On Linear stability of nonlinear Dirac equation

Pisa summer school on stability of solitary waves

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Contents

1 Examples from Quantum Mechanics

Problem 1.1. Find the eigenvalues of a particle trapped in a potential well of infinite height:

$$E\psi = -\frac{\hbar^2}{2m}\Delta\psi + V\psi, \qquad x \in \mathbb{R}, \qquad V(x) = \begin{cases} 0, & 0 \le x \le L \\ +\infty & \text{otherwise.} \end{cases}$$

That is, find the eigenvalues (the *spectrum*) of the Sturm-Liouville problem $E\psi = -\frac{\hbar^2}{2m}\Delta\psi$, $0 \le x \le L$; $\psi|_{x=0} = \psi|_{x=L} = 0$.

Problem 1.2. A particle described by the Schrödinger equation

$$i\hbar\dot{\psi}(x,t) = -\frac{\hbar^2}{2m}\Delta\psi(x,t) \tag{1.1}$$

is contained in the interior Ω of a closed circular tube of radious R and length L (assume $L \gg R$), so that $x \in \Omega$, $\psi|_{\partial\Omega} = 0$. Assume that the particle is initially in the ground state. Estimate the energy needed to squeeze the tube.

$*$ *Hint.* The energy required for squeezing the tube is equal to the difference of the ground state energies in the squeezed tube (the energy of the particle in the end of the process) and in the unsqueezed tube (the initial energy of the particle). Those, in turn, are the smallest eigenvalues of the Schrödinger operator,

$$H = -\frac{\hbar^2}{2m}\Delta.$$

The corresponding eigenfunctions are to vanish on the boundary of the tube. Since we need approximate values, we'd deform the shape of the squeezed and unsqueezed tubes so that the computations become easy.

Problem 1.3. A quantum particle of mass $m = 1/2$ sits in a well of depth V and size $2a$:

$$E\psi = -\psi'' + V(x)\psi(x), \qquad V(x) = -V_0 \text{ for } -a < x < a, \ 0 \text{ otherwise.}$$

How many *even* different eigenstates with $E < 0$ are there?

$*$ *Hint.* For $x > a$, $\psi(x) = Be^{-x\sqrt{|E|}}$; for $-a < x < a$, $\psi(x) = A\cos(x\sqrt{V_0 - |E|})$ (since we are interested in even eigenstates). Request the continuity of ψ and ψ' at $x = a$.

In the previous problem, the interval $0 \le E < \infty$ is the *essential spectrum* of the equation. To physicists, for any $E \ge 0$, there are solutions $\sim e^{\pm ix\sqrt{E}}$ for x large, called *plane waves*. In the mathematical sense, the values $E \ge 0$ do not belong to the *point spectrum* since there are no nontrivial L^2 solutions to $(-\partial_x^2 + V - E)\psi = 0$; yet, they belong to the *essential spectrum* since the operator $-\partial_x^2 + V - E$ has no bounded inverse (on L^2). To show this, take the functions $u_b(x) = \rho(x-a)\rho(b-x)\cos(x\sqrt{E})$, with $\rho \in C^\infty(\mathbb{R})$, $\rho|_{x\le 0} \equiv 0$, $\rho|_{x\ge 1} \equiv 1$; one can see that $\|u_b\|_{L^2} \to \infty$ as $b \to \infty$, while $\|(-\partial_x^2 + V - E)u_b\|_{L^2}$ remains bounded.

Problem 1.4. The stationary Schrödinger equation which describes the electron in the atom of Hydrogen has the form

$$H\psi \equiv -\frac{\hbar^2}{2m}\Delta\psi(x) + \Big(-\frac{\mathrm{ke}^2}{|x|}\Big)\psi(x) = E\psi(x), \qquad x \in \mathbb{R}^3. \tag{1.2}$$

E is the eigenvalue which corresponds to the eigenfunction ψ (or, equivalently, E is the *energy* of the electron with the *wave function* ψ). $\hbar$ is the Planck constant divided by 2π; m is the mass of electron, and $-\mathrm{e}$ is its (negative) charge; $V(x) = -\frac{k\mathrm{e}^2}{|x|}$ is the potential energy of the electron at the point $x \in \mathbb{R}^3$ in the Coulomb potential of the nucleus with charge $+\mathrm{e}$ and located in the origin. The function ψ is called the wave function of the electron. $|\psi(x)|^2$ is interpreted as the probability density (the chance to find the electron near the point x). The operator

$$H = -\frac{\hbar^2}{2m}\Delta + V(x) \tag{1.3}$$

is called the Schrödinger operator.

There are infinitely many solutions to equation (1.2). The solutions which correspond to $\psi \in L^2(\mathbb{R}^3)$ are called eigenstates. Normally, they correspond to negative eigenvalues E, and called *bound states*: *the electron can not escape the nucleus; some energy is needed to pull the electron away to infinity, where its energy would become zero*. The solutions with $E \geq 0$ normally have infinite L^2-norm. At the same time, there are examples of $V(x)$ with slow decay with L^2-eigenstates of positive energy, called *embedded eigenstates*. Such examples exist even in one dimension.

Assuming that the eigenfunction ψ that correspond to the lowest energy bound state, or *ground state*, "1s", is spherically symmetric, estimate E using the Rayleigh quotient.

$*$ *Hint.* As a sample function, take $\psi(r) = e^{-\beta r}$ and find $\beta > 0$ which gives the best (smallest) value for E.

Solution.

$$\begin{aligned}
E_0 &\leq \frac{\int_{\mathbb{R}^3}\Big(-\frac{\hbar^2}{2m}\psi\Delta\psi - \frac{\mathrm{ke}^2}{|x|}\psi^2\Big)\,d^3x}{\int_{\mathbb{R}^3}|\psi(x)|^2\,d^3x} = \frac{\int_{\mathbb{R}^3}\Big(\frac{\hbar^2}{2m}|\nabla\psi|^2 - \frac{\mathrm{ke}^2}{|x|}\psi^2\Big)\,d^3x}{\int_{\mathbb{R}^3}|\psi|^2\,d^3x} = \\
&= \frac{4\pi\int_0^\infty\Big(\frac{\hbar^2}{2m}(\partial_r\psi)^2 - \frac{\mathrm{ke}^2}{r}\psi^2\Big)\,r^2\,dr}{4\pi\int_0^\infty|\psi|^2\,r^2\,dr} = \frac{\int_0^\infty\Big(\frac{\beta^2\hbar^2}{2m}e^{-2\beta r} - \frac{\mathrm{ke}^2}{r}e^{-2\beta r}\Big)\,r^2\,dr}{\int_0^\infty e^{-2\beta r}\,r^2\,dr} = \\
&= \frac{\int_0^\infty\Big(\frac{\beta^2\hbar^2}{2m}e^{-2\beta r}r^2 - k\mathrm{e}^2e^{-2\beta r}r\Big)\,dr}{\int_0^\infty e^{-2\beta r}\,r^2\,dr} = \frac{\frac{\beta^2\hbar^2}{2m}2(2\beta)^{-3} - k\mathrm{e}^2(2\beta)^{-2}}{2(2\beta)^{-3}} = \frac{\beta^2\hbar^2}{2m} - k\mathrm{e}^2\beta
\end{aligned} \tag{1.4}$$

The minimal value is achieved when $\frac{\beta\hbar^2}{m} - k\mathrm{e}^2 = 0$; we conclude that

$$\beta = -\frac{mk\mathrm{e}^2}{\hbar^2}, \qquad E_0 \leq \frac{\beta^2\hbar^2}{2m} - k\mathrm{e}^2\beta = -\frac{mk^2\mathrm{e}^4}{2\hbar^2} = -\frac{\alpha^2}{2}mc^2, \tag{1.5}$$

where $\alpha = \mathrm{e}^2/(\hbar c) \approx 1/137$ is the fine structure constant.

Remark 1.5. $\psi(x) = e^{-\beta|x|}$, $x \in \mathbb{R}^3$ is the true eigenfunction of H; E_0 in (1.5) is the smallest eigenvalue of H.

Remark 1.6. The quantity $a = \beta^{-1} = \frac{\hbar^2}{mk\mathrm{e}^2} \approx 0.5 \cdot 10^{-8} cm$ is interpreted as the radius of the Bohr orbit. The first term in the right-hand side of (1.4) has the meaning of the kinetic energy of a particle moving with the momentum $|p| = \beta\hbar$; the second term is the potential energy of the electron at the distance a from the nucleus. Note that $\oint_{|q|=a} p \cdot dq = \beta\hbar \cdot 2\pi a = 2\pi\hbar$, in the agreement with the Bohr-Sommerfeld quantization condition.

2 Derrick's theorem and Vakhitov–Kolokolov criterion

2.1 Derrick's theorem on instability of stationary localized solutions

Let us consider the linear instability of stationary solutions to a nonlinear wave equation,

$$-\ddot{\psi} = -\Delta\psi + g(\psi), \qquad \psi = \psi(x,t) \in \mathbb{R}, \quad x \in \mathbb{R}^n, \quad n \geq 1. \tag{2.1}$$

Let the nonlinearity $g(s)$ be smooth and $g(0) = 0$. Equation (2.1) can be written as a Hamiltonian system $\dot{\pi} = -\delta_\psi E$, $\dot{\psi} = \delta_\pi E$, with the Hamiltonian

$$E(\psi,\pi) = \int_{\mathbb{R}^n} \Big(\frac{\pi^2}{2} + \frac{|\nabla\psi|^2}{2} + G(\psi) \Big)\, dx,$$

where $G(t) = \int_0^t g(s)\, ds$.

There is a well-known result [Der64] about non-existence of stable localized stationary solutions in dimension $n \geq 3$, known as *Derrick's theorem*, which we briefly recall. Denote $T(\theta) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla\theta|^2\, dx$, $V(\theta) = \int_{\mathbb{R}^n} G(\theta)\, dx$. If $\psi(x,t) = \theta(x)$ is a localized stationary solution, so that $0 = \dot{\psi} = \frac{\delta E}{\delta \pi}(\theta, 0)$, $0 = \dot{\pi} = -\frac{\delta E}{\delta\psi}(\theta,0)$, then for the family $\theta_\lambda(x) = \theta(x/\lambda)$, using the identities $T(\theta_\lambda) = \lambda^{n-2}T(\theta)$, $V(\theta_\lambda) = \lambda^n V(\theta)$, one has:

$$0 = \Big\langle \frac{\delta E}{\delta\psi}(\theta,0), \frac{\partial\theta_\lambda}{\partial\lambda}\big|_{\lambda=1} \Big\rangle = \partial_\lambda|_{\lambda=1} E(\theta_\lambda, 0) = (n-2)T(\theta) + nV(\theta); \tag{2.2}$$

this relation is known as Pokhozhaev's identity [Poh65] or the virial theorem. It follows that

$$\partial_\lambda^2|_{\lambda=1} E(\theta_\lambda) = (n-2)(n-3)T(\theta) + n(n-1)V(\theta) = -2(n-2)T(\theta),$$

which is negative as long as $n \geq 3$. That is, $\delta^2 E < 0$ for a variation corresponding to the uniform stretching, and the solution $\theta(x)$ from the physical point of view is to be unstable. We remark that the fact that $\partial_\lambda^2 E(\theta_\lambda)|_{\lambda=1}$ was not negative for $n = 1$ and 2 does not prove that in these dimensions the localized stationary solutions are stable; it just means that a particular family of perturbations failed to catch the unstable direction.

Problem 2.1. For which $p > 1$ are there localized real-valued C^2 solutions to $-u = -\Delta u - u^p$, $x \in \mathbb{R}^n$?

$*$ *Hint.* Pokhozhaev's identity (2.2) yields $(n-2)\int_{\mathbb{R}^n} \frac{1}{2}|\nabla u|^2\, dx = n\int_{\mathbb{R}^n} \Big(\frac{1}{p+1}u^{p+1} - \frac{1}{2}u^2\Big)\, dx$; one more identity is obtained by multiplying the equation by $u(x)$ and integrating. This allows to express $\int |\nabla u|^2\, dx$ in terms of $\int u^2\, dx$, leading to the restriction $p < (n+2)/(n-2)$.

More details and construction of solitary waves for $n \geq 3$ are in [BL83].

Let us modify Derrick's argument to show the linear instability of stationary solutions in any dimension.

Lemma 2.2 (Derrick's theorem for $n \geq 1$)**.** *For any $n \geq 1$, any stationary solution $\theta \in H^\infty(\mathbb{R}^n)$ to the nonlinear wave equation is linearly unstable.*

Proof. Since θ satisfies $-\Delta\theta + g(\theta) = 0$, we also have $-\Delta\partial_{x_1}\theta + g'(\theta)\partial_{x_1}\theta = 0$. Due to $\lim\limits_{|x|\to\infty} \theta(x) = 0$, $\partial_{x_1}\theta$ vanishes somewhere. According to the minimum principle, there is a nowhere vanishing smooth function $\chi \in H^\infty(\mathbb{R}^n)$ (due to Δ being elliptic) which corresponds to some smaller (hence negative) eigenvalue of $L = -\Delta + g'(\theta)$:

$$L\chi = -c^2\chi, \qquad c > 0.$$

Taking $\psi(x,t) = \theta(x) + r(x,t)$, we obtain the linearization at θ, $-\ddot{r} = -Lr$, which we rewrite as $\partial_t \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -L & 0 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}$. The matrix in the right-hand side has eigenvectors $\begin{bmatrix} \chi \\ \pm c\chi \end{bmatrix}$, corresponding to the eigenvalues $\pm c \in \mathbb{R}$; thus, the solution θ is linearly unstable. Let us also mention that $\partial_\tau^2|_{\tau=0} E(\theta + \tau\chi) < 0$, showing that $\delta^2 E(\theta)$ is not positive-definite. □

Remark 2.3. A more general result on the linear stability and (nonlinear) instability of stationary solutions to (2.1) is in [KS07]. In particular, it is shown there that the linearization at a stationary solution may be spectrally stable when this particular stationary solution is not from H^1 (such examples exist in higher dimensions).

2.2 Solitary waves in 1D

Let us construct solitary wave solutions in the simple one-dimensional case. We consider the $\mathbf{U}(1)$-invariant nonlinear Schrödinger equation,

$$i\partial_t\psi = -\partial_x^2\psi + g(|\psi|^2)\psi, \qquad \psi(x,t) \in \mathbb{C}, \quad x \in \mathbb{R},$$

with g smooth real-valued function.

Definition 2.4. The solitary waves are solutions of the form

$$\psi(x,t) = \phi_\omega(x)e^{-i\omega t}, \qquad \phi_\omega \in H^1(\mathbb{R}), \qquad \omega \in \mathbb{R}.$$

The amplitude of a solitary wave solves the stationary equation $\omega^2\phi(x) = -\phi''(x) + g(\phi^2)\phi$, which we rewrite as

$$\phi''(x) = g(\phi^2)\phi - \omega\phi(x) = -\partial_\phi \frac{\omega\phi^2 - G(\phi^2)}{2}, \tag{2.3}$$

with $G(s) = \int_0^s g(s')\,ds'$. (We will see in a moment that if there is a solitary wave, then ϕ_ω could be chosen positive.) We will interpret this equation as describing the particle in the "effective potential"

$$V_\omega(\phi) := \frac{\omega\phi^2 - G(\phi^2)}{2},$$

so that x is "the time" and ϕ is "the position" of the particle. The "mechanical" energy corresponding to the system described by equation (2.3) is $\mathcal{E}(\phi) = |\phi'|^2/2 + V_\omega(\phi)$. For a particular solution $\phi(x)$ to (2.3), $\mathcal{E}(\phi)$ is constant (it does not depend on the "time" x). We are interested in soliton-like solutions, such that $\phi \to 0$ and $\phi' \to 0$ as $|x| \to \infty$, and hence $\mathcal{E}(\phi) \equiv 0$. If there is a "turning point" $\mu_\omega > 0$ such that $V_\omega(\phi) < 0$ for $\phi \in (0, \mu_\omega)$, $V_\omega(\mu_\omega) = 0$, and $V'_\omega(\mu_\omega) > 0$, then there exists a set of solutions with zero "mechanical" energy $\mathcal{E}$, $\phi(x) = \phi_\omega(x+C)$, $C \in \mathbb{R}$, where $\phi_\omega(x)$ satisfies $\lim_{x\to\pm\infty} \phi_\omega(x) = 0$. Such a soliton is defined up to a shift along x. We fix ϕ_ω by requiring that ϕ_ω assumes its maximum value at the origin: $\phi_\omega(0) = \mu_\omega$ (then ϕ_ω is symmetric). ϕ_ω is obtained by integration from $d\phi/dx = -\sqrt{V_\omega(\phi)}$ for $x > 0$ (we assume that $\phi(0) > 0$, and hence $d\phi/dx < 0$ for positive values of x). See Figure 1.

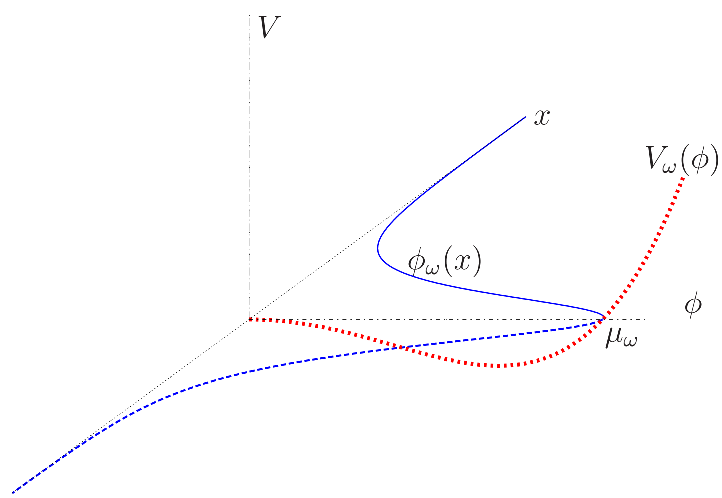

Figure 1: Solitary wave profile $\phi_\omega(x)$ as a "particle trajectory" in the effective potential $V_\omega(\phi)$

2.3 Vakhitov-Kolokolov criterion for the nonlinear Schrödinger equation

By Derrick's theorem [Der64], *any stationary localized solution* is unstable (Cf. Lemma 2.2). To get a hold of stable localized solutions, Derrick suggested that *elementary particles might correspond to stable, localized solutions which are periodic in time, rather than time-independent.* Let us show that the (generalized) nonlinear Schrödinger equation indeed could have stable solitary wave solutions.

In one dimension, the nonlinear Schrödinger equation is given by

$$i\partial_t \psi = -\partial_x^2 \psi + g(|\psi|^2)\psi, \qquad \psi = \psi(x,t) \in \mathbb{C}, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}, \tag{2.4}$$

where $g(s)$ is a smooth function. For our convenience, let us assume that $g(0) = 0$. One can easily construct solitary wave solutions $\phi_\omega(x)e^{-i\omega t}$, for some $\omega \in \mathbb{R}$ and $\phi_\omega \in H^1(\mathbb{R})$: $\phi_\omega(x)$

satisfies the stationary equation $\omega\phi = -\phi'' + g(\phi^2)\phi$, and can be chosen strictly positive, even, and monotonically decaying away from $x = 0$. The value of ω can not exceed $g(0) = 0$. We consider the Ansatz $\psi(x,t) = (\phi_\omega(x) + \rho(x,t))e^{-i\omega t}$, with $\rho(x,t) \in \mathbb{C}$. The linearized equation on ρ is called the linearization at a solitary wave:

$$\partial_t \boldsymbol{\rho} = \mathbf{JL}(\omega)\boldsymbol{\rho}, \qquad \boldsymbol{\rho}(x,t) = \begin{bmatrix} \operatorname{Re}\rho(x,t) \\ \operatorname{Im}\rho(x,t) \end{bmatrix}, \tag{2.5}$$

where

$$\mathbf{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad \mathbf{L}(\omega) = \begin{bmatrix} L_+(\omega) & 0 \\ 0 & L_-(\omega) \end{bmatrix},$$

with

$$L_-(\omega) = -\partial_x^2 + g(\phi_\omega^2) - \omega, \quad L_+(\omega) = L_- + 2g'(\phi_\omega^2)\phi_\omega^2. \tag{2.6}$$

Since $\lim\limits_{|x|\to\infty} \phi_\omega(x) = 0$, the essential spectrum of L_- and L_+ is $[g(0) - \omega, +\infty)$.

First, let us note that the spectrum of $\mathbf{JL}$ is located on the real and imaginary axes only: $\sigma(\mathbf{JL}) \subset \mathbb{R} \cup i\mathbb{R}$. To prove this, we consider $(\mathbf{JL})^2 = -\begin{bmatrix} L_-L_+ & 0 \\ 0 & L_+L_- \end{bmatrix}$. Since L_- is positive-definite ($\phi_\omega \in \ker L_-(\omega)$, being nowhere zero, corresponds to its smallest eigenvalue), we can define the selfadjoint root of $L_-(\omega)$; then

$$\sigma_d((\mathbf{JL})^2)\backslash\{0\} = \sigma_d(L_-L_+)\backslash\{0\} = \sigma_d(L_+L_-)\backslash\{0\} = \sigma_d(L_-^{1/2}L_+L_-^{1/2})\backslash\{0\} \subset \mathbb{R},$$

with the inclusion due to $L_-^{1/2}L_+L_-^{1/2}$ being selfadjoint. Thus, any eigenvalue $\lambda \in \sigma_d(\mathbf{JL})$ satisfies $\lambda^2 \in \mathbb{R}$.

Given the family of solitary waves, $\phi_\omega(x)e^{-i\omega t}$, $\omega \in \Omega \subset \mathbb{R}$, we would like to know at which ω the eigenvalues of the linearized equation with $\operatorname{Re}\lambda > 0$ appear. Since $\lambda^2 \in \mathbb{R}$, such eigenvalues can only be located on the real axis, having bifurcated from $\lambda = 0$. One can check that $\lambda = 0$ belongs to the discrete spectrum of $\mathbf{JL}$, with

$$\mathbf{JL}\begin{bmatrix} 0 \\ \phi_\omega \end{bmatrix} = 0, \qquad \mathbf{JL}\begin{bmatrix} -\partial_\omega\phi_\omega \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \phi_\omega \end{bmatrix},$$

for all ω which correspond to solitary waves. Thus, if we will restrict our attention to functions which are even in x, the dimension of the generalized null space of $\mathbf{JL}$ is at least two. Hence, the bifurcation follows the jump in the dimension of the generalized null space of $\mathbf{JL}$. Such a jump happens at a particular value of ω if one can solve the equation $\mathbf{JL}\zeta = \begin{bmatrix} \partial_\omega\phi_\omega \\ 0 \end{bmatrix}$. This leads to the condition that $\begin{bmatrix} \partial_\omega\phi_\omega \\ 0 \end{bmatrix}$ is orthogonal to the null space of the adjoint to $\mathbf{JL}$, which contains the vector $\begin{bmatrix} \phi_\omega \\ 0 \end{bmatrix}$; this results in $\langle \phi_\omega, \partial_\omega\phi_\omega \rangle = \partial_\omega \|\phi_\omega\|_{L^2}^2/2 = 0$. A slightly more careful analysis [CP03] based on construction of the moving frame in the generalized eigenspace of $\lambda = 0$ shows that there are two real eigenvalues $\pm\lambda \in \mathbb{R}$ that have emerged from $\lambda = 0$ when ω is such that $\partial_\omega \|\phi_\omega\|_{L^2}^2$ becomes positive, leading to a linear instability of the corresponding solitary wave. The opposite condition,

$$\partial_\omega \|\phi_\omega\|_{L^2}^2 < 0, \tag{2.7}$$

is the Vakhitov-Kolokolov stability criterion which guarantees the absence of nonzero real eigenvalues for the nonlinear Schrödinger equation. It appeared in [VK73, Sha83, GSS87] in relation to linear and orbital stability of solitary waves.

Lemma 2.5 (Vakhitov-Kolokolov stability criterion)**.** *There is $\lambda \in \sigma_p(\mathbf{JL})$, $\lambda > 0$, where $\mathbf{JL}$ is the linearization (2.5) at the solitary wave $\phi_\omega(x)e^{-i\omega t}$, if and only if $\frac{d}{d\omega}\|\phi_\omega\|_{L^2}^2 > 0$ at this value of ω.*

Proof. We follow [VK73]. Assume that there is $\lambda \in \sigma_d(\mathbf{JL})$, $\lambda > 0$. The relation $(\mathbf{JL}-\lambda)\Xi = 0$ implies that $\lambda^2 \Xi_1 = -L_- L_+ \Xi_1$. It follows that Ξ_1 is orthogonal to the kernel of the selfadjoint operator L_- (which is spanned by ϕ_ω):

$$\langle \phi, \Xi_1 \rangle = -\frac{1}{\lambda^2}\langle \phi, -L_- L_+ \Xi_1 \rangle = -\frac{1}{\lambda^2}\langle L_- \phi, -L_+ \Xi_1 \rangle = 0,$$

hence there is $\eta \in L^2(\mathbb{R}, \mathbb{C})$ such that $\Xi_1 = L_- \eta$ and $\lambda^2 \eta = -L_+ \Xi_1$. Thus, the inverse to L_- can be applied: $\lambda^2 L_-^{-1} \Xi_1 = -L_+ \Xi_1$. Then

$$\lambda^2 \langle \eta, L_- \eta \rangle = -\langle \Xi_1, L_+ \Xi_1 \rangle.$$

Since L_- is positive-definite and $\eta \notin \ker L_-$, it follows that $\langle \eta, L_- \eta \rangle > 0$. Since $\lambda > 0$, $\langle \Xi_1, L_+ \Xi_1 \rangle < 0$, therefore the quadratic form $\langle \cdot, L_+ \cdot \rangle$ is not positive-definite on vectors orthogonal to ϕ_ω. According to Lagrange's principle, the function r corresponding to the minimum of $\langle r, L_+ r \rangle$ under conditions $\langle r, \phi_\omega \rangle = 0$ and $\langle r, r \rangle = 1$ satisfies

$$L_+ r = \alpha r + \beta \phi_\omega, \qquad \alpha,\ \beta \in \mathbb{R}. \tag{2.8}$$

Since $\langle r, L_+ r \rangle = \alpha$, we need to know whether α could be negative. Since $L_+ \partial_x \phi_\omega = 0$, one has $\lambda_1 = 0 \in \sigma_p(L_+)$. Due to $\partial_x \phi_\omega$ vanishing at one point ($x = 0$), there is exactly one negative eigenvalue of L_+, which we denote by $\lambda_0 \in \sigma_p(L_+)$. (This eigenvalue corresponds to some non-vanishing eigenfunction.) Note that $\beta \neq 0$, or else α would have to be equal to λ_0, with r the corresponding eigenfunction of L_+, but then r, having to be nonzero, could not be orthogonal to ϕ_ω. Denote $\lambda_2 = \inf(\sigma(L_+) \cap \mathbb{R}_+) > 0$. Let us consider $f(z) = \langle \phi_\omega, (L_+ - z)^{-1} \phi_\omega \rangle$, which is defined and is smooth for $z \in (\lambda_0, \lambda_2)$. (Note that $f(z)$ is defined for $z = \lambda_1 = 0$ since the corresponding eigenfunction $\partial_x \phi_\omega$ is odd while ϕ_ω is even.) If $\alpha < 0$, then, by (2.8), we would have $f(\alpha) = \langle \phi_\omega, (L_+ - \alpha)^{-1} \phi_\omega \rangle = \frac{1}{\beta}\langle \phi_\omega, r \rangle = 0$, and since $f'(z) > 0$, one has $f(0) > 0$. On the other hand, $f(0) = \langle \phi_\omega, L_+^{-1} \phi_\omega \rangle = \langle \phi_\omega, \partial_\omega \phi_\omega \rangle = \frac{1}{2}\frac{d}{d\omega}\int_{\mathbb{R}} |\phi_\omega(x)|^2\, dx$. Therefore, the linear instability leads to $\alpha < 0$, which results in $\frac{d}{d\omega}\int_{\mathbb{R}} |\phi_\omega(x)|^2\, dx > 0$.

Alternatively, let $\frac{d}{d\omega}\|\phi_\omega\|_{L^2}^2 > 0$. We consider the function $f(z) = \langle \phi_\omega, (L_+ - z)^{-1} \phi_\omega \rangle$, $z \in \rho(L_+)$. Since $f(0) = \langle \phi_\omega, L_+^{-1} \phi_\omega \rangle > 0$, $f'(z) > 0$, and $\lim\limits_{z \to \lambda_0 +} f(z) = -\infty$ (where

$\lambda_0 < 0$ is the smallest eigenvalue of L_+), there is $\alpha \in (\lambda_0, 0) \subset \rho(L_+)$ such that $f(\alpha) = \langle \phi_\omega, (L_+ - \alpha)^{-1}\phi_\omega \rangle = 0$. Then we define $r = (L_+ - \alpha)^{-1}\phi_\omega$. Since $\langle \phi_\omega, r \rangle = f(\alpha) = 0$, there is η such that $r = L_-\eta$. It follows that the quadratic form $L_-^{1/2} L_+ L_-^{1/2}$ is not positive definite:

$$\langle L_-^{\frac{1}{2}}\eta, (L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}}) L_-^{\frac{1}{2}}\eta \rangle = \langle r, L_+ r \rangle = \langle r, (\alpha r + \phi_\omega) \rangle = \alpha \langle r, r \rangle < 0.$$

Thus, there is $\lambda > 0$ such that $-\lambda^2 \in \sigma(L_-^{1/2} L_+ L_-^{1/2})$; then also $-\lambda^2 \in \sigma(L_- L_+)$. Let ξ be the corresponding eigenvector, $L_- L_+ \xi = -\lambda^2 \xi$; then $\begin{bmatrix} 0 & L_- \\ -L_+ & 0 \end{bmatrix} \begin{bmatrix} \xi \\ -\frac{1}{\lambda} L_+ \xi \end{bmatrix} = \lambda \begin{bmatrix} \xi \\ -\frac{1}{\lambda} L_+ \xi \end{bmatrix}$, hence $\lambda \in \sigma(\mathbf{JL})$. $\square$

Problem 2.6. Find the explicit form of the solitary wave solutions $\psi(x,t) = \phi_\omega(x) e^{-i\omega t}$, $\omega < 0$, to the nonlinear Schrödinger equation

$$i\partial_t \psi = -\partial_x^2 \psi - |\psi|^2 \psi, \qquad \psi(x,t) \in \mathbb{C}, \quad x \in \mathbb{R}.$$

Problem 2.7. Let $g(s) = -s$ (cubic NLS in 1D). Use the Birman-Schwinger principle to estimate the number of discrete eigenvalues of $L_\pm(\omega)$. Hint: you will need $(-\Delta)^{-1}$ in 1D.

Problem 2.8. Let $g(s) = -s$ (cubic NLS in 1D). Use the Birman-Schwinger principle to estimate the number of discrete purely imaginary eigenvalues of $\mathbf{JL}_\pm(\omega)$ between $\lambda = 0$ and the upper end of the spectral gap.

Hint: if one removes the x-dependent parts of $\mathbf{JL}(\omega)$, the discrete spectrum disappears.

3 Limiting absorption principle

3.1 Agmon's Appendix A

We reproduce almost verbatim several results from Agmon's paper [Agm75, Appendix A].

For $s \in \mathbb{R}$, denote $L^2_s(\mathbb{R}^n) = \{u \in L^2_{loc}(\mathbb{R}^n)\colon (1+x^2)^{s/2} f \in L^2(\mathbb{R}^n)\}$.

For $m \in \mathbb{N}$, denote $H^m_s(\mathbb{R}^n) = \{u \in L^2_{loc}(\mathbb{R}^n)\colon \|u\|^2_{H^m_s} := \sum_{a \in \mathbb{Z}^n_+,\, |a| \le m} \|\partial^a_x f\|^2_{L^2_s} < \infty\}$.

Lemma 3.1 ([Agm75], Lemma A.1)**.** *Let $u \in H^1(\mathbb{R})$, $\lambda \in \mathbb{C}$, $s > 1/2$. The following inequality holds:*

$$\|u\|_{L^2_{-s}} \le c_s \|(\frac{d}{dx} - \lambda)u\|_{L^2_s}, \qquad c_s = 2\int_0^\infty (1+x^2)^{-s}\, dx. \tag{3.1}$$

Proof. We set

$$f(x) = (\frac{d}{dx} - \lambda)u(x), \qquad u \in H^1(\mathbb{R}). \tag{3.2}$$

We may assume without loss of generality that $f \in L^1(\mathbb{R})$ (or else $\|f\|_{L^2_s} = \infty$ and (3.1) holds trivially), and that $\operatorname{Re}\lambda \le 0$. Solving (3.2) for u we get

$$u(x) = \int_{-\infty}^x f(t) e^{\lambda(x-t)}\, dt. \tag{3.3}$$

From (3.3) it follows that

$$|u(x)|^2 \le \Big(\int_{-\infty}^x |f(t)|\, dt\Big)^2 \le \Big(\int_{\mathbb{R}} (1+t^2)^{-s}\, dt\Big)\Big(\int_{\mathbb{R}} (1+t^2)^s |f(t)|^2\, dt\Big). \tag{3.4}$$

We multiply (3.4) by $(1+x^2)^{-s}$ and integrate:

$$\int_{\mathbb{R}} (1+x^2)^{-s} |u(x)|^2\, dx \le \Big(\int_{\mathbb{R}} (1+t^2)^{-s}\, dt\Big)^2 \int_{\mathbb{R}} (1+t^2)^s |f(t)|^2\, dt.$$

□

Lemma 3.2 ([Agm75], Lemma A.2)**.** *Let $P(D) = P(D_1, \dots, D_n)$ be a partial differential operator of order m. Then for $\forall u \in H^m(\mathbb{R}^n)$ and any given $s > \frac{1}{2}$, the following inequality holds:*

$$\int_{\mathbb{R}^n} (1+x_j^2)^{-s} |P^{(j)}(D)u|^2\, dx \le m^2 c_s^2 \int_{\mathbb{R}^n} (1+x_j^2)^s |P(D)u|^2\, dx, \tag{3.5}$$

for $j = 1, \dots, m$, where c_s is the constant from Lemma 3.1.

Let $P(D) = P(D_1, \dots, D_n)$ be a partial differential operator with constant coefficients of order m, acting on functions on $\mathbb{R}^n$. We denote its principal part by $P_m(D)$. The operator P is said to be *of principal type* if $\operatorname{grad} P_m(\xi) \ne 0$, $\forall \xi \in \mathbb{R}^n \setminus \{0\}$; P is said to be *elliptic* if $P_m(\xi) \ne 0$, $\forall \xi \in \mathbb{R}^n \setminus \{0\}$ (an elliptic operator is thus an operator of principal type).

We shall say that a number $z \in \mathbb{C}$ is a *critical value* of P if there exists a $\xi_0 \in \mathbb{R}^n$ such that $P(\xi_0) = z$, $\operatorname{grad} P(\xi_0) = 0$. We shall denote the set of all critical values of P by $\Lambda_C(P)$.

Lemma 3.3 ([Agm75], Lemma A.3). *Let $P(D) = P(D_1, \ldots, D_n)$ be a partial differential operator of order m. Set $m' = m$ if P is elliptic, $m' = m - 1$ otherwise. Let $\mathcal{K}$ be a compact set in $\mathbb{C} \setminus \Lambda_C(P)$ and let s be a real number. The following estimate holds:*

$$\|u\|_{H_s^{m'}} \le C_s\Big(\|(P(D) - z)u\|_{L_s^2} + \sum_{j=1}^{n} \|P^{(j)}(D)u\|_{L^2}\Big) \tag{3.6}$$

for $\forall u \in H_s^m(\mathbb{R}^n)$ and $\forall z \in \mathcal{K}$, where C_s is a constant not depending on z or u.

Theorem 3.4 ([Agm75], Theorem A.1). *Let $P(D)$ be a differential operator with constant coefficients of order m and of principal type. Set $m' = m$ if P is elliptic, $m' = m-1$ otherwise. Let $\mathcal{K}$ be a compact set in $\mathbb{C} \setminus \Lambda_C(P)$ and let $s > \frac{1}{2}$. The following estimate holds:*

$$\|u\|_{H_{-s}^{m'}} \le C\|\big(P(D) - z\big)u\|_{L_s^2} \tag{3.7}$$

for $\forall u \in H^m(\mathbb{R}^n)$ where C is some constant not depending on z or u.

3.2 Improvement at the continuous spectrum

For λ at the continuous spectrum, in [Agm75, Appendix B], Agmon gives an improvement of Theorem 3.4 "moving" his estimates $\|u\|_{L^2_{s'}} \le C\|(P - z)u\|_{L^2_s}$, $s' < s$, to the region with $s' > 0$. This "shift" is very important: it implies that

$$\|u\|_{H^d_{s'}} \sim \|Pu\|_{L^2_{s'}} + \|u\|_{L^2_{s'}} \le \|(P - z)u\|_{L^2_{s'}} + (1 + |z|)\|u\|_{L^2_{s'}} \le (2 + |z|)\|(P - z)u\|_{L^2_s},$$

while for $s' > 0$ $H^d_{s'}$ is *compactly* embedded into L^2.

We will illustrate this improvement in the following lemma.

Lemma 3.5. *Let $u \in H^1(\mathbb{R})$, $\Lambda \in \mathbb{R}$, $s > 1/2$, $\epsilon > 0$. Then, for some $C_{s,\epsilon} < \infty$, the following inequality holds:*

$$\|u\|_{L^2_{s-1-\epsilon}} \le C_s\|(\frac{d}{dx} - i\Lambda)u\|_{L^2_s}. \tag{3.8}$$

Proof. Similarly to (3.4),

$$|u(x)|^2 \le \Big(\int_{-\infty}^{x} |f(t)|\,dt\Big)^2 \le \int_{\mathbb{R}} (1 + t^2)^{-s}\,dt \int_{\mathbb{R}} (1 + t^2)^s |f(t)|^2\,dt. \tag{3.9}$$

For $x \le 0$, there is the following improvement:

$$|u(x)|^2 \le \Big(\int\limits_{-\infty}^{x} |f(t)|\,dt\Big)^2 \le \int\limits_{-\infty}^{x} (1 + t^2)^{-s}\,dt \int_{\mathbb{R}} (1 + t^2)^s |f(t)|^2\,dt \le C_s\langle x\rangle^{-2s+1}\|f\|^2_{L^2_s}. \tag{3.10}$$

In the last inequality, we used the bound $s > 1/2$. Since now $\operatorname{Re}\lambda = 0$, we similarly have, for $x \geq 0$,

$$|u(x)|^2 \leq \Big(\int_{+\infty}^{x} |f(t)|\,dt \Big)^2 \leq \int_{+\infty}^{x} (1+t^2)^{-s}\,dt \int_{\mathbb{R}} (1+t^2)^s |f(t)|^2\,dt \leq C_s \langle x\rangle^{-2s+1} \|f\|_{L^2_s}^2. \tag{3.11}$$

Since $\langle x\rangle^{-s+1/2} \in L^2_{s-1-\epsilon}(\mathbb{R})$, the result follows. □

3.3 Limiting absorption principle for the Laplacian in 3D

Let us give a more general result for the Laplace operator, which is also valid in the vicinity of the threshold $\lambda = 0$.

Lemma 3.6 ([JK79]). *For any $\lambda \geq 0$, $s,\ t > 1/2$, $s + t > 2$, there is $C_{s,t,\lambda} < \infty$ such that*

$$\|u\|_{L^2_{-t}} \leq C_{s,t,\lambda} \|(-\Delta - \lambda)u\|_{L^2_s}, \qquad u \in H^2(\mathbb{R}^3).$$

Proof. Since the resolvent $(-\Delta-\lambda)^{-1}$ has the integral kernel $K(x,y) = -\frac{e^{i\sqrt{\lambda}|x-y|}}{4\pi|x-y|}$ (we assume that $\operatorname{Im}\sqrt{\lambda} \geq 0$), it is enough to prove that

$$\Big\| \frac{1}{|x|} * f \Big\|_{L^2_{-t}} \leq C \|f\|_{L^2_s}, \qquad \forall f \in L^2_s(\mathbb{R}^3).$$

Let $g \in L^2_t(\mathbb{R}^3)$; it suffices to show that

$$\Big\langle g, \frac{1}{|x|} * f \Big\rangle \leq C \|f\|_{L^2_s} \|g\|_{L^2_t}.$$

Equivalently, enough to show

$$\int_{\mathbb{R}^3} \frac{\overline{\hat{f}}(k)\hat{g}(k)}{k^2}\,dk \leq C \|\hat{f}\|_{H^s} \|\hat{g}\|_{H^t}.$$

Since $\int_{|k|>1} \frac{\overline{\hat{f}}(k)\hat{g}(k)}{k^2}\,dk \leq C\|\hat{f}\|_{L^2}\|\hat{g}\|_{L^2}$, it is enough to show that

$$\int_{|k|\leq 1} \frac{\overline{\hat{f}}(k)\hat{g}(k)}{k^2}\,dk \leq C \|\hat{f}\|_{H^s} \|\hat{g}\|_{H^t}.$$

Let χ denote the characteristic function of the unit ball in $\mathbb{R}^3$. By the Hölder inequality, the integral is bounded by

$$\|\chi\hat{f}\|_{L^p} \|\chi\hat{g}\|_{L^q} \|k^{-2}\chi\|_{L^r}, \qquad p,\ q,\ r \geq 1, \quad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1.$$

For the last factor to remain finite, we can take $r \in (1, 3/2)$; this leads to $1/r > 2/3$, hence

$$\frac{1}{p} + \frac{1}{q} < \frac{1}{3}. \tag{3.12}$$

By the Sobolev embedding, $H^s(\mathbb{R}^3) \subset L^p(\mathbb{R}^3)$ for $2 \leq p \leq p_0$ with $\frac{1}{2} - \frac{1}{p_0} = \frac{s}{n} = \frac{s}{3}$, leading to

$$\frac{1}{p} \geq \frac{1}{p_0} = \frac{1}{2} - \frac{s}{3}, \qquad \frac{1}{q} \geq \frac{1}{q_0} = \frac{1}{2} - \frac{t}{3}, \qquad \text{hence} \quad \frac{1}{p} + \frac{1}{q} \geq 1 - \frac{s+t}{3}.$$

Comparing with (3.12) yields $s + t > 2$. Besides, since $p,\ q$ are nonnegative, (3.12) implies that $p,\ q > 3$, which leads to the restriction $s,\ t > 1/2$. $\square$

Problem 3.7. Use the $\sim 1/r$ decay of the Coulomb potential to show that one can not bound $\|u\|_{L^2}$ with $\|\Delta u\|_{L^2_s}$ with arbitrarily large $s \geq 0$.

4 Bifurcations from the essential spectrum of the free Dirac equation

4.1 Limiting absorption principle for the Dirac operator

Let $n \in \mathbb{N}$. Let α^j, $1 \le j \le n$, and β be self-adjoint $N \times N$ matrices such that

$$\{\alpha^j, \alpha^k\} = \alpha^j \alpha^k + \alpha^k \alpha^j = 2\delta_{jk} I_N, \qquad \beta^2 = I_N, \qquad \{\alpha^j, \beta\} = 0.$$

If $n = 3$, we can take $\alpha^j = \begin{bmatrix} 0 & \sigma^j \\ \sigma^j & 0 \end{bmatrix}$, $\beta = \begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix}$, where $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are the Pauli matrices. The Dirac operator is defined by

$$D_m = -i \sum_{j=1}^{n} \alpha^j \frac{\partial}{\partial x^j} + m\beta, \qquad m \ge 0. \tag{4.1}$$

One can easily compute that

$$\sigma_{ess}(D_m) = \mathbb{R} \setminus (-m, m).$$

Note that $D_m^2 = -\Delta + m^2$, with $\sigma_{ess}(-\Delta + m^2) = [m^2, +\infty)$.

Theorem 4.1 ([Yam73], Theorem 3.1)**.** *Let $\mathcal{K}$ be a compact set in $\mathbb{C} \setminus \{\pm m\}$. For any $s > \frac{1}{2}$ there exists $C = C(s, \mathcal{K})$ such that*

$$\|u\|_{H^1_{-s}} \le C \|(D_m - z)u\|_{L^2_s}$$

for all $u \in H^1_s(\mathbb{R}^n, \mathbb{C}^N)$ and $z \in \mathcal{K} \setminus \mathbb{R}$.

Proof. We show that $(D_m - z)^{-1} : \ L^2_s(\mathbb{R}^n, \mathbb{C}^N) \to H^1_{-s}(\mathbb{R}^n, \mathbb{C}^N)$ is uniformly bounded for $z \in \mathcal{K} \setminus \mathbb{R}$. Indeed, by Theorem 3.4,

$$(-\Delta + m^2 - z^2)^{-1} : \ L^2_s(\mathbb{R}^n, \mathbb{C}^N) \to H^2_{-s}(\mathbb{R}^n, \mathbb{C}^N)$$

is uniformly bounded for $z \in \mathcal{K} \setminus \mathbb{R}$, and then

$$D_m + z : \ H^2_{-s}(\mathbb{R}^n, \mathbb{C}^N) \to H^1_{-s}(\mathbb{R}^n, \mathbb{C}^N)$$

is uniformly bounded for $z \in \mathcal{K} \setminus \mathbb{R}$. It follows that

$$\|(D_m + z)(-\Delta + m^2 - z^2)^{-1} f\|_{H^1_{-s}} \le C \|f\|_{L^2_s}.$$

Denote $u := (D_m + z)(-\Delta + m^2 - z^2)^{-1} f$; then

$$(D_m - z)u = (D_m - z)(D_m + z)(-\Delta + m^2 - z^2)^{-1} f = f.$$

We conclude that $\|u\|_{H^1_{-s}} \le C \|(D_m - z)u\|_{L^2_s}$. □

Problem 4.2. Prove that $D_m : \ H^2_s(\mathbb{R}^n, \mathbb{C}^N) \to H^1_s(\mathbb{R}^n, \mathbb{C}^N)$ is bounded for any $s \in \mathbb{R}$.

4.2 Nonrelativistic limit of nonlinear Dirac equation

The nonlinear Dirac equation in $\mathbb{R}^n$ has the form

$$i\partial_t\psi = D_m\psi - f(\psi^*\beta\psi)\beta\psi, \qquad \psi(x,t) \in \mathbb{C}^N, \quad x \in \mathbb{R}^n, \tag{4.2}$$

where $f \in C^\infty(\mathbb{R})$, $f(0) = 0$. We consider solitary wave solutions $\psi(x,t) = \phi_\omega(x)e^{-i\omega t}$, $\phi_\omega \in H^1(\mathbb{R}^n, \mathbb{C}^N)$. Let $\varphi_\omega, \chi_\omega \in H^1(\mathbb{R}^n, \mathbb{C}^{N/2})$ be such that

$$\phi_\omega = \begin{bmatrix} \varphi_\omega \\ \chi_\omega \end{bmatrix}. \tag{4.3}$$

Let $f(s) = s^k$; then we have:

$$\omega \begin{bmatrix} \varphi_\omega \\ \chi_\omega \end{bmatrix} = -i \begin{bmatrix} \sigma_j\partial_j\chi_\omega \\ \sigma_j\partial_j\varphi_\omega \end{bmatrix} + m \begin{bmatrix} \varphi_\omega \\ -\chi_\omega \end{bmatrix} - (|\varphi_\omega|^2 - |\chi_\omega|^2)^k \begin{bmatrix} \varphi_\omega \\ -\chi_\omega \end{bmatrix}. \tag{4.4}$$

We write this system as the following equations:

$$-(m-\omega)\varphi_\omega = -i\sigma_j\partial_j\chi_\omega - (|\varphi_\omega|^2 - |\chi_\omega|^2)^k\varphi_\omega, \tag{4.5}$$

$$-(m+\omega)\chi_\omega = -i\sigma_j\partial_j\varphi_\omega - (|\varphi_\omega|^2 - |\chi_\omega|^2)^k\chi_\omega. \tag{4.6}$$

We consider the "nonrelativistic limit" $\omega \lesssim m$; then the approximate solution is given by

$$\chi_\omega = \frac{1}{m+\omega}\sigma_j\partial_j\varphi_\omega, \tag{4.7}$$

and the first equation takes the form $-(m-\omega)\varphi_\omega = -\frac{1}{m+\omega}\Delta\varphi_\omega - |\varphi_\omega|^{2k}\varphi_\omega$. By [BL83], for $n \le 2$, $k \in \mathbb{N}$ and for $n = 3$, $k = 1$ there is a unique positive spherically symmetric function Φ from the Schwartz class which solves the equation

$$-\frac{1}{2m}\Phi = -\frac{1}{2m}\Delta\Phi - |\Phi|^{2k}\Phi.$$

Denote $\epsilon = \sqrt{m^2 - \omega^2}$. Let $\boldsymbol{n} \in \mathbb{C}^{N/2}$, $\|\boldsymbol{n}\| = 1$. It follows that there is a solution such that

$$\varphi_\omega(x) \approx \epsilon^{\frac{1}{k}}\Phi(\epsilon x)\boldsymbol{n}, \qquad \chi_\omega(x) = \frac{1}{m+\omega}\sigma_j\partial_j\varphi_\omega(x) \approx \epsilon^{1+\frac{1}{k}}\frac{1}{2m}\sigma_j(\partial_j\Phi)(\epsilon x). \tag{4.8}$$

Let us consider the Ansatz $\psi(x,t) = (\phi_\omega + \rho(x,t))e^{-i\omega t}$, and let

$$\partial_t \begin{bmatrix} \operatorname{Re}\rho \\ \operatorname{Im}\rho \end{bmatrix} = \mathbf{A}_\omega \begin{bmatrix} \operatorname{Re}\rho \\ \operatorname{Im}\rho \end{bmatrix}$$

be the linearized equation on ρ. One has:

$$\mathbf{A}_\omega = \mathbf{J}(\mathbf{D}_m - \omega + \mathbf{V}(x,\omega)),$$

where $\mathbf{J}$ and $\mathbf{D}_m$ are representations of the action of $-i$ and D_m in the space of $\mathbb{R}^{2N}$-valued functions:

$$\mathbf{J} = \begin{bmatrix} 0 & I_N \\ -I_N & 0 \end{bmatrix}, \qquad \mathbf{D}_m = \mathbf{J}\boldsymbol{\alpha}_j\partial_j + m\boldsymbol{\beta},$$

$$\mathbf{J} = \begin{bmatrix} 0 & I_N \\ -I_N & 0 \end{bmatrix}. \qquad \boldsymbol{\alpha}_j = \begin{bmatrix} \operatorname{Re}\alpha_j & -\operatorname{Im}\alpha_j \\ \operatorname{Im}\alpha_j & \operatorname{Re}\alpha_j \end{bmatrix}, \qquad \boldsymbol{\beta} = \begin{bmatrix} \operatorname{Re}\beta & -\operatorname{Im}\beta \\ \operatorname{Im}\beta & \operatorname{Re}\beta \end{bmatrix}. \tag{4.9}$$

Above, $\mathbf{V}(\cdot,\omega) \in \mathscr{S}(\mathbb{R}^n, \mathbb{C}^{2N\times 2N})$, and moreover, by (4.8),

$$\mathbf{V}(x,\omega) \approx \epsilon^2 W(\epsilon x), \tag{4.10}$$

for some $W \in \mathscr{S}(\mathbb{R}^n, \mathbb{C}^{2N\times 2N})$.

Problem 4.3. Explain (without computing anything) why $\boldsymbol{\alpha}_j$ and $\boldsymbol{\beta}$ commute with $\mathbf{J}$.

4.3 Bifurcations from the essential spectrum

Let us consider families of eigenvalues in the limit of small amplitude solitary waves, which may be present in the spectrum up to the border of existence of solitary waves: $\omega \to \omega_0 := m$. This situation could be considered as the bifurcation of eigenvalues from the continuous spectrum of the free Dirac equation.

Lemma 4.4. *Let*

$$\mathbf{L}(\omega) = \mathbf{D}_m - \omega + \mathbf{V}(x,\omega), \qquad \omega \in [-m, m],$$

with $\mathbf{V}(\cdot,\omega) \in L^\infty(\mathbb{R}^n, \operatorname{End}(\mathbb{C}^{2N}))$. *Let* $\omega_0 = m$. *Assume that there is* $s > 1/2$ *such that*

$$\lim_{\omega\to\omega_0} \|\langle x\rangle^{2s}\mathbf{V}(\omega)\|_{L^\infty} = 0. \tag{4.11}$$

Let $\omega_j \in \mathcal{O}$, $\omega_j \underset{j\to\infty}{\longrightarrow} \omega_0$. *If* $\lambda_j \in \sigma_p(\mathbf{J}\mathbf{L}(\omega_j))$, *then the only possible accumulation points of* $\{\lambda_j \colon j \in \mathbb{N}\}$ *are* $\lambda = \{0; \pm 2mi\}$.

Remark 4.5. By (4.8), the condition (4.11) is satisfied for solitary waves in the nonrelativistic limit $\omega \to m$ considered in Section 4.2.

Proof. Let $\mathcal{K} \subset \mathbb{C}$ be a compact set, $\pm m \notin \mathcal{K}$. According to [Yam73] (Cf. Theorem 4.1), there is the limiting absorption principle for the free Dirac operator $D_m = -i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta m$, so that the following action of its resolvent is uniformly bounded for $z \in \mathcal{K}\backslash\mathbb{R}$:

$$(D_m - z)^{-1} : \quad L^2_s(\mathbb{R}^n, \mathbb{C}^N) \to L^2_{-s}(\mathbb{R}^n, \mathbb{C}^N), \qquad s > 1/2, \qquad z \in \mathcal{K}\backslash\mathbb{R}. \tag{4.12}$$

Now let $\mathcal{V} \subset \mathbb{C}$ be an arbitrary compact set which does not contain $\pm 2(m \pm \omega_0)i$. To prove the theorem, we need to show that for ω sufficiently close to ω_0 there is no point spectrum of $\mathbf{J}\mathbf{L}(\omega)$ in $\mathcal{V}$. Let ω be close enough to ω_0 so that $\mathcal{V}$ does not contain $\pm i(m \pm \omega)$. One has $\lim_{|x|\to\infty} \mathbf{L}(\omega) = \mathbf{D}_m - \omega$ Since the eigenvalues of $\mathbf{J}$ are $\pm i$, the operator $\mathbf{J}(\mathbf{D}_m - \omega)$ can be represented as the direct sum of operators $i(\mathbf{D}_m-\omega)$ and $-i(\mathbf{D}_m-\omega)$, acting in some subspaces of $\mathbb{C}^{2N}$ (spectral subspaces of $\mathbf{J}$ corresponding to $\pm i$). By (4.12), the following map is bounded uniformly for $z \in \mathcal{V}\backslash i\mathbb{R}$:

$$(\mathbf{J}(\mathbf{D}_m-\omega)-z)^{-1} : \ L^2_s(\mathbb{R}^n, \mathbb{C}^{2N}) \to L^2_{-s}(\mathbb{R}^n, \mathbb{C}^{2N}), \qquad s > 1/2, \qquad z \in \mathcal{V}\backslash i\mathbb{R}. \quad (4.13)$$

The resolvent of $\mathbf{JL}(\omega)$ is expressed as

$$(\mathbf{JL}(\omega)-z)^{-1} = (\mathbf{J}(\mathbf{D}_m-\omega)-z)^{-1}\frac{1}{1+\mathbf{JV}(\mathbf{J}(\mathbf{D}_m-\omega)-z)^{-1}}. \quad (4.14)$$

Thus, the action $(\mathbf{JL}(\omega)-z)^{-1} : L^2_s(\mathbb{R}^n, \mathbb{C}^{2N}) \to L^2_{-s}(\mathbb{R}^n, \mathbb{C}^{2N})$ is uniformly bounded in $z \in \mathcal{V}(\omega)\backslash i\mathbb{R}$ as long as the operator $\mathbf{V}(\omega) : L^2_{-s}(\mathbb{R}^n, \mathbb{C}^{2N}) \to L^2_s(\mathbb{R}^n, \mathbb{C}^{2N})$ of multiplication by $\mathbf{V}(x,\omega)$ has a sufficiently small norm; it is enough to have

$$\|\mathbf{V}(\omega)\|_{L^2_{-s}\to L^2_s}\|(\mathbf{J}(\mathbf{D}_m-\omega)-z)^{-1}\|_{L^2_s\to L^2_{-s}} < 1/2. \quad (4.15)$$

Due to the bound on the action (4.13), the inequality (4.15) is satisfied since

$$\lim_{\omega\to\omega_0}\|\mathbf{V}(\omega)\|_{L^2_{-s}\to L^2_s} = \lim_{\omega\to\omega_0}\|\langle x\rangle^{2s}\mathbf{V}(\omega)\|_{L^\infty(\mathbb{R}^n,\mathrm{End}(\mathbb{C}^{2N}))} = 0$$

by the assumption of the theorem. □

Lemma 4.6. *If $\lambda \in \sigma_p(\mathbf{JL}) \setminus i\mathbb{R}$ with the corresponding eigenvector $\boldsymbol{\zeta}$, then $\langle\boldsymbol{\zeta}, \mathbf{L}\boldsymbol{\zeta}\rangle = 0$, $\langle\boldsymbol{\zeta}, \mathbf{J}\boldsymbol{\zeta}\rangle = 0$,*

Proof. One has $\mathbf{JL}\boldsymbol{\zeta} = \lambda\boldsymbol{\zeta}$, $\mathbf{L}\boldsymbol{\zeta} = -\lambda\mathbf{J}\boldsymbol{\zeta}$, hence

$$\langle\boldsymbol{\zeta}, \mathbf{L}\boldsymbol{\zeta}\rangle = -\lambda\langle\boldsymbol{\zeta}, \mathbf{J}\boldsymbol{\zeta}\rangle. \quad (4.16)$$

Since $\langle\boldsymbol{\zeta}, \mathbf{L}\boldsymbol{\zeta}\rangle \in \mathbb{R}$ and $\langle\boldsymbol{\zeta}, \mathbf{J}\boldsymbol{\zeta}\rangle \in i\mathbb{R}$, the condition $\mathrm{Re}\,\lambda \neq 0$ implies that both sides in (4.16) are equal to zero. □

Remark 4.7. If an eigenvector $\boldsymbol{\zeta}$ corresponding to $\lambda \in \sigma_p(\mathbf{JL})$ satisfies $\langle\boldsymbol{\zeta}, \mathbf{L}\boldsymbol{\zeta}\rangle = 0$, we will say that λ has zero Krein signature. The Krein signature is only interesting for $\lambda \in i\mathbb{R}$ since, according to Lemma 4.6, all eigenvalues of $\mathbf{JL}$ with nonzero real part have zero Krein signature.

Lemma 4.8. *Let $\omega_0 = \pm m$. Assume that, for each $x \in \mathbb{R}^n$ and $\omega \in [-m, m]$, $\mathbf{V}(x,\omega)$ is a self-adjoint $2N \times 2N$ matrix, and there is $C < \infty$ such that*

$$\|\mathbf{V}(\omega)\|_{L^2\to L^2} \leq C(m^2-\omega^2). \quad (4.17)$$

Let $\omega_j \in (-m, m)$, $j \in \mathbb{N}$; $\omega_j \to \omega_0$. If there is a sequence $\lambda_j \in \sigma(\mathbf{JL}(\omega_j))$, such that $\mathrm{Re}\,\lambda_j \neq 0$ and $\lim\limits_{j\to\infty}\lambda_j = 0$, then

$$|\lambda_j| = O(m^2-\omega_j^2).$$

Remark 4.9. By (4.10), for linearization at solitary waves, the condition (4.17) is satisfied as $\omega \to m$.

Proof. Without loss of generality, we will assume that $\omega_0 = m$. We have: $\mathbf{J}\mathbf{L}(\omega_j)\boldsymbol{\zeta}_j = \lambda_j \boldsymbol{\zeta}_j$, $\boldsymbol{\zeta}_j \in L^2(\mathbb{R}^n, \mathbb{C}^{2N})$, and without loss of generality we assume that $\|\boldsymbol{\zeta}_j\|_{L^2} = 1$. We write:

$$(\mathbf{D}_m - \omega_j + \mathbf{J}\lambda_j)\boldsymbol{\zeta}_j = -\mathbf{V}(\omega_j)\boldsymbol{\zeta}_j. \tag{4.18}$$

Let $\Pi^{\pm}$ be orthogonal projections onto eigenspaces of $\mathbf{J}$ corresponding to $\pm i \in \sigma(\mathbf{J})$. Applying $\Pi^{\pm}$ to (4.18) and denoting $\boldsymbol{\zeta}_j^{\pm} = \Pi^{\pm}\boldsymbol{\zeta}_j$, we get:

$$(\mathbf{D}_m - \omega_j + i\lambda_j)\boldsymbol{\zeta}_j^{+} = -\Pi^{+}\mathbf{V}(\omega_j)\boldsymbol{\zeta}_j, \qquad (\mathbf{D}_m - \omega_j - i\lambda_j)\boldsymbol{\zeta}_j^{-} = -\Pi^{-}\mathbf{V}(\omega_j)\boldsymbol{\zeta}_j. \tag{4.19}$$

Since $\omega_j \to m$, without loss of generality, we can assume that $\omega_j > m/2$ for all $j \in \mathbb{N}$. Since the spectrum $\sigma(\mathrm{JL})$ is symmetric with respect to real and imaginary axes, we may assume, without loss of generality, that $\operatorname{Im}\lambda_j \ge 0$ for all $j \in \mathbb{N}$, so that $\operatorname{Re}(i\lambda_j) \le 0$ (see Figure 2). At the same time, since $\lambda_j \to 0$, we can assume that $|\lambda_j| < m/2$.

Figure 2: The closest point from $\sigma(\mathbf{D}_m - \omega_j)$ to $i\lambda_j$ is $m - \omega_j$.

With $\mathbf{D}_m - \omega_j$ being self-adjoint, one has

$$\|(\mathbf{D}_m - \omega_j - i\lambda_j)^{-1}\| = \frac{1}{\operatorname{dist}(i\lambda_j, \sigma(\mathbf{D}_m - \omega_j))} = \frac{1}{|m - \omega_j - i\lambda_j|}. \tag{4.20}$$

Combining (4.19) and (4.20), we get

$$\|\boldsymbol{\zeta}_j^{-}\|_{L^2} \le \frac{\|\Pi^{-}\mathbf{V}(\omega_j)\boldsymbol{\zeta}_j\|}{|m - \omega_j - i\lambda_j|} \le \frac{C(m^2 - \omega_j^2)}{|m - \omega_j - i\lambda_j|}. \tag{4.21}$$

We used the normalization $\|\boldsymbol{\zeta}_j\| = 1$ and the bound $\|\Pi^{-}\mathbf{V}(\omega_j)\|_{L^2 \to L^2} \le C(m^2 - \omega_j^2)$ (Cf. (4.17)). At the same time, due to $\operatorname{Re}\lambda_j \ne 0$, Lemma 4.6 yields

$$0 = \langle \boldsymbol{\zeta}_j, \mathbf{J}\boldsymbol{\zeta}_j \rangle = i\|\boldsymbol{\zeta}_j^{+}\|_{L^2}^2 - i\|\boldsymbol{\zeta}_j^{-}\|_{L^2}^2,$$

hence $\|\boldsymbol{\zeta}_j^{+}\|^2 = \|\boldsymbol{\zeta}_j^{-}\|^2 = \frac{1}{2}\|\boldsymbol{\zeta}_j\|^2 = \frac{1}{2}$, thus (4.21) yields

$$|m - \omega_j - i\lambda_j| \le \sqrt{2}C(m^2 - \omega_j^2),$$

leading to

$$|\lambda_j| \le \sqrt{2}C(m^2 - \omega_j^2) + |m - \omega_j| \le \big(\sqrt{2}C + \frac{1}{2m}\big)(m^2 - \omega_j^2). \qquad \square$$

5 The Paley-Wiener theorem

Theorem 5.1 (Paley-Wiener)**.** *(1) Let $\varphi \in \mathscr{S}(\mathbb{R}^n)$. If $\operatorname{supp}\varphi \subset B_R \subset \mathbb{R}^n$ (ball of radius R), then $\hat\varphi(\zeta)$ is an* entire function *of $\zeta \in \mathbb{C}^n$ (analytic function in the whole space $\mathbb{C}^n$) and for any $N \in \mathbb{N}$ there is $C_N < \infty$ such that*

$$|\hat\varphi(\zeta)| \le C_N \langle\zeta\rangle^{-N} e^{R|\operatorname{Im}\zeta|}. \tag{5.1}$$

(2) Conversely, if $\hat\varphi \in \mathscr{S}(\mathbb{R}^n)$ has a holomorphic extension to $\mathbb{C}^n$ (also denoted $\hat\varphi$) which satisfies (5.1) with some $R < \infty$, for any $N \in \mathbb{N}$, then $\varphi \in C^\infty(\mathbb{R}^n)$, $\operatorname{supp}\varphi \subset B_R$.

Above, $\operatorname{Im}\xi = (\operatorname{Im}\xi_1, \dots, \operatorname{Im}\xi_n) \in \mathbb{R}^n$.

Proof. The first part is immediate: integrate by parts in x in the integral $\hat\varphi(\xi) = \int e^{-ix\cdot\xi}\varphi(x)\,dx$.
For the second part, we pick $x \ne 0$ and define $\omega = x/|x|$. Then, due to analyticity of $\hat\varphi$,

$$\varphi(x) = \int_{\mathbb{R}^n} \hat\varphi(\xi) e^{i\xi\cdot x}\,\frac{d\xi}{(2\pi)^n} = \int_{\mathbb{R}^n} \hat\varphi(\xi + i\tau\omega) e^{i(\xi+i\tau\omega)\cdot x}\,\frac{d\xi}{(2\pi)^n},$$

$$|\varphi(x)| \le C_N \int_{\mathbb{R}^n} \langle\xi\rangle^{-N} e^{R\tau} e^{-\tau|x|}\,d\xi.$$

Taking $N = n + 1$ and sending τ to $+\infty$, we see that for $|x| > R$ the integral is arbitrarily small, hence $\varphi(x) = 0$ for $|x| > R$. $\square$

Problem 5.2. Let f be smooth and compactly supported. Prove that if $r > 0$ is the smallest value such that $\operatorname{supp} f \subset [-r, r]$, then $R = 2r$ is the smallest value such that $\operatorname{supp} f * f \subset [-R, R]$.

Hint: Apply the Paley-Wiener theorem to $\widehat{f * f} = \hat f^2$.

Problem 5.2 is a particular case of the Titchmarsh convolution theorem:

Theorem 5.3 (Titchmarsh convolution theorem)**.** *For any $f,\ g \in \mathscr{E}'(\mathbb{R})$ (compactly supported distributions), there are the following relations:*

$$\sup\operatorname{supp} f * g = \sup\operatorname{supp} f + \sup\operatorname{supp} g, \qquad \inf\operatorname{supp} f * g = \inf\operatorname{supp} f + \inf\operatorname{supp} g.$$

Problem 5.4. Let $u \in L^2(\mathbb{R})$ satisfy

$$(-\Delta - \lambda^2)u(x) = f(x)u(x),$$

where $\lambda > 0$ and $f \in C(\mathbb{R})$, $|f(x)| < e^{-\epsilon|x|}$ for some $\epsilon > 0$. Prove that for any $N > 0$ there is $C_N < \infty$ such that $|u(x)| < C_N e^{-N|x|}$, $x \in \mathbb{R}$.

Hint: $|f(x)u(x)| \le Ce^{-\epsilon|x|}$, hence $\widehat{fu}(\xi)$ is analytic for $|\operatorname{Im}\xi| < \epsilon$.

Hint: So is $\hat u(\xi) = \frac{\widehat{fu}(\xi)}{\xi^2-\lambda^2}$ since $\widehat{fu}(\pm\lambda) = 0$, or else $\hat u = \frac{\widehat{uf}(\xi)}{\xi^2-\lambda^2} \notin L^2(\mathbb{R})$.

Hint: By Paley-Wiener, $|u(x)| \le Ce^{-\epsilon|x|}$; then $|f(x)u(x)| \le Ce^{-2\epsilon|x|}$.

6 Carleman estimates

We will illustrate the technique of Carleman estimates on the following easy result:

Proposition 6.1. *Let $L = i\partial_x - W(x)$, $D(L) = H^1(\mathbb{R})$, with $W \in C(\mathbb{R})$ such that $|W(x)| \le Ce^{-\epsilon|x|}$ with some $\epsilon > 0$ and $C < \infty$. Then there are no eigenvalues embedded into $\sigma_{ess}(L) = \mathbb{R}$.*

The argument goes like this: $|\psi| \le C_N e^{-N|x|}$, $\forall N > 0$; ψ has compact support; $\psi \equiv 0$.

Lemma 6.2. *If $(i\partial_x - W(x))\psi = \lambda\psi$ with $\lambda \in \mathbb{R}$, $\psi \in L^2(\mathbb{R})$, then for any $N > 0$ there is $C_N < \infty$ such that*

$$|\psi(x)| \le C_N e^{-N|x|}, \qquad x \in \mathbb{R}.$$

This lemma immediately follows from Problem 5.4. Note that $\psi \in L^2(\mathbb{R})$ implies $\psi \in H^1(\mathbb{R}) \subset C(\mathbb{R})$.

Lemma 6.3 (Hardy-type estimate; by N. Boussaïd)**.** *Let $\varphi \in C^1(\mathbb{R})$, $\varphi' > 0$. Then one has*

$$\|\sqrt{\varphi'} e^{\varphi} u\| \le \|\frac{1}{\sqrt{\varphi'}} e^{\varphi} u'\| \qquad \textit{for any } u \in H^1(\mathbb{R}) \textit{ with compact support.}$$

Proof. It is enough to consider $u \in C^1_{\mathrm{comp}}(\mathbb{R}, \mathbb{R})$. We integrating by parts and apply the Cauchy-Schwarz inequality:

$$\left| \int_{\mathbb{R}} \varphi' e^{2\varphi} u^2 \, dx \right| = \left| \int_{\mathbb{R}} e^{2\varphi} u u' \, dx \right| \le \Big(\int_{\mathbb{R}} \varphi' e^{2\varphi} u^2 \, dx \Big)^{1/2} \Big(\int_{\mathbb{R}} \frac{1}{\varphi'} e^{2\varphi} (u')^2 \, dx \Big)^{1/2}. \qquad \square$$

Let $\varphi(x) = x\tau$, $\tau > 0$, $\operatorname{supp} u \subset (0, \infty)$. Then Lemma 6.3 yields $\tau\|e^{\varphi}u\| \le \|e^{\varphi}u'\|$, hence

$$(\tau - |\lambda|)\|e^{\tau x}u\| \le \|e^{\tau x}(u' - \lambda u)\|, \qquad \operatorname{supp} x \subset \mathbb{R}_+. \tag{6.1}$$

Here $\tau > 0$ could be arbitrarily large; such estimates are called *Carleman estimates*.

Now we prove Proposition 6.1. Let $\rho \in C^\infty(\mathbb{R})$ be such that $\rho|_{(-\infty,0)} = 0$, $\rho|_{(1,+\infty)} = 1$, $\sup|\rho'| \le 2$. Let $a \ge 0$, $b \ge a + 2$. Denote

$$\rho_{a,b} = \rho(x-a)\rho(b-x), \qquad \operatorname{supp} \rho_{a,b} \subset [a,b]; \qquad \sup_{x\in\mathbb{R}} |\partial_x \rho_{a,b}| \le 2.$$

Since $(i\partial_x - \lambda)\psi = W\psi$, the function $u = \psi\rho_{a,b} \in C^1_{\mathrm{comp}}(\mathbb{R})$ satisfies $(i\partial_x - \lambda)u = Wu + i\psi\partial_x\rho_{a,b}$. Therefore, by (6.1),

$$(\tau - |\lambda|)\|e^{\varphi}u\| \le \|e^{\varphi}(Wu + i\psi\partial_x\rho_{a,b})\|.$$

Let $a > 0$ be large enough so that $|W|\big|_{(a,\infty)} \le 1$. Then, due to $\operatorname{supp} \partial_x\rho_{a,b} \subset [a, a+1] \cup [b-1, b]$,

$$(\tau - |\lambda| - 1)\|e^{\varphi}u\| \le \|e^{\varphi}\psi\partial_x\rho_{a,b}\| \le 2\|e^{\varphi}\psi\|_{L^2(a,a+1)} + 2\|e^{\varphi}\psi\|_{L^2(b-1,b)}.$$

Fix $\tau \geq |\lambda| + 2$, so that the coefficient on the left is not smaller than 1. Sending $b \to \infty$ and noticing that $\|e^{\varphi}\psi\|_{L^2(b-1,b)} \to 0$ due to Lemma 6.2, we conclude by the monotone convergence theorem that $v(x) = |\psi(x)|\rho(x-a)$ satisfies $\|e^{\varphi} v\| \leq 2\|e^{\varphi}\psi\|_{L^2(a,a+1)}$, and moreover we have

$$e^{(a+2)\tau}\|v\|_{L^2(a+2,\infty)} \leq \|e^{\varphi}v\|_{L^2(a+2,\infty)} \leq \|e^{\varphi}v\| \leq 2\|e^{\varphi}\psi\|_{L^2(a,a+1)} \leq 2e^{(a+1)\tau}\|\psi\|_{L^2(a,a+1)}.$$

Since τ could be arbitrarily large, $\|v\|_{L^2(a+2,\infty)} = 0$. Since $v \in H^1(\mathbb{R}) \subset C^1(\mathbb{R})$, one has $\operatorname{supp}\psi \subset \operatorname{supp} v \subset B_{a+2}$, which is the ball of radius $a+2$.

Finally, one needs the *unique continuation principle*; this is a property of certain equations that once a solution to such an equation vanishes on an open interval then it is identically zero. In one dimension, this is immediate due to the local well-posedness: Since the solution to $i\psi' - W\psi = \lambda\psi$ vanishes identically on an open interval (in the 1D case, cancellation at a single point would be enough), one has $\psi(x) \equiv 0$.

By [BG87], there are the following *Carleman estimates* for the Dirac operator: for $R > 0$ sufficiently large,

$$\|\sqrt{\tau}e^{\tau r}u\| \leq C_{\lambda,R}\|\sqrt{r}e^{\tau r}(D_m - \lambda)u\|, \quad \lambda \in \mathbb{R}\setminus[-m,m], \quad \forall\tau > 0, \quad u \in H^1, \quad u|_{\mathbb{B}_R} \equiv 0.$$

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