D12 Final Conference

Nitra, 1 days in the period September 22 – 26, 2013

Venue: Constantine the Philosopher University in Nitra, SK

Purpose of D12

Final project conference to present the project outcomes to members of target groups (pre- and in-service mathematics teachers, teacher trainers, mathematics educators), also allowing members of target groups to meet and get in contact with the partnership. The international workshop (together with the final conference) will allow teachers and teacher trainers to work with the materials and try out the E-Learning course, as well as discuss about them and give feedback to the partners.

Programme

Sunday 22 September 2013

All teams will arrive. Sona Ceretkova will wait at the airport and will organize the transport from Vienna Airport-Nitra for all members of the team who arrives to Vienna. Details will be communicated via e-mail(s). Accommodation, hotel River Nitra, http://www.hotelriver.sk/

Monday 23 September 2013

10:00 – 11:30 Ceremony of the Academic Year 2013-2014 Opening

11:30 – 12:30 Lunch

Programme of Final meeting of project team

13:00 – 14:30 Discussions on organization of the homepage, preparation of outputs and reports for E book, descriptions of the courses, didactic materials. Possible next steps after the end of the project.

14:45 – 16:00 Remarks from the evaluator. Concrete plans for improvement of the homepage

Tuesday 24 September 2013

Social programme – excursion, Bratislava, the capital of Slovakia, Danubiana Gallery http://www.danubiana.sk/eng/index.html, transport to gallery probably by boat

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Wednesday 25 September 2013

Final project conference and International workshop for teachers and educators

9:15-11:00 Presentations at the special conference session, Session 6.

The total number of participants can be evaluated from the pisctures of the opening ceremony.

List of registered participants includes 103 participants (see the attached file)

The content of Dynamat part of the proceedings is given below

13:00-16:00 Round tables and discussions with teachers participating at the conference.

Possible results of D13: projects started after the end of the course and developed on the basis of the course.

The work of Marco Ferrigo (student at University of Pisa) is one typical example for the results and explorations of the e-book after the end of e-course. It is based on an open problem proposed in the e-book. In the case of Diophantine equations with 3 variables the problem is completely solved in the work of Ferrigo. The topic attracted the interests of some motivated students in mathematics at the University of Pisa in the beginning of the standard lecture course in October 2013. A preprint on the subject can be dowloaded. Further extention is connected with the Napoleon problem, namely the generalization for the case of Napoleon polygons is considered and the article is accepted for publication in American Mathematical Monthly. Another interesting output in the short period after

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the end of the e-course was the project of Gergana Georgieva (BG student): Crimes and Probability, awarded at the session of HSSI

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8 PROJECT COMENIUS DYNAMAT

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FINDING GEOMETRIC PATTERNS AS A GAME OF DYNAMIC EXPLORATIONS

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Abstract

The paper deals with the inquiry based style of learning as applied to traditional and open geometry problems by means of dynamic geometry software. The so called what-if strategy *(i.e. exploring what happens if the formulation of the original problem varies) is demonstrated in the context of a well know problem from the geometry textbook – to find the locus of the midpoints of the segments joining a fixed point within a circle with the points of that circle. After making a dynamic construction for the locus the students are offered additional tools in support of the rigorous proof. The exploration game continues with varying the initial conditions of the problem (e.g. replacing the circle by other figures and the midpoint with a point dividing the segment in a fixed ratio). Then the well known problem of finding the locus of the centers of the equilateral triangles inscribed in an equilateral triangle is considered together with its ambitious generalization, viz. to find the locus of the centers of the regular m-gons inscribed in a regular n-gon ($m \leq n$). The process of generalization leading to open problems is considered together with the construction of appropriate dynamic tools for explorations. It is the very process rather than the description of the results which is of primary interest since it illustrates how the atmosphere around the working mathematicians could be transferred into a class setting. The expectation is that some teachers and students would gain motivation in attacking the considered open problems themselves.*

Keywords

Inquiry based learning, dynamic geometry software, loci related problems, what-if strategy

INTRODUCTION

Many interesting geometric problems deal with finding a locus ─ the set of points satisfying a particular condition. The traditional problems on loci are limited to finding simple curves. Language based computers environments allow for much more sophisticated explorations (Sendov, Sendova, 1995). While the computer language offers a vast spectrum of expressive means, enabling the user to enlighten the finest details of his thought, it is often found to be a great obstacle for the math teachers. Thus, the inquiry based learning in mathematics has been recently promoted within a number of European educational projects (DALEST, *Meeting in Mathematics*, *Math2Earth*, *InnoMathEd*, *Fibonacci*, DynaMAT) by means of dynamic geometry software offering direct manipulation of geometric objects (Christou et al., 2007, Georgiev et al., 2008, Bianco and Ulm, 2010, Baptist and Raab, 2013, Andersen et al., 2010).

In this paper we shall demonstrate how the inquiry based style of learning could be applied in the context of traditional and open geometry problems.

LOOKING AT THE CLASSICS WITH A DYNAMIC EYE

A very important component of the inqury-based mathematics learning is the *what-if* strategy, i.e. to explore what will happen if we vary the formulation of the original problem. Let us illustrate this strategy in the context of a well know problem from the geometry textbooks:

A traditional geometry problem: *What is the locus of the midpoints* ***M*** *of the segments joining a fixed point* ***P*** *within a circle with the points of that circle?*

To solve this problem by means of dynamic geometry software (say *GeoGebra*) the students study the behaviour of the midpoint under question while moving the endpoint of the segment on the circle along it (Fig. 1a):

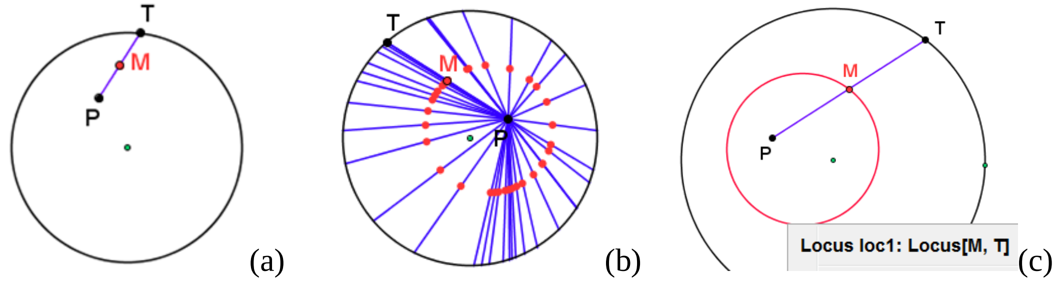

Fig. 1 The first steps of the exploration

They could strengthen their conjectures about the shape of locus by following the the trace of the midpoint's path (possibly with the segment) (Fig. 1b) and finaly chek experimentally their conjectures by constructing the locus of the midpoint by the inbuilt tool (Fig. 1c).

The game is not over, however. It is time to ask some *What-if* questions, e.g. *What if **M** is not the midpoint, but divides the segment at a fixed ratio? What if **P** is outside of the circle?.*

The typical conjecture of the students is that in this case the locus would look like a more general curve of a second degree, e.g. an ellipse. The teacher guides the explorations by suggesting to make the ratio a variable **e** in which **M** is dividing the segment (i.e. to create a slider in our case) (Fig. 2a and 2b):

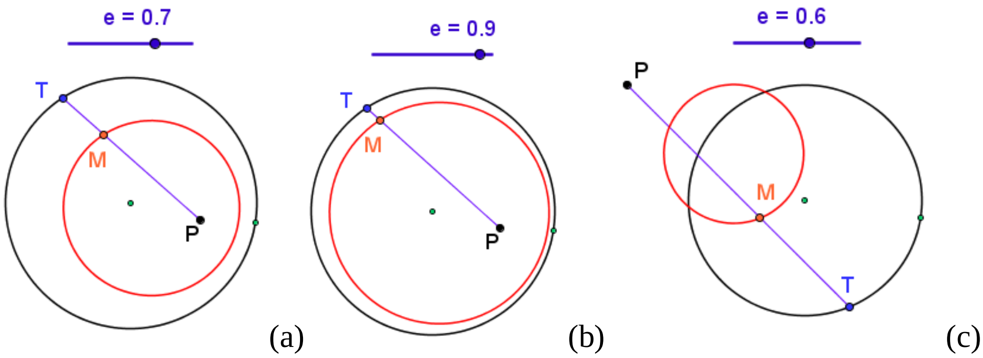

Fig. 2 Changing the ratio of the division

The students are genuinely surprised to find that the locus remains a circle. A further idea arises — to explore the situation when the point **P** is outside of the circle (Fig. 2c) — a circular shape once again! Then a new idea is suggested bringing an interesting effect — to trace the segment for a point outside the circle:

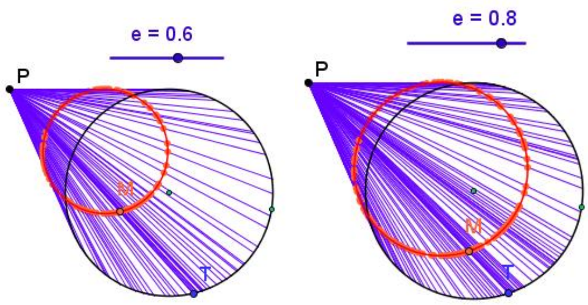

Fig. 3 Changing the position of point P

Now is the time for the teacher to raise students' suspition - could they be absolutely sure that **M** describes a circle? Couldn't it be in fact an ellipse which is very close to a circle…

One way to verify their conjecture (still experimentally) is to construct 3 points (**H**, **I**, **J**) on the locus, pass a circle through them and check if this circle concides with the locus. Another way which could help them prove the conjecture rigorously is to observe some interesting properties of the construction enriched with some auxiliary elements (Fig.4):

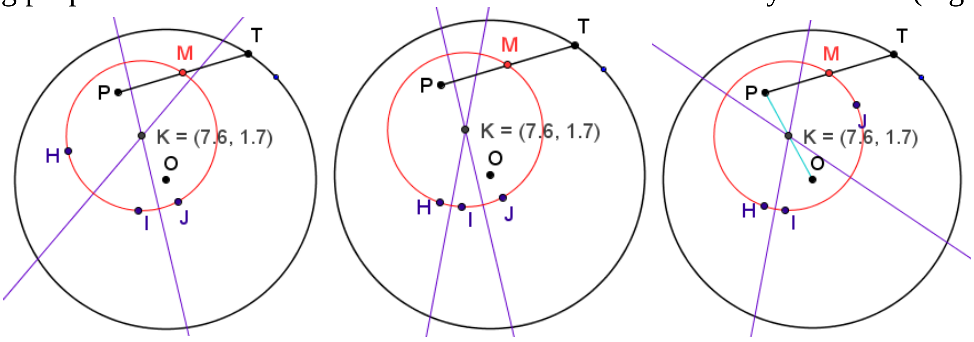

Fig. 4 Auxiliary construction in support of the rigorous proof

It is easy for the students to see that that the center **K** of the locus keeps its coordinates the same. In addition the segments **KJ**, **KI** and **KH** have equal legth which shows experimentally (but **with a greater degree of conviction**) that the locus is a circle. Furthermore, they would notice that **K** is the midpoint of the segment **PO**, where **O** is the center of the original circle. Now they are ready to prove rigorously that the locus is a circle with a center the midpoint **K** of the segment **PO** and a radius ─ half of the radius of the given circle.

The exploration game can continue with replacing the circle by a square, a triangle, an arbitrary regular polygon, a curve of their own choice.

If the students have studied *dilation* (in the Bulgarian curriculum it is introduced a year after the first occurrence of *loci*) they could use it to solve the problems but it is very appropriate for them to get used to generalizing their findings. Applying the *What-if* strategy could cultivate an exploratory spirit in mathematics classes - the students are encouraged to explore interesting partial cases, to generalize relatively simple problems in various directions, and even to attack and generalize challenging problems of Olympic level (Atanasova, 2011).

FROM A WELL KNOWN PROBLEM TO AN OPEN ONE

Here we demonstrate a process which is typical for the working mathematicians – we generalise a well-known problem, then we attack it with tools we believe are the most appropriate for the purpose (in our case with dynamic constructions we have specially designed in a *step-by-step refinement and enrichment* spirit). We systemize our explorations and reflect on the ideas we get. It is the very process that will be of our primary interest rather than the description of the results. In addition, we expect some teachers and students to get motivated in attacking some of the open problems themselves.

A well-known problem:
Find the locus of the centers of the equilateral triangles inscribed in an equilateral triangle.

An ambitions generalization of this problem could be formulated as follows.

An ambitious generalization:
*Find the locus of the centers of the regular **m**-gons inscribed in a regular **n**-gon,* $m \leq n$.

Further below we shall write **(*m*;*n*)** to denote the construction of a regular ***m***-gon inscribed in a regular ***n***-gon. Note that we are not even sure for which ***m*** and ***n*** the **(*m*;*n*)** constructions are possible. Let us start our *attack* with a more modest problem, dealing with the case **(3;*n*)** for ***n*** = 3, 4, …

The first attack – the (3;*n*) case:
*Find the locus of the centers of the equilateral triangles inscribed in a regular **n**-gon.*

A primitive (hand-made) dynamic model

We construct an equilateral triangle two of whose vertices are on the ***n***-gon and move the third one so as to get an inscribed triangle**.** To get the flavor of the dynamic construction to be then generalized it is natural to start with the simplest case (***n*=3)**, and proceed in what could be called a *hand-made* model (Fig.5):

- We select two arbitrary points **M** and **N** on different sides of the given (the *blue*) triangle.
- Then we construct an equilateral (*red*) triangle with a side **MN.**
- Next we move **N** (keeping **M** at irs current position) so that the red triangle becomes inscribed in the blue one. The center of the red triangle is a point of the locus sought.
- Now we repeat the above proces for a new position of **M.**

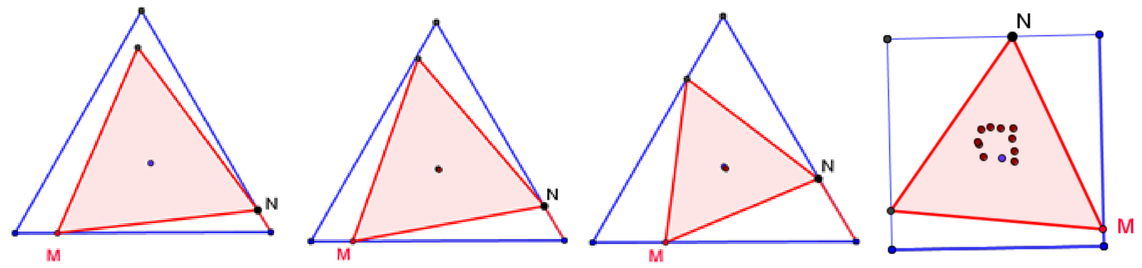

Fig. 5 The hand made **(3;3)** and **(3;4)** dynamic models

Thus, using consecutive positions of point **M** we get an approximate idea about the locus **–** in the **(3;3)** case the centers seem to coincide (or are at least close enough)… If we apply a similar procedure for the **(3;4)** case the centers appear to be on a square. But inscribing the triangle *by hand* is a time-consuming method (still better than constructing on a paper and considering just one possibly misleading case due to imprecision (Pehova, 2011).

To automize the construction let us take a better look at the **(3;3)** construction. It is natural to conjecture that in this case the locus is a single point coinciding with the center of the given triangle (Fig.6).

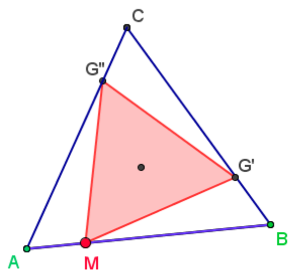

Fig. 6 The **(3;3)** dynamic construction

The congruence of the triangles **AMG"** and **BG'M** implies **AM**=**BG'**. Therefore, we can use in this particular case a dynamic construction based on the congruence.

An automized dynamic model for (3;n) constructions

There are various methods of creating automized models for the **(3;3)** constructions. Here is one of them:

- We construct a point **M** on the contour of a regular **3**-gon`(the triangle **ABC**)

- We construct the image **G'** of **M** under a rotation of 120° about the center of **ABC**
- We construct the image **G''** of **G'** under a rotation of 120° about the center of **ABC**
- We connect the points **M, G'** and **G''** in a triangle.

For **n**>3 we can proceed as follows:

- We construct a point **M** on the contour of a regular **n**-gon.
- Then we construct the image of the **n**-gon under a rotation **ρ** of 60° about **M**.
- We construct their interestion point **F.** (It will be another vertex of the equilateral triangle whose first vertex is **M**, and which is inscribed in the ***n***-gon.)
- Then we construct the thrid vertex as the pre-image **F'** of **F**.
- We connect **M**, **F'** and **F** to get the equilateral triangle inscribed in the ***n***-gon.

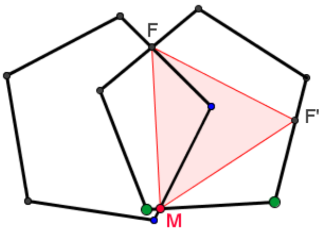

Fig. 7 Constructing a (**3;5**) dynamic model

Here are some snap-shots of the trace the triangle's center in the **(3;4)** construction leaves during the movement of the inscribed triangle:

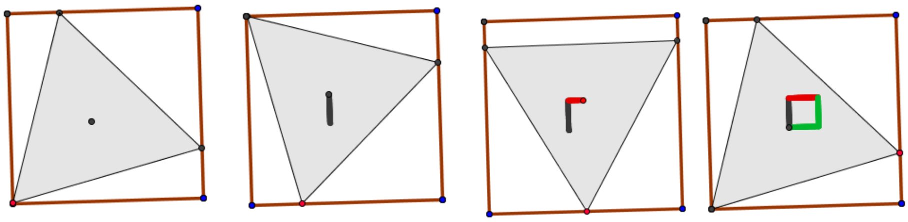

Fig. 8 The **(3;4)** dynamic construction

When we move the red point (**M**) until the next vertex of the triangle coincides with a vertex of the square (i.e. takes its initial position) we observe the trace becoming a shape which looks as a half of square. By analogy, when moving the point **M** along the rest of the sides of the square the center of the triangle will leave a trace which completes a square-like shape and after which it will start repeating the trace (three times). If the considered locus of the **(3;4)** construction is a square indeed could we conjecture that the corresponding locus of the **(3;5)** construction would be a regular pentagon? In the latter case it is sufficient to observe the effect of the movement of the red point on a part of the pentagon only.

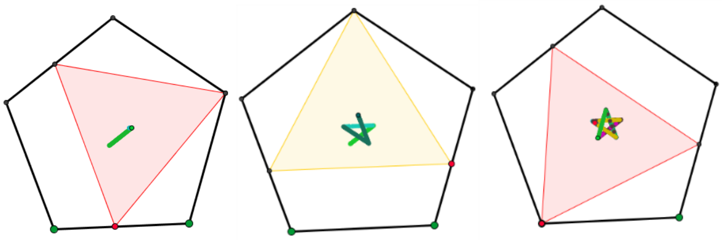

Fig. 9 The **(3;5)** dynamic construction

A-a-ah! Still 5 sides but it does not look like a pentagon – rather like a pentagram! Then what we suspected to be a square could be considered maybe as a „4-side star"...

Again, the center of the triangle describes the locus three times while the red point makes a full round along the original pentagon.

In the **(3;6)** case the locus appears to be a single point:

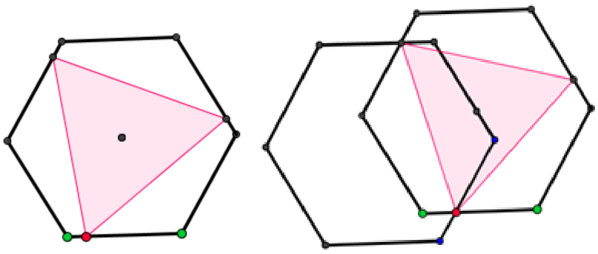

Fig. 10 The **(3;6)** dynamic construction

Such was the locus in the **(3;3)** case. By analogy we could conjecture that the same would hold for **(3;9),** and more general – for **(3;3*k*).** We could make separate construction for the **(*m*;*km*)**.

Further explorations providing insight

The (m; km) model

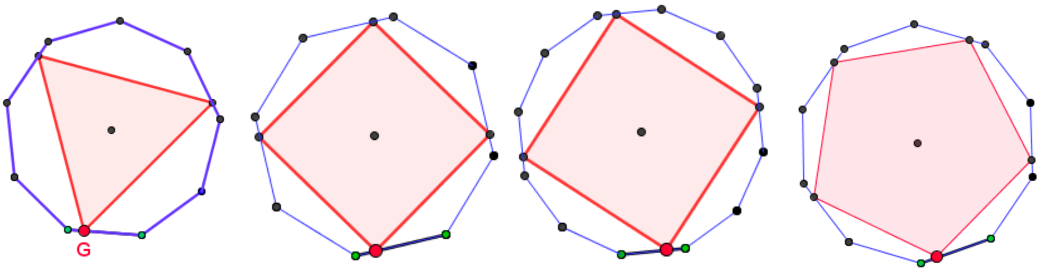

Fig. 11 The **(*m*;*km*)** dynamic constructions

The **(*m*;*km*)** constructions could be also achieved by analogy of the methods in Fig.6. The general conjecture we could draw after exploring the **(*m*;*km*)** model is that *for every point* ***G*** *on the* ***n****-gon (****n****=****km****) there exists an inscribed* ***m****-gon with a vertex* ***G*** *and the locus under consideration is a single point coinciding with the center of the* ***n****-gon.*

Let us continue our explorations with the (**3;*n***) model. In the case of **(3;7)** for instance we are expecting a star with its generating module emerging when going along one of the heptagon's sides. Indeed (Fig. 12)!

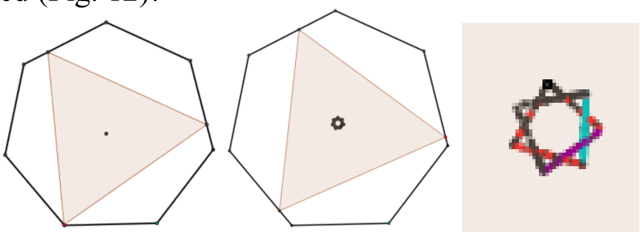

Fig. 12 The **(3;7)** dynamic model

Exploring further the (**3;*n***) model leads us to the conjecture that it is possible to inscribe an equilateral triangle in every regular ***n***-gon, i.e. (**3;*n***) *is always a possible construction*.

It is interesting to see what is the situation in the case of the (**4;*n***) model (Fig. 13).

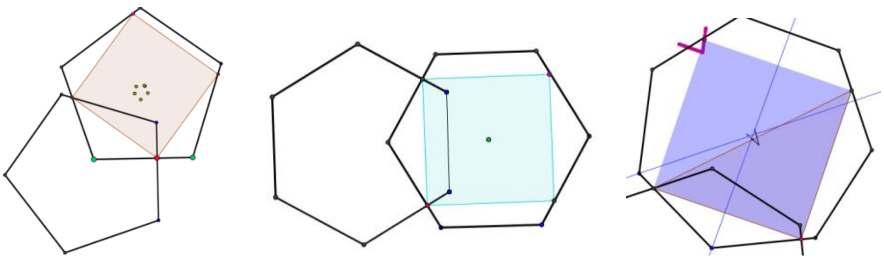

Fig. 13 The **(4;5), (4;6)** and **(4;7)** dynamic models

For a number of specific cases for $m > 4$, it is easy to make the conjecture that the construction is not always possible. In some cases additional means are needed for the inquiry. For example, in the **(5;6)** model it appears at first glance that the fifth vertex is on the hexagon (Fig.14 *a*). But a more careful exploration (Fig. 14 *b* and 14 *c*) shows that this is not so.

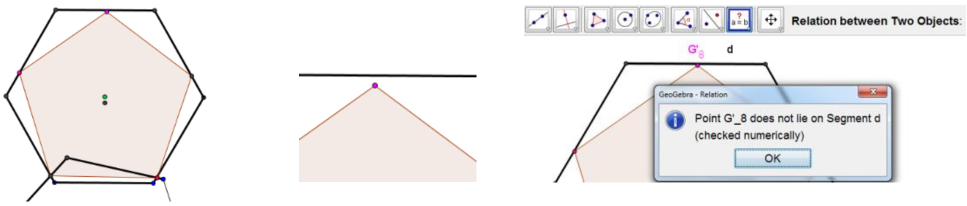

Fig. 14 The **(5;6**) dynamic model

At this point it is a good idea to stop and take a look around – *what is known in relation to our explorations?* We entered the magic phrase ***a regular m-gon inscribed in a regular n-gon*** and here it appeared (Dilworth, Mane, 2010)! Almost the same title and the same denotation showing how natural it is in its simplicity and conciseness when exploring various cases and describing the conjectures and results. Dilworth and Mane present there the necessary and sufficient conditions on ***m*** and ***n*** for inscribing a regular ***m***-gon in a regular ***n***-gon. It is interesting to note that *naively* (their own phrasing) they expected *this problem to be solved in the time of Euclid, but it seems to be not completely solved.*

Here is what Dilworth and Mane prove in (Dilworth and Mane, 2010) by means of complex numbers:

> **Theorem.** *Suppose that m, n* ≥ 3. *A regular m-gon can be inscribed in a regular n-gon if and only if one of the following mutually exclusive conditions is satisfied:*
>
> (a) *m* = 3*;*
> (b) *m* = 4*;*
> (c) *m* ≥ 5 *and m divides n;*
> (d) *m* ≥ 6 *is even and n is an odd multiple of m/*2. *(Note that this includes the case n = m/*2.*)*
>
> *In (c) and (d) the polygons are necessarily concentric and in (d) they share a common axis of symmetry. In case (d) we insist that n be an odd multiple of m/2 because if n is an even multiple of m/2, then n is a multiple of m, which is already covered in case (c).*

Thus it follows from the Theorem that the locus we are interested is a single point in the cases (c) and (d). The last examples of our explorations belong to (d).

Had we seen this article before attacking it with dynamic means we would feel very reluctant to offer it to students (even if they were very motivated to explore new mathematical territories). However, the explorations themselves harnessed mathematical skills accessible to students knowing about geometric transformations. Furthermore, the patterns and the relationships observed during these explorations gave rise to other interesting questions.

What really matters for us in relation to this problem is not even the solution itself but the whole process of creating a good platform for explorations, enhancing our intuition and understanding about some patterns among the constructions, designing a more systematic approach of explorations, realizing that not all combinations of inscribing a regular ***m***-gon in a regular ***n***-gon are possible, and finally – the belief in teachers' ability to promote the inquiry-based learning of mathematics. In a nut shell, to illustrate the „grook“ (*Grooks*, 2013) of the great Danish mathematician, architect and poet Piet Hein: *Problems worthy of attack, prove their worth by hitting back.*

ACKNOWLEDGEMENT

We express our deepest gratitude to Prof. Oleg Mushkarov for suggesting the general problem and for his helpful comments.

LITERATURE

Atanasova, S. 2011. *An Olympic problem as a source of inquiry based learning*, <http://www.dynamatadmin.oriw.eu/upload_pdf/20121022_161113__0.pdf> [Accessed April 29, 2013]

Christou, C., et al. 2007. *Stereometry Activities with Dalest.* University of Cyprus: Nicosia.

Dilworth, S.J., Mane, S.R. 2010. Inscribing a regular *m*-gon in a regular *n*-gon, *Journal of Geometry* 97, 59–67, c 2010 Springer Basel AG 0047-2468/10/010059-9, published online August 4, 2010, DOI 10.1007/s00022-010-0043-1 <http://www.math.sc.edu/~dilworth/preprints_files/DilworthManeJOGpublished.pdf> [Accessed October 25, 2011]

Grooks, <http://www.archimedes-lab.org/grooks.html> [Accessed May 8, 2013]

Bianco. T., Ulm V. (Eds.) 2010. *Mathematics Education with Technology - Experiences in Europe*. University of Augsburg, Augsburg

Pehova, Y. 2011. The Story of a Project... ... or how GeoGebra can help in a difficult situation <http://www.dynamatadmin.oriw.eu/upload_pdf/20121022_161236__0.pdf> [Accessed April 29, 2013]

Sendov, B., Sendova, E. 1995. East or West - GEOMLAND is BEST, or Does the Answer Depend on the Angle? (with B. Sendov), in A.A. diSessa, C. Hoyles, R. Noss (Eds.) Computers and Exploratory Learning , NATO ASI Series, Series F; Computer and Systems Sciences, Vol. 146, Berlin: Springer - Verlag, 1995, pp. 59 – 79

Baptist, P., D. Raab (Eds.) 2012. *Implementing Inquiry in Mathematics Education*, Bayreuth.

Georgiev, V., Ulovec, A., Mogensen, A., Mushkarov, O., Dimitrova, N., Sendova, E. 2008. *Meeting in Mathematics*, Demetra Publishing House, Sofia, ISBN 978-954-9526-49-3

Andersen, J. et al. 2010. *Math2Earth – Bringing Mathematics to Earth.* PRVOKRUH, Prague, ISBN 978-901470-2-7

STUDYING FINE-ART COMPOSITIONS BY MEANS OF DYNAMIC GEOMETRY CONSTRUCTIONS

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Abstract

The paper deals with integrating the study of art and mathematics by exploring the balance and the logical emphasis of paintings by means of dynamic geometry constructions developed in mathematics classes. A dynamic scenario developed and experimented in the context of the DynaMAT Comenius project is discussed as an illustration of how students could be encouraged to apply their mathematical knowledge for gaining a deeper insight in art compositions. Several methods suggested by art specialists are considered (e.g. rabatment, the rule of thirds, the golden section) together with appropriate dynamic geometry implementations. Ideas for further activities with students formulated in terms of long-term projects are offered. Although the considered scenario is still in its early phase of experimentation (mainly at teacher training courses) the first impressions are shared as being promising – the teachers become aware that they could attract more students to mathematics when showing its application in various contexts. The experience gained by the authors on a broader scale ─ in the context of visual modeling, is reported to have contributed to building new strategies in teacher education, which could prepare teachers for their changing role of partners in a creative process.

Keywords

dynamic geometry software, art, rabatment

INTRODUCTION

Seeing is not as simple as it looks
Ad Reinhardt

Many artists claim that they could explain nothing about their works, that their paintings came upon them by inspiration. The founder of the abstract art however, expresses in his book (Kandinsky, 2011) his theory of painting and sums up ideas that influenced his contemporaries. Kandinsky makes the brave prediction that *we are fast approaching the time of reasoned and conscious composition, when the painter will be proud to declare his work constructive*.

To motivate better the study of geometry for students with interests in art we could reveal for them the strong relation between the esthetics of artistic compositions and some geometric principles. When reading the works of art critics we come across notions such as *harmony, style, rhythm, balance* (not necessarily the better defined *rules, symmetry, geometry*). Perhaps they think that if revealed the rules behind a balance composition would trivialize the art. To us, revealing certain patterns and rules would in contrast raise the level of appreciation of an observer. The modern fine art tries to speak about things which *will be seen*, that is why its language is not understandable for many. But this language could be better learned if we try to study it together with the language of geometry.

VISUAL MODELING

Exploring the properties of geometric shapes in a computer environment has proven to be more exciting for students of different age if made part of a visual modeling of some works of art (Sendova and Girkovska, 2005, Nikolova et al, 2011). By building computer models of a given painting the students can gain deeper insight in its structure and motivation to elaborate their knowledge in mathematics and informatics.

When analyzing an abstract painting from mathematical point of view it is interesting to discuss its basic elements and to classify them. From an artistic point of view, though, the problem is not only to understand the elements of a composition, but also to understand its balance. In pre-service and in-service teacher training courses on using language-based computer environments for education we used a specially designed Logo microworld (Sendova, 2001) in which it was easy to experiment with figures of various sizes, colors and degrees of complexity, i.e. to verify different definitions of *balance*. In addition, the participants in the courses could play with Kandinsky's ideas concerning the relation between geometric shape and color and study the effect of both components in various combinations.

We could qualify the following factors as the most relevant ones in the study of an abstract painting:

- The character of the objects and their composition in terms of clustering, overlapping, isolation, balance, relationship between size, shape and color
- Main categories of the objects
- Establishing hierarchy related to the distance of the center, the size, the color, etc.
- Functional associations (which objects occur in combination in the work of a given author).

The visual modeling could be used not only to study a specific painting, or a specific artist, or more general – the style of a certain artistic movement, but also to bring possibly new creative ideas. Thus, products of the visual modeling should be judged with respect not only to the closeness between the original and the generated works but also to their potential to generate works bringing the spirit of the original together with new, unexpected ideas – a potential that depends on the user, of course. After leaving the frames of the strict imitation some of the future teachers were inspired by new combinations of forms and colors and got new insight, which in turn led to new formalization.

These visual modeling activities were carried out by means of programming which might create certain psychological problems among the typical mathematics teachers. Still combing art with geometry seems a very natural way of motivating the students to enhance their understanding in both fields. Some inspirational sources for integrating mathematics and art include (Ghyka, 1946, Livio, 2002, Hemneway, 2005, Olsedn, 2006, Skinner, 2009). More recent developments offer various dynamic geometry constructions as tools for analyzing works of art and appreciating the esthetics of well known paintings (Sánchez, 2013). The ideas presented below are based on a dynamic scenario (Sendova and Chehlarova, 2011) developed and experimented in the context of the *DynaMAT* Comenius project (http://www.dynamatadmin.oriw.eu) with the intention to encourage students in applying their mathematical knowledge for gaining a deeper insight in art compositions.

CREATING DYNAMIC CONSTRUCTIONS OF COMPOSITION TOOLS

The dynamic scenario deals with several relatively simple geometric constructions which have proved useful in creating and studying the balance of the fine-art compositions. After describing them we present their implementation in a dynamic software environment (*GeoGebra* in our case) so as to illustrate how they could be applied to exploring various paintings (classical and more modern alike). A further step offered to the students is to apply their newly gained art-evaluation competencies in the context of taking and editing photographs.

RABATMENT

The first (relatively less known) compositional method we introduce is *rabatment* which has been broadly used in the 19th century. This method is applicable to paintings in rectangular shape. It consists of taking the shorter side of a rectangle and placing it against the longer side (rotating the shorter side along the corner), creating points along the edge that can be connected directly across the canvas as well as a diagonal from these points to the corners. In a rectangle whose longer side is horizontal, there is one implied square for the left side and one for the right; for a rectangle with a vertical longer side, there are upper and lower squares. In traditions in which people read left to right, the attention is mainly focused inside the left-hand rabatment, or on the line it forms at the right-hand side of the image (Fig. 1).

Fig. 1 The left rabatment and its appearance in the Monet's painting *Red Poppy-field*

To achieve a more powerful composition one could add the diagonals of the rectangle and the two squares. Here is how the rabatment applied to the painting *A Sunday Reading in a Village School* of Bogdanov-Belsky looks like (Fig. 2).

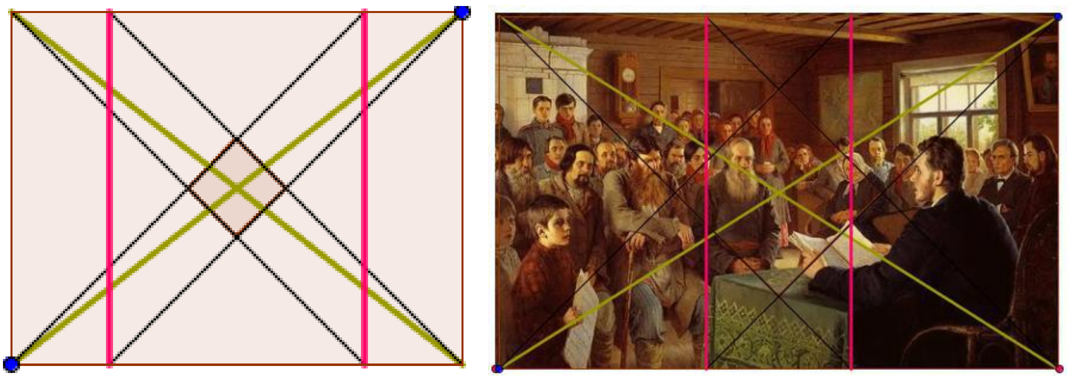

Fig. 2 The rabatment as applied to a Bogdanov-Belsky painting

More information about the rabatment method can be found at http://emptyeasel.com/2009/01/27/how-to-use-rabatment-in-your-compositions/ (Mize, 2013).

To enable the analysis of paintings of any rectangular shape it is convenient to have a universal rabatment tool. It could be created as a *dynamic geometrical construction* which means a construction which saves its main properties under movement of some of its objects defined as independent. The rabatment construction is introduced in our dynamic scenario by means of *GeoGebra* as follows.

CREATING A DYNAMIC RABATMENT TOOL

We start with constructing a dynamic rectangle. If the students are novices to using the software they should be encouraged to suggest and try out various ways of constructing a rectangle and discuss which of their constructions are in fact dynamic ones. After the discussion we could consider the following construction as appropriate for our purpose.

We construct one of the corners of the rectangle as an independent object – point *A*, and two variables (sliders) for its base *a* and height *b* specifying the range of their values. Next we construct point *B* - the opposite corner of the diagonal through *A* as a point whose coordinates depend on the coordinates of *A*, *a* and *b*). Then we construct lines through *A* and *B* parallel to the coordinate axes. The remaining two vertices of the rectangle could be obtained as intersection points of these lines. Then we connect the four points to get a rectangle (which is dynamic with respect to the size of its sides but always with a horizontal base, the normal position of a painting's frame).

Let $b < a$. Now we construct circles with centers the four vertices of the rectangle and a radius *b* – the length of the shorter side of the rectangle. We find the intersection points of the four circles with a side of the rectangle and construct two of the rabatment segments. Then we complete the construction with the diagonals.

What is left is to construct the square in the center which appears when $b < a < 2b$. We hide the auxiliary objects (the lines and the circles) and we explore the construction for various values of *a* and *b*.

Similarly, we make a workable construction for a rectangle with a shorter base.

It is worth mentioning here that a good educational quality of *GeoGebra* is the opportunity for the users to enrich the toolkit with their own tools. This facilitates the implementation of our rabatment tool, viz. we show to the students how to make the rabatment construction a part of the toolbar. For the purpose a suitable name, an icon and the inputs of the construction (a point and two numbers in our case) should be specified.

In our scenario we have considered in fact two constructions appropriate for a rabatment tool, the second one having as inputs the ends of one of the rectangles diagonals. Thus the students could create and use two rabatment buttons in the same *GeoGebra* file (named RabatmanPNN and RabatmanPP after the necessary inputs for the respective construction). As a further step in our scenario we show how images could be studied by means of the composition tools being created, i.e. how to display and how to resize (if necessary) the inserted image by preserving its proportions. Now the ground for explorations is set – the students could use the rabatment button, place the rabatment construction on the image and look for interesting properties of the composition of a specific painting (Fig.3).

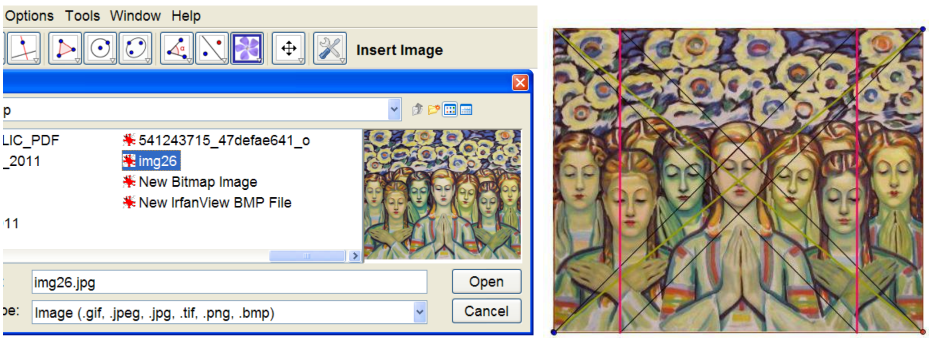

Fig. 3 Inserting the painting *Prayer* by Maystora and applying the Rabatment button to it

Depending on the emphasis the teacher would like to make, s/he could encourage the students to continue with exploring the created dynamic tool with other paintings and to formulate their findings. Alternatively s/he could enhance their mathematics skills of implementing other composition tools. Here is what we have suggested further in our scenario.

THE RULE OF THIRDS

The *rule of thirds* is a simple method that can be used not only as a tool for exploring the paintings of famous artists but also to enhance and improve our own compositions (when we draw or take pictures). In the diagram below, a rectangle has been divided horizontally and vertically by four lines. The rule of thirds states that the points of interest for any rectangle are determined by those lines. The intersections of the lines are considered by some specialists (Maze, 2013) to be *power points* (the black dots in Fig. 4).

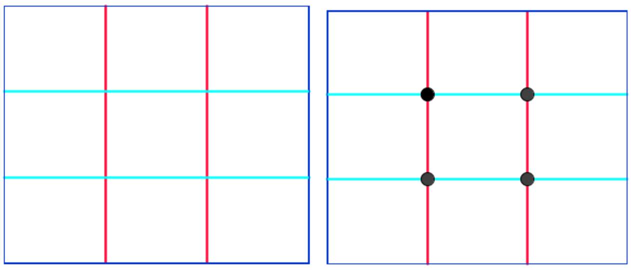

Fig. 4 The rule of thirds and the *power points*

Here is the rule of thirds in action (in horisontal version and in vertical one):

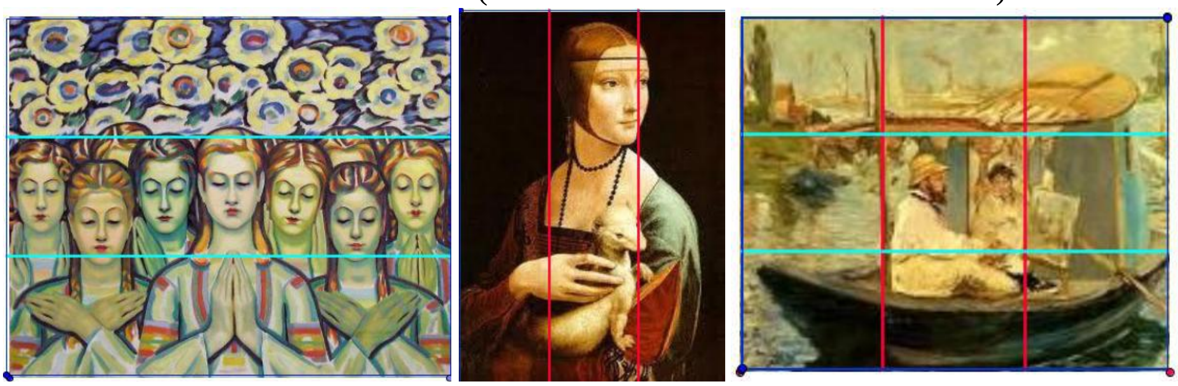

Fig. 5 The *rule of thirds* applied to Maystora's *Prayer,* to Leonardo's *Lady with an Ermine,* and to Manet's *Monet painting in his floating studio*

For the students it is of essential importance to use the rule of thirds not only when studying famous paintings but also when taking (or editing) photographs of a scenery (Fig. 6).

Fig. 6 The *rule of thirds* in photography

The teachers could create a list of tasks for the students taking into account their interests and mathematics background. Some examples of tasks we have offered in our dynamic scenario deal with

- Making several digital pictures of a scenery by applying the rule of thirds in just one of them and explain which version seems to be the most balanced one
- Creating *Thirds* buttons (a vertical and a horizontal versions)
- Exploring some classical and some modern paintings by various composition buttons.

In addition to using some classical composition tools the mathematics teachers could suggest geometric constructions of their own, possibly jointly with the art teachers. Here is an example from our scenario.

THE CENTRAL RHOMBUS

The logical emphasis of a painting is often located in a rhombus with vertices the midpoints of the sides of the rectangle:

Fig. 7 The central rhombus applied to Maystora's *The Girl with the Dahlias*, and to Mrkvička's *Ruchenitsa*.

To make a dynamic construction and turn it into a rhombus button in *GeoGebra* for exploring images is another activity offered to the students.

Of course, the most popular notion combining art and mathematics is the *golden section*.

A DYNAMIC GOLDEN SECTION CONSTURCTION

The most famous mathematical composition tool, though, is the *Golden Section* (also known as the *Golden Mean* or the *Golden Ratio*) defined as the point at which a segment can be divided in two parts *a* and *b*, so that $a/a+b = b/a$. We introduce the notion of a *golden rectangle* as a rectangle whose side lengths are in the golden ratio. The golden ratio is often depicted as a single large rectangle formed by a square and another rectangle. What is unique about this is that we can repeat the sequence infinitely within each section. If in addition we draw an arc of 90º in the consecutive squares we get the so called *golden spiral* (Fig. 8).

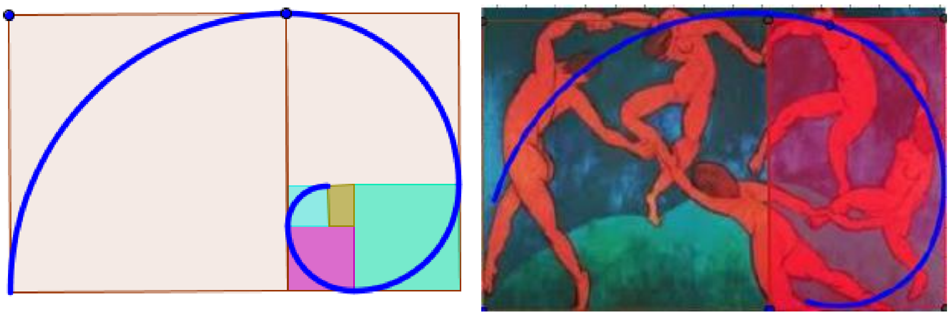

Fig. 8 A sequence of golden rectangles and the golden spiral applied to *The Dance* by Matisse

We give in our scenario the following algorithm for constructing a dynamic golden spiral

- *Construct a unit square (blue).*
- *Draw a segment from the midpoint of one side to an opposite corner.*
- *Use that segment as the radius of an arc that defines the longer dimension of the rectangle (Fig. 9).*
- *Construct an arc of 90° in each square so as to get a golden spiral:*

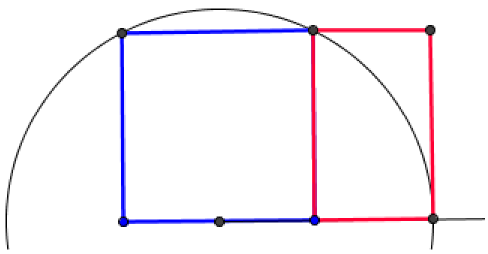

Fig. 9 Constructing a dynamic golden rectangle

Then we suggest to students to construct *GeoGebra* buttons based on the golden ratio and to explore various paintings with all the composition tools they have created.

It is very important to extend these activities by assigning long-term projects to the students. Here are some examples of *dynamic mini-projects* included in the scenario:

- *Take a picture of a scenery in two ways so that they reflect specific goals. Explore the result by means of dynamic constructions and edit the pictures correspondingly by cutting out.*
- *Arrange for a picture in two ways (according to two composition methods): 6 persons at a birthday party sitting around a round table; a class of 24 pupils and their teacher; flowers and fruits; perfumes and an advertisement. Explore the result with dynamic constructions and make corrections if necessary.*
- *Make an advertisement in two ways of: your school; your hobby; natural juices; an old town. Explore the result with dynamic constructions and make corrections if necessary.*
- *Make in two ways a design of an invitation cart for: a fest of mathematics (physics, music, the flowers, athletics); a ball with masques; a birthday party. Explore the result with dynamic constructions and make corrections if necessary.*
- *Explore the rotational dynamic constructions by means of the sliders so as to create models similar to the pictures of rotational objects (Fig. 10).*

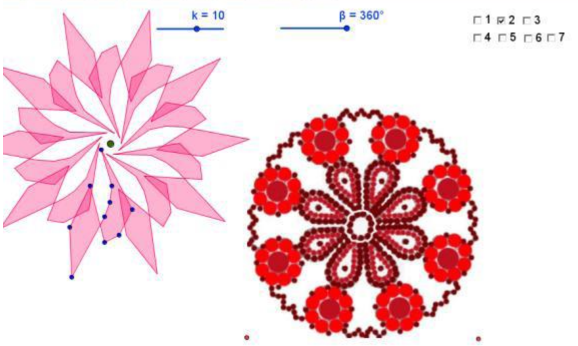

Fig. 10 Rotational dynamic construction

- *Create models of objects around you based on rotational symmetry (wood carved ceilings, embroidered table clothes, etc.)*

CONCLUSIONS

Our overall experience in educating students and teachers alike shows that the integration of the learning and creative processes by means of visual modeling could contribute to a new learning style in mathematics education. Such type of activities sensitizes students to looking at not only the art but also at the world around them in a more meaningful way.

Although the considered scenario is still in its early phase of experimentation (mainly at teacher training courses) the first impressions are promising – the teachers become aware that they could attract more students to mathematics when showing its application in various contexts (often unexpected for them as art is).

A famous quote by the american poet Robert Frost reads: *Writing free verse is like playing tennis with the net down.* We could extend this quote to art in general. But an important point we make to the teachers is that every rule can and should be broken for artistic effect, from time to time. This should be done however not because we don't know the rules but rather when we are looking for new ideas. This is for example how some stunning photographs are made.

The experience gained leads us naturally to building new strategies in teacher education, which could prepare teachers for their changing role of partners in a creative process.

LITERATURE

Chehlarova, T., Sendova, E. 2010. Stimulating different intelligences in a congruence context. In: *Constructionist approaches to creative learning, thinking and education: Lessons for the 21st century*. Proceedings for Constructionism 2010. ISBN 978-80-89186-65-5

Chehlarova, T., Sendova, E., Stefanova, E. 2012. Dynamic tessellations insupport of the inquiry-based learning of mathematics and arts, In Kynigos, C., Clayson, J., Yiannoutsou, N. (Eds) *Theory, Practice and Impact* - Proceedings of Constructionism 2012, Athens, August 21-25, pp.570-574. ISBN: 978-960-88298-3-1

Chehlarova, T. 2013. Problems with fractions in the style of Escher. in Kenderov, P. and Sendova, E. (Eds.) *Inquiry-based learning in mathematics education*, Regalia 6, Sofia, pp. 127-132 (in Bulgarian)

Chehlarova, T. 2013. Problems in the style of Mondrian. In Kenderov, P. and Sendova, E. (Eds.) *Inquiry-based learning in mathematics education*, Regalia 6, Sofia, pp. 121-126

DynaMAT Comenius Project. <http://www.dynamatadmin.oriw.eu> [Accessed May 5, 2013].

Ghyka, M. 1946. *The geometry of art and life*. New Dover publications, Ins, York, 1977

Kandinsky, W. (2011) *Concerning the Spiritual in Art*, Dover Publications, Inc. 1977, ISBN 0-486-23411-8

Hemneway, P. 2005. *Divine Proportion*, Sterling Publishing, New York,

Livio, M. 2002. *The Golden Ratio*, Broadway Books, New York,

Mize, D. *How to Use Rabatment in Your Compositions*, Art tutorial [online] <http://emptyeasel.com/2009/01/27/how-to-use-rabatment-in-your-compositions/> [Accessed May 5, 2013].

Nikolova, N., Stefanova, E., Sendova, E. 2011. Op Art or the Art of Object-Oriented Programming. In *Situation, Evolution and Perspectives*, Proceedings of the International Conference on Informatics in Schools (ISSEP), October 26-29, Bratislava

Olsedn, S. 2006. *The Golden Section*, Walker&Company, New York

Sánchez, J. A. M. Geometría dinámica y Arte. *(*Dynamic Geometry and Art*)*. <http://jmora7.com/Arte/arte.htm> [Accessed May 5, 2013].

Sendova, E., Chehlarova, T. (2011) *Creating dynamic geometry constructions as composition tools in art and photography*. <http://www.dynamatadmin.oriw.eu/upload_pdf/20121022_160507__0.pdf> [Accessed May 5, 2013].

Sendova, E., 2001. Modeling Creative processes in Abstract Art and Music, In G. Futschek (Ed.) *EUROLOGO 2001 A Turtle Odyssey*, Proceedings of the 8 th European Logo Conference 21-25 August, Linz , Austria

Sendova, E., Grkovska, S. Vsisual Modelling as a Motivation for Studying Mathematics and Art, in Gregorczyk, G et al (Eds.) *EUROLOGO'2005*, Proceedings of the 10 th European Logo Conference, pp. 12-23. <http://eurologo2005.oeiizk.waw.pl/PDF/E2005SendovaGrkovska.pdf> [Accessed May 5, 2013].

Skinner, S. 2009. *Sacred Geometry*, Sterling, New York/London

FIELDWORK AS A TEACHING METHOD - A CASE STUDY USING GPS

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Abstract

In this paper we want to demonstrate a case study showing how fieldwork can be used as a teaching method. We will provide a definition of this teaching method, explain its advantages and disadvantages, describe the background and the setting of the case study, and report about the evaluation results. Fieldwork is rarely used in several school systems, both for legal and practical reasons. Yet it can increase students' motivation, particularly in mathematics, as students can experience first-hand "what can mathematics be used for?" We will present a teaching unit using GPS as a tool to construct or measure geometric figures in the field, i.e. outside the classroom. This gives students an opportunity to learn about geometric figures not only in an abstract setting, but as shapes used in the real world. The questionnaires and interviews that we conducted show that this teaching unit improved students' motivation to find out more about real-life uses of mathematics, as well as the possibility of increasing students' attitudes towards learning mathematics by providing possible applications.

Keywords

Fieldwork. Real-life tasks. Electronic media as tools for learning

DEFINITION AND INTRODUCTION

By fieldwork we mean work of students outside the classroom. It may involve work in the school grounds or further afield. It can vary in duration – part of a lesson, a half day, or longer. It involves live collection of primary data by means of observation, experiment or survey (Ulovec et al., 2007). In this way, students can experience familiar and unfamiliar phenomena beyond the normal confines of the classroom (Dillon et al., 2005).

However, fieldwork is not frequently used in many classrooms. This might be because of practical reasons, because of legal hassles, or simply because of a lack of teaching materials with proper suggestions. As for the practical reasons, some tips and hints can be found in Simperler (2012). As for the legal hassles, it might be true that using fieldwork requires some form-filling, parents to be contacted etc. But this is also true for other out-of-school activities, e.g. ski courses, swimming weeks, excursions etc., and so should not prevent one from using this method. As for the lack of teaching materials, this is the main reason for writing this paper. We developed a number of out-of-classroom activities in an EU-funded project called DynaMAT. One of the materials is presented here, together with an evaluation in the form of questionnaires and interviews, to serve as a case study about the usefulness of this teaching method.

ADVANTAGES AND DISADVANTAGES OF FIELDWORK

Several authors have already dealt with this issue, and a number of advantages and disadvantages have been listed in the literature. The following table presents a summary of this work (cf. Sauerborn and Brühne, 2009):

Tab. 1 Advantages and disadvantages of fieldwork as a teaching method

Advantages	**Disadvantages**
Action-oriented	Difficult with large number of students
Reality-related	Organisational effort
Physical activities	Risks of injury
Self-responsible learning	Difficult assessment

New method for most students	Students not used to this activity
Addresses several cognitive learning levels	Hard for students to concentrate
Often interdisciplinary	Hard to place in curriculum

In a 1999-study of the University of Regensburg about out-of-classroom learning activities, the most frequently named disadvantage was “costs” (53.3%), closely followed by “time pressure by curriculum” (51.7%). In our own study (see below), costs were not an issue to teachers, as most activities took place either on the school grounds or in walking distance of the school. Time pressure by curriculum was the most frequently named reason (65.7%), followed by fear of disciplinary issues (37.1%), and organisational effort (28.6%).

CASE STUDY: USING GPS IN FIELDWORK

Teaching material

This teaching unit (cf. Andersen, 2012) consists of two parts: In the first part, students are asked to use the tracking function of a GPS receiver to measure the geometric shape of a given outdoor feature. In the second part, the students are given a certain geometric figure and are asked to “walk along” this figure outdoors, i.e. to use the GPS receiver to navigate in such a way as to produce a track in the form of the given geometric figure.

Part 1: Measuring a geometric figure in the field

Task: Go to Heldenplatz in Vienna (or a park nearby the school) and stand on one corner of the rectangle that is shown in the map below (or another suitable rectangular figure in the park). Switch on the tracking function of your GPS receiver. Now walk along the edges of the rectangle until you are back at the original point. Then switch off the GPS receiver. Compare the resulting track with the original rectangle. Use the obtained data to calculate the side lengths of the rectangle and the length of the diagonal. Then go back to Heldenplatz (or the chosen park) and measure the length of the diagonal with the GPS receiver.

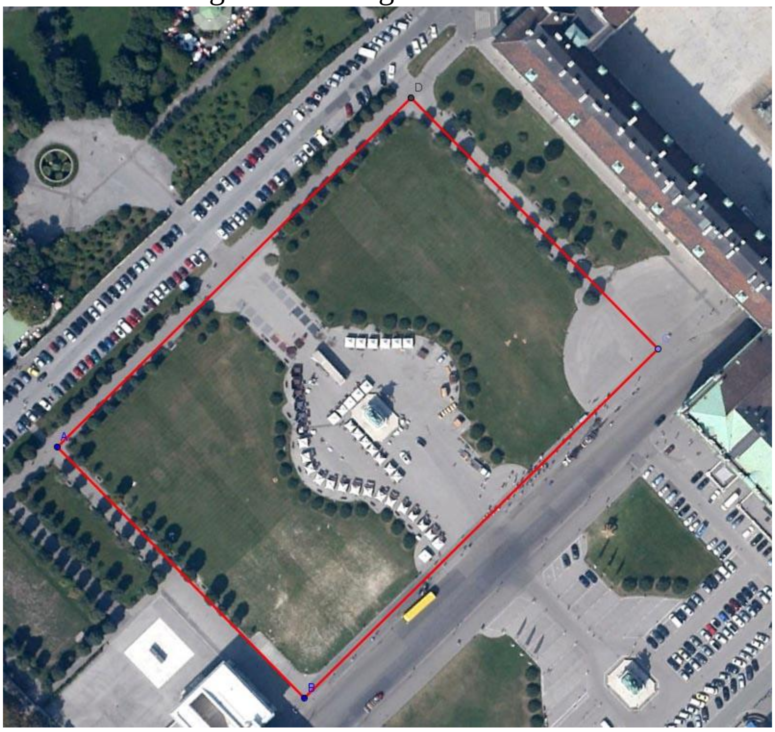

Fig. 1 Measurement of rectangle – ideal track

Part 2: Creating a geometric figure in the field

Go to Heldenplatz (or a suitable nearby park or field) again and use your GPS receiver to walk along an equilateral triangle with a base length of 40 m. Before you set off, think about how to do this, what strategies are possible, what their advantages and disadvantages are, and which one you will choose. Switch on the tracking function of the GPS receiver and record your “triangular walk”. Transfer the data into Google Earth and check with GeoGebra how close your track comes to an exact equilateral triangle. Compare your results with those of your classmates, particularly with those who have chosen another strategy than yourself.

Setting

The teaching unit was performed in 13 secondary school classes in 8 schools in Vienna, with a total number of 223 students and 35 teachers involved. The teachers received the teaching materials and – if required – a number of GPS receivers. The students received an instruction into GPS as such (using Ulovec, 2012a) and an instruction on how to transfer and interpret GPS data with Excel (using Ulovec, 2012b), Google Earth and GeoGebra (using Andersen, 2012). These instructions took two lessons (50 minutes each) per class. The fieldwork as such was led by the mathematics teacher with the support of 1 – 2 colleagues (also teachers, but mostly of other subjects). Part 1 took one lesson, part 2 took two lessons of 50 minutes each. Part 2 was usually (with 2 exceptions) done in a double lesson of 100 minutes in one piece. After the teaching units were conducted, the teachers and students were given questionnaires about the concrete teaching units and the teaching method “fieldwork”. The students’ questionnaires did not contain mathematical tasks (i.e. it was not a pre-post-test setting), but did make some references to the geometrical content. 5 teachers and 22 students were also interviewed after the teaching units.

Description

As to part 1, students were usually able to walk the path as described and record the data. A typical track looked like this:

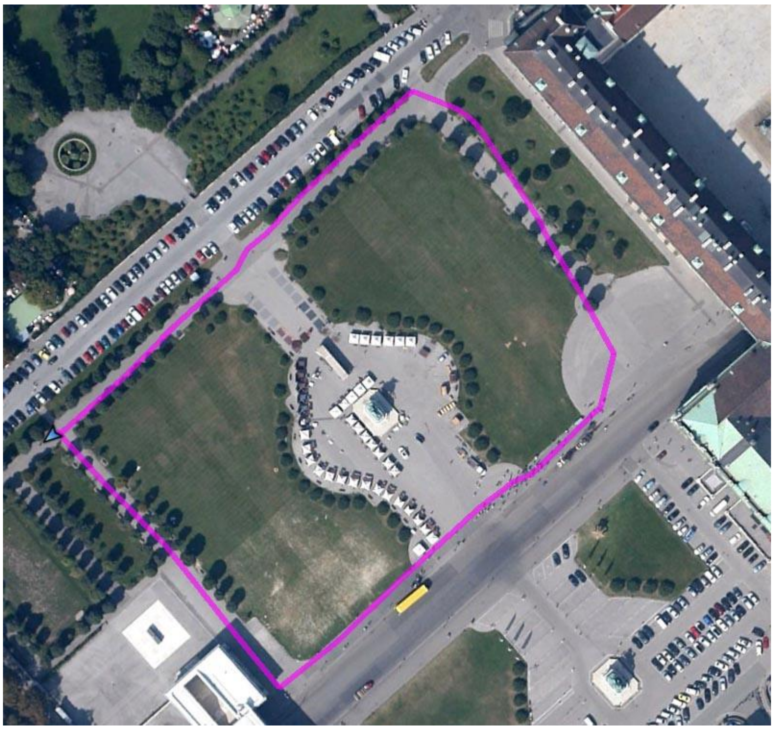

Fig. 2 Measurement of rectangle – real-life GPS track

Measurements of the diagonal length resulted in fairly exact data, 210 (of 223) students were within a 15% error margin.

In part 2, we could observe mainly two strategies: Using angles, and using vertices as waypoints:

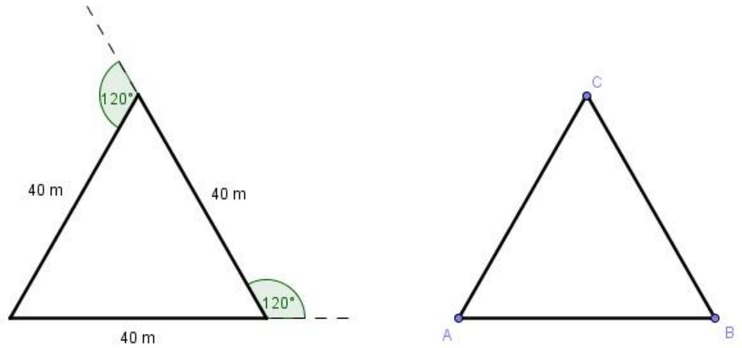

Fig. 3 Angle strategy waypoint strategy

Strategy 1 calls for the student to walk 40 m, then turn 120° anti-clockwise, walk another 40 m, turn again 120° anti-clockwise, and walk another 40 m. Strategy 2 calls for the student to calculate (or construct in GeoGebra) the coordinates of the vertices and set them as waypoints in the GPS receiver before starting out, then walk to the first waypoint, the second waypoint, the third waypoint, and then back to the first waypoint.

Typical results of strategies 1 and 2 look like this:

Fig. 4 Triangle – strategies 1 and 2

Teachers and students then discussed the advantages and disadvantages of the two main strategies that occurred. Main points that were mentioned were: Strategy 1 has the advantage of not requiring any pre-fieldwork calculations (except for figuring out that the outer angle of an equilateral triangle is 120°). It has the disadvantage of not being too accurate in the field. Strategy 2 requires some calculations and constructions with GeoGebra or similar tools, and additional operations with the GPS receiver. However, it leads to better results.

Questionnaire and interviews

After finishing the activities of the teaching units, both the teachers and the students received (different) questionnaires. Aside from personal data (grade for students, teaching experience for teachers etc.) there were the following quantitative questions for students, to be answered on a scale from 4 (very much) to 1 (not at all):

1s) Was the teaching material adequate for the lessons?
2s) Did you know/learn all the technologies that you needed for these lessons?
3s) Did these lessons change your attitude towards mathematics in a positive way?
4s) Did these lessons increase your motivation to find out more about practical uses of mathematics?

Teachers received the following quantitative questions:

1t) Was the teaching material sufficient for the lessons and their preparation?
2t) Did you know/learn all the technologies that you needed for the preparation and execution of the lessons?
3t) Did these lessons change your attitude towards fieldwork as a teaching method in a positive way?

Both groups received the following qualitative questions:

5) What did you like the most about the teaching material?
6) What did you like the least about the teaching material?
7) What did you like the most about the teaching method "fieldwork"?
8) What did you like the least about the teaching method "fieldwork"?

Teachers were additionally asked:

9t) What do you see as the three biggest advantages of the teaching method "fieldwork"?
10t) What do you see as the three biggest disadvantages of the teaching method "fieldwork"?

After the analysis of the questionnaires, we chose 5 teachers and 22 students, and interviewed them about their answers to some of the quantitative questions, and all qualitative questions. The interviews lasted about 20 minutes per interviewee.

Results

207 students and 35 teachers handed in the questionnaires. Here are the results of the quantitative questions. Given is the percentage of answers on a scale of "4" (very much) to "1" (not at all).

Tab. 2 Results of questionnaires, questions 1 – 4

	Teachers (n = 35)				Students (n = 207)			
	4	3	2	1	4	3	2	1
Question 1	80	17	3	0	84	9	5	2
Question 2	74	20	6	0	77	10	8	5
Question 3	49	28	17	6	38	18	28	16
Question 4					30	39	16	15

As to the teaching materials and the technologies used, we can clearly see that they were very well accepted. Also the interviewees confirmed this observation. The only issue for students was the inaccuracy of the GPS receiver, which particularly occurred when students did not use a stand-alone receiver but their smartphone or similar device.

As to the attitude aspects, there was no significant improvement by the teaching units alone. However, in the interviews those students who claimed that their attitude towards mathematics has changed in a positive way, almost unanimously stated that this is because the teaching unit showed "what mathematics can be used for, except in school" or "real-life applications".

The teaching units definitely increased students' motivation to find out more about practical uses of mathematics (69% of students answered either with 4 or 3 to this question).

27 out of the 35 teachers answered with 4 or 3 to whether the lessons changed their attitude towards fieldwork in a positive way. This was also confirmed in the interviews, where teachers (most of whom have never used fieldwork as a teaching method in a regular lesson) stated that this was a good opportunity for students to use mathematics outside the classroom, and experience geometric figures that are not just drawn in their notebooks or displayed on a computer screen. Also, in the interviews many teachers stated that fieldwork either requires very well prepared teaching unit descriptions, as delivered here, or a lot of effort from the teacher to develop and prepare suitable units themselves.

In the qualitative questions with respect to the teaching materials, in question 5 both teachers and students commented positively on the use of GPS technology, which is not usual in mathematics teaching, and seemed to be very motivating for students (as was mentioned in several interviews). Also "practical example" and "good instructions" have been mentioned frequently, both by teachers and students. In question 6, "too technology-centred" and "hard to fit into curriculum" was mentioned by teachers, "better use rectangle instead of triangle, for comparison" and "would have been better if everyone would have their own GPS unit" was mentioned.

In the qualitative fieldwork questions, at question 7 students mostly answered "to work outside" and "it is not boring", teachers answered "seems to be motivating for students" and "allows the teachers to show application in real life instead of just explaining it in classroom". At question 8, only few students and teachers gave any answers, mostly along the lines of "a lot of work for a maths class".

In the final two questions for teachers, the three most frequently named advantages were "opportunity to show real-life applications" (51.4%), "motivation for students" (37.1%), and "physical exercise" (22.8%). The three most frequently named disadvantages were "time pressure by curriculum" (65.7%), "fear of disciplinary issues" (37.1%), and "organisational effort" (28.6%).

CONCLUSION

Fieldwork is a teaching method that can help students to see possible applications of mathematics in real life, outside their classrooms, and by that increase their motivation to look for more applications of mathematics in their lives. It is clear that it requires appropriate preparation, both with respect to the actual teaching unit, and with respect to organisation. However, as this case study shows, and other authors confirm (e.g. Scherer and Rasfeld, 2010), it is a good opportunity for students to widen their views of mathematics and prevent it to become a classroom-only activity.

In any case, most of the literature in this field concerns work with very young children (e.g. Dühlmeier, 2008; Stevens and Scott, 2002), and there is not all too much about fieldwork in mathematics with secondary school students, so more work needs to be done for this particular age group.

ACKNOWLEDGEMENT

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LITERATURE

Andersen, J. 2012. *GPS – Geometry in the Landscape*. [online] Available at <http://www.dynamathmat.eu> [Accessed 2 May 2013].

Dillon, J. et al. 2005. *Engaging and Learning with the Outdoors: The Final Report of the Outdoor Classroom in a Rural Context Action Research Project*. [online] Available at <http://www.nfer.ac.uk/nfer/publications/OCR01/OCR01.pdf> [Accessed 2 May 2013].

Dühlmeier, B. 2008. *Außerschulische Lernorte in der Grundschule*. Baltmannsweiler: Schneider Verlag Hohengehren.

Sauerborn, P., Brühne, T. 2009. *Didaktik des außerschulischen Lernens*. Baltmannsweiler: Schneider Verlag Hohengehren.

Scherer, P., Rasfeld, P. 2010. Außerschulische Lernorte. In *mathematik lehren*, vol. 160, 2010, p. 4 - 10.

Simperler E. 2012. *Feldarbeit im Mathematikunterricht*. Vienna: Diplomarbeit an der Universität Wien.

Stevens, J., Scott, K. 2002. Developing mathematics out-of-doors. In *mathematics teaching*, vol. 180, 2002, p. 20 - 22.

Ulovec A. et al. 2007. *Motivating and Exciting Methods in Mathematics and Science – Glossary of Terms*. Olomouc: Palacký University Press.

Ulovec, A. 2012a. *GeoCaching – How to find it … using satellites*. [online] Available at <http://www.dynamathmat.eu> [Accessed 2 May 2013].

Ulovec, A. 2012b. *Fly, fly away … and bring back data*. [online] Available at <http://www.dynamathmat.eu> [Accessed 2 May 2013].

PROBLEM POSING IN MATHEMATICAL EDUCATION: DIOPHANTINE EQUATIONS AND A PROBLEM IN GEOGEBRA

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Abstract
This paper is part of Project Comenius DynaMAT, so it shares with it motivation and scope: to create problems aimed at high school students in order to introduce them to mathematical arguments and reasoning, and to, via problem-solving, advanced topics in an elementary way; all of this is done using computer software to help visualisation and intuition. In this paper we particularly concentrate on the problem-conjecture-proof-problem process of mathematical reasoning, and on the limits showed by ICT, which helps intuition, but has to be supported by rigorous demonstration. In particular we will deal with two problems, one arithmetical problem based on Diophantine equations, and one geometrical problem analysed with GeoGebra. Solutions are given, but the real protagonists of the paper are the many possibilities of deeper study and exploration offered to the students along the solution and proposed generalisation of the problems. No conclusions are given, but offers of further evolution, the basis of which consists in actual experimentation of the materials produced in classrooms, followed by data collection and analysis, in a collaboration between researchers and teachers.

Keywords
Dynamical Approach, Divulgation And Didactics, High School Problems, Diophantine Equation, Geogebra, Excel

INTRODUCTION

In this paper we will analyse two problems prepared for high school students: we will describe solutions to these problems which can be presented to such an audience. Our main interest is to propose approaches that can, in our opinion, help the students understanding and interiorising the processes and ideas typical of mathematical reasoning. Moreover, we will introduce some topics and problems connected with Diophantine equations and modular arithmetics following problem-solving approach.

In doing so, computer programs like GeoGebra and Excel can be a great help, and we will describe their use. Moreover, we will also describe some risks that computers programs can hide, following the idea that while a computer graphic program can be a wonderful source of intuitions in the hands of an expert mathematician, it can also be a hint to wrong conclusions for a young student: we will therefore underline the importance of the formal demonstration part in mathematics, that has to follow necessarily the intuition.

More motivations and information about this approach can be found in (Georgiev, 2012).

ARITHMETIC PROBLEM: PRESENTATION AND SOLUTION

The problem is well introduced in form of real life stories or games in (Dimitrova et al., 2008), (Georgiev, Kurokawa, 2012-1), (Georgiev, Kurokawa, 2012-2) and (Anderson et al., 2010).

We briefly recall the game formulation: consider the equation $ax = by + c$, where a, b, c are natural number. x, y can be interpreted as two buttons (green and red) : if one presses the green button this corresponds to the operation $x = x + 1$, if one presses the red button, then $y = y + 1$ is performed. At first $x = 0$ and $y = 0$, who obtains the equality pressing the buttons as few times as possible wins.

After having played long enough, the students can be lead to a formal exposition of the problem, and its solution, which is what we are interested in, in this paper. We are particularly interested in the possibilities to introduce mathematical concepts and methods given by this elementary solution.

We start by suggesting one possible translation of the problem in mathematical language:

Problem. Given $a, b, c \in \boldsymbol{N}$ (set of natural numbers), consider the following Diophantine equation:

$$ax - by = c \qquad (1)$$

Find, if it exists, among the solutions of equation (1) the one, denoted by (x_0, y_0), such that $x_0 + y_0$ is minimum.

In other words, find (x_0, y_0) solution of equation (1) such that (x_1, y_1) solution of (1) implies:

$$x_0 + y_0 \leq x_1 + y_1.$$

Please notice that we just consider non-negative numbers.

First of all it is important to find out whether a solution always exists or not.
It is useful to this purpose to recall the concept of greatest common divisor, or simply gcd, of two integer numbers.
If we consider $m = \gcd(a, b) = (a, b)$, then it is clear that:

$$m \mid ax - by \qquad \forall x, y \in N;$$

So if m is not a divisor of c, there is no solution at all to equation (1).

It is an elementary fact that if m divides c, then equation (1) has solutions, and moreover all solutions are given by a closed formula, check (Herstein, 1972) for some details.
Anyway, our goal is to introduce young students to these topics, using this problem as an excuse, so let us show a possible solution which covers almost every aspect of the basic arithmetics.
We can recall the concept of Euclidean division between two integers, and the so called extended Euclidean algorithm to find the gcd and a linear combination of the two integer numbers that gives the gcd itself, namely two integers k, h such that:

$$ak + bh = (a, b).$$

An interesting way of explaining the Euclidean algorithm is using a spreadsheet application such as Excel.
A quick explanation of how the algorithm works can be found at (Wikipedia, *Extended Euclidean Algorithm*), while a good description of the implementation is given by (Mounth Olyoke College, *The Euclidean Algorithm in Excel*).
Anyway a quick recall is given in Fig1.

	A	B	C	D
1		a	b	
2				N = ax+by
3				
4		N	x	y
5		=B2	1	0
6	=INT(B5/B6)	=C2	0	1
7	=INT(B6/B7)	=B5-A6*B6	=C5-A6*C6	=D5-A6*D6
8	=INT(B7/B8)	=B6-A7*B7	=C6-A7*C7	=D6-A7*D7
9	=INT(B8/B9)	=B7-A8*B8	=C7-A8*C8	=D7-A8*D8
10	=INT(B9/B10)	=B8-A9*B9	=C8-A9*C9	=D8-A9*D9
11	=INT(B10/B11)	=B9-A10*B10	=C9-A10*C10	=D9-A10*D10
12	=INT(B11/B12)	=B10-A11*B11	=C10-A11*C11	=D10-A11*D11
13	=INT(B12/B13)	=B11-A12*B12	=C11-A12*C12	=D11-A12*D12
14		=B12-A13*B13	=C12-A13*C13	=D12-A13*D13

	A	B	C	D
1		a	b	
2		5915	2317	N = ax+by
3				
4		N	x	y
5		5915	1	0
6	2	2317	0	1
7	1	1281	1	-2
8	1	1036	-1	3
9	4	245	2	-5
10	4	56	-9	23
11	2	21	38	-97
12	1	14	-85	217
13	2	7	123	-314
14		0	-331	845

Fig. 1 Euclidean algorithm in Excel: formulae and an example

A very good way to help the students become acquaintance with both the algorithm and the potential of a spreadsheet application is to explain the algorithm and the basic functions of the

software, and then ask them to find an implementation of the algorithm: they have a quick way to check its correctness: just trying with some couples of numbers, they learn how to use a spreadsheet "naturally", i.e. actually using it to solve some problem they should care about. Moreover they discover the usefulness of abstract reasoning and formulae expression, over the simple analysis of some special cases.
Going back to the problem, thanks to the Euclidean algorithm it is now easy to obtain any multiple of (a, b) with combinations of a and b. Consider $d = n \cdot (a, b)$, then:

$$a(nk) + b(nh) = (a, b)n = \text{d}.$$

We can therefore simplify the problem, assuming $(a, b) = 1$, since if (a, b) does not divide c, then the problem has no solution, if (a, b)divides c, then call:
$m = (a, b), a = m \cdot a_1, b = m \cdot b_1, c = m \cdot c_1$, so that:

$$ax - by = c \leftrightarrow m(a_1x + b_1y) = mc_1 \leftrightarrow a_1x + b_1y = c_1$$

Thanks to these preliminary considerations, and especially to the programming of the Euclidean algorithm, a student should become acquaintance with the tools involved in this formalization of the problem.
Therefore, we can now go on proposing a solution.
Theorem 1. Given $a, b, c \in \boldsymbol{N}$ (set of natural numbers), the following Diophantine equation:

$$ax - by = c \qquad (1)$$

has one and one only solution (x_0, y_0) such that y_0 is an element of the set $\{0, \dots, a-1\}$ and x_0 is a positive number.
Proof. Since $(a, b) = 1$, also$(a, -b) = 1$, and we already proved via Euclidean Algorithm that exist k, h integer numbers such that:

$$ak - bh = 1$$

In general $k, h \in Z$, but considering the following auxiliary equation:

$$ax - by = 0,$$

which has solutions: $x = bt, y = -at$ for any $t \in Z$, it is easy to notice that any couple $(k + bt, h - at)$ is a solution to equation (1).

Finally we can consider the Euclidean division $h = qa + r$, with $0 \leq r < a$, and find out that the couple:

$$(k + bq, h - aq) = (k + bq, r)$$

Is our solution, since r is in the set considered and is unique.
We just have to check x_0 is non-negative, and this is a simple computation:

$$x_0 = \frac{c + by_0}{a} \geq 0.$$

This concludes the proof.

Notice that what we did here is a simplification of the construction of general solutions to a Diophantine linear equation, so such topic could easily follow.

Notice also that the proof could be easily expressed in modular arithmetic terms, and this could be a fine way to introduce such formalism among young students too. In fact, we could have exposed the proof in this way:
Consider the modular equation:

$$-bY \equiv c \quad mod(a)$$

Since$(a, -b) = 1$, the Euclidean algorithm tells us that exists k such that:

$$ah - bk = 1.$$

This means by definition: $-bk \equiv 1 \ mod(a)$, we could write $k = b^{-1} \ mod(a)$.

So it is easy to compute $Y \equiv -cb^{-1} \quad mod(a)$.
Then it is obvious by definition that exists one and one only $y \in [0, a-1]$ such that $[y]_a = Y$.

To solve our problem we have to show that (x_0, y_0), the solution to equation (1) given by the last theorem is also a solution to our original problem.

Theorem 2. In the same hypothesis of the previous theorem, given $(x_1, y_1) \in N^2$solution to equation (1), different from (x_0, y_0) then:

$$y_1 > y_0 \; and \; x_1 > x_0$$

Proof. The first observation is that (x_1, y_1) different from (x_0, y_0) implies both $x_1 \neq x_0$ and $y_1 \neq y_0$, quite obviously. Then since $y_1 \in \boldsymbol{N}$ and y_0 is the only possible y in $\{0, \ldots, a-1\}$ part of a solution to equation (1), we deduce $y_1 > y_0$.

It follows immediately:

$$x_1 = \frac{c+by_1}{a} > \frac{c+by_0}{a} = x_0.$$

Our proof is now complete, and the problem is solved.

DESCRIPTION OF THE FINAL ALGORITHM

Given equation:

$$ax - by = c \qquad (1)$$

We can find (x_0, y_0) solution to equation (1), such that $x_0 + y_0$ is minimum following these steps:

1) If (a, b)does not divide c, then the problem has no solution.
   If it does:
2) Compute, via Euclidean algorithm k, h such that $ak - bh = (a, b)$. Then consider the Euclidean division $h = qa + r$. Put $y_0 = r$.
3) Compute $x_0 = (c + by\_0)/a$.
4) (x_0, y_0) is the only solution to our problem.

Now that we solved this problem, it is possible to propose to the students some modification or generalization of it. The aim is to involve the students in the typically mathematical circle of problem-conjecture-proof-problem: once we have concluded a demonstration of our conjectures, we are pushed to analyze it and wonder: “What did we really show?” “What results similar to this could I face now?” “How could I generalize my results?”
Some of the more immediate and interesting generalization are:

Change of sign.

What if we considered the same problem, but with the sign “+”?
It is now clear that in some simple examples there is no solution (among positive numbers):

$$2x + 3y = 1,$$

even if $(2,3) = 1$. So we are facing a different kind of problem, and a first natural question is: will the approach we adopted in the preceding problem still work? If not, where does it fail? And how could we solve this new problem?

Problem with three variables.

What if we considered an equation with three variables instead of two, i.e.:

$$ax = by + cz + d,$$

and we tried to find a solution made of natural numbers, such that $x + y + z$ is minimum?
This particular example is interesting for the following reason: while it is “easy” to occur in the solution in the case of two variables linear equation, since in some sense it is the first

solution one finds (remember we proved y_0, part of the solution, is the only y in $\{0, \dots, a-1\}$ part of a solution) if tries with $x = 0$, then $x = 1$ and so on, the first solution one finds is the good one. This does not always happen in the three variables linear equation problem.
Consider the following equation:

$$29x = 4y + 53z + 5$$

If one looks for a solution with $x = 1$, finds the solution $(1,6,0)$, which has sum $S_1 = 7$. But if we consider $x = 2$, then we find a better solution $(2,0,1)$, which has sum $S_2 = 3$.
A good work would be to formalize the "rational" algorithm suggested for the two unknowns problem (trying for successive values of x) and find out why it does not work with the three variables problem.

The difficulty of these generalizations, the solutions of which we do not take in consideration here, should help the students realize how easy and fascinating it is to find problems to face, and how research in mathematics could work.

GEOMETRIC PROBLEM: PRESENTATION AND SOLUTION

We do not care about fascinating formulations of this problem either, but again about the possibilities given by the solution. In particular, we will describe how easily a well-driven student can find by himself new problems and attempts to generalize or modify the problem given, having a concrete experience of investigation in mathematics.

DESCRIPTION OF THE PROBLEM

Given a rectangle ABCD, find, if it exists, a rectangle EFGH circumscribed to the previous one, such that the area of the EFGH is twice the area of ABCD. An example of circumscribed rectangles is given by Fig. 2.

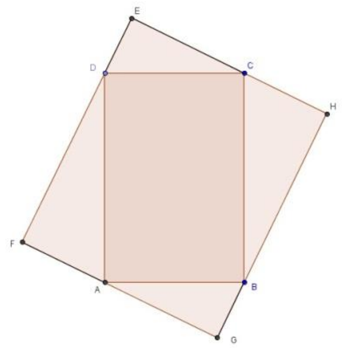


Fig. 2 Example of circumscribed rectangles

A good way to attack the problem is by considering it in GeoGebra. Consult (GeoGebrawiki, *GeoGebra manual*) for some help about the full potential of this tool. Drawing perpendicular lines passing for each point A,B,C,D, we can represent the problem as explained in Fig. 3: the only free parameter is the position of point P:

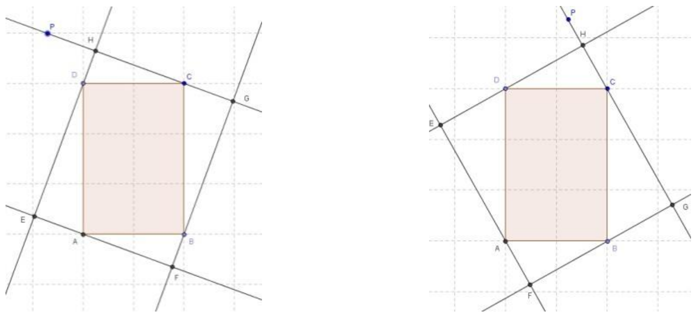


Fig. 3 The problem in GeoGebra, different positions of point P

Through the exploration of the problem via GeoGebra, a student can find a good way to conjecture that a solution always exists, and an hypothesis of how to find it. The main point is to help the student to verify whether his/her intuition is correct or not, by formal demonstration. This way we have a possibility to emphasize the importance of both the intuitive and formal part of mathematics.

The perfect situation arises when some students have “wrong” intuitions (i.e. that are not correct, or that do not lead to the solution) and other students have “right” intuitions (i.e. that are correct and lead to the solution).

A good teacher should analyze deeply any suggest from his/her students, in order to help them comprehend why they are wrong, or to fully comprehend why they are right.

Anyway, we present two possible ways to find a solution.

1) Solution by intuition: with the help of GeoGebra, one could decide to see what happens if the point P is put inside the rectangle ABCD: if P coincides with A, we find the inclination we want, as shown in Fig. 4:

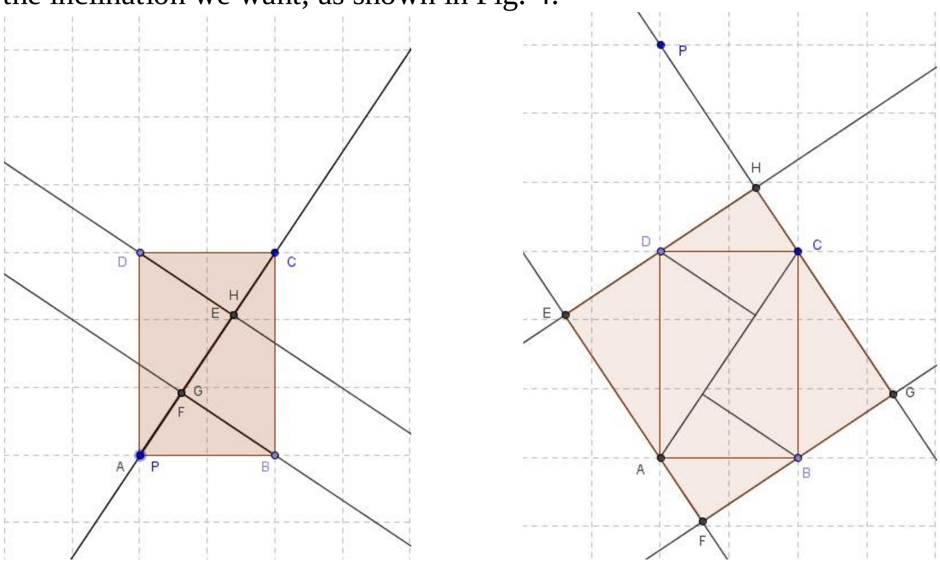
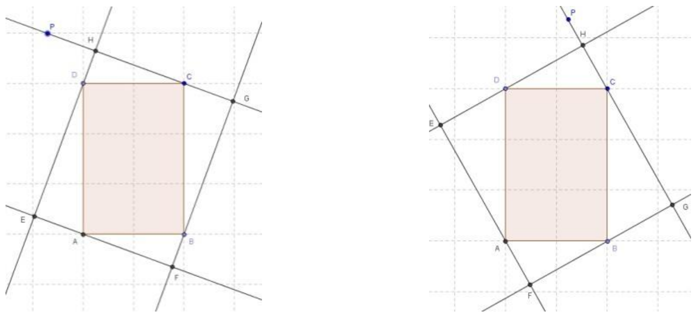


Fig. 4 Idea of the solution to the problem

2) Solution with help of other tools: using GeoGebra at its full potential, we can create a spreadsheet that calculates the area of our rectangles. Moving the point P, we find that at some points the area is very small (even 0), while it easily assumes values greater than double the area of ABCD. One can therefore guess where to put the point P to obtain its solution, and then try to prove it works.

Some problems that may arise (or that should be proposed) during the investigation are here described:



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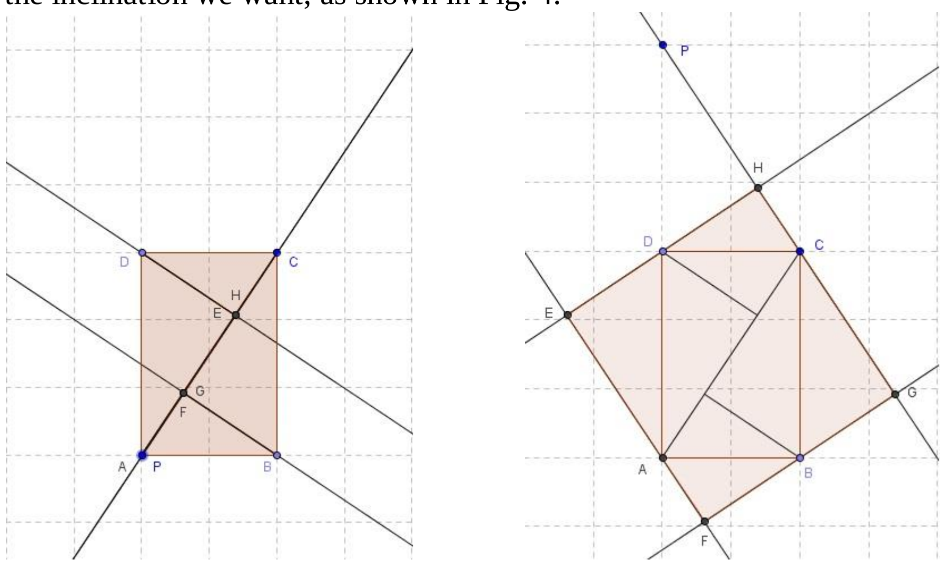

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Some problems that may arise (or that should be proposed) during the investigation are here described:

1) What rectangle EFGH has a maximum area? Especially when using a spreadsheet, a student can guess on how to build it. There are many ways to see that maximum area is obtained when considering a square circumscribed: a nice proof starts from noticing that the vertices move along semicircles, and showing that the maximum area is obtained when the angles formed are of 45°.
2) In the second solution proposed we used a certain concept of "continuity" of the area of the rectangle, if our point P moves in the plane. It is a simple (but full of meaning) concept: if the points moves "a little", then the area changes "a little".
   But is this function really continuous, even in this imprecise sense? What happens if we place the point P in C, and then move it "a little distance" from C? Does it still follow the rule of continuity? Exploring with GeoGebra, it is possible to actually prove it is not: if P moves along the line between A and C, then the area of EFGH is constantly 0, while if P moves along the line between D and C, then EFGH constantly coincides with ABCD, as Fig.5 shows:

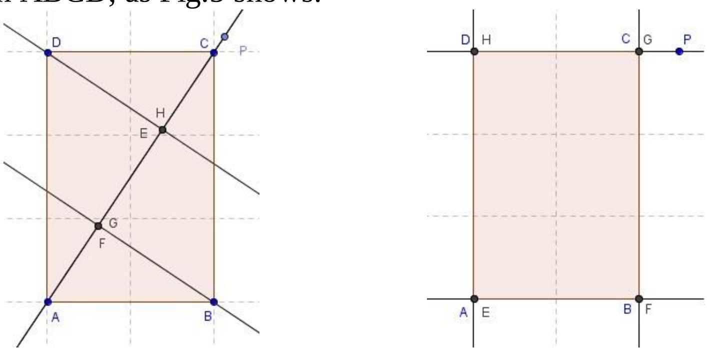

Fig. 5 How to show the function is not continuous

3) Always talking about continuity, we could decide to force the point P to move along a circumference, for example centered in the center of ABCD, with radius greater that the semi-diagonal of ABCD. This is a "right" idea, since we still consider all cases, and now we obtained a continuous function. This can help us in all our investigations.

   But what would happen if we decided to place point P on another geometrical locus? For example, consider the problem of finding the maximum area rectangle, and fix point P on the line given by $x = -y$. Which rectangle has maximum area? If we trust the spreadsheet given by GeoGebra, it seems that the value of the area of EFGH is definitely constant at its maximum value, and therefore there are infinitely many rectangles with the same maximum area!
   (Definitely means from a point on, this point is found taking bigger and bigger values of x).
   This cannot be true, since moving "a little" the point P should change the area. (Is it so obvious?)

   First of all, the problem we found is about approximation. Depending on our choices, the computer considers 0,9999=1, and therefore from a point on, all the areas look the same to it. A human instead should always notice the difference!
   It can be very interesting to help the students to prove that as x grows, the area grows, and therefore it cannot attain its maximum.
   Then why does it happen that, placing point P on that line, we can no longer reach exactly the maximum value, but we can get as close as we want to? The answer lies in the fact that choosing that specific line we took away the maximum rectangle: we took away a point from a circumference (which is compact, so a locus where

continuous functions always attain their maximum) obtaining a line, which does not have this compactness property.

SOME OBSERVATIONS AND FURTHER INVESTIGATIONS PROPOSED

These problems were analyzed during a course held by Professor V. Georgiev in Pisa in 2013, designed for both mathematics students and future mathematics teachers; the first problem was proposed by Prof. Georgiev, and the solution described here is my own, while I invented ex-novo the second problem. Both problems/solutions received a good response by the audience of the course, in particular were appreciated the many possible applications and the strict contact among various areas of mathematics. Unfortunately there is still no experience of proposing these topics to young students, so no further conclusions are allowed.

Hopefully, the strict collaboration created by these mixed courses (aimed to students and future teachers) will produce occasions of experimenting the concepts expressed in this paper in classrooms.

LITERATURE

Dimitrova, N., Georgiev, V., Morgensen, A., Mushkarov, O., Sendova, E., Ulovec, A. *MEETING in Mathematics*, Demetra Publishing House, Sofia, 2008.

Anderson, J., Ceretkova, S., Dimitrova, N., Georgiev, V., Mushkarov, O., Sendova, E., Ulovec, A. *MATH to EARTH*, 2010.

Georgiev, V. 2012. *Problem posing and dynamical approach*. [online] Available at <http://www.dynamatadmin.oriw.eu/upload_pdf/20121022_152508__0.pdf> [Accessed 01 May 2013].

Georgiev, V., Kurokawa. 2012. Y. *From static to dynamical problem posing.*[online]. Available at <http://www.dynamatadmin.oriw.eu/upload_pdf/20121022_153348__0.pdf> [Accessed 02 May 2013].

Georgiev, V., Kurokawa. 2012. Y. *Math problems of samurai period.*[online]. Available at <http://www.dynamatadmin.oriw.eu/upload_pdf/20121022_153129__0.pdf> [Accessed 02 May 2013].

Herstein, I.N. 1972. *Topics in Algebra*. 2nd edition. John Wiley & Sons Inc. 1975

GeoGebra Wiki. 2013. *GeoGebra Manual*. <http://wiki.geogebra.org/en/Manual%3AMain_Page> [Accessed 07 May 2013].

Wikipedia. *Extended Euclidean Algorithm.* <http://en.wikipedia.org/wiki/Extended_Euclidean_algorithm>

Mounth Olyoke College. *The Euclidean Algorithm in Excel*. <https://www.mtholyoke.edu/courses/quenell/s2002/ma319/demos/euclid.html>

INVESTIGATION WITH CIRCLES

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Abstract

In our contribution we describe an activity that supports development of geometrical thinking for new mathematical concept based on previous learners' models. We used circles as a base for students' investigation within conic sections. We presented several dichotomies or possible approaches to the learning of new mathematical concept using inquiry based learning. Based on the van Hiele theory of levels of geometrical theory (van Hiele, 1957) we describe two didactical environments for development of new knowledge. First we use inductive reasoning based on the dynamical manipulation with circles and second is the activity done by the dynamical geometrical applets that were prepared using deductive reasoning .We suggest that mentioned task is a good example for pre-service and in-service mathematics teachers to support their reasoning on several levels (van Hiele, 1957) with use of dynamical geometrical software GeoGebra.

Keywords

Inquiry based learning, locus of points, conic sections, GeoGebra

INTRODUCTION

Secondary mathematics education in Slovakia has been going through reform changes since 2008. Students were supposed to study less mathematical content but with deeper understanding. On the other site the requirements for final exam - Matura remain mostly the same as it was in the pre-reform period. Teachers usually deal with this situation that they remain within the traditional pedagogy and teach the extended content as it was in the pre-reform period. This lack of the time leads to learning several contents without deeper understanding. As it was put by (P. van Hiele, 1959 in van Hiele, 1985) "student knows only what has been taught to him and what has been deduced from it. He has not learned to establish connections between the system and the sensory world. He will not know how to apply what he has learned in a new situation." We can characterize that "teachers and students speak a very different languages". To characterize more the origin of those language we can use the commonly accepted theory of five levels of geometrical thinking defined by van Hiele (van Hiele, 1985).

In our contribution we analyse one prepared activity from the didactical point of view, where we would like to characterize important bits in students modelling and problem solving processes that are more present in Slovak reform curriculum. Better understanding to these processes can be a base for better differentiated instruction for students as well as for pre-service teachers' preparation. As a concept for our contribution we use manipulation with circles. Based on the van Hiele levels of geometrical thinking we assume that students should be able to reason on third or above level. We encourage students to use their knowledge in new situation where they need to use both inductive as well as deductive reasoning.

These activities can be presented with or without the real life concept. Within the Comenius project DYNAMAT we chose to present one activity using the concept of finding the place with the best view angle. We would characterise our aim as written in (Georgiev, 2012)

> "In order to stimulate ”nonstandard” thinking, research of original solutions, the capacity to model real life phenomena, we are trying to prepare didactic units for teachers giving them some concrete examples that can be implemented in everyday work in class or in some extracurricular activities."

THEORETICAL FRAMEWORK

For better orientation in the processes and reasoning that students and mathematics teachers need to go through we will use van Hiele levels of geometrical thinking (van Hiele, 1985). The five levels are Level 1 (Visualisation), Level 2 (Analysis), Level 3 (Abstraction), Level 4 (Deduction), Level 5 (Rigor). According to this model, learners have to master one level to be able move to a higher level. We will focus on the second and the third level of geometrical thinking, because these two levels are the most used in our contribution.

Level 2 (Analysis)

> "Students see figures as collection of properties. They can recognise and name properties of geometric figures, but they do not see relationships between these properties. When describing an object, a student operating at this level might list all the properties he/she knows, but may not discern which properties are necessary and are sufficient to describe the object."

Level 3 (Abstraction)

> "Students perceive relationships between properties and between figures. At this level, students can create meaningful definitions and give informal arguments to justify their reasoning. Logical implications and class inclusions, such as squares being a type of rectangle, are understood. The role and significance of formal deduction are, however, not understood."

As we mentioned all students should be approximately on the third level of "Abstraction" level or above. Within these prerequisites we can lead students to use and deepen their reasoning skills. In learning of new concept students should be able to decrease their level as well as develop new content within the inquiry based learning.

SETTING THE PROBLEM IN CONCEPT

In the article (Šunderlík, Barcíková, 2011) we introduce a real life problem in which we describe an inquiry approach to solve of the several stated problems connected to the best viewing point at the Appolo Bridge in Bratislava. This concept can be easily transferred to almost any observation in nature. We can look for set of points that have the same distance from the point and line or to a two circles and so on. Our context helped us to state the possible problems based on some real life situations that cause the need to investigate and looking for new properties. In the process of looking for solution we use only the knowledge that we have formed on primary and lower secondary education and the mathematical language and reasoning on the appropriate level of upper secondary education.

DEVELOPMENT OF CONCEPT

Within the set of problems in (Šunderlík, Barcíková, 2011) students should develop appropriate language and reasoning that will lead to more complex solution. Within setting up this problem we used real life situation. Students should find the "best spot", place on the Harbor Bridge (straight line) from where we can see the Appolo Bridge (two endpoints of the bridge) under the biggest angle (Fig. 1) (Šunderlík, Barcíková, 2011). Based on the previous investigation students are supposed to find the center of a circle that will be circumscribed to the bridge and tangent to the Harbor Bridge (straight line). The intersection of this circle and the line give us the "best spot". To define the wanted locus we would like to use knowledge that we already have gained about circles and geometrical construction.

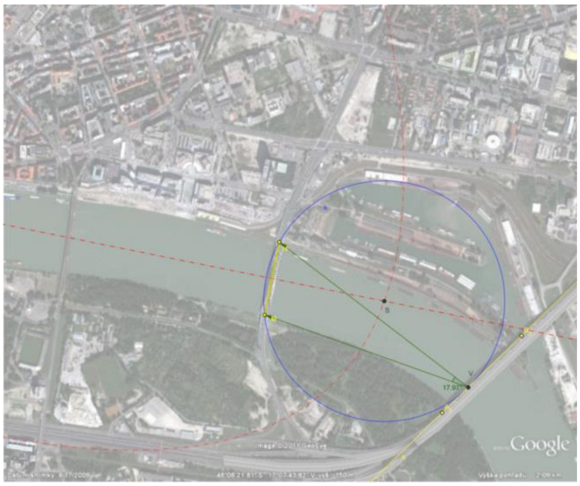

Fig. 1 Visualisation and graphical solution of the problem

We can make some arrangements and use GeoGebra to model the situation.

Modeling real life situation - Mathematisation of the situation

We put the Google Map of bridges in GeoGebra and investigate this problem. At the beginning we construct the line *c* on one side of the bridge and a point *X* on this line. Then we press the angle button in GeoGebra and measure the angle *MXN*. What if the point *X* (lying on line s) can move along line *c*? Then move with the point *X* along the line and observe what will happen with the angle. Can you explain why?

How can we find the place where the angle is the biggest?

How is the view angle connected to the distance from the object? Where is the place on the Harbour Bridge with the biggest view angle depending on the distance from the object – Apollo Bridge?

Imagine for example any point in the river. How is it connected to the circles with centres lying on the perpendicular bisector? We have investigated that in problem 2 (Šunderlík, Barcíková, 2011). From problem 2 we have gained the knowledge that the centre of circle lies on the perpendicular bisector, but we don't exactly know where. We are looking for the centre S_x of circle k_x that will touch the line s and pass through the points M and N.

How can we find the place where the angle is the biggest?

We can let students to give some ideas or use some prior suggestions. For example let the students investigate with circle in GeoGebra, try to approach the circle to points *MN* and line *c*. Than we can discuss with students that it probably satisfies our purposes, but it is not mathematically exact.

Let backtrack – to move our thinking from the final result backwards. Such steps will lead us to a solution. We need to investigate the final solution (Fig. 1). We need to look for a locus that has the same distance of line *c* and points *M* or *N*. What is the locus equally distant from point *M* and also from line *c*? To come up with this question is the hardest part of the mathematisation of the real problem. After this we can move from real concept to the geometrical figures.

Metathinking and metacomments are very important in the process of investigation. We can divide the problem into two problems.

A) Which locus do the centers of the circles touching two fixed points describe?

B) Which locus do the centers of the circles touching a fixed line and passing through a fixed point describe?

Within the inductive reasoning we touched the first level of visualization and did the appropriate analysis. We needed to move down in the language and reasoning to better understand the whole problem.

Solving the problem within the mathematics

Now we can move from the environment of our map to universal model, because we don't exactly know where the point *S* (centre of wanted circle) is located.

Within the approach we need to apply inductive as well as deductive reasoning skills that were already developed in different mathematical context and need to be applied to new situation.

In this situation we need to distinguish the role of teacher and the learner. In some cases both role can be parallel within one student. We will distinguish them as two starting points, inductive and deductive. Before we start to produce a suitable mathematical model that will help us solve the problem and may be visualized by the GeoGebra we can let students to give some ideas or use some prior suggestions. For example let the students investigate with circle in GeoGebra, try to approach the circle to points MN and line c. Then we can discuss with students that it probably satisfies our purposes, but it is not mathematically exact. We can use models of circles touching the line and a given point.

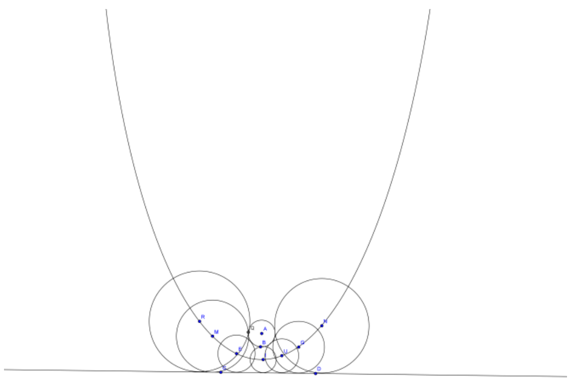

Fig. 2 Investigating parabola in GeoGebra

Fig. 3 Investigating parabola on piece of paper

Decreasing the level of geometrical thinking – Preparation of tool for analysis

Now we can move to universal model, because we don't exactly know where the point S (centre of wanted circle) is located.

1: We choose one point and call it point *M* that is given in the plane. (We cannot move it. In reality it is one of the endpoints of our bridge.) What is the set of points which are equidistant from a given point?

It is the circle with centre in point *M* and constant distance *r* - radius.

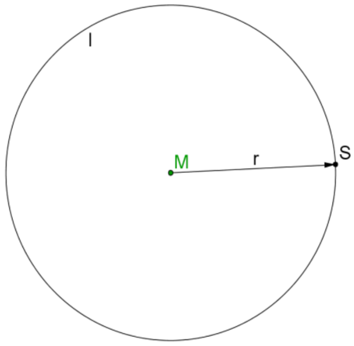

Fig. 4 Investigating parabola: Step one

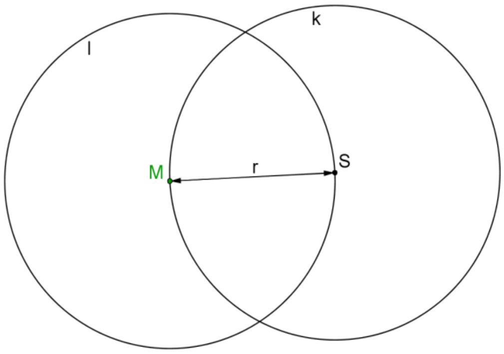

Fig. 5 Investigating parabola: Step two

Now imagine one point from the set which can be freely moved round the circle. Let choose one point and name it point *S*. This point lies on circle *l*. Then also point *M* lies on the circle *k*, which has centre in point *S*.

Now add to our thoughts line *p*. This line is tangent to the circle in point T which is different from the point M and lies on the circle *k*. Because of the tangent p to the circle *k* we know that ST is perpendicular to the line *p*. From the definition of the circle that we were talking about by the beginning of our thoughts, we also know that SM = ST.

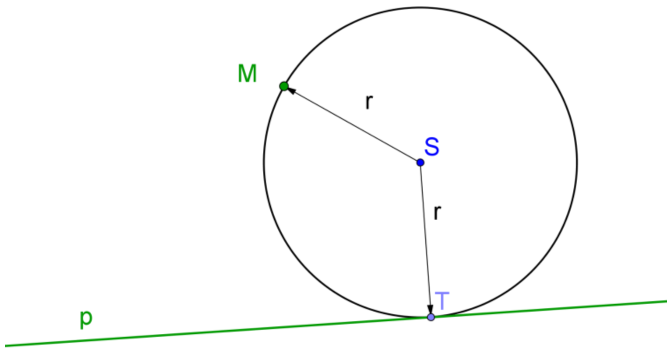

Fig. 6 Investigating parabola: Step three

What is important is that point T is not fixed and so we can freely move with him along the line p. On the other side, point M is fixed.

Now we prepared the tool for investigation and students analysis. There are several options how it can be used. It is good that students could use the similar deductive reasoning to construct the model for analysis. In other case it is the task for the teacher to decrease the level of reasoning suitable for analysis and abstraction.

Investigation with prepared tool

What can we say about the distance from the point S to M and the distance from the point S to line p when we move the circle (we move it by the point T)? Move the point T and try to predict the results. Observe what will happen.

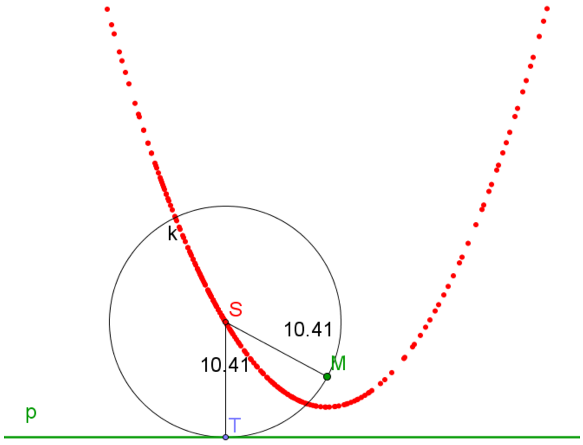

Fig. 7 Conjecture about the locus line and point.

Imagine that point S traces red points. Once again what is the name of this locus?

- This curve is parabola. It is the set of all points in the plane whose distance from a fixed point *M* is equal to their distance from the fixed line *p*.

How many circles are there that touch point M and line p and their radius is *r*? How many points are on the line?

Demathematisation of the solution

Now we should be ready to move back to our map and find our best spot. We will construct two loci, the perpendicular bisector to segment MN and parabola to line *s* and point M. The intersection of these curves gives us the centre of circle *k*, that is circumscribed to points M, N and line *s*. The intersections of this circle and line *s* give us the wanted “best spot” to look at the Apollo Bridge from the Harbour Bridge.

DISCUSSION

We could characterize the mentioned process with the transition between the levels that are necessary to develop a coherent base for general solution. We try to describe the learning trajectory that student needs to go through to solve the problem. What mathematical concepts as well as reasoning skills are necessary to solve the problem. This information then can be useful for designing a mini investigation for students as well as for professional development of in-service and training of pre-service mathematics teachers. The main instruction should then be given in appropriate level and the language should be more intuitive for lower levels. There have been several statements that the levels of van Hiele geometrical thinking are not discrete (Gutiérrez & all, 1991) so we also need to consider when students have already moved to higher level and we may apply different teaching strategies as well as language and reasoning.

According to our suggested solution we design the possible learning trajectories within the levels of geometrical thinking (Fig. 2).

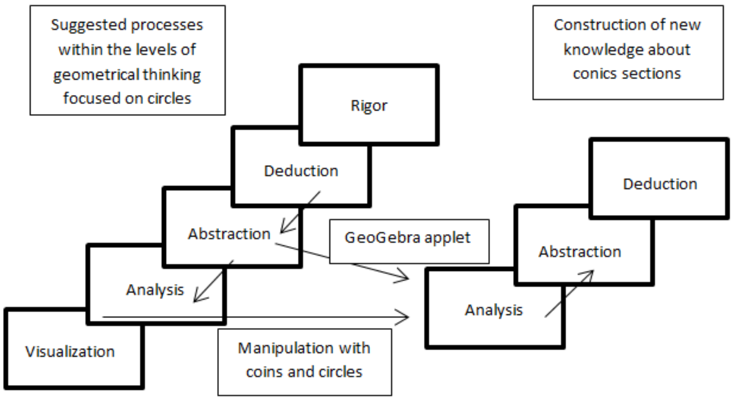

Fig. 8 Suggested learning trajectories within the levels of geometrical thinking.

We can develop new knowledge and reasoning mostly up to the level of previous geometrical thinking. But we also suggested that we may increase the acquisition for the next upper level of geometrical thinking.

At the beginning they decrease the level of geometrical thinking, where they need to deeper understand the known concept in a new situation.

ACKNOWLEDGEMENT

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LITERATURE

Barcíková, E. (2011) *Apollonius problem solved by using analytical geometry methods.* In: Young Researchers 2011 : PhD Students, Young Scientists and Pedagogues Conference Proceedings. Nitra, 29. June 2011 / David Turčáni et al. - Nitra: UKF, 2011. - ISBN 978-80-8094-946-4, P. 723-730

Gutiérrez, A., Jaime, A., and Fortuny, J.M. (1991). An alternative paradigm to evaluate the acquisition of the Van Hiele levels. Journal for Research in Mathematics Education, 21 (3), 237-251

Barcíková, E., Šunderlík J. (2012) *Objavovanie elipsy manipuláciou s kružnicami,* In: Acta Mathematica 14 : zborník príspevkov z IX. nitrianskej matematickej konferencie organizovanej Katedrou matematiky FPV UKF v Nitrev dňoch 22. - 23. septembra 2011. - Nitra : UKF, 2011. - ISBN 9788080949587. - S. 41-46

Georgiev, V. (2011) Introduction: Problem posing and dynamical approach citied 15.5.2013] available online: <http://www.dynamatadmin.oriw.eu/upload_pdf/20121022_152508__0.pdf >

Šunderlík, J. Barcíková E. (2011) *Best spot - investigation with circles,*[citied 15.5.2013] available online: <http://www.dynamatadmin.oriw.eu/upload_pdf/20121022_155122__0.pdf>

van Hiele, P. (1985) [1959], The Child's Thought and Geometry, Brooklyn, NY: City University of New York, pp. 243–252

SOME EXAMPLES ON USING GEOGEBRA TO TEACH CALCULUS

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Abstract
In this paper we consider some examples showing how the dynamic properties of Geogebra or similar software can be used to investigate certain concepts from Calculus. This is based on material created during the project DynaMat. The examples grew out of attempts to try to increase visualization done by students and possibly increase their understanding of these concepts.

Keywords
GeoGebra, derivative,functions, piecewise defined functions, DynaMAT

INTRODUCTION

During the last 10 years there has been much discussion on the problems many students face when entering University and learning mathematics at University level (Thomas et al 2012). The topics that have been mentioned in particular are understanding the real numbers, understanding the concept of a function, limits, continuity, sequence and series etc.

Tall and Vinner (1981) define for a certain mathematical concept the term *Concept Image* to describe the cognitive structure associated with the concept in the students mind, i.e. visual representations, mental pictures, expressions and thoughts that the student relates to the concept. They use this definition to discuss problems that students may have with certain mathematical concepts such as functions, limits and continuity. In a questionnaire answered by 41 students they examine the concept image the students have of a continuous function and reveal misconceptions such as *the given function is continuous because it is given by a single formula.*

Thompson (1994) discusses research on students' initial understanding of functions and has found indications that the concept image of a function is *something given by a single formula*. This formula is considered as a recipe that is applied to some number to get another number. In other words the student has a *process conception* of a function.

The mathematical definition of a function is as follows: A function f from X to Y is a subset of the Cartesian product X × Y subject to the following condition: every element of X is the first component of one and only one ordered pair in the subset.

This definition is hard to grasp for many students and in fact Vinner (1983) found that even students who could give a correct set-theoretic definition of a function didn't necessary use it when needed in answering questions about functions. Instead, they relied on some earlier concept image concerning formulas. In Petterson et al. (2013) the concept of a function is discussed as a threshold concept

Apart from the difficulty in understanding basic concepts many students view mathematics as a set of rules on how to manipulate symbolic expressions. For a discussion on this see e.g. Tall (1997). This view is of course strengthened by the many exercises in textbooks that mostly require some algebraic calculations as well as an imitation of an example given in the book (Lithner 2004).

In this paper we discuss some possibilities to use GeoGebra to potentially increase the understanding of the concepts mentioned above and give some models for visualisation. The approach is mostly to have students work directly on graphs and thus gain some insights into functions, continuity and derivatives. The hope is that perhaps such examples will help eliminate certain misconceptions such as *a function is defined by a single formula*. Most of the examples given are a part of the project DynaMAT.

ELEMENTARY USES OF GEOGEBRA

If a function is given by a single formula it is very easy to graph it in GeoGebra, the formula is simply typed into the input field. It is also quite simple to define piecewise defined functions by using the command *Function* or by using the command *If*. The *If* command has the advantage that the derivative of the function can be calculated directly. If there are three different intervals a nested *If* command can be used e.g. *If[x < 2, x², If[x < 3, x + 2, 10 - x]]* results in the figure below:

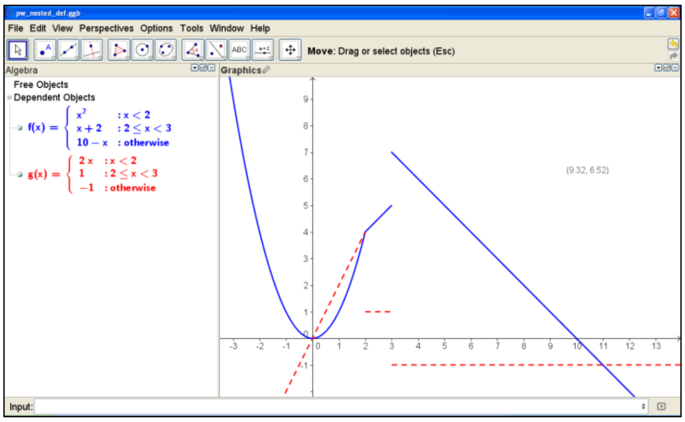

Fig. 1 A function *f(x)* defined on three intervals. Its derivative *g(x)* is shown in red. The derivative is undefined at $x = 2$ and at $x = 3$ but the program does not indicate that in any manner

It is also very easy to reflect, rotate and translate graphs using tools or dragging by the mouse in GeoGebra. Since the algebraic equations of everything in the graphics view are shown in the algebra view, this can be used to e.g. examine the formula of a function after it has been translated horizontally by a constant or to determine when a particular rotated graph is the graph of a function or not.

In the most recent version of GeoGebra there is a tool called the *Function Inspector* which can be used to examine points on graphs, tangent, second derivative at a point etc. Also new is the possibility of *Slowplot* which slowly draws the graph of a function.

THE DERIVATIVE AND THE SHAPE OF A FUNCTION

Typical use of GeoGebra or similar software is to draw the graph of a function, mark a point on the graph and then make the software draw a tangent to the graph at that point.

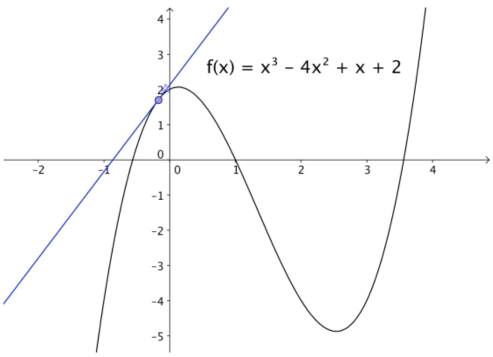

Fig. 2 Tangent through the graph of a function at a point

After this is done the student can drag the point along the graph and watch the changes in the slope of the tangent.

In many textbooks on differentiation there are exercises of the type where a student is asked to find the equation of the tangent line to the graph of a given function at a given value, e.g. at $x = 1$. Usually these problems are solved by differentiating the function, finding the value of the derivative at the given x – value, is this case $f\,'(1)$, to get the slope of the tangent line, and then finding the equation of a line through the given point i.e. $(1, f(1))$.

Software like GeoGebra or a graphics calculator is very useful in such situations to check if the calculated answer makes sense and also to connect the algebraic calculations just completed to the actual geometric object that has been found.

Another kind of problem is of the type: *The slope of the tangent to* $f(x) = ax^3 - 3x^2 + x + 3$ *at the point* $(-1, f(-1))$ *is* 1. *Find the value of a.*

These problems are usually solved algebraically only, i.e. the student differentiates the function, substitutes the value -1 for *x* and solves $f\,'(-1) = 1$ to get the value of *a*:

$f'(x) = 3ax^2 - 6x + 1$ so $f\,'(-1) = 3a + 6 + 1 = 1 \Rightarrow 3a = -6 \Rightarrow a = -2$.

Once the problem has been translated to an algebra problem it is often not considered necessary to graph the function or check the answer.

In most calculus books, there are also more complicated problems of the same type involving the location of relative minima and maxima, points of inflection, integrals etc. Most often the assumption is that the calculations are done by hand and the problem is converted into an algebra problem, i.e. into solving a system of equations. The focus is then more on algebra than on the actual shape of the function in question.

The following problem is taken from a handout for students at the gymnasium level (Menntaskólinn við Hamrahlíð, 2010): *Find a third degree polynomial* $f(x) = x^3 + bx^2 + cx + d$ *such that it has a local minimum at* $x = 1$ *and its graph has an inflection point at* $(-1, 12)$.

It is assumed that the students will solve it using algebraic techniques after differentiation i.e. to differentiate $f(x)$ twice and then use the information on the inflection point to get the equation $-6 + 2b = 0$, the information on the minimum to get $3 + 2b + c = 0$ and finally to get $-1 + b - c + d = 12$ since $(-1, 12)$ is on the graph. The equation system is then solved in the usual way.

In GeoGebra this can be solved in a graphical way by working directly with the parameters b, c and d. The user defines three sliders with integer values, b, c and d and the third degree polynomial $f(x) = x^3 + bx^2 + cx + d$ (for details on how this is done see Hreinsdóttir 2012a).

Anyone who tries this out will quickly find that it is impossible to find the solution by just playing with the values of the sliders even if the effect of changing the value of *d* should become clear.

The student working on this therefore needs some guidance and is advised to use the graphing possibilities of GeoGebra to get the graphs of $f\,'(x)$ and $f\,''(x)$. These graphs are then given different colours and the effect of changing the values of *b* and *c* is examined. This information can then be used to find a value of *b* that gives a point of inflection at $x = -1$ and subsequently a value of *c* such that $f(x)$ has a minimum at $x = 1$.

Now the graph of *f* should have the desired shape but the user still needs to find a value of *d* that gives the correct location of the graph. The point $(-1, 12)$ is defined in the input field and the value of d is modified until the graph of $f(x)$ passes through the point.

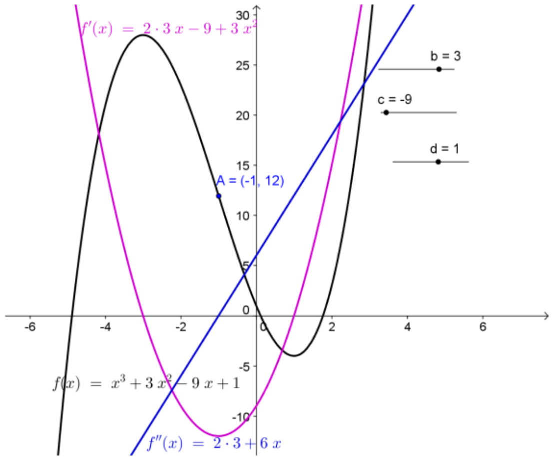

Fig. 3 The solution to the problem above

For similar problem see Hreinsdóttir (2012a).

From a pedagogical point of view one may wonder how useful this is to students, i.e. what is there to gain by solving such a problem? It is the experience of many teachers that once the student has transformed the problem into an algebraic problem they are unaware of what they really have calculated, i.e. they focus on the algebraic steps needed for the solution and then give their answer in terms of a = value, b = value etc. Therefore, such tasks become an exercise in algebra rather than a Calculus exercise.

Working on a problem in this graphical way can possibly encourage the students to think about what it is they are calculating. Thus the students might be less likely to just form a rule about an algebraic procedure and more likely to connect such a problem to the interpretation of the derivative as the slope of a tangent line at a certain point.

The students' mathematical maturity or understanding of derivatives may benefit from thinking about the task as finding certain shapes of objects rather than following a rule and apply algebraic procedures.

LEARNING ABOUT CONTINUITY AND DIFFERENTIABILITY

In most Calculus books, the definition of continuity is given by: A function f is continuous at a point c in its domain if $\lim_{x \to c} f(x) = f(c)$. This is then sometimes explained by saying something like "you can draw the graph of the function through the point without lifting your pen". See e.g. Adams (2004).

The problem with this explanation is for some students that most of the functions they have seen are continuous so in a sense this definition does not mean anything to them. It is therefore important that they work actively with discontinuous functions such as piecewise defined functions.

USING SLIDERS TO DEFINE CONTINUOUS FUNCTIONS

Consider the following problem: determine values of the parameters a, b and c such that the function

$$h(x) = \begin{cases} x - a & \text{if } x < -1 \\ x^2 + b & \text{if } -1 \leq x \leq 3 \\ c - x & \text{otherwise} \end{cases}$$

is continuous.

This problem can be solved in GeoGebra by defining three sliders *a*, *b*, *c* and using a nested *If* command to define *h(x)*. The values of the sliders can now be changed to make the function continuous.

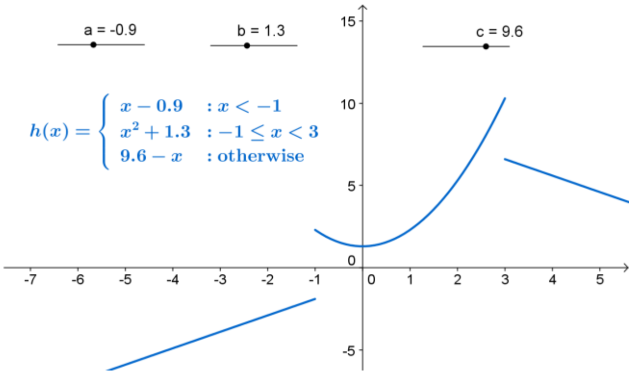

Fig. 4 We change the definition of the function by moving the sliders.

If we change the values of *a*, *b* and *c*, the graphs move up and down so it is very easy to find values such that the graphs are connected.

A more complicated situation arises if the parameters not only control the location of the graph but also its shape (see Hreinsdóttir 2012b).

DIFFERENTIABLE FUNCTIONS

In problems like the ones given above it is very noticeable that at the connecting points we get "a corner point" in the graph of the function. This is because even though the function is continuous at the points it is not *differentiable*.

In GeoGebra it is quite easy to demonstrate the tangent at a point as the limit of secants through the point as is shown in figure 5.

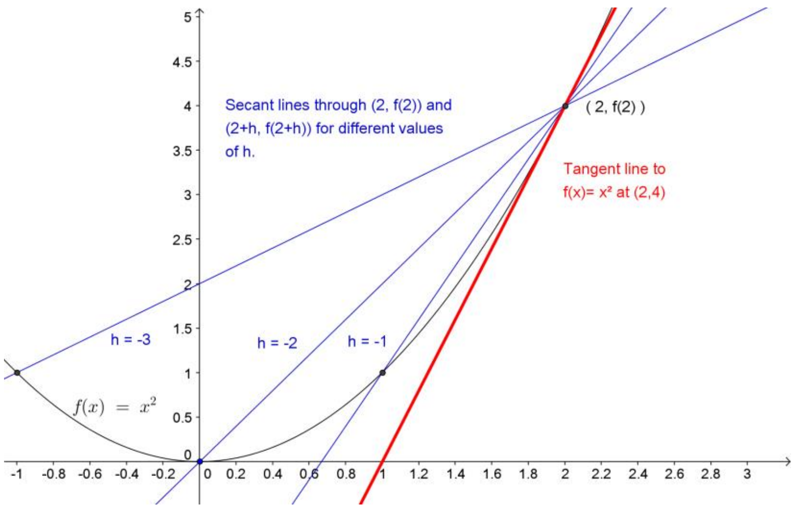

Fig. 5 Secant lines approaching the tangent line at a certain point

Below we describe how we can join two different quadratic functions to get a piecewise defined function that is both continuous and differentiable at the meeting point.

We define four sliders *b*, *c*, *d* and e in GeoGebra and use them to define two quadratic functions $f(x) = -x^2 + bx + c$ and $g(x) = x^2 + dx + e$. We then find values of the sliders such that $(1, f(1)) = (1, g(1))$. This ensures that the graphs of the two functions intersect at this point. Now we use the tangent tool to get the tangents to both functions at the meeting point.

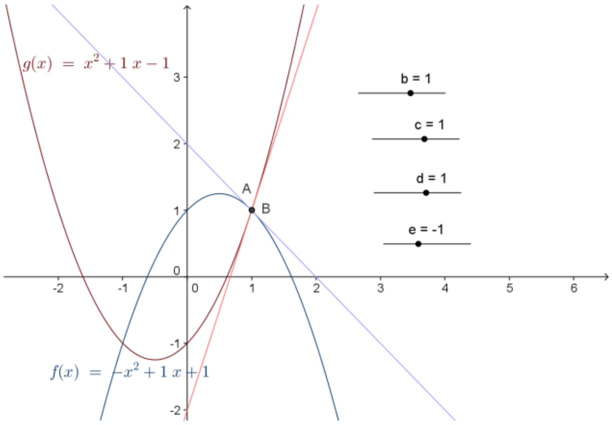

Fig. 6 The graphs intersect at the $x = 1$ but the tangents are not the same.

If we now define a function $h(x)$ such that

$$h(x) = \begin{cases} f(x) & \text{if } x \le 1 \\ g(x) & \text{otherwise} \end{cases}$$

we get the function below:

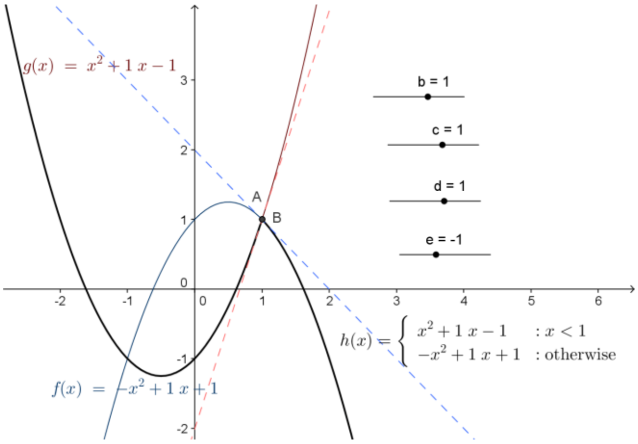

Fig. 7 The function $h(x)$ (black graph) is not differentiable at $x = 1$.

To redefine the function $h(x)$ so that it is differentiable we need to adjust the values of the sliders such that the tangents are the same.

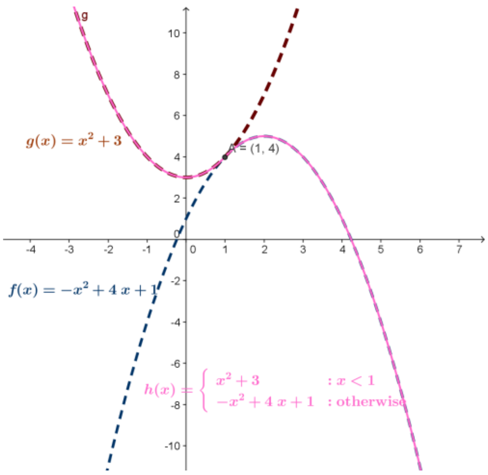

Fig. 8 Here we have one solution to the problem. The function $h(x)$ (pink graph) is differentiable everywhere.

REMOVING CORNER POINTS

We can use a similar method to redefine a function on a small interval in order to remove a corner point in the graph of a function and thus create a function that is differentiable at that point.

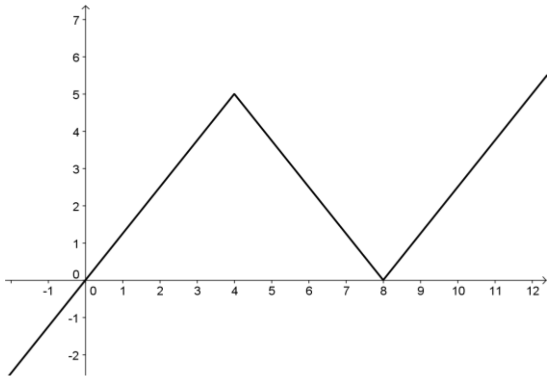

Fig. 9 A simple example of a function that has corner points at several points

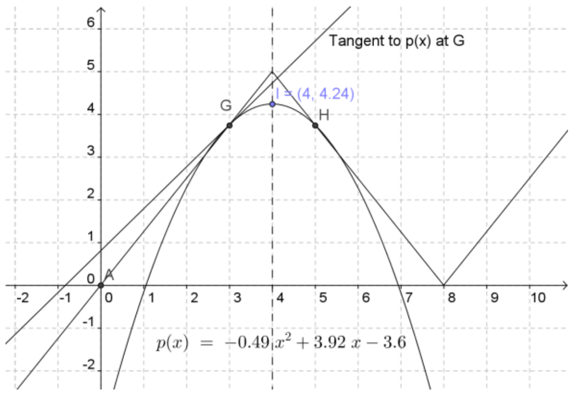

Fig. 10 Attempt to remove a corner in the graph by redefining the function on a small interval

We define two points *G* and *H* on opposite sites of the corner point we want to remove and a point *I* on the line *x* = *4*. We then use the command *FitPoly[{G,I,H},2]* to get a second degree polynomial that goes through these three points and the tangent tool to get a tangent to the graph of this polynomial at the point G. We then move the point *I* (it is fixed on the line $x = 4$) until this tangent coincides with the segment from $(0,0)$ to $(4,5)$. After removing help lines, changing colours etc we get the graph below.

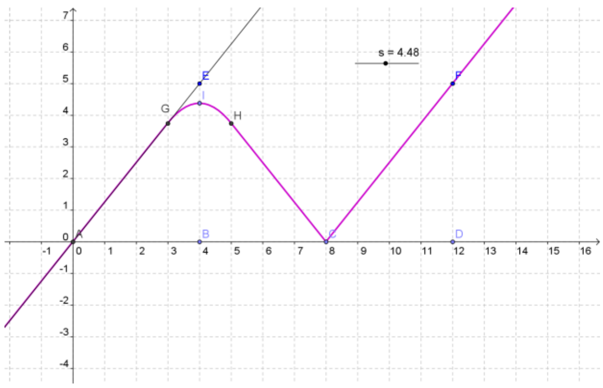

Fig. 11 One of the corner points has been removed

FINAL COMMENTS

The examples above are suggestions on how to use the dynamical possibilities of GeoGebra to increase the understanding of certain concepts in Calculus. They are not intended to replace in any way regular paper and pencil exercises that are necessary to master the techniques. It is also very important that students see mathematically correct definitions of the concepts used.

LITERATURE

Adams, R. 2004. *Calculus.* 4th ed. Ontario: Addison-Wesley.

GeoGebra, downloadable from <http://www.geogebra.org>

Hreinsdóttir, F. 2012a. Using sliders to investigate functions, tangents and integrals. Available at: <http://www.dynamathmat.eu/>

Hreinsdóttir, F. 2012b. Piecewise defined functions. Available at: <http://www.dynamathmat.eu/>

Lithner, J. 2004. Mathematical reasoning in calculus textbook exercises. *The Journal of Mathematical Behavior*, *23*(4), p. 405-427.

Menntaskólinn við Hamrahlíð. 2010. Stærðfræði 403.

Pettersson, K., Stadler, E., & Tambour, T. 2013 Development of Students' Understanding of the Threshold Concept of Function. Available at: <http://cerme8.metu.edu.tr/wgpapers/WG14/WG14_Pettersson.pdf>

Tall, D., Vinner, S. 1981. Concept Image and Concept Definition in Mathematics with Particular Reference to Limits and Continuity. In *Educational Studies in Mathematics*, vol. 12, No.2, p. 151-169.

Tall, D. 1997. Functions and calculus. In *International handbook of mathematics education* (p. 327-370). Springer Netherlands.

Thomas, M., de Freitas Druck, I., Huillet, D., Ju, M. K., Nardi, E., Rasmussen, C., & Xie, J. 2012. Survey team 4: Key mathematical concepts in the transition from secondary to university. Available at: <http://www.math.auckland.ac.nz/~thomas/ST4.pdf>

Thompson, P. W. 1994. Students, functions, and the undergraduate curriculum. In *Research in collegiate mathematics education, 1* (Issues in Mathematics Education Vol. 4) p. 21-44.

Vinner, S. (983. Concept definition, concept image and the notion of function. *International Journal of Mathematical Education in Science and Technology*,*14*(3), p. 293-305

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