\psi(\sigma_s)$, where σ_s is a curve in U so that $\sigma_{s=0} = u$ and $\frac{\partial \sigma_s}{\partial s} = \xi$.

The tangent space, T_xS, to S at x is defined as the vector space of velocities at x of all curves in S passing through x. We have $T_xS = d\psi(u)(\mathbb{R}^n)$ for $x = \psi(u)$. The inner product on T_xS is defined in the usual way: $g(v, w) = \langle v, w\rangle$, $\forall v, w \in T_xS$. Here $\langle \cdot, \cdot\rangle$ is the inner product in $\mathbb{R}^{n+1}$.

Let $\nu(x)$ be the unit outward normal vector to S at x. The map $\nu : S \to S^n$ given by $\nu : x \to \nu(x)$ is called the Gauss map and the map $W_x := d\nu(x) : T_xS \to T_{\nu(x)}S^n$ is called the Weingarten map or the shape operator. Here $d\nu(x)\eta = \partial_s|_{s=0}\nu(\gamma_s)$, where γ_s is a curve on S so that $\gamma_{s=0} = x$ and $\frac{\partial \gamma_s}{\partial s} = \eta$. We have the following theorem.

Theorem 38. $\mathrm{Ran}(W_x) \subset T_xS$ *and* W_x *is self-adjoint, i.e.* $g(W_x\xi, \eta) = g(\xi, W_x\eta)$.

Proof. Let γ_s be as above. Then

$$\langle W_x\xi, \nu(x)\rangle = \langle \partial_s|_{s=0}\nu(\gamma_s), \nu(x)\rangle = \frac{1}{2}\partial_s|_{s=0} \langle \nu(\gamma_s), \nu(\gamma_s)\rangle = 0.$$

This implies that $W_x\xi \in T_xS$.

Let $\varphi = \varphi_{st}$ be a parametrization of a two-dimensional surface in S such that $\varphi_{st}|_{s=t=0} = x$, $\partial_s|_{s=t=0}\varphi_{st} = \xi$ and $\partial_t|_{s=t=0}\varphi_{st} = \eta$. Then

$$\langle \xi, W_x\eta\rangle = \langle \partial_s|_{s=t=0}\varphi, \partial_t|_{s=t=0}\nu(\varphi)\rangle = \partial_t|_{s=t=0} \langle \partial_s\varphi, \nu(\varphi)\rangle - \langle \partial_t\partial_s|_{s=t=0}\varphi, \nu(x)\rangle .$$

Since $\langle \partial_s\varphi, \nu(\varphi)\rangle = 0$, this implies $\langle \xi, W_x\eta\rangle = -\left\langle \frac{\partial^2\varphi}{\partial s\partial t}(0, 0), \nu(x)\right\rangle$. Similarly we have

$$\langle W_x\xi, \eta\rangle = -\left\langle \frac{\partial^2\varphi}{\partial s\partial t}(0, 0), \nu(x)\right\rangle,$$

which implies that $\langle \xi, W_x\eta\rangle = \langle W_x\xi, \eta\rangle$. □

The principal curvatures are all the eigenvalues of the Weingarten map W_x. The mean curvature $H(x)$ is the trace of W_x, which is the sum of all principal curvatures,

$$H(x) = \mathrm{Tr}\, W_x. \tag{A.1}$$

The Gaussian curvature is the determinant of W_x, which is the product of the principal curvatures.

If S is a level set of some function $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}$, i.e. $S = \varphi^{-1}(0) = \{x \in \mathbb{R}^{n+1} : \varphi(x) = 0\}$, then $\nu(x) = \frac{\nabla\varphi(x)}{|\nabla\varphi(x)|}$. Note that $\nu(x)$ is defined on the entire $\mathbb{R}^{n+1}$ and therefore we can introduce $A_x^T = (\partial_{x_i}\nu^j(x))$. The associated linear map has the following properties: a) $A_x^T\nu = 0$ and b) $W_x = A_x|_{T_xS}$. Indeed, since $0 = \partial_{x^i}\nu^j\nu^j = 2\nu^j\partial_{x^i}\nu^j$, we have $A_x^T\nu = 0$. Furthermore, let γ_s be a curve such through a point x, i.e. $\gamma_s(0) = x$, and with $\partial_s\gamma_s|_{s=0} = \xi$. We have $d\nu(x)\xi = \partial_s|_{s=0}\nu(\gamma_s) = \partial_{x^i}\nu(x)\frac{\partial\gamma_s^i}{\partial s}|_{s=0} = \partial_{x^i}\nu\xi^i = A_x\xi$. This implies that $W_x = A_x|_{T_xS}$. The properties a) and b) yield that $\text{Tr}(W_x) = \text{Tr}(A_x) = \partial_{x^i}\nu^i(x) = \text{div}\,\nu(x)$, which gives

$$H(x) = \text{div}\,\nu(x) = \text{div}(\frac{\nabla\varphi(x)}{|\nabla\varphi(x)|}). \tag{A.2}$$

If S is a graph of some function $f : U \to \mathbb{R}$, then it is a zero level set of the function $\varphi(x) = x_{n+1} - f(u),\ x = (u, x_{n+1})$, and therefore by the formula above $\nu(x)|_{x=\psi(u)} = \frac{(-\nabla f(u),1)}{\sqrt{1+|\nabla f|^2}}$ and

$$H(x) = -\,\text{div}(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}). \tag{A.3}$$

A.2 First and second fundamental forms

We denote by TS the collection of all tangent planes, the tangent bundle. A vector field V is a map $V : S \to TS$ assigning to every $x \in S$ a tangent vector $V_x \in T_xS$. Given an immersion $\psi : U \to S$, $\{\frac{\partial\psi}{\partial u^i}(u)\}$ is a basis in T_xS, where $x = \psi(u)$. Varying $u \in U$, we obtain the basis, $\{\frac{\partial\psi}{\partial u^i}\}$, for vectors fields on S (if the immersion ψ is fixed, one writes this basis as $\{\frac{\partial}{\partial u^i}\}$). This allows to write vector fields in the coordinate form as $v = v^i\frac{\partial\psi}{\partial u^i}$. Plug the coordinate form of v and w into $g(v, w)$ to obtain $g(v, w) = g_{ij}v^iw^j$, where

$$g_{ij} := \left\langle \frac{\partial\psi}{\partial u^i}, \frac{\partial\psi}{\partial u^j} \right\rangle.$$

The matrix (tensor) $\{g_{ij}\}$ is called the metric on S or the first fundamental form of S. If S is a graph of some function $f : U \to \mathbb{R}$, i.e. S can be parameterized as $\psi(u) = (u, f(u))$. Then $g_{ij}(u) = \delta_{ij} + \frac{\partial f}{\partial u^i}\frac{\partial f}{\partial u^j}$.

Notation. $\{g^{ij}\}$ denotes the inverse matrix. The summation is understood over repeated indices. The indices are raised and lowered by applying g^{ik} or g_{ik} as in $b^i_j = g^{ik}b_{kj}$. Unless we are dealing with the Euclidian metric, one of the indices are upper and the other lower. (For the Euclidian metric, δ_{ij}, the position of the indices is immaterial.) The quadratic form $\langle W_xv, w\rangle$, where $v, w \in T_xS$, is called the second fundamental form of S. We have

Lemma 39. *Recall, ψ is a parametrization of S. Define the matrix elements of W_x in the basis $\frac{\partial\psi}{\partial u^i}$,*

$$b_{ij} := \langle\frac{\partial\psi}{\partial u^i}, W_x, \frac{\partial\psi}{\partial u^j}\rangle. \tag{A.4}$$

Then

$$b_{ij} = -\langle\frac{\partial^2\psi}{\partial u_i\partial u_j}, \nu\rangle \quad \textit{and} \quad H = g^{ij}b_{ji}. \tag{A.5}$$

Proof. In the proof of Theorem 40, we have shown that $\langle\xi, W_x\eta\rangle = -\left\langle \frac{\partial^2\varphi}{\partial s\partial t}_{s=t=0}, \nu(x)\right\rangle$, where $\varphi = \varphi_{st}$ is a parametrization of a two-dimensional surface in S such that $\varphi_{st}|_{s=t=0} = x$, $\partial_s|_{s=t=0}\varphi_{st} = \xi$ and $\partial_t|_{s=t=0}\varphi_{st} =$

η. Now take $\varphi_{st} = \psi(u+se_i+te_j)$, where e_i are the co-ordinate unite vectors in U. Then $\frac{\partial^2 \varphi}{\partial s \partial t}_{s=t=0} = \frac{\partial^2 \psi}{\partial u_i \partial u_j}$, which proves the first equality. To show the second equality, we write $H = \mathrm{Tr}(W_x) = \sum \langle e_i, W_x e_i \rangle$, where $\{e_i\}$ is an orthonormal basis of T_xS. Let $e_i = \mu_{ij}\psi_j$, where $\psi_j = \frac{\partial \psi}{\partial u^j}$. Let $U = (\mu_{ij})$. Then

$$\langle e_i, W_x e_i \rangle = \mu_{ik}\mu_{il} \langle \psi_k, W_x \psi_l \rangle = \mu_{ik}\mu_{il} b_{kl} = \mathrm{Tr}(UBU^T) = \mathrm{Tr}(U^T UB),$$

where $B = (b_{ji})$. Since $\delta_{ij} = \langle e_i, e_j \rangle = \mu_{ik}\mu_{jl} g_{kl} = (UGU^T)_{ij}$, where $G = (g_{ij})$. Hence $UGU^T = I$ and therefore $G^{-1} = U^T U$. So $H = \mathrm{Tr}(G^{-1}B)$ and therefore the second equation in (A.5) follows. □

By (A.4) and the general formula $v = g^{ij} \langle \frac{\partial \psi}{\partial u^i}, v \rangle \frac{\partial \psi}{\partial u^j}$, we have the following useful expression

$$W_x \frac{\partial \psi}{\partial u^i} = g^{ij} b_{jk} \frac{\partial \psi}{\partial u^k}. \tag{A.6}$$

Examples. 1) The $n-$sphere S^n_R of radius R in $\mathbb{R}^{n+1}$. We can define S^n_R by the immersion $\psi(u) = R\hat{\psi}(u)$, where $\hat{\psi}(u)$ is the immersion for the standard $n-$sphere $\mathbb{S}^n = S^n_1$. Then $\nu(\psi(u)) = \hat{\psi}(u)$ and Lemma 41 gives $b_{ij} = -R\langle \frac{\partial^2 \hat{\psi}}{\partial u_i \partial u_j}, \hat{\psi} \rangle = R\langle \frac{\partial \hat{\psi}}{\partial u_i}, \frac{\partial \hat{\psi}}{\partial u_j} \rangle = R^{-1} g_{ij} = R g^{\mathrm{stand}}_{ij}$, where g_{ij} and g^{stand}_{ij} are the metrics on S^n_R and $\mathbb{S}^n = S^n_1$, respectively. This in particular implies $H = g^{ij} b_{ji} = nR^{-1}$.

2) The $n-$cylinder $C^n_R = S^{n-1}_R \times \mathbb{R}$ of radius R in $\mathbb{R}^{n+1}$. We can define C^n_R by the immersion $\phi(u,x) = (R\hat{\psi}(u), x)$, where $\hat{\psi}(u)$ is the immersion for the standard $(n-1)-$sphere $\mathbb{S}^{n-1} = S^{n-1}_1$. Then $\nu(\phi(u,x)) = (\hat{\psi}(u), 0)$ and Lemma 41 gives $b_{ij} = -R\langle \frac{\partial^2 \hat{\psi}}{\partial u_i \partial u_j}, \hat{\psi} \rangle = R\langle \frac{\partial \hat{\psi}}{\partial u_i}, \frac{\partial \hat{\psi}}{\partial u_j} \rangle = R^{-1} g_{ij} = R g^{\mathrm{stand}}_{ij}$, where g_{ij} and g^{stand}_{ij} are the metrics on S^{n-1}_R and $\mathbb{S}^{n-1} = S^{n-1}_1$, respectively, for $i,j = 1,\dots,n-1$, and $b_{nn} = 0$. This in particular implies $H = g^{ij} b_{ji} = (n-1)R^{-1}$.

A.3 Integration.

The first fundamental form allows us to define the integration on S. Using a partition of unity one can reduce the integral $\int_S h$ of a function h over a surface S to the integral $\int_\psi h$ over an immersion patch, $\psi : U \to S$, defined as $\int_\psi h := \int_U h \circ \psi \sqrt{g} d^n u$, where $g := \det(g_{ij})$. ($\sqrt{g} d^n u$ is the infinitesimal volume spanned by the basis vectors $\frac{\partial \psi}{\partial u^j}$.)

If S is locally a graph, $S = \mathrm{graph}\, f$, of some function $f : U \to \mathbb{R}$, so that one can take the immersion $\psi(u) = (u, f(u))$, then $\sqrt{g} d^n u = \sqrt{1 + |\nabla f|^2} d^n u$.

A.4 Connections

The connection, ∇, is probably a single, most important notion of Differential Geometry. It generalizes the map

$$V \to \nabla^{\mathbb{R}^{n+1}}_V$$

from $\mathbb{R}^{n+1}$ to manifolds (in our case surfaces). Here V is a vector field on $\mathbb{R}^{n+1}$, i.e. $V : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, and $\nabla^{\mathbb{R}^{n+1}}_V$ is the directional derivative which is defined as $\nabla^{\mathbb{R}^{n+1}}_V : T(x) \mapsto \partial_s|_{s=0} T(\gamma_s)$, where $\gamma_{s=0} = x$,

$\frac{\partial\gamma_s}{\partial s}|_{s=0} = V$ and T is either a function or a vector field. (We will not use tensors.) Note that $\nabla_V^{\mathbb{R}^{n+1}} T = \sum_i V^i \frac{\partial T}{\partial u^i}$. However, if W is a vector field on S, in general $\nabla_V^{\mathbb{R}^{n+1}} W(x) \notin T_x S$, i.e. " does not belong to S ". This suggests the following definition.

If f is a function on S, we define $\nabla_V f(x) = \nabla_V^{\mathbb{R}^{n+1}} f(x)$. If W is a vector field on S, we define $\nabla_V W(x)$ as the projection of $\nabla_V^{\mathbb{R}^{n+1}} W(x)$ into $T_x S$,

$$\nabla_V W(x) := (\nabla_V^{\mathbb{R}^{n+1}} W)^T(x). \tag{A.7}$$

∇_V is called the covariant derivative on $S \subset \mathbb{R}^{n+1}$. So in the latter case the connection $\nabla : V \mapsto \nabla_V$ is a linear map from space $\mathcal{V}(S)$, of vector fields on S, to the space of first order differential operators on $\mathcal{V}(S)$.

The covariant derivative, ∇_V, has the following properties:

- ∇_V is linear, i.e. $\nabla_V(\alpha W + \beta Z) = \alpha \nabla_V W + \beta \nabla_V Z$;
- $\nabla_{fV+gW} = f\nabla_V + g\nabla_W$;
- $\nabla_V(fW) = f\nabla_V W + W\nabla_V f$ (Leibnitz rule).

Homework: show these properties hold.

∇ is called a Levi-Civita or Riemann connection if

$$\nabla_V g(W, Z) = g(\nabla_V W, Z) + g(W, \nabla_V Z) \tag{A.8}$$

and is called symmetric if

$$\nabla_V W - \nabla_W V = [V, W].$$

Homework: Show that our connection is a Levi-Civita symmetric connection.

Claim: for a Levi-Civita symmetric connection, one has

$$\langle Z, \nabla_Y X\rangle = \frac{1}{2}\{X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle - \langle [X, Z], Y\rangle - \langle [Y, Z], X\rangle - \langle [X, Y], Z\rangle\}. \tag{A.9}$$

Thus ∇ depends only on the first fundamental form g_{ij}. To prove this relation we add the permutations of $X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$.

Consider a vector field V on S, i.e. a map $V : S \to TS$. So if γ_s is a path in S s.t. $\gamma_s|_{s=0} = x$ and $\partial_s \gamma_s|_{s=0} = V(x)$, then

$$Vf(x) = \nabla_V f(x) = \frac{\partial}{\partial s}|_{s=0} f(\gamma_s).$$

Let $\sigma : U \to S \subset \mathbb{R}^{n+1}$ be a local parametrization of S. Then $\{\frac{\partial\sigma}{\partial u^i}\}$ is a basis in $T_{\sigma(u)}S$ (generally not orthonormal). We can write $V(x) = \partial_s \gamma_s|_{s=0} = \partial_s \sigma \circ \sigma^{-1} \circ \gamma_s|_{s=0}$. By the chain rule we have $V(x) = \sum_i \frac{\partial\sigma}{\partial u^i} \frac{\partial}{\partial s}(\sigma^{-1} \circ \gamma_s)_i|_{s=0}$. Hence we can write $V = V^i \frac{\partial\sigma}{\partial u^i}$, where $V^i = \frac{\partial((\sigma^{-1}\circ\gamma_s)_i)}{\partial s}|_{s=0}$.

A vector field V is defined as a map $V : S \to TS$, but is also identified with an operator $Vf = \nabla_V f$. Let $f : S \to \mathbb{R}$ be a function on S. Then

$$\begin{aligned} Vf(x) &= \tfrac{\partial}{\partial s}|_{s=0} f \circ \sigma \circ \sigma^{-1} \circ \gamma_s \\ &= \tfrac{\partial(f\circ\sigma)}{\partial u^i} \tfrac{\partial((\sigma^{-1}\circ\gamma_s)_i)}{\partial s}|_{s=0}, \end{aligned}$$

which by the above relations can be rewritten as

$$Vf(x) = V^i \frac{\partial f}{\partial u^i}.$$

We write $V = V^i \frac{\partial}{\partial u^i}$. We think of $\{\partial_{u^i}\}$ as a basis in TS ($\{(\partial_{u^i})_x\}$, as a basis in T_xS). We think of ∂_{u^i} are either operators acting on functions on S or as vectors $\partial_{u^i}\sigma$. (Note that the vector $\frac{\partial\sigma}{\partial u^i}$ has components δ_{ij} in the basis $\{\frac{\partial\sigma}{\partial u^j}\}$ and therefore the operator associated with it is $\delta_{ij}\frac{\partial}{\partial u^j} = \frac{\partial}{\partial u^i}$.)

Since $\{\partial_{u^i}\}$ is a basis, then $\nabla_{\partial_{u^i}}\partial_{u^j}$ can be expanded in it: $\nabla_{\partial_{u^i}}\partial_{u^j} = \Gamma^k_{ij}\partial_{u^k}$, for some coefficients Γ^k_{ij}, called the Christofel symbols. If we let $X = \partial_{u^i}$, $Y = \partial_{u^j}$ and $Z = \partial_{u^k}$ in (A.9), then we find

$$\Gamma^k_{ij} = \frac{1}{2}g^{kl}(\frac{\partial g_{lj}}{\partial u^i} + \frac{\partial g_{li}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^l}).$$

This can be also shown directly.

Since $\nabla_V = \nabla_{V^i\partial_{u^i}} = V^i\nabla_{\partial_{u^i}}$ and $\nabla_{\partial_{u^i}}W^j = \frac{\partial}{\partial u^i}W^j$, we have

$$\begin{aligned}\nabla_V W &= \nabla_{V^i\partial_{u^i}}(W^j\partial_{u^j}) \\ &= V^i\nabla_{\partial_{u^i}}(W^j\partial_{u^j}) \\ &= V^i[(\nabla_{\partial_{u^i}}W^j)\partial_{u^j} + W^j\nabla_{\partial_{u^i}}\partial_{u^j}] \\ &= V^i(\partial_{u^i}W^j\partial_{u^j} + W^j\Gamma^k_{ij}\partial_{u^k}),\end{aligned}$$

which gives $\nabla_V W = V^i\nabla_i W^k\partial_{u^k}$, or $(\nabla_V W)^k = V^i\nabla_i W^k$, where

$$\nabla_i W^k \equiv \nabla_{\partial_{u^i}}W^k := \frac{\partial W^k}{\partial u^i} + \Gamma^k_{ij}W^j.$$

A.5 Various differential operators

We give definitions of various differential operators used in geometry.
(1) Divergence $\operatorname{div} V = \operatorname{Tr}\nabla V$, where ∇V is the map $W \mapsto \nabla_W V$.
(2) Gradient. The vector field $\operatorname{grad}(f)$ is defined by $g(\operatorname{grad}(f), W) = \nabla_W f$ $\forall$ vector field W, where $\nabla f : V \mapsto \nabla_V f$ is considered as a functional on vector fields.
(3) Laplace-Beltrami $\Delta := \operatorname{div}\ \operatorname{grad}$.
(4) Hessian $\operatorname{Hess}(f) := \nabla^2 f$. Explicitly, $\nabla^2 f(V, W) = (VW - \nabla_V W)f$.
Here ∇f is viewed as $(0,1)$ tensor and ∇_V is defined on $(0,1)$ tensors $T : TS \to \mathbb{R}$ as $(\nabla_V T)(W) = \nabla_V(T(W)) - T(\nabla_V W)$.

Recall the notation $g = \det(g_{ij})$. In local coordinates, these differential operators have the following expressions:

$$\operatorname{div} V = \nabla_i V^i = \frac{1}{\sqrt{g}} \partial_{u^i}(\sqrt{g} V^j), \tag{A.10}$$

$$(\operatorname{grad}(f))^i = \nabla^i f = g^{ij} \frac{\partial f}{\partial u^j}, \tag{A.11}$$

$$(\operatorname{Hess}(f))_{ij} = \nabla_i \nabla_j f = \frac{\partial^2 f}{\partial u^i \partial u^j} - \Gamma^k_{ij} \frac{\partial f}{\partial u^k}, \tag{A.12}$$

$$\Delta f = \nabla^i \nabla_i f = \frac{1}{\sqrt{g}} \partial_{u^i}(\sqrt{g} g^{ij} \frac{\partial f}{\partial u^j}). \tag{A.13}$$

Exercise: (1) Show (A.10)- (A.13); (2) Show that grad $= -(\operatorname{div})^*$ and $\Delta^* = \Delta$.

Proof of (A.10)**-** (A.13)**.** We have, for an orthonormal basis $\{e_i\}$,

$$\begin{aligned} \operatorname{div} V &= \langle \nabla_{e_i} V, e_i \rangle = g^{ij} \left\langle \nabla_{\partial_{u^i}} V, \partial_{u^j} \right\rangle \\ &= g^{ij} \nabla_i V^k \langle \partial_{u^k}, \partial_{u^j} \rangle = g^{ij} g_{kj} \nabla_i V^k, \end{aligned}$$

which gives

$$\operatorname{div} V = \nabla_i V^i \tag{A.14}$$

To show $\operatorname{div} V = \frac{1}{\sqrt{g}} \partial_{u^i}(\sqrt{g} V^j)$, we use the representation $\det(g_{ij}) = e^{\operatorname{Tr} \ln(g_{ij})}$ to obtain the formula (cf. (1.11))

$$\frac{\partial g}{\partial u^k} = g g^{ij} \frac{\partial g_{ij}}{\partial u^k}. \tag{A.15}$$

From (A.15), we have $\frac{1}{\sqrt{g}} \partial_{u^k} \sqrt{g} = \frac{1}{2g} \partial_{u^k} g = \frac{1}{2} g^{ij} \frac{\partial g_{ij}}{\partial u^k}$. On the other hand, $\Gamma^m_{mk} = \frac{1}{2} g^{ml}(\partial_{u^m} g_{kl} + \partial_{u^k} g_{ml} - \partial_{u^l} g_{mk})$. It follows that $\Gamma^m_{mk} = \frac{1}{2} g^{ml} \partial_{u^k} g_{ml}$ and therefore $\frac{1}{\sqrt{g}} \partial_{u^k} \sqrt{g} = \Gamma^m_{mk}$. So we have

$$\operatorname{div} V = \frac{\partial V^i}{\partial u^i} + \Gamma^m_{mi} V^i = \frac{\partial V^i}{\partial u^i} + \frac{1}{\sqrt{g}} \partial_{u^i} \sqrt{g} V^i = \frac{1}{\sqrt{g}} \frac{\partial}{\partial u^i} (\sqrt{g} V^i).$$

which gives (A.10).

If $\operatorname{grad}(f) = (\operatorname{grad}(f))^i \partial_{u^i}$, then $\frac{\partial f}{\partial u^j} = \nabla_{\partial_{u^j}} f = \left\langle (\operatorname{grad}(f))^i \partial_{u^i}, \partial_{u^j} \right\rangle = (\operatorname{grad}(f))^i \langle \partial_{u^i}, \partial_{u^j} \rangle = g_{ij} (\operatorname{grad}(f))^i$, which gives (A.11).

Since $\nabla_i W^j \equiv \nabla_{\partial_{u^i}} W^j := \frac{\partial W^j}{\partial u^i} + \Gamma^j_{ik} W^k$, we have $(\operatorname{Hess}(f))^j_i = \frac{\partial}{\partial u^i} g^{jk} \frac{\partial f}{\partial u^k} + \Gamma^j_{ik} g^{k\ell} \frac{\partial f}{\partial u^\ell}$.

Furthermore, $(\operatorname{Hess}(f))_{ij} := \nabla^2 f(\partial_{u^i}, \partial_{u^j}) = \frac{\partial^2 f}{\partial u^i \partial u^j} - (\nabla_{\partial_{u^i}} \partial_{u^j}) f$, which gives (A.12).

Finally combining the previous results, we obtain $\Delta f = \operatorname{div} \operatorname{grad}(f) = \operatorname{div}(g^{ij} \frac{\partial f}{\partial u^j} \partial_{u^i}) = \frac{1}{\sqrt{g}} \partial_{u^i}(\sqrt{g} g^{ij} \frac{\partial f}{\partial u^j})$. □

Proposition 40. *Let $\sigma : U \to S \subset \mathbb{R}^{n+1}$ be a local parametrization of S. Then*

$$\Delta_S \sigma = -H\nu. \tag{A.16}$$

Proof. We will prove it at a point $x_0 \in S$. Since (A.16) is invariant under translations and rotations (show this), we can translate and rotate S so that x_0 becomes a critical point in the sense that $S \bigcap W = \operatorname{graph} f$ and $\operatorname{grad}(f)(x_0) = 0$. Now $\sigma(u) = (u, f(u))$ and $g_{ij} = \langle \frac{\partial \sigma}{\partial u^i}, \frac{\partial \sigma}{\partial u^j} \rangle = \delta_{ij} + \frac{\partial f}{\partial u^i}\frac{\partial f}{\partial u^j}$. Then $\nu = \frac{(-\nabla f, 1)}{\sqrt{1+|\nabla f|^2}}$ and $\nu|_{x=x_0} = (0, 1)$. Note that

$$\begin{aligned} \Delta_S f &= \nabla_i (\operatorname{grad}(f))^i = \nabla_i g^{ij} \tfrac{\partial f}{\partial u^j} \\ &= \tfrac{\partial}{\partial u^i}(g^{ij} \tfrac{\partial f}{\partial u^j}) + \Gamma^m_{mi} g^{ij} \tfrac{\partial f}{\partial u^j} \\ &= g^{ij} \tfrac{\partial^2 f}{\partial u^i u^j} + \tfrac{\partial g^{ij}}{\partial u^i}\tfrac{\partial f}{\partial u^j} + \Gamma^m_{mi} g^{ij} \tfrac{\partial f}{\partial u^j}. \end{aligned}$$

Since $g_{ij}|_{x=x_0} = \delta_{ij}$ and therefore $g^{ij}|_{x=x_0} = \delta_{ij}$, we have $\Delta_S f|_{x=x_0} = g^{ij}\frac{\partial^2 f}{\partial u^i u^j}|_{x=x_0} = \sum_i \frac{\partial^2 f}{\partial {u^i}^2}|_{x=x_0}$. Hence

$$\Delta_S \sigma|_{x=x_0} = (0, \sum_i \frac{\partial^2 f}{\partial {u^i}^2})|_{x=x_0} = (0,1)\Delta^{\mathbb{R}^{n+1}} f|_{x=x_0} = \nu|_{x=x_0} \Delta^{\mathbb{R}^{n+1}} f|_{x=x_0}.$$

But $H(x_0) = -\operatorname{div}^{\mathbb{R}^{n+1}}(\frac{\nabla^{\mathbb{R}^{n+1}} f}{\sqrt{1+\nabla^{\mathbb{R}^{n+1}} f^2}})|_{x=x_0} = -\operatorname{div}^{\mathbb{R}^{n+1}} \nabla^{\mathbb{R}^{n+1}} f|_{x=x_0} = -\Delta^{\mathbb{R}^{n+1}} f|_{x=x_0}$, this completes the proof of the proposition. □

A.6 Submanifolds

Recall that the Weingarten map is defined as $W_x V = d\nu(x)V$, which can be written in terms of the Euclidean connection $\nabla^{\mathbb{R}^{n+1}}$ as

$$W_x V = \nabla^{\mathbb{R}^{n+1}}_V \nu(x). \tag{A.17}$$

For the second fundamental form, we have consequently that for $\xi, \eta \in TR^{n+1}$,

$$W_x(\xi, \eta) = \langle \nabla^{\mathbb{R}^{n+1}}_\xi \nu(x), \eta \rangle = -\langle \nu(x), \nabla^{\mathbb{R}^{n+1}}_\xi \eta \rangle.$$

If S is a hypersurface in an $(n+1)-$ dimensional Riemannian manifold (M, s), rather than in the Euclidean space $\mathbb{R}^{n+1}$, we define the Weingarten map by

$$W_x \xi = \nabla^M_\xi \nu(x), \tag{A.18}$$

where ν is the outward normal vector field on S in the metric s and ∇^M is a (Levi-Civita symmetric) connection on M, the second fundamental form by

$$W_x(\xi, \eta) = \langle \nabla^M_\xi \nu(x), \eta \rangle_s = -\langle \nu(x), \nabla^M_\xi \eta \rangle_s, \tag{A.19}$$

where we used the relation (A.8) for Levi-Civita connections, and the mean curvature, still by (A.1).

As an example, we consider the Euclidean space $\mathbb{R}^{n+1}$, with the conformally flat metric $\sigma^2 dx^2$, where $\sigma > 0$. Let $\sigma = e^{-\varphi}$ and denote ∇^M by ∇^σ. Then we have

$$\nabla^\sigma_\xi \eta = \nabla^{\mathbb{R}^{n+1}}_\xi \eta - (\xi\varphi)\eta - (\eta\varphi)\xi + \langle \xi, \eta \rangle_\sigma \operatorname{grad} \varphi,$$

where $\xi f = \nabla_\xi f$ is the the Euclidean directional derivative (application of the vector field to a function). Moreover, let ν and ν^σ be the outward normal vector fields on S in the euclidean metric and in the metric s. Using that $\langle \xi, \nu\rangle_\sigma = \sigma^2\langle \xi, \nu\rangle = 0$ for $\forall \xi \in TS$, we have $\nu^\sigma = \sigma^{-1}\nu$ and $W_x^\sigma \xi = \nabla_\xi^\sigma \nu^\sigma(x) = \nabla_\xi^{\mathbb{R}^{n+1}}(\sigma^{-1}\nu) - (\xi\varphi)\sigma^{-1}\nu - \sigma^{-1}(\nu\varphi)\xi$. Since $\nabla_\xi^{\mathbb{R}^{n+1}}(\sigma^{-1}\nu) = (\xi\sigma^{-1})\nu + \sigma^{-1}\nabla_\xi^{\mathbb{R}^{n+1}}\nu$, $\nabla_\xi^{\mathbb{R}^{n+1}}\nu = W_x\xi$ and $\xi\sigma^{-1} = \sigma^{-1}\xi\varphi$, this gives $W_x^\sigma\xi = \sigma^{-1}W_x\xi - \sigma^{-1}(\nu\varphi)\xi$, which implies

$$W_x^\sigma\xi = \sigma^{-1}W_x\xi - (\nu\sigma^{-1})\xi = \sigma^{-1}(W_x - (\nu\varphi))\xi.$$

(Here as before, $\nu f = \nabla_\nu f$.) This gives the expression for the second fundamental form $b_{ij}^\sigma = \langle \partial_{x^i}, W_x \partial_{x^j}\rangle_\sigma = \sigma\langle \partial_{x^i}, (W_x - (\nu\varphi)\partial_{x^j}\rangle$ and therefore

$$b_{ij}^\sigma = e^{-\varphi}(b_{ij} - (\nu\varphi)g_{ij}). \tag{A.20}$$

For the mean curvature $H_\sigma = \sigma^{-2}g_{ij}b_{ij}^\sigma$, we obtain

$$H_\sigma = \frac{1}{\sigma}(H + \frac{n}{\sigma^2}\nabla_\nu\sigma) = e^\varphi(H - n\nabla_\nu\varphi). \tag{A.21}$$

We derive (A.21) differently using that the mean curvature arises in normal variations, $\eta = f\nu$, of the surface volume functional $V(\psi)$ (see (1.9)). In the conformal case, the analogue of (1.9) is $dV_\sigma(\psi)\eta = \int_U H_\sigma\nu\cdot\eta\ \sqrt{g_\sigma}d^nu$, where $\eta = f\nu_\sigma$. We use that $\sqrt{g_\sigma} = \sigma^n\sqrt{g}$, $(d\sqrt{g})\eta = (d\sqrt{g})(\sigma^{-1}f\nu) = H\nu\cdot\sigma^{-1}f\nu$ and $\nu_\sigma = \sigma^{-1}\nu$ to compute $dV_\sigma(\psi)\eta = \int_U(H\nu + n\sigma^{-1}\nabla\sigma)\cdot\eta\ \sqrt{g_\sigma}d^nu = \int_U(H + n\sigma^{-1}\nabla\sigma\cdot\nu)\sigma^{-1}f\ \sqrt{g_\sigma}d^nu$, which implies $H_\sigma = \sigma^{-1}(H + n\sigma^{-1}\nabla_\nu\sigma)$.

Our next goal is to compute the normal hessian, $\mathrm{Hess}^N A$, of the area functional A, for a hypersurface S immersed in a manifold $M = (\mathbb{R}^{n+1}\sigma dx^2)$, the Euclidean space $\mathbb{R}^{n+1}$, with the conformally flat metric σdx^2, $\sigma = e^{-2\varphi}$. Recall that the normal hessian, $\mathrm{Hess}^N A$, of the area functional A, for a hypersurface S immersed in a manifold M is given by the expression (2.30). Thus to to compute $\mathrm{Hess}^N A$, we have to compute the trace norm $|W_x^M|_s^2$, the Ricci curvature $Ric^M(\nu,\nu)$ of M evaluated on the outward normal vector field, ν,on S and the Laplace-Beltrami operator Δ_S on S in the metric induced by the metric on M.

We begin with the trace norm $|W_x^\sigma|_\sigma^2 = |W_x^M|_s^2$. Since by the definition $|W_x^\sigma|_\sigma^2 = \sigma^{-2}g^{ik}g^{jm}b_{ij}^\sigma b_{km}^\sigma$, we have $|W_x^\sigma|_\sigma^2 = \sigma^{-1}g^{ik}g^{jm}(b_{ij} - \nabla_\nu\varphi g_{ij})(b_{km} - \nabla_\nu\varphi g_{km})$, which gives

$$|W_x^\sigma|_\sigma^2 = e^{2\varphi}(|W_x|^2 - 2\nabla_\nu\varphi H + n(\nabla_\nu\varphi)^2). \tag{A.22}$$

Furthermore, to compute $Ric^\sigma = Ric^M$, we use standard formulae, which give (see e.g. [18, 24])

$$Ric_{ij}^\sigma = (n-1)(\nabla_i\nabla_j\varphi - \nabla_i\varphi\nabla_j\varphi) + (\Delta\varphi + (n-1)|\nabla\varphi|^2)\delta_{ij}. \tag{A.23}$$

(Recall that $\nabla_i,\ \nabla,\ \Delta$ are the operators in the euclidean metric.)

Finally, we compute the change in the Laplace-Beltrami operator. By the equation (A.13) below, we have

$$\Delta_S^\sigma f = \frac{1}{\sqrt{g^\sigma}}\partial_{u^i}(\sqrt{g^\sigma}g_\sigma^{ij}\frac{\partial f}{\partial u^j}) = \frac{1}{\sigma^{n/2}\sqrt{g}}\partial_{u^i}(\sigma^{\frac{n}{2}-1}\sqrt{g}g^{ij}\frac{\partial f}{\partial u^j}) \tag{A.24}$$

$$= \sigma^{-1}\Delta_S + \frac{n-2}{2}\sigma^{-2}(\partial_{u^i}\sigma)g^{ij}\frac{\partial f}{\partial u^j} = \sigma^{-1}\Delta_S f + \frac{n-2}{2}\sigma^{-2}(\nabla^j\sigma)\nabla_j f \tag{A.25}$$

$$= e^{2\varphi}(\Delta_S - (n-2)(\nabla^j\varphi)\nabla_j)f. \tag{A.26}$$

In the example of the $n-$sphere S^n_R of radius R in $\mathbb{R}^{n+1}$ considered in Subsection A.2, we computed $b_{ij} = R^{-1} g_{ij} = R g^{\text{stand}}_{ij}$, where g_{ij} and g^{stand}_{ij} are the metrics on S^n_R and $\mathbb{S}^n = S^n_1$, respectively, which implies, in particular, that $H = g^{ij} b_{ji} = nR^{-1}$. In this case, since $\nu = \omega$, (A.22) gives

$$|W^\sigma_x|^2_\sigma = e^{2\varphi}(nR^{-2} - 2nR^{-1}\nabla_\omega\varphi + n(\nabla_\omega\varphi)^2) = ne^{\varphi}(R^{-1} - \nabla_\omega\varphi)^2. \tag{A.27}$$

To compute $Ric^\sigma(\nu^\sigma, \nu^\sigma)$, we use (A.23) and $\nu^\sigma = \sigma^{-1/2}\nu = \sigma^{-1/2}\omega$ to obtain $Ric^\sigma(\nu^\sigma, \nu^\sigma) = \sigma^{-1} Ric^\sigma(\omega, \omega) = e^{2\varphi}[(n-1)(\nabla_i\nabla_j\varphi - \nabla_i\varphi\nabla_j\varphi)\omega^i\omega^j + (\Delta\varphi + (n-1)|\nabla\varphi|^2)]$.

Hence, due to (2.30), $\text{Hess}^N A$ is given by

$$\text{Hess}^N A_\sigma(S^n_R) = -e^{2\varphi}(R^{-2}\Delta_{\mathbb{S}^n} - (n-2)(\nabla^j\sigma)\nabla_j) - ne^{2\varphi}(R^{-1} - \nabla_\omega\varphi)^2 \tag{A.28}$$

$$- e^{2\varphi}[(n-1)\big(\nabla_i\nabla_j\varphi\omega^i\omega^j - (\nabla\varphi\cdot\omega)^2 + |\nabla\varphi|^2\big) + \Delta\varphi], \tag{A.29}$$

which gives

$$\text{Hess}^N A_\sigma(S^n_R) = -e^{2\varphi}\big[R^{-2}(\Delta_{\mathbb{S}^n} + n) - (n-2)(\nabla^j\sigma)\nabla_j - 2nR^{-1}\nabla_\omega\varphi - (\nabla_\omega\varphi)^2 \tag{A.30}$$

$$+ (n-1)\big(\nabla_\omega\nabla_\omega\varphi + |\nabla\varphi|^2\big) + \Delta\varphi\big]. \tag{A.31}$$

A.7 Riemann curvature tensor for hypersurfaces

A.8 Mean curvature of normal graphs over the round sphere $\mathbb{S}^n$ and cylinder $\mathcal{C}^{n+1}$

Let $\hat{S}$ be a fixed convex $n-$dimensional hypersurface in $\mathbb{R}^{n+1}$. We consider hypersurfaces S given as normal graphs $\theta(\hat{x}, t) = u(\hat{x})\hat{\nu}(\hat{x})$, over a given hypersurface $\hat{S}$. Here $\hat{\nu}(\hat{x})$ is the outward unit normal vector to $\hat{S}$ at $\hat{x} \in \hat{S}$. We would like to derive an expression for the mean curvature of S in terms of the graph function ρ.

To be more specific, we are interested in $\hat{S}$ being the round sphere $\mathbb{S}^n$. Then a normal graph graph is given by

$$\theta : \omega \mapsto \rho(\omega)\omega. \tag{A.32}$$

The result is given in the next proposition.

Proposition 41. *If a surface S, defined by an immersion $\theta : \mathbb{S}^n \to \mathbb{R}^{n+1}$, is a graph* (A.32) *over the round sphere $\mathbb{S}^n$, with the graph function $\rho : \mathbb{S}^n \to \mathbb{R}_+$, then the mean curvature, H, of S is given in terms of the graph function u by the equation*

$$H = p^{-1/2}(n - \rho^{-1}\Delta\rho) + p^{-3/2}\left[|\nabla\rho|^2 + \rho^{-1} g^{ik} g^{j\ell} \nabla_i\nabla_j\rho\nabla_k\rho\nabla_\ell\rho\right], \tag{A.33}$$

where ∇ denotes covariant differentiation on $\mathbb{S}^n$, with the components ∇_i in some local coordinates and

$$p := \rho^2 + |\nabla\rho|^2. \tag{A.34}$$

Proof. We do computations locally and therefore it is convenient to fix a point $\omega \in \mathbb{S}^n$ and choose normal coordinates at this point. Let $(e_1, \dots, e_n)$ denote an orthonormal basis of $T_\omega\mathbb{S}^n \subset T_\omega\mathbb{R}^{n+1}$ in this coordinates, i.e. the metric on $\mathbb{S}^n$ at $\omega \in \mathbb{S}^n$ is $g^{\text{sphere}}_{ij} = g^{\text{sphere}}(e_i, e_j) = \delta_{ij}$ and the Christphel symbols vanish, $\Gamma^k_{ij} = 0$ at $\omega \in \mathbb{S}^n$. By ∇_i we denote covariant differentiation on $\mathbb{S}^n$ w.r.to this basis (frame). First, we compute the metric g^{ij} on S_t. To this end, we note that, for the map (A.32), one has

$$\nabla_i \theta = (\nabla_i \rho)\omega + u e_i,\ i = 1, \dots, n, \tag{A.35}$$

where we used the relation $W \frac{\partial \psi}{\partial u^i} = g^{ij} b_{jk} \frac{\partial \psi}{\partial u^k}$ (see (A.6) of Appendix **??**), where W is the Weigarten map, which in our situation (i.e. $b^{\text{sphere}}_{jk} = g^{\text{sphere}}_{ij} = \delta_{ij}$ and $\nu(\omega) = \omega$) gives $\nabla_i \omega = e_i$. It follows that the Riemannian metric g induced on S has components

$$g_{ij} = \langle \nabla_i \theta, \nabla_j \theta \rangle = \rho^2 \delta_{ij} + \nabla_i \rho \nabla_j \rho.$$

Recall that the symbol $\langle \cdot, \cdot \rangle$ denotes the pointwise Euclidean inner product of $\mathbb{R}^{n+1}$. As can be easily checked directly, the inverse matrix is

$$g^{ij} = \rho^{-2}(\delta_{ij} - p^{-1} \nabla_i \rho \nabla_j \rho). \tag{A.36}$$

It is easy to see that the following normalized vector field on S is orthogonal to all vectors (A.35) and therefore is the outward unit normal to $\mathcal{M}$ is

$$\nu = p^{-1/2}(\rho\omega - e_i \nabla_i \rho) = p^{-1/2}(\rho\omega - \nabla\rho). \tag{A.37}$$

Straightforward calculations show that

$$p^{1/2} \nabla_j \nu = 2(\nabla_j \rho)\omega + \rho e_j - \nabla_j \nabla \rho + \{\cdots\}\nu,$$

where the terms in braces are easy to compute but irrelevant for what follows. One thus finds that the matrix elements $b_{ij} = \langle \nabla_i \theta, \nabla_j \nu \rangle$ of the second fundamental form are

$$b_{ij} = p^{-1/2}(\rho^2 \delta_{ij} + 2\nabla_i \rho \nabla_j \rho - \rho \nabla_i \nabla_j \rho). \tag{A.38}$$

Since $H = \text{tr}_g(h) = g^{ji} b_{ij}$, we use (A.36) and (A.38) and $\delta_{ji}\delta_{ij} = n$ to compute:

$$\begin{aligned}
H &= p^{-1/2}[\delta_{ji}\delta_{ij} + 2\rho^{-2}\delta_{ij}\nabla_i\rho\nabla_j\rho - \rho^{-2}\delta_{ij}\rho\nabla_i\nabla_j\rho - p^{-1}\nabla_i\rho\nabla_j\rho\delta_{ij} - 2(\rho^2 p)^{-1}\nabla_i\rho\nabla_j\rho\nabla_i\rho\nabla_j\rho \\
&\quad + (\rho p)^{-1}\nabla_i\rho\nabla_j\rho\nabla_i\nabla_j\rho] \\
&= p^{-1/2}[n + 2\rho^{-2}\nabla_i\rho\nabla_i\rho - \rho^{-1}\nabla_i\nabla_i\rho - p^{-1}\nabla_i\rho\nabla_i\rho - 2(\rho^2 p)^{-1}|\nabla\rho|^4 + (\rho p)^{-1}\nabla_i\rho\nabla_j\rho\nabla_i\nabla_j\rho] \\
&= p^{-1/2}[n + 2\rho^{-2}|\nabla\rho|^2 - \rho^{-1}\Delta\rho - p^{-1}|\nabla\rho|^2 - 2(\rho^2 p)^{-1}|\nabla\rho|^4 + (\rho p)^{-1}\nabla_i\rho\nabla_j u\nabla_i\nabla_j\rho] \\
&= p^{-1/2}[n - \rho^{-1}\Delta\rho + (\rho p)^{-1}\nabla_i\rho\nabla_j\rho\nabla_i\nabla_j\rho + p^{-1}\rho^{-2}(2p|\nabla\rho|^2 - \rho^2|\nabla\rho|^2 - 2|\nabla\rho|^4)] \\
&= p^{-1/2}[n - \rho^{-1}\Delta\rho + \rho^{-1}p^{-1}\nabla_i\nabla_j\rho\nabla_i\rho\nabla_j\rho + p^{-1}|\nabla\rho|^2].
\end{aligned}$$

which implies (A.33). □

A different derivation is given below. **(to be done)**

B Some general results using maximum principle

Theorem 42. *If $S_0 \subset B_{\rho_0}$ (compact and without boundary?) for some $\rho_0 > 0$, where B_ρ is the ball with radius ρ, center at 0, then $S_t \subset B_{\sqrt{\rho_0^2 - 2nt}}$. In particular, (MCF) exists for time $t_* < \frac{\rho_0^2}{2n}$.*

Proof of Theorem 44. Let $f(x,t) = |x|^2 + 2nt$. Compute $\frac{df}{dt} = \frac{\partial f}{\partial t} + \text{grad}\, f \cdot \frac{\partial x}{\partial t} = 2n + 2x \cdot \overrightarrow{H}$, where $\overrightarrow{H} = -H\nu$. We also have

$$\begin{aligned} \Delta_S |x|^2 &= \tfrac{1}{\sqrt{g}} \partial_{u^i} (\sqrt{g} g^{ij} 2x^k \tfrac{\partial x^k}{\partial u^j}) \\ &= 2x^k \Delta_S x^k + \tfrac{2}{\sqrt{g}} \sqrt{g} g^{ij} \tfrac{\partial x^k}{\partial u^i} \tfrac{\partial x^k}{\partial u^j} \\ &= 2x \cdot \overrightarrow{H} + 2g^{ij} g_{ij} \\ &= 2x \cdot \overrightarrow{H} + 2n. \end{aligned}$$

So we obtain $\frac{df}{dt} = \Delta_S f$.

Let $\tilde{f} = f - \epsilon t$ for $\epsilon > 0$, then $(\frac{d}{dt} - \Delta_S)\tilde{f} = -\epsilon < 0$. We claim that $\max_{S_t} \tilde{f} \leq \max_{S_0} f$. Otherwise let t_* be the first time when $\max_{S_t} \tilde{f}$ reaches a value larger than $\max_{S_0} f$ and $\tilde{f}(x_*) = \max_{S_t} \tilde{f}$. Then at (x_*, t_*), $d_t \tilde{f} \leq 0$, $\partial_{u^i} \tilde{f} = 0$ and $(\partial_{u^i} \partial_{u^j} \tilde{f}) \leq 0$. Hence $\Delta_S \tilde{f} = (\partial_{u^j} + \Gamma^i_{ij}) g^{jk} \frac{\partial \tilde{f}}{\partial u^k} = g^{jk} \frac{\partial^2 \tilde{f}}{\partial_{u^j} \partial_{u^k}} \leq 0$. So $\frac{df}{dt} - \Delta_S f \geq 0$, contradiction! This proves our claim $\max_{S_t} \tilde{f} \leq \max_{S_0} f \;\forall\; \epsilon > 0$. Let $\epsilon \to 0$, we have $\max_{S_t} f \leq \max_{S_0} f$, i.e. $|x|^2 \leq \rho_0^2 - 2nt$ at time t. □

How the collapse take place? Some indications come from the following estimates.

Theorem 43. *Let $n \geq 2$ and $\min H > 0$ on a closed surface S_0. Then under the MCF, the mean curvature $H(t)$ increases monotonically.*

Proof. Let A be the second fundamental form of the surface. **(change of notation!)** The relations $H = g^{ij} A_{ij}$, $\partial_t g^{ij} = 2HA^{ij}$ and $\partial_t A_{ij} = \nabla_i \nabla_j H - A_{ik} A^k_j$ (see [Eckert,Regularity Theory of MCF, App B]) imply the following differential inequality for the mean curvature

$$\partial_t H \geq \triangle H + \frac{1}{n} H^3.$$

Now we prove that

$$H \to \infty \text{ as } t \to T_0,$$

with $T_0 := (\frac{2}{n} H^2_{min})^{-1}$. Indeed, let φ satisfy $\partial_t \varphi = \frac{1}{n} \varphi^3$ and $\varphi|_{t=0} < \min H|_{t=0}$. Then, since φ is independent of x,

$$\partial_t (H - \varphi) \geq \triangle (H - \varphi) + \frac{1}{n} (H^3 - \varphi^3) \quad \text{and} \quad (H - \varphi)|_{t=0} > 0, \tag{B.1}$$

which implies

$$H - \varphi \geq 0 \quad \text{for} \quad \forall t \in [0, T). \tag{B.2}$$

Indeed, let $H = \varphi$ at $x = \bar{x}, t = \bar{t}$ and $H > \varphi$ for either $x \neq \bar{x}, t = \bar{t}$ or $t < \bar{t}$. Then $H|_{t=\bar{t}}$ has a minimum at $x = \bar{x}$. So we obtain

$$\begin{aligned} &\text{Hess}\, H(\bar{x}, \bar{t}) > 0 \quad \Rightarrow \quad \triangle H(\bar{x}, \bar{t}) = \text{Tr}\, \text{Hess}\, H(\bar{x}, \bar{t}) > 0 \\ \Rightarrow &\partial_t (H - \varphi) > \frac{1}{n} (H^3 - \varphi^3) = 0 \quad \text{at} \quad (\bar{x}, \bar{t}) \quad \Rightarrow \quad (H - \varphi)(\bar{x}, t) \quad \text{grows in} \quad t \quad \text{at} \quad \bar{t}. \end{aligned}$$

This implies (B.2). But $\varphi(t) = H_{min}(1 - \frac{2}{n}H_{min}^2 t)^{-\frac{1}{2}}$ which implies $H \geq \varphi \to \infty$ as $t \to T_0$ with $T_0 := (\frac{2}{n}H_{min}^2)^{-1}$. Hence we have the claimed relations. □

Now, by a local existence result we know that S_t exists on some time interval. Let $[0, T)$ be the maximal time interval for the existence of the MCF. The above result implies that under the assumptions of the theorem

Proposition 44. *$T \leq T_0 := (\frac{2}{n}H_{min}^2)^{-1} < \infty$.*

C Elements of spectral theory

C.1 A characterization of the essential spectrum of a Schrödinger operator

In this appendix we present a result characterizing the essential spectrum of a Schrödinger operator in a manner similar to the characterization of the discrete spectrum as a set of eigenvalues. We follow [**?**].

Theorem 45 (Schnol-Simon)**.** Let H be a Schrödinger operator with a bounded potential. Then

$$\sigma(H) = \text{ closure } \{\lambda \mid (H-\lambda)\psi = 0 \text{ for } \psi \text{ polynomially bounded } \}.$$

So we see that the essential spectrum also arises from solutions of the eigenvalue equation, but that these solutions do not live in the space $L^2(\mathbb{R}^3)$.

Proof. We prove only that the right hand side $\subset \sigma(H)$, and refer the reader to [**?**] for a complete proof. Let ψ be a polynomially bounded solution of $(H-\lambda)\psi = 0$. Let C_r be the box of side-length $2r$ centred at the origin. Let j_r be a smooth function with support contained in C_{r+1}, with $j_r \equiv 1$ on C_r, $0 \leq j_r \leq 1$, and with $\sup_{r,x,|\alpha|\leq 2} |\partial_x^\alpha j_r(x)| < \infty$. Our candidate for a Weyl sequence is

$$w_r := \frac{j_r\psi}{\|j_r\psi\|}.$$

Note that $\|w_r\| = 1$. If $\psi \notin L^2$, we must have $\|j_r\psi\| \to \infty$ as $r \to \infty$. So for any R,

$$\int_{|x|<R} |w_r|^2 \leq \frac{1}{\|j_r\psi\|^2} \int_{|x|<R} |\psi|^2 \to 0$$

as $r \to \infty$. We show that

$$(H-\lambda)w_r \to 0.$$

Let $F(r) = \int_{C_r} |\psi|^2$, which is monotonically increasing in r. We claim there is a subsequence $\{r_n\}$ such that

$$\frac{F(r_n+2)}{F(r_n-1)} \to 1.$$

If not, then there is $a > 1$ and $r_0 > 0$ such that

$$F(r+3) \geq aF(r)$$

for all $r \geq r_0$. Thus $F(r_0 + 3k) \geq a^k F(r_0)$ and so $F(r) \geq (const)b^r$ with $b = a^{1/3} > 1$. But the assumption that ψ is polynomially bounded implies that $F(r) \leq (const)r^N$ for some N, a contradiction. Now,

$$(H - \lambda)j_r\psi = j_r(H - \lambda)\psi + [-\Delta, j_r]\psi.$$

Since $(H - \lambda)\psi = 0$ and $[\Delta, j_r] = (\Delta j_r) + 2\nabla j_r \cdot \nabla$, we have

$$(H - \lambda)j_r\psi = (-\Delta j_r)\psi - 2\nabla j_r \cdot \nabla\psi.$$

Since $|\partial^\alpha j_r|$ is uniformly bounded,

$$\|(H - \lambda)j_r\psi\| \leq (const)\int_{C_{r+1}\setminus C_r} (|\psi|^2 + |\nabla\psi|^2) \leq (const)\int_{C_{r+1}\ C_r} |\psi|^2.$$

So

$$\|(H - \lambda)w_r\| \leq C\frac{F(r+2) - F(r-1)}{F(r)} \leq C(\frac{F(r+2)}{F(r-1)} - 1)$$

and so $\|(H - \lambda)w_{r_n}\| \to 0$. Thus $\{w_{r_n}\}$ is a Weyl sequence for H and λ. □ □

C.2 Perron-Frobenius Theory

Consider a bounded operator T on the Hilbert space $X = L^2(\Omega)$.

Definition 6. *An operator T is called* positivity preserving/improving *if and only if* $u \geq 0,\ u \neq 0 \implies Tu \geq 0/Tu > 0$.

Note if T is positivity preserving, then T maps real functions into real functions.

Theorem 46. *Let T be a bounded positive and positivity improving operator and let λ be an eigenvalue of T with an eigenvector φ. Then*

a) $\lambda = \|T\| \Rightarrow \lambda$ *is simple and* $\varphi > 0$ *(modulo a constant factor).*

b) $\varphi > 0$ *and* $\|T\|$ *is an eigenvalue of* $T \Rightarrow \lambda$ *is simple and* $\lambda = ||T||$.

Proof. a) Let $\lambda = ||T||$, $T\psi = \lambda\psi$ and ψ be real. Then $|\psi| \pm \psi \geq 0$ and therefore $T(|\psi| \pm \psi) > 0$. The latter inequality implies that $|T\psi| \leq T|\psi|$ and therefore

$$\langle|\psi|, T|\psi|\rangle \geq \langle|\psi|, |T\psi|\rangle \geq \langle\psi, T\psi\rangle = \lambda||\psi||^2.$$

Since $\lambda = ||T|| = \sup_{||\psi||=1}\langle\psi, T\psi\rangle$, we conclude using variational calculus (see e.g. [**?**] or [28]) that

$$T|\psi| = \lambda|\psi| \tag{C.1}$$

i.e., $|\psi|$ is an eigenfunction of T with the eigenvalue λ. Indeed, since $\lambda = ||T|| = \sup_{||\psi||=1}\langle\psi, T\psi\rangle$, $|\psi|$ is the maximizer for this problem. Hence $|\psi|$ satisfies the Euler-Lagrange equation $T|\psi| = \mu|\psi|$ for some μ. This implies that $\mu||\psi||^2 = \langle|\psi|, T|\psi|\rangle = \lambda||\psi||^2$ and hence $\mu = \lambda$. Equation (C.1) and the positivity improving property of T imply that $|\psi| > 0$.

Now either $\psi = \pm|\psi|$ or $|\psi| + \psi$ and $|\psi| - \psi$ are nonzero. In the latter case they are eigenfunctions of T corresponding to the eigenvalue $\lambda : T(|\psi| \pm \psi) = \lambda(|\psi| \pm \psi)$. By the positivity improving property of T this implies that $|\psi| \pm \psi > 0$ which is impossible. Thus $\psi = \pm|\psi|$.

If ψ_1 and ψ_2 are two real eigenfunctions of T with the eigenvalue λ then so is $a\psi_1 + b\psi_2$ for any $a, b \in \mathbb{R}$. By the above, either $a\psi_1 + b\psi_2 > 0$ or $a\psi_1 + b\psi_2 < 0$ $\forall a, b \in \mathbb{R} \setminus \{0\}$, which is impossible. Thus T has a single real eigenfunction associated with λ.

Let now ψ be a complex eigenfunction of T with the eigenvalue λ and let $\psi = \psi_1 + \imath\psi_1$ where ψ_1 and ψ_2 are real. Then the equation $T\psi = \lambda\psi$ becomes

$$T\psi_1 + iT\psi_2 = \lambda\psi_1 + i\lambda\psi_2.$$

Since $T\psi_2$ and $T\psi_2$ and λ are real (see above) we conclude that $T\psi_i = \lambda\psi_2, i = 1, 2$, and therefore by the above $\psi_2 = c\psi_1$ for some constant c. Hence $\psi = (1+\imath c)\psi_1$ is positive and unique modulo a constant complex factor.

b) By a) and eigenfunction, ψ, corresponding to $\nu := ||T||$ can be chosen to be positive, $\psi > 0$. But then

$$\lambda\langle\psi, \varphi\rangle = \langle\psi, T\varphi\rangle = \langle T\psi, \varphi\rangle = \nu\langle\psi, \varphi\rangle$$

and therefore $\lambda = \nu$ and $\psi = c\varphi$. □

Question: Can the condition that $||T||$ is an eigenvalue of T (see b) be removed?

Now we consider the Schrödinger operator $H = -\Delta + V(x)$ with a real, bounded potential $V(x)$. The above result allows us to obtain the following important

Theorem 47. *Let $H = -\Delta + V(x)$ have an eigenvalue E_0 with an eigenfunction $\varphi_0(x)$ and let $\inf \sigma(H)$ be an eigenvalue. Then*

$$\varphi_0 > 0 \quad \Rightarrow \quad E_0 = \inf\{\lambda | \lambda \in \sigma(H)\} \text{ and } E_0 \text{ is non-degenerate}$$

and, conversely,

$$E_0 = \inf\{\lambda | \lambda \in \sigma(H)\} \quad \Rightarrow \quad E_0 \text{ is non-degenerate and } \varphi_0 > 0$$

(modulo multiplication by a constant factor).

Proof. To simplify the exposition we assume $V(x) \leq 0$ and let $W(x) = -V(x) \geq 0$. For $\mu > \sup W$ we have

$$(-\Delta - W + \mu)^{-1} = (-\Delta + \mu)^{-1} \sum_{n=0}^{\infty} [W(-\Delta + \mu)^{-1}]^n \tag{C.2}$$

where the series converges in norm as

$$\|W(-\Delta+\mu)^{-1}\| \le \|W\|\|(-\Delta+\mu)^{-1}\| \le \|W\|_{L^\infty}\mu^{-1} < 1$$

by our assumption that $\mu > \sup W = \|W\|_{L^\infty}$. To be explicit we assume that $d = 3$. Then the operator $(-\Delta+\mu)^{-1}$ has the integral kernel

$$\frac{e^{-\sqrt{\mu}|x-y|}}{4\pi|x-y|} > 0$$

while the operator $W(-\Delta+\mu)^{-1}$ has the integral kernel

$$W(x)\frac{e^{-\sqrt{\mu}|x-y|}}{4\pi|x-y|} \ge 0.$$

Consequently, the operator

$$(-\Delta+\mu)^{-1}f(x) = \frac{1}{4\pi}\int \frac{e^{-\sqrt{\mu}|x-y|}}{|x-y|}f(y)\,dy$$

is positivity improving ($f \ge 0, f \ne 0 \Rightarrow (-\Delta+\mu)^{-1}f > 0$) while the operator

$$W(-\Delta+\mu)^{-1}f(x) = \frac{1}{4\pi}\int W(x)\frac{e^{-\sqrt{\mu}|x-y|}}{|x-y|}f(y)\,dy$$

is positivity preserving ($f \ge 0 \Rightarrow W(-\Delta+\mu)^{-1}f \ge 0$). The latter fact implies that the operators $[W(-\Delta+\mu)^{-1}]^n$, $n \ge 1$, are positivity preserving (prove this!) and consequently the operator

$$(-\Delta+\mu)^{-1} + \sum_{n=1}^{\infty}[W(-\Delta+\mu)^{-1}]^n$$

is positivity improving (prove this!).

Thus we have shown that the operator $(H+\mu)^{-1}$ is positivity improving. Series (C.2) shows also that $(H+\mu)^{-1}$ is bounded. Since

$$\langle u, (H+\mu)u\rangle \ge (-\sup W + \mu)\|u\|^2 > 0,$$

we conclude that the operator $(H+\mu)^{-1}$ is positive (as an inverse of a positive operator). Finally, $\|(H+\mu)^{-1}\| = \sup\sigma((H+\mu)^{-1}) = (\inf\sigma(H)+\mu)^{-1}$ is an eigenvalue by the condition of the theorem. Hence the previous theorem applies to it. Since $H\varphi_0 = E_0\varphi_0 \Leftrightarrow (H+\mu)^{-1}\varphi_0 = (E_0+\mu)^{-1}\varphi_0$, the theorem under verification follows. ***This paragraph needs details!***. □

D Proof of the Feynmann-Kac Formula

In this appendix we present, for the reader's convenience, a proof of the Feynman-Kac formula (5.28)-(5.29) and the estimate (5.31).

Let $L_0 := -\partial_y^2 + \frac{\alpha^2}{4}y^2 - \frac{\alpha}{2}$ and $L := L_0 + V$ where V is a multiplication operator by a function $V(y,\tau)$, which is bounded and Lipschitz continuous in τ. Let $U(\tau,\sigma)$ and $U_0(\tau,\sigma)$ be the propagators generated by the operators $-L$ and $-L_0$, respectively. The integral kernels of these operators will be denoted by $U(\tau,\sigma)(x,y)$ and $U_0(\tau,\sigma)(x,y)$.

Theorem 48. *The integral kernel of $U(\tau,\sigma)$ can be represented as*

$$U(\tau,\sigma)(x,y) = U_0(\tau,\sigma)(x,y)\int e^{\int_\sigma^\tau V(\omega_0(s)+\omega(s),s)ds}d\mu(\omega) \tag{D.1}$$

where $d\mu(\omega)$ is a probability measure (more precisely, a conditional harmonic oscillator, or Ornstein-Uhlenbeck, probability measure) on the continuous paths $\omega : [\sigma,\tau] \to \mathbb{R}$ with $\omega(\sigma) = \omega(\tau) = 0$, and the pathe $\omega_0(\cdot)$ is given by

$$\omega_0(s) = e^{\alpha(\tau-s)}\frac{e^{2\alpha\sigma} - e^{2\alpha s}}{e^{2\alpha\sigma} - e^{2\alpha\tau}}x + e^{\alpha(\sigma-s)}\frac{e^{2\alpha\tau} - e^{2\alpha s}}{e^{2\alpha\tau} - e^{2\alpha\sigma}}y. \tag{D.2}$$

Remark 7. *$d\mu(\omega)$ is the Gaussian measure with mean zero and covariance $(-\partial_s^2 + \alpha^2)^{-1}$, normalized to 1. The path $\omega_0(s)$ solves the boundary value problem*

$$(-\partial_s^2 + \alpha^2)\omega_0 = 0 \text{ with } \omega(\sigma) = y \text{ and } \omega(\tau) = x. \tag{D.3}$$

Proof of Theorem 50. We begin with the following extension of the Ornstein-Uhlenbeck process-based Feynman-Kac formula to time-dependent potentials:

$$U(\tau,\sigma)(x,y) = U_0(\tau,\sigma)(x,y)\int e^{\int_\sigma^\tau V(\omega(s),s)\,ds}d\mu_{xy}(\omega). \tag{D.4}$$

where $d\mu_{xy}(w)$ is the conditional Ornstein-Uhlenbeck probability measure, which is the normalized Gaussian measure $d\mu_{xy}(\omega)$ with mean $\omega_0(s)$ and covariance $(-\partial_s^2 + \alpha^2)^{-1}$, on continuous paths $\omega : [\sigma,\tau] \to \mathbb{R}$ with $\omega(\sigma) = y$ and $\omega(\tau) = x$ (see e.g. [**?**, **?**, **?**]). This formula can be proven in the same way as the one for time independent potentials (see [**?**], Equation (3.2.8)), i.e. by using the Kato-Trotter formula and evaluation of Gaussian measures on cylindrical sets. Since its proof contains a slight technical wrinkle, for the reader's convenience we present it below.

Now changing the variable of integration in (D.4) as $\omega = \omega_0 + \tilde{\omega}$, where $\tilde{\omega}(s)$ is a continuous path with boundary conditions $\tilde{\omega}(\sigma) = \tilde{\omega}(\tau) = 0$, using the translational change of variables formula $\int f(\omega)\,d\mu_{xy}(\omega) = \int f(\omega_0 + \tilde{\omega})\,d\mu(\tilde{\omega})$, which can be proven by taking $f(\omega) = e^{i\langle\omega,\zeta\rangle}$ and using (D.3) (see [**?**], Equation (9.1.27)) and omitting the tilde over ω we arrive at (D.1). □

There are at least three standard ways to prove (D.4): by using the Kato-Trotter formula, by expanding both sides of the equation in V and comparing the resulting series term by term and by using Ito's calculus (see [**?**, **?**, 27, **?**]). The first two proofs are elementary but involve tedious estimates while the third proof is based on a fair amount of stochastic calculus. For the reader's convenience, we present the first elementary proof of (D.4).

Lemma 49. *Equation* (D.4) *holds.*

Proof. In order to simplify our notation, in the proof that follows we assume, without losing generality, that $\sigma = 0$. We divide the proof into two parts. First we prove that for any fixed $\xi \in \mathcal{C}_0^\infty$ the following Kato-Trotter type formula holds

$$U(\tau,0)\xi = \lim_{n\to\infty} \prod_{0\le k\le n-1} U_0(\frac{k+1}{n}\tau, \frac{k}{n}\tau)e^{\int_{\frac{k\tau}{n}}^{\frac{(k+1)\tau}{n}} V(y,s)ds}\xi \tag{D.5}$$

in the L^2 space. We start with the formula

$$\begin{aligned} & U(\tau,0) - \prod_{0\le k\le n-1} U_0(\frac{k+1}{n}\tau, \frac{k}{n}\tau)e^{\int_{\frac{k\tau}{n}}^{\frac{(k+1)\tau}{n}} V(y,s)ds} \\ = & \prod_{0\le k\le n-1} U(\frac{k+1}{n}\tau, \frac{k}{n}\tau) - \prod_{0\le k\le n-1} U_0(\frac{k+1}{n}\tau, \frac{k}{n}\tau)e^{\int_{\frac{k\tau}{n}}^{\frac{(k+1)\tau}{n}} V(y,s)ds} \\ = & \sum_{0\le j\le n} \prod_{j\le k\le n-1} U_0(\frac{k+1}{n}\tau, \frac{k}{n}\tau)e^{\int_{\frac{k\tau}{n}}^{\frac{(k+1)\tau}{n}} V(y,s)ds} A_j U(\frac{j}{n}\tau, 0) \end{aligned}$$

with the operator

$$A_j := U_0(\frac{j+1}{n}\tau, \frac{j}{n}\tau)e^{\int_{\frac{j\tau}{n}}^{\frac{(j+1)\tau}{n}} V(y,s)ds} - U(\frac{j+1}{n}\tau, \frac{j}{n}\tau).$$

We observe that $\|U_0(\tau,\sigma)\|_{L^2\to L^2} \le 1$, and moreover by the boundness of V, the operator $U(\tau,\sigma)$ is uniformly bounded in τ and σ in any compact set. Consequently

$$\begin{aligned} & \|[U(\tau,0) - \prod_{0\le k\le n-1} U_0(\frac{k+1}{n}\tau, \frac{k}{n}\tau)e^{\int_{\frac{k\tau}{n}}^{\frac{(k+1)\tau}{n}} V(y,s)ds}]\xi\|_2 \\ \le & \max_j n\| \prod_{j\le k\le n-1} U_0(\frac{k+1}{n}\tau, \frac{k}{n}\tau)e^{\int_{\frac{k\tau}{n}}^{\frac{(k+1)\tau}{n}} V(y,s)ds} A_j U(\frac{j}{n}\tau, 0)\xi\|_2 \\ \lesssim & \max_j n\|A_j U(\frac{j}{n}\tau,0)\xi\|_2. \end{aligned} \tag{D.6}$$

We show below that

$$\max_j n\|A_j U(\frac{j}{n}\tau,0)\xi\|_2 \to 0, \tag{D.7}$$

as $n \to \infty$. Equations (D.7), (D.6), (D.13) and imply (D.5). This completes the first step.

In the second step we compute the integral kernel, $G_n(x,y)$, of the operator

$$G_n := \prod_{0\le k\le n-1} U_0(\frac{k+1}{n}\tau, \frac{k}{n}\tau)e^{\int_{\frac{k\tau}{n}}^{\frac{(k+1)\tau}{n}} V(\cdot,s)ds}$$

in (D.5). By the definition, $G_n(x,y)$ can be written as

$$G_n(x,y) = \int\cdots\int \prod_{0\le k\le n-1} U_{\frac{\tau}{n}}(x_{k+1},x_k)e^{\int_{\frac{k\tau}{n}}^{\frac{(k+1)\tau}{n}} V(x_k,s)ds} dx_1\cdots dx_{n-1} \tag{D.8}$$

with $x_n := x,\ x_0 := y$ and $U_\tau(x,y) \equiv U_0(0,\tau)(x,y)$ is the integral kernel of the operator $U_0(\tau,0) = e^{-L_0\tau}$. We rewrite (D.8) as

$$G_n(x,y) = U_\tau(x,y) \int e^{\sum_{k=0}^{n-1} \int_{\frac{k\tau}{n}}^{\frac{(k+1)\tau}{n}} V(x_k,s)\,ds} d\mu_n(x_1,\dots,x_n), \tag{D.9}$$

where

$$d\mu_n(x_1,\dots,x_n) := \frac{\prod_{0\le k\le n-1} U_{\frac{\tau}{n}}(x_{k+1},x_k)}{U_\tau(x,y)} dx_1 \dots dx_{k-1}.$$

Since $G_n(x,y)|_{V=0} = U_\tau(x,y)$ we have that $\int d\mu_n(x_1,\dots,x_n) = 1$. Let $\Delta := \Delta_1 \times \dots \times \Delta_n$, where Δ_j is an interval in $\mathbb{R}$. Define a cylindrical set

$$P^n_\Delta := \{\omega : [0,\tau] \to \mathbb{R} \mid \omega(0) = y,\ \omega(\tau) = x,\ \omega(k\tau/n) \in \Delta_k,\ 1 \le k \le n-1\}.$$

By the definition of the measure $d\mu_{xy}(\omega)$, we have $\mu_{xy}(P^n_\Delta) = \int_\Delta d\mu_n(x_1,\dots,x_n)$. Thus, we can rewrite (D.9) as

$$G_n(x,y) = U_\tau(x,y) \int e^{\sum_{k=0}^{n-1} \int_{\frac{k\tau}{n}}^{\frac{(k+1)\tau}{n}} V(\omega(\frac{k\tau}{n}),s)\,ds} d\mu_{xy}(\omega), \tag{D.10}$$

By the dominated convergence theorem the integral on the right hand side of (D.10) converges in the sense of distributions as $n \to \infty$ to the integral on the right hand side of (D.4). Since the left hand side of (D.10) converges to the left hand side of (D.4), also in the sense of distributions (which follows from the fact that G_n converges in the operator norm on L^2 to $U(\tau,\sigma)$), (D.4) follows. □

Now we prove (D.7). We have

$$\max_j n\|A_j U(\frac{j}{n}\tau,0)\xi\|_2 \lesssim n \max_j \|A_j + \mathcal{K}(\frac{k}{n}\tau, \frac{1}{n}\tau)\|_{L^2\to L^2} + \max_j n\|\mathcal{K}(\frac{j}{n}\tau,\frac{1}{n}\tau)U(\frac{j}{n}\tau,0)\xi\|_2, \tag{D.11}$$

where the operator $\mathcal{K}(\sigma,\delta)$ is defined as

$$\mathcal{K}(\sigma,\delta) := \int_0^\delta U_0(\sigma+\delta,\sigma+s)V(\sigma+s,\cdot)U_0(\sigma+s,\sigma)ds - U_0(\sigma+\delta,\sigma)\int_0^\delta V(\sigma+s,\cdot)ds \tag{D.12}$$

Now we claim that

$$\|A_j + \mathcal{K}(\frac{k}{n}\tau,\frac{1}{n}\tau)\|_{L^2\to L^2} \lesssim \frac{1}{n^2}, \tag{D.13}$$

and

$$\sup_{0\le\sigma\le\tau} \|\frac{1}{\delta}\mathcal{K}(\sigma,\delta)U(\sigma,0)\xi\|_2 \to 0. \tag{D.14}$$

We begin with proving the first estimate. By Duhamel's principle we have

$$U(\frac{j+1}{n}\tau,\frac{j}{n}\tau) = U_0(\frac{j+1}{n}\tau,\frac{j}{n}\tau) + \int_{\frac{j}{n}\tau}^{\frac{j+1}{n}\tau} U_0(\frac{j+1}{n}\tau,s)V(y,s)U(s,\frac{j}{n}\tau)ds.$$

Iterating this equation on $U(s,\frac{k}{n}\tau)$ and using the fact that $U(s,t)$ is uniformly bounded if s,t is on a compact set, we obtain

$$\|U(\frac{j+1}{n}\tau,\frac{j}{n}\tau)-U_0(\frac{j+1}{n}\tau,\frac{j}{n}\tau)-\int_0^{\frac{1}{n}\tau}U_0(\frac{j+1}{n}\tau,s)V(y,s)U_0(s,\frac{j}{n}\tau)ds\|_{L^2\to L^2}\lesssim\frac{1}{n^2}.$$

On the other hand we expand $e^{\int_{\frac{j\tau}{n}}^{\frac{(j+1)\tau}{n}}V(y,s)ds}$ and use the fact that V is bounded to get

$$\|U_0(\frac{j+1}{n}\tau,\frac{j}{n}\tau)e^{\int_{\frac{j\tau}{n}}^{\frac{(j+1)\tau}{n}}V(y,s)ds}-U_0(\frac{j+1}{n}\tau,\frac{j}{n}\tau)-U_0(\frac{j+1}{n}\tau,\frac{j}{n}\tau)\int_{\frac{j\tau}{n}}^{\frac{(j+1)\tau}{n}}V(y,s)ds\|_{L^2\to L^2}\lesssim\frac{1}{n^2}.$$

By the definition of $\mathcal{K}$ and A_j we complete the proof of (D.13). A proof of (D.14) is given in the lemma that follows.

Lemma 50. *For any $\sigma\in[0,\tau]$ and $\xi\in\mathcal{C}_0^\infty$ we have, as $\delta\to0^+$,*

$$\sup_{0\le\sigma\le\tau}\|\frac{1}{\delta}\mathcal{K}(\sigma,\delta)U(\sigma,0)\xi\|_2\to0. \tag{D.15}$$

Proof. If the potential term, V, is independent of τ, then the proof is standard (see, e.g. [27]). We use the property that the function V is Lipschitz continuous in time τ to prove (D.15). The operator $\mathcal{K}$ can be further decomposed as

$$\mathcal{K}(\sigma,\delta)=\mathcal{K}_1(\sigma,\delta)+\mathcal{K}_2(\sigma,\delta)$$

with

$$\mathcal{K}_1(\sigma,\delta):=\int_0^\delta U_0(\sigma+\delta,\sigma+s)V(\sigma,\cdot)U_0(\sigma+s,\sigma)ds-\delta U_0(\sigma+\delta,\sigma)V(\sigma,\cdot)$$

and

$$\mathcal{K}_2(\sigma,\delta):=\int_0^\delta U_0(\sigma+\delta,\sigma+s)[V(\sigma+s,\cdot)-V(\sigma,\cdot)]U_0(\sigma+s,\sigma)ds-U_0(\sigma+\delta,\sigma)\int_0^\delta[V(\sigma+s,\cdot)-V(\sigma,\cdot)]ds.$$

Since $U_0(\tau,\sigma)$ are uniformly L^2-bounded and V is bounded, we have $U(\tau,\sigma)$ is uniformly L^2-bounded. This together with the fact that the function $V(\tau,y)$ is Lipschitz continuous in τ implies that

$$\|\mathcal{K}_2(\sigma,\delta)\|_{L^2\to L^2}\lesssim2\int_0^\delta sds=\delta^2.$$

We rewrite $\mathcal{K}_1(\sigma,\delta)$ as

$$\mathcal{K}_1(\sigma,\delta)=\int_0^\delta U_0(\sigma+\delta,\sigma+s)\{V(\sigma,\cdot)[U_0(\sigma+s,\sigma)-1]-[U_0(\sigma+s,\sigma)-1]V(\sigma,\cdot)\}ds.$$

Let $\xi(\sigma)=U(\sigma,0)\xi$. We claim that for a fixed $\sigma\in[0,\tau]$,

$$\|\mathcal{K}_1(\sigma,\delta)\xi(\sigma)\|_2=o(\delta). \tag{D.16}$$

Indeed, the fact $\xi_0 \in \mathcal{C}_0^\infty$ implies that $L_0\xi(\sigma),\ L_0V(\sigma)\xi(\sigma) \in L^2$. Consequently (see [?])

$$\lim_{s\to 0^+} \frac{(U_0(\sigma+s,\sigma)-1)g}{s} \to L_0 g,$$

for $g = \xi(\sigma)$ or $V(\sigma,y)\xi(\sigma)$ which implies our claim. Since the set of functions $\{\xi(\sigma)|\sigma \in [0,\tau]\} \subset L_0L^2$ is compact and $\|\frac{1}{\delta}K_1(\sigma,\delta)\|_{L^2\to L^2}$ is uniformly bounded, we have (D.16) as $\delta \to 0$ uniformly in $\sigma \in [0,\tau]$.

Collecting the estimates on the operators $\mathcal{K}_i,\ i = 1,2$, we arrive at (D.15). □

Note that on the level of finite dimensional approximations the change of variables formula can be derived as follows. It is tedious, but not hard, to prove that

$$\prod_{0\le k\le n-1} U_n(x_{k+1},x_k) = e^{-\alpha\frac{(x-e^{-\alpha\tau}y)^2}{2(1-e^{-2\alpha\tau})}} \prod_{0\le k\le n-1} U_n(y_{k+1},y_k)$$

with $y_k := x_k - \omega_0(\frac{k}{n}\tau)$. By the definition of $\omega_0(s)$ and the relations $x_0 = y$ and $x_n = x$ we have

$$G_n(x,y) = U_\tau(x,y)G_n^{(1)}(x,y) \tag{D.17}$$

where

$$G_n^{(1)}(x,y) := \frac{1}{4\pi\sqrt{\alpha}(1-e^{-2\alpha\tau})} \int\cdots\int \prod_{0\le k\le n-1} U_n(y_{k+1},y_k)e^{\int_{\frac{k\tau}{n}}^{\frac{(k+1)\tau}{n}} V(y_k+\omega_0(\frac{k\tau}{n}),s)ds} dy_1\cdots dy_{k-1}. \tag{D.18}$$

Since $\lim_{n\to\infty} G_n\xi$ exists by (D.15), we have $\lim_{n\to\infty} G_n^{(1)}\xi$ (in the weak limit) exists also. As shown in [?], $\lim_{n\to\infty} G_n^{(1)} = \int e^{\int_0^\tau V(\omega_0(s)+\omega(s),s)ds} d\mu(\omega)$ with $d\mu$ being the (conditional) Ornstein-Uhlenbeck measure on the set of path from 0 to 0. This completes the derivation of the change of variables formula.

Remark 8. *In fact, Equations (D.5), (D.17) and (D.18) suffice to prove the estimate in Corollary 33.*

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