Mini-course "Analysis of nonlinear evolution equations with the help of integrability"

Lecture #1 "Energy method in orbital stability of solitary waves"

We are concerned here with energy preserving evolution equations. As a prototypical example, we consider the cubic focussing NLS equation in one spatial dimension, although the methods can be applied to many other equations as well. Consider the Cauchy problem

$$\begin{cases} i\Psi_t + \Psi_{xx} + 2|\Psi|^2\Psi = 0, & t>0 \quad (NLS) \\ \Psi|_{t=0} = \Psi_0 \end{cases}$$

where Ψ_0 is defined in a subset of Sobolev space $H^s(\mathbb{R})$. Since (NLS) is a semi-linear equation, a contraction method can be applied for $s > 1/2$ to obtain a local solution $\Psi \in C((-t_0, t_0), H^s(\mathbb{R}))$: $\Psi|_{t=0} = \Psi_0 \in H^s(\mathbb{R})$ where $t_0 > 0$ and $s > 1/2$.

(energy space)

Lemma. $\forall \Psi_0 \in H^1(\mathbb{R})$, $\exists!\ \Psi \in C(\mathbb{R}, H^1(\mathbb{R}))$: Ψ depends continuously on Ψ_0.

Proof: (NLS) admits conservation of $N(\Psi) = \|\Psi\|_{L^2}^2 < \infty$ and $E(\Psi) = \|\Psi_x\|_{L^2}^2 - \|\Psi\|_{L^4}^4 < \infty$ if $\Psi \in H^1(\mathbb{R})$.

By Gagliardo-Nirenberg inequality, we have

$$\exists C>0: \|\Psi\|_{L^4} \le C\|\partial_x\Psi\|_{L^2}^{1/4}\|\Psi\|_{L^2}^{3/4}, \quad \forall \Psi \in H^1(\mathbb{R})$$

so that

$$E(\Psi) \ge \|\Psi_x\|_{L^2}^2 - C\|\Psi_x\|_{L^2}\|\Psi\|_{L^2}^3$$
$$\ge \left(\|\Psi_x\|_{L^2} - \tfrac{1}{2}CN^{3/2}_{(\Psi)}\right)^2 - \tfrac{1}{4}C^2N^3_{(\Psi)}$$

Since $N(\Psi) = N(\Psi_0) \equiv N_0$, then $\|\Psi(t)\|_{L^2} = N_0^{1/2}$

$E(\Psi) = E(\Psi_0) \equiv \mathcal{E}_0$ $\quad \|\Psi_x(t)\|_{L^2} \le \frac{1}{2}CN_0^{3/2} + \sqrt{\mathcal{E}_0 + \frac{1}{4}C^2N_0^3}$

and $\|\Psi(t)\|_{H^1} \le C(\|\Psi_0\|_{H^1})$, $\forall t \in \mathbb{R}$. ∎

We can now consider a family of solitary waves

$$\Psi(x,t) = Q_\omega(x - x_0 - vt)\, e^{i\theta_0 + \frac{i}{2}vx + i(\omega - v^2/4)t}$$

where x_0, θ_0, v are arbitrary, $\omega \in \mathbb{R}_+$ is arbitrary and $Q_\omega(x) := \sqrt{\omega}\,\mathrm{sech}(\sqrt{\omega}\,x)$ is a solution of

$$Q_\omega''(x) - \omega Q_\omega(x) + 2 Q_\omega^3(x) = 0, \quad x \in \mathbb{R}.$$

In what follows, we can neglect $x_0 = v = 0$ by considering even functions. Parameters $\theta_0 \in \mathbb{R}$ and $\omega > 0$ are important to define the concept of orbital stability of solitary waves.

Lemma. Q_ω is a critical point of the energy functional

$$\Lambda_\omega(\psi) := E(\psi) + \omega N(\psi)$$

Proof:

$$\Lambda_\omega'(\psi) = -\frac{d^2\psi}{dx^2} - 2|\psi|^2\psi + \omega\psi = 0 \quad \text{at } \psi = Q_\omega. \ \blacksquare$$

To study stability of Q_ω, we can consider the second variation of Λ_ω that determine convexity of Λ_ω at Q_ω.

$\Psi = Q_\omega + u + iw$:

$$\Lambda_\omega(Q_\omega + u + iw) = \int_{\mathbb{R}} \left[(Q_\omega + u)_x^2 + w_x^2 + \omega\left((Q_\omega + u)^2 + w^2\right) - \left((Q_\omega + u)^2 + w^2\right)^2 \right] dx$$

$$= \Lambda_\omega(Q_\omega) + \langle L_+ u, u\rangle_{L^2} + \langle L_- w, w\rangle_{L^2} + R(u,w)$$

where $R(u,w)$ is cubic in $\|u\|_{H^1} + \|w\|_{H^1}$. We obtain

$$L_+ := -\partial_x^2 + \omega - 6 Q_\omega^2(x)$$

$$L_- := -\partial_x^2 + \omega - 2 Q_\omega^2(x)$$

Note that as $|x| \to \infty$, $Q_\omega(x) \to 0$ and $L_\pm \longrightarrow -\partial_x^2 + \omega$ with $\sigma_c(\omega - \partial_x^2) = [\omega, \infty)$

Furthermore, symmetries of NLS yield

$$\begin{cases} L_+ Q_\omega'(x) = 0 \\ L_- Q_\omega(x) = 0 \end{cases}$$

By Sturm's theorem, $\sigma(L_-) \geq 0$, in fact $\sigma(L_-) = \{0\} \cup [\omega, \infty)$ and $\sigma(L_+) \ngeq 0$, in fact $\sigma(L_+) = \{\lambda_{-1}\} \cup \{0\} \cup [\omega, \infty)$ with $\lambda_{-1} = -3\omega$. Hence, Q_ω is not a minimum but a saddle point of Λ_ω

Nevertheless, we can show the following

Lemma. $E(Q_\omega) \le E(\Psi) \quad \forall \Psi \in M_\omega \subset H^1(\mathbb{R})$,
where
$$M_\omega := \{ \Psi \in H^1(\mathbb{R}) : \|\Psi\|_{L^2} = \|Q_\omega\|_{L^2} \}.$$

Proof. ~~Be~~ let us define a constrained L^2 space
$$L^2_c = \{ u \in L^2(\mathbb{R}) : \langle u, Q_\omega \rangle_{L^2} = 0 \}.$$
By the second derivative test, it suffices to show that $\sigma(L_+|_{L^2_c}) \ge 0$, where $L_+|_{L^2_c} = P_c L_+ P_c$
if $\frac{d}{d\omega} N(Q_\omega) > 0$. and $P_c : L^2 \to L^2_c$ is an orthogonal projection
The result follows from the Vakhitov-Kolokolov method.
$$(\mu - L_+) u = Q_\omega, \quad \mu \in \mathbb{R}.$$
$\forall \mu \notin \sigma(L_+)$, let us define $F(\mu) := \langle (\mu - L_+)^{-1} Q_\omega, Q_\omega \rangle_{L^2}$

Then, ① $F(\mu) = 0$ if $\mu \in \sigma(L_+|_{L^2_c}) \setminus \sigma(L_+)$

② $F(\mu)$ is smooth if $\mu \notin \sigma(L_+)$ with $F'(\mu) < 0$

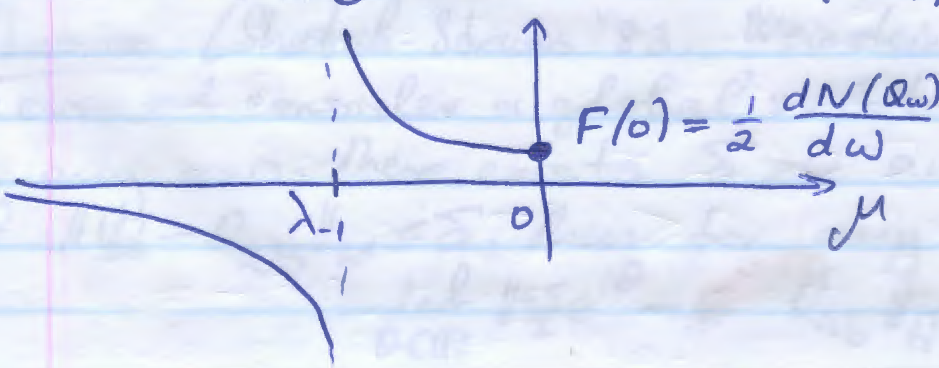
③ $F(\mu) \to -0$ as $\mu \to -\infty$

④ $F(\mu) \to -\langle L_+^{-1} Q_\omega, Q_\omega \rangle_{L^2}$ as $\mu \to 0$

⑤ $F(\mu)$ has an infinite jump at $\mu = \lambda_{-1} < 0$

Now we compute
$$-\frac{d^2 Q_\omega}{dx^2} - 2 Q_\omega^3 + \omega Q_\omega = 0$$
$$L_+ \frac{\partial Q_\omega}{\partial \omega} = -Q_\omega$$
so that
$$F(0) = -\langle L_+^{-1} Q_\omega, Q_\omega \rangle_{L^2} = \langle \frac{\partial Q_\omega}{\partial \omega}, Q_\omega \rangle_{L^2} = \frac{1}{2} \frac{dN(Q_\omega)}{d\omega}$$

Since $N(Q_\omega) := \int \omega \operatorname{sech}^2(\sqrt{\omega} x) dx = 2\sqrt{\omega}$
we have $F(0) > 0$ so that $\sigma(L_+|_{L^2_c}) \ge 0$. ∎



To prove orbital stability, we can consider a decomposition of solutions in a small neighborhood of a solitary wave.

Fix $\omega_0 > 0$ and

Lemma. Assume that for small $\varepsilon > 0$, $\Psi \in H^1(\mathbb{R})$ satisfies
$$\inf_\theta \|\Psi e^{i\theta} - Q_{\omega_0}\|_{H^1} \le \varepsilon \quad \text{with } \theta_0 = \arg\inf_\theta \|\Psi e^{i\theta} - Q_{\omega_0}\|$$
Then, Ψ can be uniquely represented by
$$e^{i\theta}\Psi = Q_\omega + u + iw.$$
such that $\langle u, Q_{\omega_0}\rangle_{L^2} = \langle w, Q_{\omega_0}\rangle_{L^2} = 0$
where the unique values of θ and ω satisfy
$$|\omega - \omega_0| + |\theta - \theta_0| \le C\varepsilon, \quad \text{for some } C > 0.$$

Proof: Let us define
$$F(\omega, \theta) := \langle e^{i\theta}\Psi, Q_{\omega_0}\rangle_{L^2} - \langle Q_\omega, Q_{\omega_0}\rangle_{L^2}$$
We are looking for zeros of $F(\omega, \theta)$.
Note that $|F(\omega_0, \theta_0)| = |\langle e^{i\theta_0}\Psi - Q_{\omega_0}, Q_{\omega_0}\rangle_{L^2}| \le C_0\varepsilon$ is small.
By differentiating, we obtain
$$\partial_\omega F(\omega_0, \theta_0) = -\langle \frac{\partial Q_\omega}{\partial \omega}, Q_{\omega_0}\rangle_{L^2} = -\frac{1}{2}\frac{dN(Q_\omega)}{d\omega}\Big|_{\omega=\omega_0} \neq 0$$
and
$$\partial_\theta F(\omega_0, \theta_0) = i\langle e^{i\theta_0}\Psi - Q_{\omega_0}, Q_{\omega_0}\rangle_{L^2} + i\|Q_{\omega_0}\|_{L^2}^2 \neq 0$$
By the implicit function theorem, $\exists!\,(\omega, \theta)$ near (ω_0, θ_0) such that the assertion of the lemma is satisfied. ●

With this decomposition, we can now prove the main result on the orbital stability of a solitary wave in $H^1(\mathbb{R})$.

Theorem (Shatah-Strauss '83; Weinstein '86) Fix $\omega_0 > 0$. Let $\Psi_0 \in H^1(\mathbb{R})$ be even and consider a global solution Ψ of (NLS) in $H^1(\mathbb{R})$. For any $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|\Psi_0 - Q_{\omega_0}\|_{H^1} \le \delta$, then for any $t \in \mathbb{R}$,
$$\inf_{\theta \in \mathbb{R}} \|\Psi e^{i\theta} - Q_{\omega_0}\|_{H^1} < \varepsilon.$$

Remark. The assumption that Ψ_0 is even is added to eliminate $\mathrm{Ker}(L_+) = \mathrm{span}\{\frac{dQ_{\omega_0}}{dx}\}$.

Define

$$H'_c = \{u \in H'(\mathbb{R}) : \langle u, Q_{\omega_0}\rangle_{L^2} = 0\}$$

We have shown that $\exists C > 0$ such that

$$\left.\begin{array}{l} \langle L_+ u, u\rangle_{L^2} \geq C\|u\|_{H'}^2 \\ \langle L_- w, w\rangle_{L^2} \geq C\|w\|_{H'}^2 \end{array}\right\} \quad u, w \in H'_c.$$

Let $D(\omega) := \Lambda_{\omega}(Q_\omega) = E(Q_\omega) + \omega N(Q_\omega)$

Then $D'(\omega) = N(Q_\omega)$ and $D''(\omega) = \frac{dN(Q_\omega)}{d\omega} > 0$.

Now we substitute a decomposition

$\Psi e^{i\theta} = Q_\omega + u + iw$ for any $t \in \mathbb{R}$ as long as $\|\Psi e^{i\theta} -$

without writing modulation equations.

We have

$$\Lambda_\omega(\Psi e^{i\theta}) = D(\omega) + \langle L_+ u, u\rangle_{L^2} + \langle L_- w, w\rangle$$
$$+ O\left(|\omega - \omega_0|(\|u\|_{H'} + \|w\|_{H'})^2, (\|u\|_{H'} + \|w\|_{H'})^3\right)$$

Now, $\Lambda_\omega(\Psi e^{i\theta}) - \Lambda_\omega(Q_{\omega_0}) = D(\omega) - D(\omega_0) - (\omega - \omega_0) N(Q_{\omega_0})$

$$+ \langle L_+ u, u\rangle_{L^2} + \langle L_- w, w\rangle_{L^2} + O(3)$$

$$\geq \frac{1}{2} D''(\omega_0)(\omega - \omega_0)^2 + C(\|u\|_{H'}^2 + \|w\|_{H'}^2) + O(3)$$

$$= \frac{1}{2} D''(\omega_0)(\omega - \omega_0)^2 + C\|\Psi e^{i\theta} - Q_\omega\|_{H'}^2 + O(3).$$

On the other hand

$\Lambda_\omega(\Psi e^{i\theta}) - \Lambda_\omega(Q_{\omega_0}) = E(\Psi e^{i\theta}) - E(Q_{\omega_0}) + \omega N(\Psi e^{i\theta}) - \omega N(Q_{\omega_0})$

$= E(\Psi_0) - E(Q_{\omega_0}) + \omega_0(N(\Psi_0) - N(Q_{\omega_0})) + (\omega - \omega_0)(N(\Psi_0) - N$

From the initial data, we have $\|\Psi_0 - Q_{\omega_0}\| \leq \delta$,

hence

$$|\Lambda_\omega(\Psi e^{i\theta}) - \Lambda_\omega(Q_{\omega_0})| \leq C\delta^2 + C\delta^2|\omega - \omega_0|$$

From the lower bound, we have

$$\begin{cases} |\omega - \omega_0| \leq C\delta \\ \|e^{i\theta}\Psi - Q_\omega\|_{H'} \leq C\delta. \end{cases}$$

Then, the triangle inequality yields

$$\|\Psi e^{i\theta} - Q_{\omega_0}\|_{H'} \leq \|e^{i\theta}\Psi - Q_\omega\|_{H'} + \|Q_\omega - Q_{\omega_0}\|_{H'} \leq C\delta \equiv \varepsilon. \blacksquare$$

Hence $\boxed{\delta = \varepsilon/C}$

Lecture #2 "Backlund transformation and stability of a solitary wave in L^2"

Global well-posedness of (NLS) in L^2 was established by Tsutsumi by using Strichartz estimates.

Th (Tsutsumi, 1987) Given $\Psi_0 \in L^2(\mathbb{R})$, there exists a unique solution $\Psi \in C(\mathbb{R}, L^2(\mathbb{R})) \cap L^4_{loc}(\mathbb{R}, L^\infty(\mathbb{R}))$ such that

$$\|\Psi(t)\|_{L^2} = \|\Psi_0\|_{L^2}, \quad \forall t \in \mathbb{R}.$$

We can now try studying stability of a solitary wave of (NLS): $\Psi(x,t) = Q_\omega(x) e^{i\theta_0 + i\omega t}$, $Q_\omega(x) = \sqrt{\omega}\,\mathrm{sech}(\sqrt{\omega}\, x)$ in $L^2(\mathbb{R})$ (space of low regularity), hence, conservation of energy is neglected.

Theorem (Mizumachi-Pelinovsky '2012) Fix $\omega_0 > 0$. Let $\Psi_0 \in L^2(\mathbb{R})$ yield a global solution $\Psi \in C(\mathbb{R}, L^2(\mathbb{R}))$ of (N
There exist $C > 0$ and $\delta > 0$ such that if $\|\Psi_0 - Q_{\omega_0}\|_{L^2} \leq \delta$,
Then there exists ω satisfying

$$|\omega - \omega_0| \leq C \|\Psi_0 - Q_{\omega_0}\|_{L^2}$$

such that for all $t \in \mathbb{R}$,

$$\inf_{(a,\theta)\in\mathbb{R}^2} \| e^{i\theta} \Psi(\cdot + a, t) - Q_\omega e^{i\omega_0 t} \|_{L^2} \leq C \|\Psi_0 - Q_{\omega_0}$$

Remarks:

1. One can formulate this result explicitly in ε and δ.
2. By triangle inequality, one can replace Q_ω with Q_{ω_0}.
3. (a,θ) can be characterized more precisely.
4. Ψ_0 may support several solitons with small L^2-nor[m] hence, no asymptotic stability result can be stated without additional requirements such as $\|\Psi_0 - Q_{\omega_0}\|_{L^1}$ is small.

The technique to prove the theorem relies on the integrability properties of the cubic (NLS).

Let $Q(\psi) = \begin{bmatrix} 0 & \psi \\ -\bar{\psi} & 0 \end{bmatrix}$ and consider a solution of the linear system $\vec{\Psi} \in C^2(\mathbb{R}^2, \mathbb{C}^2)$

$$\begin{cases} \dfrac{\partial \vec{\Psi}}{\partial x} = \lambda \sigma_3 \vec{\Psi} + Q(\psi)\vec{\Psi} & \text{(Lax)} \\ \dfrac{\partial \vec{\Psi}}{\partial t} = 2i\lambda^2 \sigma_3 \vec{\Psi} + i|\psi|^2 \sigma_3 \vec{\Psi} + 2i\lambda Q(\psi)\vec{\Psi} + Q(i\psi_x)\vec{\Psi} \end{cases}$$

Then, ~~[illegible]~~

$\dfrac{\partial^2 \vec{\Psi}}{\partial t \partial x} = \dfrac{\partial^2 \vec{\Psi}}{\partial x \partial t}$ is equivalent to $i\dfrac{\partial \psi}{\partial t} + \dfrac{\partial^2 \psi}{\partial x^2} + 2|\psi|^2\psi = 0$ where $\psi, \psi_t, \psi_{xx} \in C^0(\mathbb{R}^2)$.

Remark. If $\psi \in C(\mathbb{R}, H^3(\mathbb{R})) \cap C^1(\mathbb{R}, H^1(\mathbb{R}))$ for $\psi_0 \in H^3$ then ψ is a classical solution of (NLS).

Backlund transformation: Fix $\lambda > 0$ and let $\vec{\Psi} \neq \vec{0}$ be a solution of (Lax) for a given ψ. Then,

$$\tilde{\psi} := -\psi - \frac{4\lambda \Psi_1 \bar{\Psi}_2}{|\Psi_1|^2 + |\Psi_2|^2}$$ is another solution of (NLS)

and

$$\tilde{\Psi} := \frac{1}{|\Psi_1|^2 + |\Psi_2|^2} \begin{bmatrix} \bar{\Psi}_2 \\ \bar{\Psi}_1 \end{bmatrix}$$ is a solution of (Lax) for the same λ and different $\tilde{\psi}$

Ex. $\psi = 0$. We obtain a solution of (Lax):

$$\Psi = \begin{bmatrix} e^{\lambda x + 2i\lambda^2 t} \\ -e^{-\lambda x - 2i\lambda^2 t} \end{bmatrix}$$

Then, (BT) gives

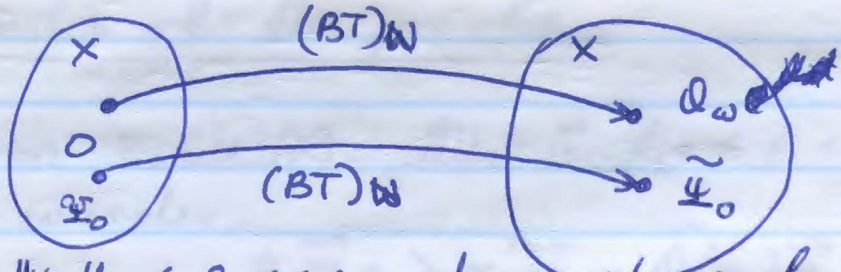
$$\tilde{\psi} = 2\lambda e^{4i\lambda^2 t} \operatorname{sech}(2\lambda x) = Q_\omega(x) e^{i\omega t}, \quad \omega = 4\lambda^2 > 0.$$

Moreover, we obtain

$$\tilde{\Psi} = \frac{1}{2} \operatorname{sech}(2\lambda x) \begin{bmatrix} -e^{-\lambda x + 2i\lambda^2 t} \\ e^{\lambda x - 2i\lambda^2 t} \end{bmatrix}$$

is a decaying solution of (Lax) for $\psi = Q_\omega(x) e^{i\omega t}$. Additionally, if $\tilde{\Psi}$ is taken for $\tilde{\Psi}$ and is used in (BT), then (BT) results in the solution $\tilde{\psi} = 0 = \psi$.

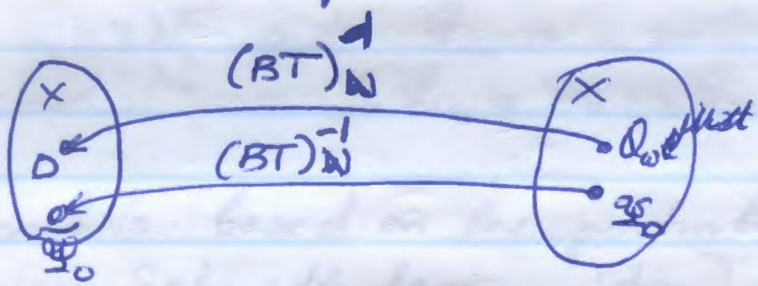
Questions:



Q1: If $\|\psi_0\|_{L^2} \le \varepsilon \ll 1$, does it imply that $\|\tilde{\psi}_0 - Q_\omega\|_{L^2} \le C\varepsilon$?

A1: YES and several parameters can be used in this transf

Q2:



Q2: If $\|\psi_0 - Q_\omega\|_{L^2} \le \varepsilon \ll 1$, does it imply that $\|\tilde{\psi}_0\|_{L^2} \le C\varepsilon$?

A2. NO! - because a 2-soliton solution is generated by $(BT)_\omega$. One needs to find a decaying eigenfu of (Lax) for ψ_0 close to Q_ω, which corresponds to a different eigenvalue $\tilde{\omega}$ close to ω.

Scheme of the proof of the Theorem:

#1: Given $\psi_0 \in H^3(\mathbb{R})$: $\|\psi_0 - Q_{\omega_0}\|_{L^2} \le \varepsilon \ll 1$, find ω and $(BT)^{-1}_\omega$ such that $|\omega - \omega_0| \le C\varepsilon$ and $\|(BT)^{-1}_\omega \psi_0\|_{L^2} \le C\varepsilon$.

#2: Obtain a solution of (NLS), starting with $(BT)^{-1}_\omega \psi_0$, denoted by $T(t)\,(BT)^{-1}_\omega \psi_0$.

#3: Given a solution $T(t)(BT)_\omega \Psi_0$ use $(BT)_\omega$ to construct $\Psi(t) = (BT)_\omega \, T(t) \, (BT)_\omega^{-1} \Psi_0$ and prove that $\inf_{(a,\theta)\in\mathbb{R}^2} \| \Psi(\cdot + a) e^{i\theta} - Q_\omega e^{i\omega t} \|_{L^2} \leq C\varepsilon$.
Here (a,θ) arise in the construction of $(BT)_\omega$.

#4: Run an approximation argument for a sequence $\{\Psi_0^{(n)}\}_{n\in\mathbb{N}}$ in $H^3(\mathbb{R})$ so that $\Psi_0^{(n)} \xrightarrow[n\to\infty]{} \Psi_0$ in $L^2(\mathbb{R})$.

Details of the proofs:

#1: Given $\Psi_0 \in H^3(\mathbb{R})$: $\|\Psi_0 - Q_{\omega_0}\|_{L^2} \leq \varepsilon \ll 1$ and consider

$$\partial_x \vec{\Psi} = \lambda \sigma_3 \vec{\Psi} + Q(\Psi_0) \vec{\Psi}$$

$\exists \lambda > 0$ and $\vec{\Psi} \in H^1(\mathbb{R})$ such that

$$|\omega - \omega_0| + \| \vec{\Psi} - \vec{\Psi}_{\omega_0} \|_{H^1} \leq C \|\Psi_0 - Q_{\omega_0}\|_{L^2}$$

where $\omega = 4\lambda^2$ and $\vec{\Psi}_{\omega_0}$ is a solution of
$\omega_0 = 4\lambda_0^2$

$$\partial_x \vec{\Psi}_{\omega_0} = \lambda_0 \sigma_3 \vec{\Psi}_{\omega_0} + Q(Q_{\omega_0}) \vec{\Psi}_{\omega_0}.$$

The proof is based on the perturbation theory (Lyapunov-Schmidt decomposition) for linear systems.

Now set $\tilde{\Psi}_0 := -\Psi_0 - \dfrac{4\lambda \Psi_1 \overline{\Psi}_2}{|\Psi_1|^2 + |\Psi_2|^2}$
and prove with estimates in exponentially weighted spaces that

$$\| \tilde{\Psi}_0 \|_{L^2} \leq C \|\Psi_0 - Q_{\omega_0}\|_{L^2}.$$

#2: This step is trivially adopted from the well-posedness theory in $H^3(\mathbb{R})$ and $L^2(\mathbb{R})$.

#3. We now have $\tilde{\Psi}(t) \in C(\mathbb{R}, H^1(\mathbb{R}))$: $\|\tilde{\Psi}(t)\|_{L^2} \le C \epsilon$
and consider the (Lax) system

$$\partial_x \vec{\psi} = \lambda \sigma_3 \vec{\psi} + Q(\tilde{\Psi}(t)) \vec{\psi}$$

If $\tilde{\Psi}(t) \equiv 0$, then two fundamental solutions exist

$$\vec{\psi}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{\lambda x + 2i\lambda^2 t} \qquad \vec{\psi}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-\lambda x - 2i\lambda^2 t}$$

Proposition. For $\tilde{\Psi}(t)$ with ϵ-small L^2-norm, there exist two fundamental solution in the form

$$\vec{\psi}_1 = \vec{\varphi}\, e^{\lambda x + 2i\lambda^2 t}, \quad \vec{\psi}_2 = \vec{\chi}\, e^{-\lambda x - 2i\lambda^2 t}$$

where $\|\varphi_1 - 1\|_{L^\infty} + \|\varphi_2\|_{L^\infty \cap L^2} \le C \|\tilde{\Psi}\|_{L^2}$
and

$$\|\chi_1\|_{L^\infty \cap L^2} + \|\chi_2 - 1\|_{L^\infty} \le C \|\tilde{\Psi}\|_{L^2}$$

The proof is based on the integral formulation of (Lax) and a contraction principle in $L^\infty \times (L^2 \cap L^\infty)$.

General solution is $\vec{\psi} = c_1 \vec{\varphi}\, e^{\lambda x + 2i\lambda^2 t} + c_2 \vec{\chi}\, e^{-\lambda x - 2i\lambda^2 t}$
where (c_1, c_2) are x-independent but may depend on
These constants determine a, θ and are excluded if we define the $\inf\limits_{(a,\theta)} \|\cdot\|_{L^2}$.

Finally, let $\Psi(t) = -\tilde{\Psi}(t) - \dfrac{4\psi_1 \bar{\psi}_2}{|\psi_1|^2 + |\psi_2|^2}$
and prove that

$$\|\Psi(t) - Q_\omega(\cdot - a(t))\, e^{i\omega t + i\theta(t)}\|_{L^2} \le C \|\tilde{\Psi}(t)\|_{L^2}$$

where

$$c_1 = e^{a + i\theta}, \quad c_2 = e^{-(a + i\theta)}$$

This estimate is again obtained in the exponentially weighted spaces.

#4: Approximation arguments work because the bounds are independent
$|\omega_n - \omega_0| + \inf\limits_{(a,\theta)} \|\Psi_n(\cdot + a) e^{i\theta} - Q_{\omega_n}\|_{L^2} \le C \|\Psi_0 - Q_{\omega_0}\|_{L^2}$

Lecture #3 "Inverse scattering and stability of multiple solitary waves in L^2"

We continue working with the cubic focussing NLS equation in L^2

$$\begin{cases} i\Psi_t + \Psi_{xx} + 2|\Psi|^2\Psi = 0, & t > 0 \qquad \text{(NLS)} \\ \Psi|_{t=0} = \Psi_0 \in L^2(\mathbb{R}) \end{cases}$$

Besides a single solitary wave $\Psi(x,t) = Q_\omega(x)e^{i\omega t}$, this equation has a family of multiple solitary waves. The n-soliton solutions are known explicitly from available algebraic methods. If 1-soliton has 4 parameters, then n-solitons have $(4n)$-parameters.

Ex. A two-soliton solutions has two phase parameters θ_1, θ_2; two translational parameters x_1, x_2; two velocities v_1, v_2; and two frequencies ω_1, ω_2. If $v_1 = v_2$, then the 2-soliton solution is a periodic breather. If $v_1 \neq v_2$, then it describes a collission of two solitons.

Functional analytical methods to study multiple solitons we employed by Perelman (2004), Martel-Merle-Tsai (200… Holmer-Lin (2012). These results are usually weak because large separation between solitary waves is required for analysis and the justification is limited to finite time intervals. Using inverse scattering, we can get either orbital or asymptotic stability results ~~for the~~ all times.

Orbital stability of n solitons in NLS is proved by T. Kapitula (2006) in $H^n(\mathbb{R})$ by using $(n+1)$ conserved quantities and the energy method. Here, with inverse scattering, we can work in a subset of $L^2(\mathbb{R})$, namely, in $L^2_s(\mathbb{R}) := \{u \in L^2 : (1+x^2)^{s/2} u \in$

<u>Theorem</u> (S. Cuccagna – D.P. '2014) Fix $s \in (\frac{1}{2}, 1]$ and consider $\Psi = \Phi_{\omega_0, v_0}(\cdot - v_0 t - x_0) e^{i\omega_0 t + i\theta_0}$. There are $\varepsilon_0 > 0$, $C > 0$, and $T > 0$ such that if $\Psi_0 \in L^2_s(\mathbb{R})$ satisfies $\varepsilon := \| \Psi_0 - Q_{\omega_0} \|_{L^2_s} \le \varepsilon_0$, then $\exists (\omega, v)$ and $(x_\pm, \theta_\pm)$ such that

$$|\omega - \omega_0| + |v - v_0| + |x_\pm - x_0| + |\theta_\pm - \theta_0| \le C\varepsilon$$

and $\forall \pm t \geq T$,

$$\| \Psi(\cdot, t) - \Phi_{\omega, v}(\cdot - vt - x_\pm) e^{i\omega t + i\theta_\pm} \|_{L^\infty} \le \frac{C\varepsilon}{(1+|t|)^{1/2}}$$

<u>Theorem</u> (A. Contreras – D.P. '2014) Fix $s \in (\frac{1}{2}, 1]$ and consider n-soliton Ψ_n with $4n$ parameters $\{\omega_k, v_k, x_k, \theta_k\}$ such that $(\omega_k, v_k) \neq (\omega_m, v_m)$ for $k \neq m$. Then, there are $\varepsilon_0 > 0$ and $C > 0$ such that if $\Psi_0 \in L^2_s(\mathbb{R})$ satisfies

$$\varepsilon := \| \Psi_0 - \Psi_n|_{t=0} \|_{L^2_s} \le \varepsilon_0$$

then $\exists \{\omega_k', v_k', x_k', \theta_k'\}_{k=1}^n$ such that

$$\max_{1 \le k \le n} |\omega_k' - \omega_k| + |v_k' - v_k| + |x_k' - x_k| + |\theta_k' - \theta_k| \le C\varepsilon$$

and for all $t \in \mathbb{R}$,

$$\| \Psi(\cdot, t) - \Psi_n'(\cdot, t) \|_{L^2} \le C\varepsilon.$$ ~~t∈R~~

<u>Remarks</u>: ① $\| \Psi(\cdot, t) \|_{L^2_s}$ is expected to grow as $t \to \infty$ because $\| x \Psi \|_{L^2_1} = O(t)$ generally in (NLS)

② The orbital stability result is expected if

$$\| \Psi_0 - \Psi_n|_{t=0} \|_{L^2} \le \varepsilon_0$$

similar to the 1-soliton result of Mizumachi-P. (2012)

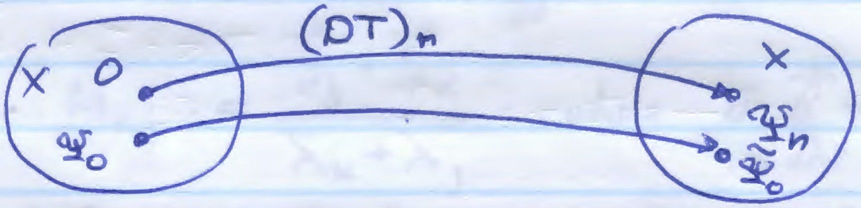
③ The asymptotic stability result is expected under the same condition with

$$\| \Psi(\cdot, t) - \Phi_n(\cdot, t) \|_{L^\infty} \le \frac{C\varepsilon}{(1+|t|)^{1/2}}, \quad |t| \geq T.$$

The improvements ② and ③ would require lengthy arguments to control the inverse scattering technique.

Outline of the proof is based on the same 4-step scheme as before, but it includes two new developments.

#1: dressing transformation that enables us to construct n-solitons from zero solution directly.



As before, if $\|\Psi_0\|_{L^2} \le \varepsilon \ll 1$, then

$$\|(DT)_n \Psi_0 - \Psi_n\|_{L^2} \le C \|\Psi_0\|_{L^2}.$$

However, if $\|\tilde\Psi_0 - \Psi_n\|_{L^2} \le \varepsilon \ll 1$, then $(DT)_n^{-1}\tilde\Psi_0$ does not produce a L^2-small function.

#2: inverse scattering that enables us to invert the dressing transformation:

if $\|\tilde\Psi_0 - \Psi_n\|_{L^2_3} \le \varepsilon \ll 1$, then $\tilde\Psi_0$ produces exactly n isolated eigenvalues in the Lax system and $\exists \{\omega_k', v_k'\}_{k=1}^{n}$ such that

$$\|(DT)_{n'}^{-1}\tilde\Psi_0\|_{L^2} \le C\|\tilde\Psi_0 - \Psi_n\|_{L^2}.$$

Let us explain new ingredients in more details.

#1 <u>dressing transformation</u>

The original technique was developed by Zakharov and Shabat in 1970s. We have just formalized this purely algebraic technique

Recall the Lax equation

$$\frac{\partial}{\partial x}\vec{\psi} = \lambda\sigma_3\vec{\psi} + Q(\psi)\vec{\psi}, \qquad Q(\psi) := \begin{bmatrix} 0 & \psi \\ -\bar\psi & 0 \end{bmatrix}$$

Fix $\{\lambda_k\}_{k=1}$ with $\lambda_k \neq \lambda_m$ for $k \neq m$ and $Re(\lambda_k) > 0$.
Define $\{\vec{s}_k\}_{k=1}^n$ ~~all~~ to be nonzero solutions of

$$\frac{\partial}{\partial x}\vec{s}_k = \bar{\lambda}_k \sigma_3 \vec{s}_k + Q(\Psi)\vec{s}_k$$

such that the Gramian-type matrix M is invertible with

$$M_{k,j} := \frac{\vec{s}_j \cdot \vec{s}_k}{\bar{\lambda}_k + \lambda_j}, \text{ where } \vec{a}\cdot\vec{b} = \bar{a}_1 b_1 + \bar{a}_2 b_2$$

is the inner product of $\vec{a}, \vec{b}$

Then, find a unique solution of the linear system

$$\sum_{j=1}^{n} M_{k,j}\vec{r}_j = \vec{s}_k, \quad 1 \le k \le n$$

and define

$$Q(\tilde{\Psi}) = Q(\Psi) + \sum_{k=1}^{n} \vec{r}_k \otimes \vec{s}_k \sigma_3 - \sigma_3 \vec{s}_k \otimes \vec{r}_k$$

where

$$\vec{a} \otimes \vec{b} = \begin{bmatrix} a_1\bar{b}_1 & a_1\bar{b}_2 \\ a_2\bar{b}_1 & a_2\bar{b}_2 \end{bmatrix}$$ is the outer product of $\vec{a}, \vec{b} \in \mathbb{C}^2$

Then, it is guaranteed that $\{\vec{r}_k\}_{k=1}^n$ are solutions of

$$\frac{\partial}{\partial x}\vec{r}_k = \lambda_k \sigma_3 \vec{r}_k + Q(\tilde{\Psi})\vec{r}_k$$

so that $\tilde{\Psi}$ is a new solution of (NLS)

Remarks: ① Same transformation holds for t-dependent equation of the Lax operator

② The dressing transformation is symmetric for maps $\Psi \longmapsto \tilde{\Psi}$ and $\tilde{\Psi} \longmapsto \Psi$.

$\lambda_1 \in \mathbb{R}_+$ $\quad \vec{s}_1 = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} : \frac{\partial}{\partial x}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \lambda_1 \begin{bmatrix} b_1 \\ -b_2 \end{bmatrix}$

$\Psi = 0$

Then, we look for $\vec{r}_1 = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} : \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \frac{2\lambda_1}{|b_1|^2 + |b_2|^2}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

Then,

$$Q(\tilde{\Psi}) = \begin{bmatrix} 0 & \tilde{\Psi} \\ -\bar{\tilde{\Psi}} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -a_1\bar{b}_2 - \bar{a}_2 b_1 \\ -a_2\bar{b}_1 - \bar{a}_1 b_2 & 0 \end{bmatrix}$$

$$\Rightarrow \tilde{\Psi} = -\frac{4\lambda_1 b_1 \bar{b}_2}{|b_1|^2 + |b_2|^2}$$

This expression is similar to what we have as a Bäcklund transformation but it works now for n solitons as well.

#2: ~~In~~ inverse scattering

This technique for NLS was developed by P. Deift and X. Zhou in 1990s. It relies on analysis of Jost functions and reflection coefficients.

Jost function: let $\lambda = -iz$ with $z \in \mathbb{R}$ and solve

$$\frac{\partial}{\partial x}\vec{\Phi} = -iz\sigma_3\vec{\Phi} + Q(\Psi)\vec{\Phi}$$

with boundary conditions

$$\vec{\Phi} \to \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-ixz} \quad \text{as } x \to -\infty$$

If $\Psi \in L^2_s(\mathbb{R})$ with $s > 1/2$, then there is a unique solution with $\vec{\Phi} \to a(z)\begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-ixz} + b(z)\begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{ixz}$ as $x \to \infty$

where a, b are continuous on $\mathbb{R}$.

Reflection coefficient:

$a(z)$ is analytically extended in $\mathbb{C}_+$ and may have up to n zeros in $\mathbb{C}_+$ (eigenvalues of Lax operator)

$$r(z) := \frac{b(z)}{a(z)}, \quad z \in \mathbb{R}.$$

Lemma 1 (direct scattering) Let $\Psi \in L^2_s(\mathbb{R})$, $s > \frac{1}{2}$ and $a(z) \neq 0$ for $z \in \mathbb{R}$ with n single zeros in $\mathbb{C}_+$. Then, $r \in H^s(\mathbb{R})$ and the map $\Psi \mapsto r$ is Lipschitz.

Lemma 2 (inverse scattering) Let $r \in H^s(\mathbb{R})$ and $n=0$. Then, $\Psi \in L^2_s(\mathbb{R})$ is uniquely recovered and $\exists C>0: \ \|\Psi\|_{L^2_s} \leq C\|r\|_{H^s}$.

Implementation of these results for the proof:
By inverse scattering, we start with $\Psi_0 \in L^2_s(\mathbb{R})$ such that $\varepsilon := \|\Psi_0 - \Psi_n|_{t=0}\|_{L^2_s} \leq \varepsilon_0 << 1$.
By Lipschitz continuity in Lemma 1, we have

$$\|r\|_{H^s} \leq C\|\Psi_0 - \Psi_n|_{t=0}\|_{L^2_s}$$

The initial data Ψ_0 supports exactly n zeros of a in $\mathbb{C}_+$ close to zeros corresponding for $\Psi_n|_{t=0}$.

By dressing transformation, we remove zeros of a and obtain $\tilde{\Psi}_0$ with $\tilde{r}$ such that $|\tilde{r}(z)| = |r(z)|, z\in$
By Lemma 2, we obtain

$$\|\tilde{\Psi}_0\|_{L^2_s} \leq C\|\tilde{r}\|_{H^s} = C\|r\|_{H^s} \leq C\|\Psi_0 - \Psi_n|_{t=0}\|$$

Then, we add time evolution and the dressing transformation ~~[illegible]~~ with exactly the same parameters. In this way, we obtain a solution of (NLS)

$$\Psi := (DT)_{n'} \, T(t) \, (DT)^{-1}_{n'} \, \Psi_0$$

such that

$$\|\Psi - \Psi_n\|_{L^2} \leq C\|\Psi_0 - \Psi_n|_{t=0}\|_{L^2_s}$$

for all times $t \in \mathbb{R}$. ∎

Lecture #4 "Higher-order conserved quantities and global existence in Sobolev spaces of higher regularity"

Associated with integrable equations, one can construct an infinite set of conserved quantities. The set is constructed by an algebraic algorithm from the Lax system. The set is useful for many purposes:

1. global existence of derivative NLS in H^2 (Tsutsumi '19.. or ~~...~~ massive Dirac equations in H^1 (P, Shimbakuro '201..
2. orbital stability of n solitons in H^n for KdV (Maddock-Sachs '1993) and NLS (Kapitula..
3. orbital stability of Dirac solitons in H^1 (P., Shimbakuro '14) or breathers of mKdV in H^2 (Alejo-Munoz '13)
4. orbital stability of periodic waves in KdV (Deconinck-Kapitula '2010) or in NLS defocussing (Bottman et al, 2011; Gallay-P, 2014).

Here we shall illustrate these methods with two examples related to the NLS equations.

Example 1: derivative NLS equation

$$i\psi_t + \psi_{xx} + i(|\psi|^2\psi)_x = 0 \qquad \text{(dNLS)}$$

Local existence was proved in $H^s(\mathbb{R})$ for $s > 3/2$ (Tsutsumi-Fukada '1981) and $s > 1/2$ (Colliander, Keel, Staffilani-Takaoka-Tao, 2001). To prove global existence, we use conserved quantities

$$\begin{cases} N(\psi) = \|\psi\|^2_{L^2} \\ E(\psi) = i\int_{\mathbb{R}}(\bar\psi\psi_x - \psi\bar\psi_x)\,dx - \int_{\mathbb{R}}|\psi|^4 dx \end{cases}$$

However, we can see that the Hamiltonian $E(\Psi)$ is not so useful. It is defined in $H^{1/2}(\mathbb{R})$ but it is sign-indefinite. Nevertheless, because of integrability of derivative NLS, we obtain

$$R(\Psi) = \int |\Psi|^6 dx + 2\int |\Psi_x|^2 dx + \frac{3i}{2}\int |\Psi|^2 (\Psi \bar{\Psi}_x - \bar{\Psi}\Psi_x) dx$$

which is defined in $H^1(\mathbb{R})$ and is sign-definite for the first two terms.

<u>Theorem</u>: $\exists N_0 : \forall \Psi_0 \in H^1(\mathbb{R}) : \|\Psi_0\|_{L^2}^2 \le N_0$, there is a unique global solution Ψ of (dNLS) with $\|\Psi(t)\|_{H^1} \le C(\|\Psi_0\|_{H^1})$, $\forall t \in \mathbb{R}$.

<u>Proof</u>. By ~~Gagliardo-Nirenberg~~ Cauchy-Schwarz inequality, we have

$$R(\Psi) = \|\Psi\|_{L^6}^6 + 2\|\Psi_x\|_{L^2}^2 + \frac{3i}{2}\int |\Psi|^2 (\Psi \bar{\Psi}_x - \bar{\Psi}\Psi_x) dx$$

$$\ge \|\Psi\|_{L^6}^6 + 2\|\Psi_x\|_{L^2}^2 - 3\|\Psi\|_{L^6}^3 \|\Psi_x\|_{L^2}$$

Unfortunately, the quadratic form

$$a^2 + 2b^2 - 3ab = \left(a - \frac{3b}{2}\right) - \frac{1}{4}b^2$$

is not sign-definite. Nevertheless, using Gagliardo-Nirenberg inequality

$$\exists C > 0 : \|\Psi\|_{L^6}^3 \le C \|\Psi\|_{L^2}^{2/3} \|\Psi_x\|_{L^2}^{1/3}$$

we obtain

$$R(\Psi) \ge \|\Psi\|_{L^6}^6 + [2 - 3C\, N(\Psi)] \|\Psi_x\|_{L^2}^2$$

and the bound holds for $N(\Psi_0) \le N_0 := \dfrac{2}{3C}$.

Here we recall that
$$R(\Psi(t)) = R(\Psi_0)$$
$$N(\Psi(t)) = N(\Psi_0).$$

Example 2: (defocusing NLS)

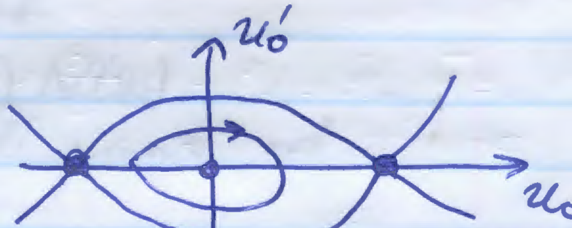
$$i\Psi_t + \Psi_{xx} - |\Psi|^2\Psi = 0 \qquad \text{(NLS)}$$

Here we can study periodic waves

$$\Psi(x,t) = e^{-it}\, u_0(x)$$

where $u_0: \mathbb{R} \to \mathbb{R}$ satisfies

$$\boxed{\frac{d^2u_0}{dx^2} + u_0 - u_0^3 = 0}$$



Periodic waves have period $\approx 2\pi$ near the center and $\approx \infty$ near the saddle points. Integrating once, we have

$$\left(\frac{du_0}{dx}\right)^2 = \frac{1}{2}\left[(1-u_0^2)^2 - \mathcal{E}^2\right]$$

where $\mathcal{E} \in [0,1]$ depends on the period. As $\mathcal{E} \to 1$, we have small-amplitude waves

$$u_0(x) \sim a\cos(x) + O(a^3), \text{ where } a \text{ depends on } \mathcal{E}$$

As $\mathcal{E} \to 0$, we have a black soliton

$$u_0(x) = \tanh\left(\frac{x}{\sqrt{2}}\right).$$

Why is the orbital stability result challenging for the periodic waves? A simple answer is provided by the following computation:

u_0 is a critical point of

$$E(\Psi) = \int\left[|\Psi_x|^2 + \frac{1}{2}(1-|\Psi|^2)^2\right]dx$$

Set $\Psi = u_0 + u + iv$ with real (u,v), we obtain,

$$E(\Psi) - E(u_0) = \langle L_+ u, u\rangle_{L^2} + \langle L_- v, v\rangle_{L^2} + N(u,v)$$

where $N(u,v)$ is cubic in (u,v).

Here $L_+ := -\partial_x^2 - 1 + 3u_0^2(x)$

$L_- := -\partial_x^2 - 1 + u_0^2(x)$

Hence as $\mathcal{E} \to 1$, $u_0(x) \longrightarrow 0$ and

$$\langle L_+ u, u\rangle_{L^2} \Longrightarrow \int\left[(u_x)^2 - u^2\right]dx$$

Hence, the periodic wave is a saddle point of energy E.

However, to resque the proof of stability, let us consider the next conserved quantity:

$$R(\psi) = \int \left[|\psi_{xxx}|^2 + 3|\psi|^2|\psi_x|^2 + \tfrac{1}{2}(\bar{\psi}\psi_x + \psi\bar{\psi}_x)^2 + \tfrac{1}{2}|\psi|^6 \right]$$

Then, u_0 is a critical point of

$$S(\psi) := R(\psi) - \tfrac{1}{2}(3-\varepsilon^2)\, N(\psi)$$

in addition to being a critical point of $E(\psi)$.

Let

$$\Lambda_c(\psi) := S(\psi) - c\, E(\psi), \quad c \in \mathbb{R}.$$

Then, we have

<u>Theorem</u> (Gallay-P'2014) There exists $\varepsilon_0 \in (0,1)$ such that for every $\varepsilon \in (\varepsilon_0, 1)$, the periodic wave u_0 is a minimum of Λ_c for $c \in (c_-, c_+)$, among all perturbations in $H^2(\mathbb{R})$, where $0 < c_- < 2 < c_+ < 3$. Moreover, the black soliton u_0 is a minimum point of Λ_c for $c = 2$.

To give the idea as $\varepsilon \to 1$, we write

$$S(\psi) - S(u_0) = \langle M_+ u, u \rangle_{L^2} + \langle M_- v, v \rangle_{L^2} + \tilde{N}(u,v)$$

where

$$M_+ := \partial_x^4 - 5\,\partial_x u_0^2\, \partial_x - 5u_0^4 + 15u_0^2 - 4 + 3\varepsilon^2$$
$$M_- := \partial_x^4 - 3\,\partial_x u_0^2\, \partial_x + u_0^2 - 1.$$

As $u_0(x) \to 0$, we have

$$\langle L_+ u, u \rangle_{L^2} = \int (u_x^2 - u^2)\,dx$$
$$\langle M_+ u, u \rangle_{L^2} = \int (u_{xx}^2 - u^2)\,dx$$

but

$$\langle (M_+ - c L_+) u, u \rangle_{L^2} = \int \left[u_{xx}^2 - c\, u_x^2 + (c-1) u^2 \right] dx$$

$$= \int \left[u_{xx} + \tfrac{c}{2} u \right]^2 dx - (1 - c/2)^2 \int u^2\,dx$$

This is sign-positive if $c = 2$.

In fact, Floquet-Bloch analysis yields

$$c_\pm(\varepsilon) = 2 \pm \sqrt{2(1-\varepsilon)} + O(1-\varepsilon) \text{ as } \varepsilon \to 1$$

and $K_+(c) := M_+ - cL_+$ ~~are~~ Both non-negative.
$K_-(c) := M_- - cL_-$

Another interesting representation holds in the case of black soliton ($\varepsilon = 0$).

$c=2$: $\langle K_-(c)v, v\rangle_{L^2} = \|L_- v\|_{L^2}^2 + \|u_0 v_x - u_0' v\|_{L^2}^2 \geq 0$
Note that $L_- u_0 = 0$ and $u_0 u_0' - u_0' u_0 = 0$.

$c \in \mathbb{R}$: $\langle K_+(c)u, u\rangle_{L^2} = \|w_x\|_{L^2}^2 + (3-c)\|w\|_{L^2}^2 \geq 0$ if $c < 3$
where $w := u_x + \sqrt{2}\, u_0 u$, with $u_0(x) = \tanh\left(\frac{x}{\sqrt{2}}\right)$.
Note again $w = 0$ if $u = u_0'(x)$.
Also note that
$\langle K_+(c)u, u\rangle_{L^2} = \|L_+ u\|_{L^2}^2 + \mathrm{Rem}(u)$ is not sign-definite even for $c=2$ because $\mathrm{Rem}(u)$ changes sign.

It follows from positivity of the Lyapunov functional that the periodic wave is ~~orbitally~~ stable with respect to periodic perturbations in $H^2_{per}(0,T)$ with any T multiple to period of u_0.

Also, the black soliton is stable with respect to perturbations in $H^2(\mathbb{R})$ with the control on $\|u\|_{H^2}$ and $\|v_{xx}\|_{L^2}$, $\|v_x\|_{L^2}$, and $\|v\|_{L^\infty(-A,A)}$ for any fixed $A > 0$. See Bethuel et al (2008) and Gallay-P (2014) for further details.