Lifelong Learning Programme DynaMAT

Napoleon polygons

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Abstract. An n-gon is called Napoleon if the centers of the regular n- gons erected outwardly on its sides are vertices of a regular n-gon. In this note we obtain a new geometric characterization of Napoleon n- gons and give a new proof of the well-known theorem of Barlotti - Greber ([**1**], [**3**]) that an n-gon is Napoleon if and only if it is affine - regular. Moreover, we generalize this theorem by obtaining an analytic characterization of the n-gons leading to a regular n-gon after iterating the above construction k times.

1. INTRODUCTION. A popular topic in plane geometry is to study configurations obtained by constructing polygons on the sides of a given polygon. The most classical result in this direction is the so-called Napoleon's theorem which states that if equilateral triangles are erected outwardly(inwardly) on the sides of an arbitrary triangle then their centers are vertices of an equilateral triangle. There are various interesting generalizations of this beautiful theorem (see e.g. [**2**] and the literature cited there) of which we mention that obtained first by Barlotti [**1**] in 1955 and then by Greber [**3**] in 1980. It says that if regular n-gons are erected outwardly(inwardly) on the sides of an n-gon P, then their centers are vertices of a regular n-gon if and only if P is affine-regular, i.e. it is the image of a regular n-gon under an affine transformation of the plane. We call the polygons having this property Napoleon polygons.

In this paper, we obtain a new geometric characterization of Napoleon polygons. Namely, we proved in Theorem 1 that any such an n-gon is obtained by fixing two consecutive vertices of a regular n-gon and translating the remaining $(n-2)$ vertices by collinear vectors with lengths whose ratios we compute explicitly. As an application we give a new proof of the theorem of Barlotti-Greber (Theorem 2). Moreover, we examine the polygons obtained by iterating the above construction. Given an n-gon P denote by $P^{(1)}$ the n-gon whose vertices are the centers of the regular n-gons erected outwardly on its sides. Then we define recursively the sequence $P^{(k)}$ of n-gons by

$$P^{(0)} = P,\ P^{(k+1)} = (P^{(k)})^{(1)},\ k \geq 0.$$

We say that a polygon P is k-step Napoleon if the polygon $P^{(k)}$ is regular. For example, the Barlotti-Greber theorem says that a polygon is 1-step Napoleon if and only if it is affine-regular. In Theorem 3 we generalize this result by obtaining an analytic characterization of the k-step Napoleon polygons for all $k \geq 1$.

2. NAPOLEON POLYGONS. In what follows we denote a point on the plane and the complex number it represents by the same symbol. Also we always assume that all polygons under consideration are simple and non-degenerate plane polygons.

Given an n-gon P with vertices $z_1, z_2, \ldots, z_n$ (as usual all subscripts are taken modulo n) we denote by $P^{(1)}$ the n-gon whose vertices $z_1^{(1)}, z_2^{(1)}, \ldots, z_n^{(1)}$ are the centers of the regular n-gons erected outwardly on its sides $z_1z_2, z_2z_3, \ldots, z_nz_1$, respectively.

Definition. We say that a polygon P is Napoleon if the polygon $P^{(1)}$ is regular.

In this section we give a new proof of the Barlotti-Greber theorem mentioned in Introduction. To do this we first prove the following analytic characterization of Napoleon polygons.

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Theorem 1. *Let P be an n-gon with vertices $z_1, z_2, \ldots, z_n$ and let $z_1^0, z_2^0, \ldots, z_n^0$ be the vertices of the regular n-gon erected inwardly on the side $z_1 z_2$ of P. Then P is a Napoleon n-gon if and only if*

$$z_k = z_k^0 + p_k.u \tag{1}$$

where u is a complex number and

$$p_k = \frac{\sin\frac{(k-2)\pi}{n}\sin\frac{(k-1)\pi}{n}}{\sin\frac{\pi}{n}\sin\frac{2\pi}{n}} \tag{2}$$

for all $1 \le k \le n$.

Proof. Set $\omega = e^{i\frac{2\pi}{n}}$. Since $z_k^{(1)}$ is the center of the regular n-gon erected outwardly on the side $z_k z_{k+1}$ of P we have

$$z_k - z_k^{(1)} = \omega(z_{k+1} - z_k^{(1)})$$

and we get

$$z_k^{(1)} = \frac{z_k - \omega.z_{k+1}}{1-\omega},\ 1 \le k \le n. \tag{3}$$

On the other hand the n-gon $P^{(1)}$ is regular if and only if

$$z_{k+1}^{(1)} - z_k^{(1)} = \omega^{k-1}(z_2^{(1)} - z_1^{(1)}),\ 1 \le k \le n. \tag{4}$$

Hence it follows from (3) and (4) that P is a Napoleon n-gon if and only if its vertices satisfy the following recursive relation

$$\omega.z_{k+2} - (1+\omega)z_{k+1} + z_k = \omega^{k-1}(\omega.z_3 - (1+\omega)z_2 + z_1),\ 1 \le k \le n. \tag{5}$$

We now set $z_k = z_k^0 + u_k, 1 \le k \le n$ and notice that $u_1 = u_2 = 0$. Set also $u_3 = u$. Since z_k^0 satisfy the relation (5) and

$$\omega.z_3^0 - (1+\omega)z_2^0 + z_1^0 = 0$$

plugging z_k in (5) gives the following relation for u_k:

$$\omega.u_{k+2} - (1+\omega)u_{k+1} + u_k = \omega^k.u,\ 1 \le k \le n. \tag{6}$$

If $u = 0$ then $u_1 = u_2 = \cdots = u_n = 0$. Hence we may set $u_k = p_k.u$, where $p_1 = p_2 = 0, p_3 = 1$ and p_k satisfy the recurence relation

$$\omega.p_{k+2} - (1+\omega)p_{k+1} + p_k = \omega^k,\ 1 \le k \le n. \tag{7}$$

Now setting $q_k = p_{k+1} - p_k$ we obtain from (7) that

$$q_k - \omega.q_{k+1} = -\omega^k,\ 1 \le k \le n.$$

It follows by induction on k that

$$q_k = \frac{1-\omega^{2k-2}}{(1-\omega^2)\omega^{k-2}},\ 1 \le k \le n. \tag{8}$$

Now taking into account that

$$p_k = q_2 + q_3 + \cdots + q_{k-1},\ 3 \le k \le n$$

and using (8) we get

$$p_k = \sum_{s=2}^{k-1} \frac{\omega^{-s+2} - \omega^s}{1-\omega^2} = \frac{(1-\omega^{k-2})(1-\omega^{k-1})}{\omega^{k-3}(1-\omega)(1-\omega^2)} \tag{9}$$

for all $1 \le k \le n$. Finally, to obtain (2) it is enough to apply the formula

$$1 - \omega^s = -2i \sin \frac{s\pi}{n} \omega^{\frac{s}{2}}$$

in the above identity.

■

The above theorem gives a nice geometric description of the Napoleon n-gons. Namely, it shows that each of them can be obtained by fixing two consecutive vertices of a regular n-gon and translating the remaining $n-2$ vertices by collinear vectors with lengths in ratio $p_3 : p_4 : \cdots : p_n$, where p_k is given by (2).

Now we can prove the Barlotti-Greber theorem by using Theorem 1.

Theorem 2. *(Barlotti-Greber) A polygon is Napoleon if and only if it is affine-regular.*

Proof. Note first that using complex numbers every affine transformation of the (complex) plane has the form $w = az + b\overline{z} + c$, where a, b, c are complex numbers. Hence an n-gon P with vertices $z_1, z_2, \ldots, z_n$ is affine-regular if and only if there are complex numbers a, b, c such that

$$z_k = a\omega^k + b\overline{\omega}^k + c,\ 1 \le k \le n,$$

where $\omega = e^{i\frac{2\pi}{n}}$. One can see easily that P is a regular n-gon if and only if there are complex numbers a, c such that

$$z_k = a\omega^k + c,\ 1 \le k \le n.$$

Let now P be an n-gon with vertices $z_1, z_2, \ldots, z_n$ and let $z_1^0, z_2^0, \ldots, z_n^0$ be the vertices of the regular n-gon erected inwardly on the side z_1z_2 of P. Then there are complex numbers a, c such that

$$z_k^0 = a\omega^k + c,\ 1 \le k \le n.$$

Hence by the first remark above and Theorem 1 it follows that to prove the theorem it is enough to show that there are complex numbers α, β, γ such that

$$a\omega^k + c + p_k u = \alpha\omega^k + \beta\overline{\omega}^k + \gamma \tag{10}$$

for all $1 \le k \le n$. Plugging the formula for p_k given in (9) and $\overline{\omega} = \frac{1}{\omega}$ in (10), and then clearing the denominators we can write both sides of the resulting equality as quadratic functions with respect to ω^k. Now comparing the coefficients leads to

$$\alpha = a + \frac{u}{(1-\omega)(1-\omega^2)},\ \beta = \frac{\omega^3.u}{(1-\omega)(1-\omega^2)},\ \gamma = c - \frac{\omega.u}{(1-\omega)^2}.$$

Hence α, β, γ are uniquely determined by a, c, u and vice versa the theorem is proved. ■

Remark 1. Theorems 1 and 2 are true also if regular n-gons are erected inwardly on the sides of the given n-gon.

3. A GENERALIZATION OF BARLOTTI-GREBER THEOREM. Given an n-gon P we can iterate the construction of the n-gon $P^{(1)}$ k times to obtain an n-gon denoted by $P^{(k)}$. More precisely, this sequence of polygons can be defined recursively as follows:

$$P^{(0)} = P,\ P^{(s+1)} = (P^{(s)})^{(1)},\ s \geq 0.$$

Definition. A polygon P is said to be k-step Napoleon if the polygon $P^{(k)}$ is regular.

For instance, a polygon is 0-step Napoleon iff it is regular and 1-step Napoleon if it is affine-regular. The next theorem gives an analytic characterization of k-step Napoleon polygons for every $k \geq 0$.

Theorem 3. *An n-gon with vertices $z_1, z_2, \ldots, z_n$ is k-step Napoleon if and only if there are complex numbers a, c and a degree $k-1$ polynomial $b_{k-1}(x)$ with complex coefficients such that*

$$z_m = a\omega^m + b_{k-1}(m)\overline{\omega}^m + c, 1 \leq m \leq n. \tag{11}$$

Here $b_{-1} \equiv 0$ and $b_0 \equiv const$.

Proof. We proceed by induction on k. For $k = 0$ the statement follows by the characterization of regular n-gons used in the proof of Theorem 1 and for $k = 1$, by Theorem 1. Suppose it is true for some k and let P be a $(k+1)$-step Napoleon n-gon. This means that $P^{(1)}$ is a k-step Napoleon polygon and it follows by the inductive assumption that

$$z_m^{(1)} = a\omega^m + b_{k-1}(m)\overline{\omega}^m + c,\ 1 \leq m \leq n, \tag{12}$$

where a, c are complex numbers and $b_{k-1}(x)$ is a degree $k-1$ polynomial with complex coefficients. On the other hand we know that

$$z_m^{(1)} = \frac{z_m - \omega.z_{m+1}}{1-\omega},\ 1 \leq m \leq n$$

and we obtain from (12) that

$$z_m - \omega z_{m+1} = a(1-\omega)\omega^m + (1-\omega)b_{k-1}(m)\overline{\omega}^m + c(1-\omega),\ 1 \leq m \leq n. \tag{13}$$

Denote the right hand side of (13) by A_m. Then it follows by induction on m that

$$z_m = \sum_{s=m}^{n} \omega^{s-m} A_s + \omega^{n-m+1} z_1.$$

Now plugging the expression for A_s in the above identity and summing up we find

$$z_m = \frac{a\omega^m(1-\omega^{2(n-m+1)})}{1+\omega} + (1-\omega)\overline{\omega}^m \sum_{s=m}^{n} b_{k-1}(s) + c(1-\omega^{n-m+1}) + \omega^{n-m+1} z_1.$$

Notice that $\omega^n = 1$ and $\sum_{s=m}^{n} b_{k-1}(s)$ is a polynomial of degree k on m. Hence we can write z_m in the form

$$z_m = A\omega^m + b_k(m)\overline{\omega}^m + c$$

where

$$A = \frac{a}{1+\omega},\ b_k(m) = (z_1 - c - \frac{a}{1+\omega})\omega + (1-\omega)\sum_{s=m}^{n} b_{k-1}(s).$$

Conversely, suppose that P is an n-gon whose vertices are given by (11). Then using (3) it follows easily by induction on s that for all $1 \le s \le k$ we have

$$z_m^{(s)} = a(1+\omega)^s\omega^m + d_{k-s-1}(m)\overline{\omega}^m + c,\ 1 \le m \le n,$$

where $d_{k-s-1}(x)$ is a polynomial of degree $k-s-1$. Hence

$$z_m^{(k)} = a(1+\omega)^k\omega^m + c, 1 \le m \le n$$

and therefore $P^{(k)}$ is a regular polygon. Thus the theorem is proved. ∎

Finally, let us note that Theorem 3 holds also true if we construct regular n-gons inwardly on the sides of the given n-gon P. In this case one has to switch the roles of ω and $\overline{\omega}$ in the formula for the vertices of P.

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