Can Equations Be Exciting?

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- *deterministic* vs *predictable*
- *The butterfly effect*

Discrete predator-prey models

A Lotka-Volterra model (proposed in the year 1926):

$$\begin{aligned} x_{n+1} &= x_n(1+p_1-p_2x_n-p_3y_n) \\ y_{n+1} &= y_n(1-q_1+q_2x_n), \quad n=0,1,2,\ldots \end{aligned}$$

Having some values at the initial time $n=0$, say (x_0,y_0), we can consecutively compute by means of (1) a sequence of points in the plane

$$(x_1,y_1),\ (x_2,y_2),\ldots,(x_n,y_n),\ (x_{n+1},\ y_{n+1}),\ldots$$

This sequence describes the evolution of the populations as time increases and is called a trajectory of (x_0,y_0) for (1). Obviously, the values of the sequence members depend on the choice of the constants p_1, p_2, p_3, q_1 and q_2.

Figure 1(a) presents three trajectories within

$$p_1 = 0.05, \;\; p_2 = 0.0001, \;\; p_3 = 0.001, \;\; q_1 = 0.03, \;\; q_2 = 0.0002, \tag{2}$$

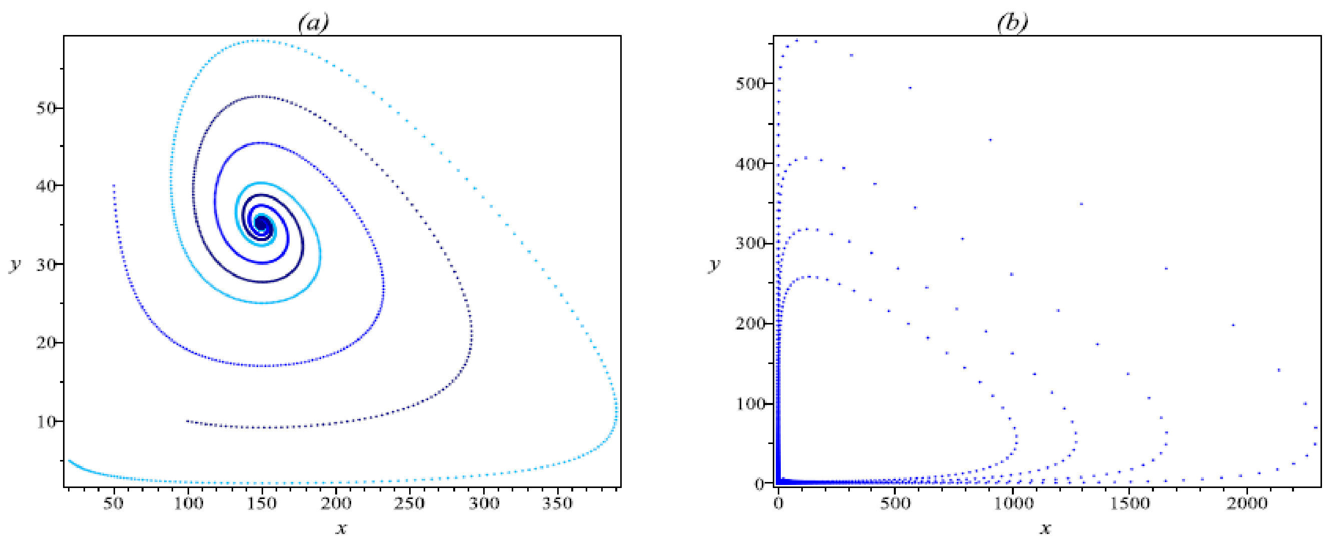
for three different initial conditions $(x_0, y_0) = (20, 5)$, $(x_0, y_0) = (100, 10)$ and $(x_0, y_0) = (50, 40)$. As n increases, the three trajectories approach one point in the plane and and remain very close to it for all over the time. Such a point is called a *stable* steady state or an *attractor*.

Now set $p_2 = 0$ in the model (1):

$$\begin{aligned} x_{n+1} &= x_n(1 + p_1 - p_3 y_n) \\ y_{n+1} &= y_n(1 - q_1 + q_2 x_n), \quad n = 0, 1, 2, \ldots \end{aligned} \tag{3}$$

Figure 1(b) presents trajectories with the same coefficient values for p_1, p_3, q_1 and q_2 from (2).

We see totally different behavior of the trajectories. The two populations oscillate, building *cycles*.



Iterated quadratic maps

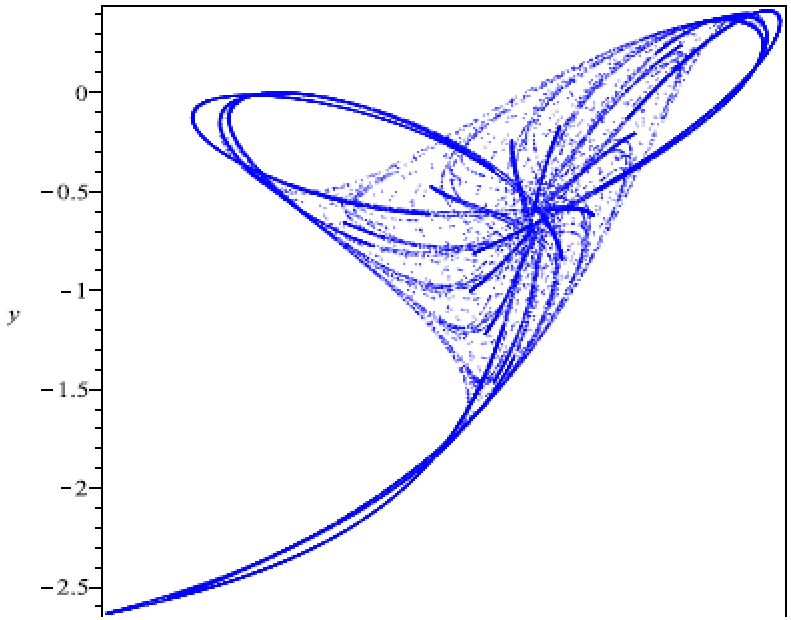
The iterated map (1) has a term of the form x_n^2 as its highest order terms, thus it is a map of order 2 or a quadratic map. The most general discrete quadratic map is

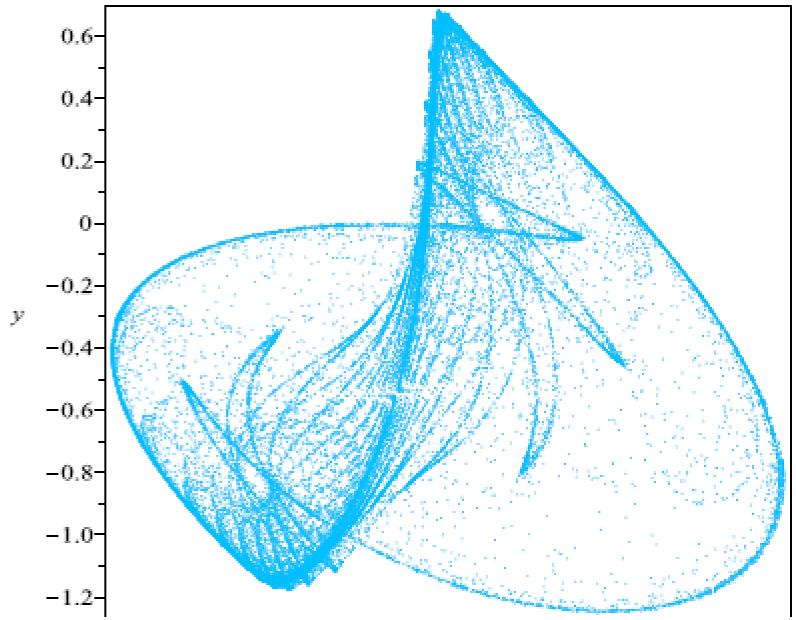
$$\begin{aligned} x_{n+1} &= a_1 + a_2 x_n + a_3 x_n^2 + a_4 x_n y_n + a_5 y_n + a_6 y_n^2 \\ y_{n+1} &= b_1 + b_2 x_n + b_3 x_n^2 + b_4 x_n y_n + b_5 y_n + b_6 y_n^2, \quad n = 0, 1, 2, \ldots \end{aligned} \tag{4}$$

Table 1: Examples of quadratic maps with chaotic attractors

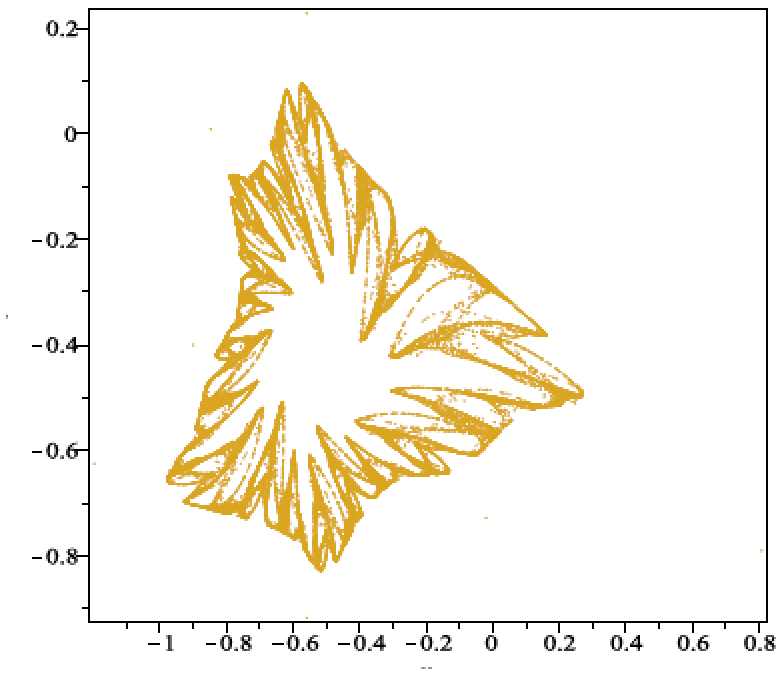
Example	Coefficient vectors a and b	FD	LE	Figure
1	$a = (-1.2, -0.6, -0.5, 0.1, -0.7, 0.2)$ $b = (-0.9, 0.9, 0.1, -0.3, -1, 0.3)$	1.36	0.18 −0.45	2 (left)
2	$a = (-1.1, -1, 0.4, -1.2, -0.7, 0)$ $b = (-0.7, 0.9, 0.3, 1.1, -0.2, 0.4)$	1.31	0.096 −0.098	2 (right)
3	$a = (-0.9, 0.6, 1.2, 0.8, -0.8, -1)$ $b = (-0.4, 0.1, -0.6, 0.4, 0.1, 0.9)$	1.42	0.027 −0.025	3 (left)
4	$a = (-0.3, 0.7, 0.7, 0.6, 0, -1.1)$ $b = (0.2, -0.6, -0.1, -0.1, 0.4, -0.7)$	1.37	0.16 −0.32	3 (right)
5	$a = (0.2, 0.8, -0.6, -0.7, -0.3, -0.2)$ $b = (-0.9, -0.5, 0.6, -1.2, -0.3, 0.8)$	1..38	0.18 −0.37	4 (left)

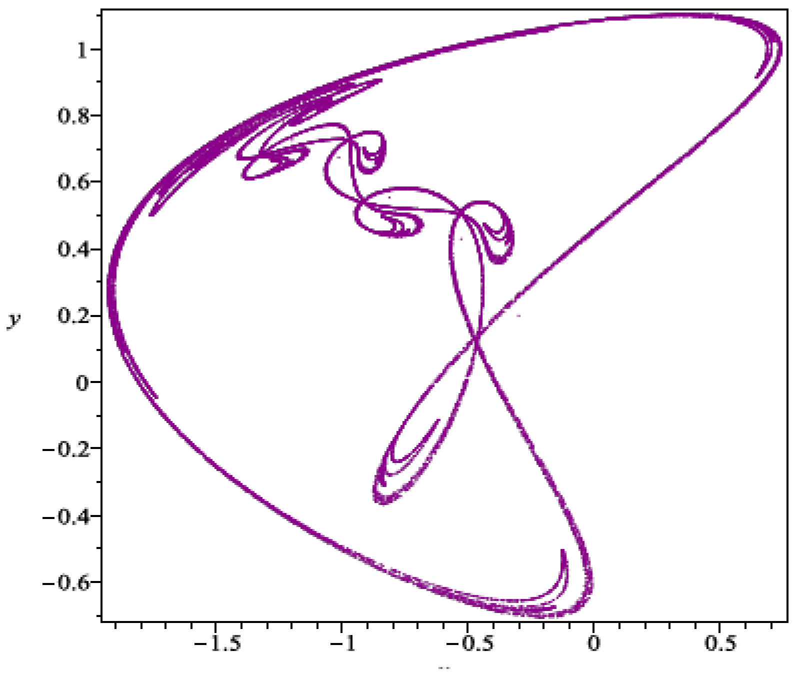
Chaotic attractors



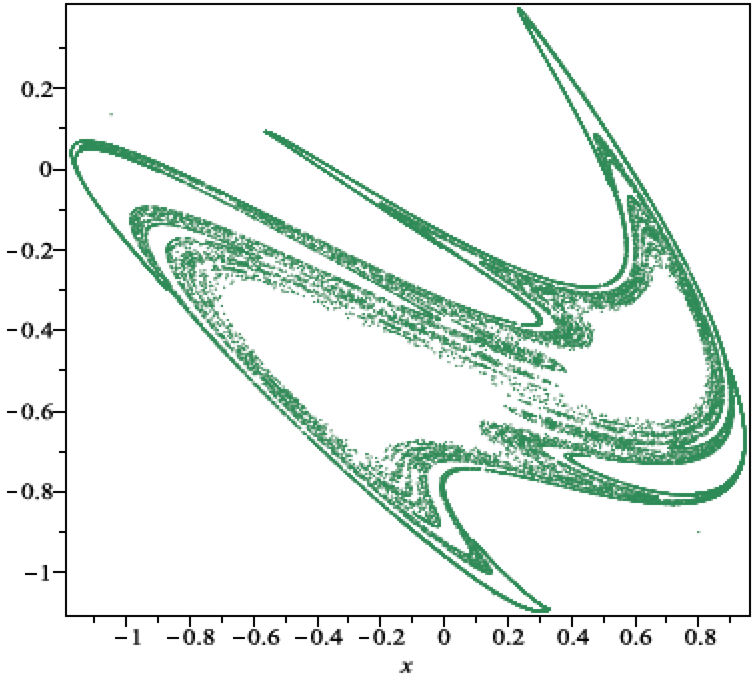


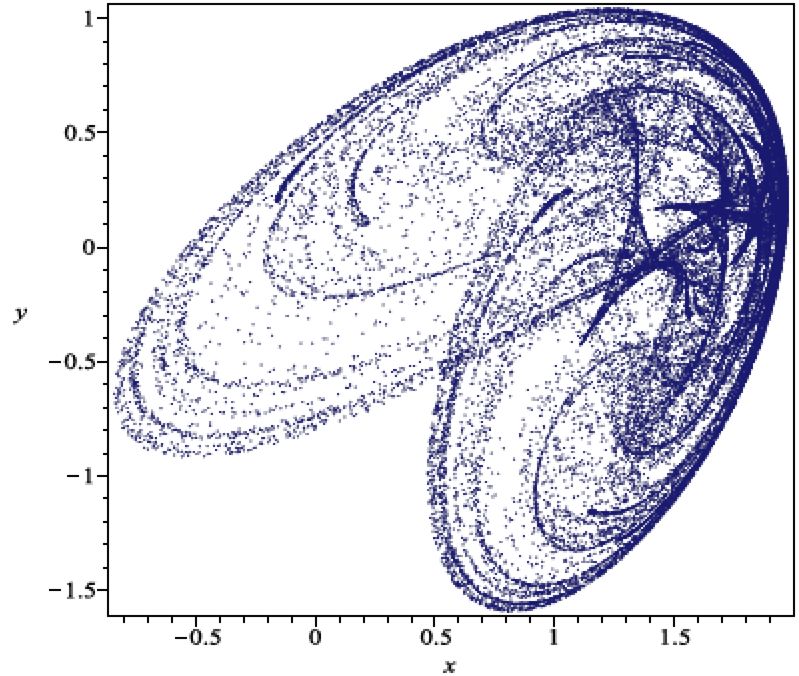
Chaotic attractors





Chaotic attractors





The idea of the fractal dimension

The fractals shown on Figures 2 to 5 and further in the paper do not display exactly self-similarity; they have only regions that are self-similar. The main ingredient in the definition of a fractal is its *fractal dimension*. Isolated points have dimension zero, line segments have dimension one, surfaces have dimension two and solids have dimension three. Without going into detail, this is their "usual" or so called *topological dimension*. A fractal has a dimension that exceeds its topological dimension. Fractal dimension gives finer information about the density or roughness or complexity of the set. In most cases, fractals possess non-integer dimension. However, there are fractals with integer dimension; in this case the fractal dimension must exceed their topological dimension. For example the Sierpinski tetrahedron (a tetrahedral analogue of the triangle) has fractal dimension two, but topological dimension one.

The idea behind Lyapunov exponents

Chaotic iterated maps exhibit *sensitive dependance* on initial conditions. Imagine a set of initial conditions filling a small circular region in the (x, y)-plane. After one iteration the points will move to a new position in the plane, but they now occupy an elongated region like an ellipse. The circle has contracted in one direction and expanded in the other. With each iteration the ellipse gets longer and narrower. The orientation of the ellipse also changes with each iteration and it wraps up like a ball of taffy. The quantitative interpretation of these effects are given by the so called Lyapunov exponents. The name comes from the late 19th-century Russian mathematician Aleksandr M. Lyapunov. The quadratic iterated maps possess two Lyapunov exponents – a positive one, corresponding to the direction of expansion, and a negative one corresponding to the direction of contraction. The signature of chaos is that at least one of the Lyapunov exponents is positive.

Iterated cubic maps

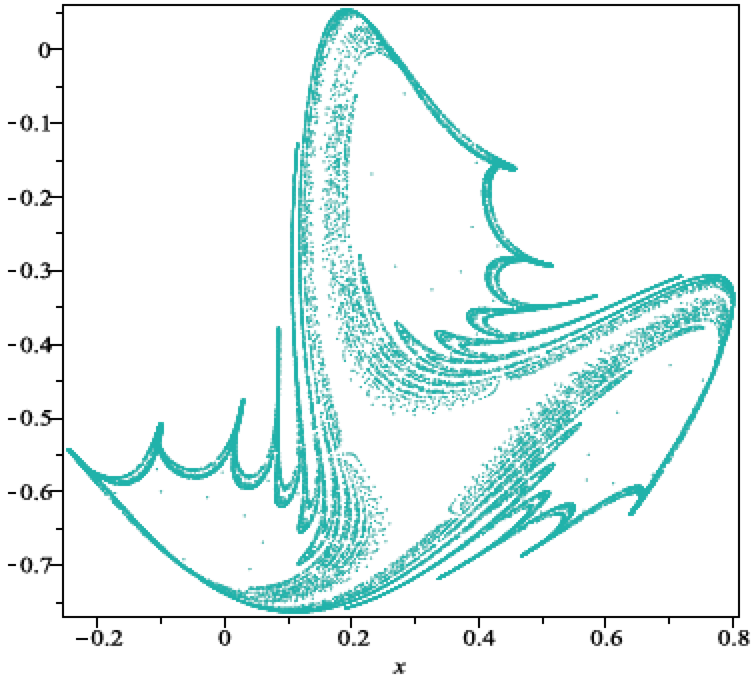
$$
\begin{aligned}
x_{n+1} &= a_1 + a_2 x_n + a_3 x_n^2 + a_4 x_n^4 + a_5 x_n^2 y_n + a_6 x_n y_n + a_7 x_n y_n^2 + a_8 y_n + a_9 y_n^2 + a_{10} y_n^3 \\
y_{n+1} &= b_1 + b_2 x_n + b_3 x_n^2 + b_4 x_n^4 + b_5 x_n^2 y_n + b_6 x_n y_n + b_7 x_n y_n^2 + b_8 y_n + b_9 y_n^2 + b_{10} y_n^3, \quad n = 0, 1, 2, \ldots
\end{aligned} \tag{5}
$$

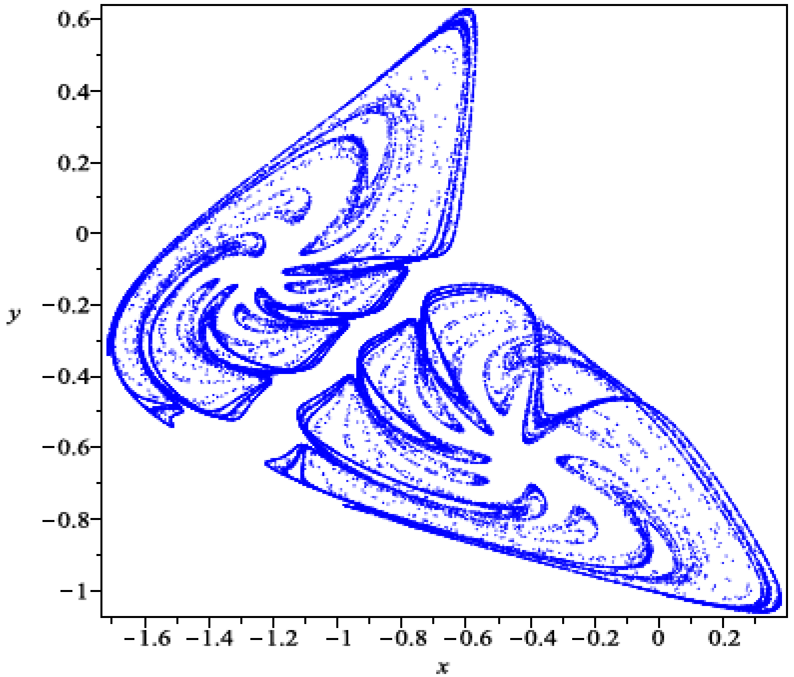
Denote by $a = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10})$ and $b = (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10})$ the vectors of coefficients in the cubic iterated system (5).

Table 2: Examples of cubic maps with chaotic attractors

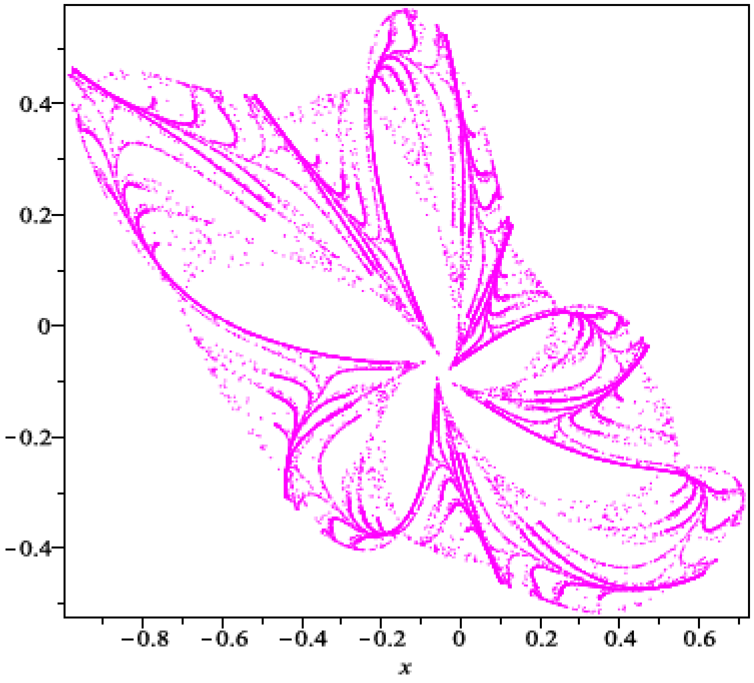
Example	Coefficient vectors a and b	FD	LE	Figure
1	$a = (-0.1, -0.6, 0.5, 0.2, -0.2, -0.3, -0.7, -0.8, -0.1, -0.9)$ $b = (-0.6, -0.2, 1.1, 0.6, 0.8, -0.8, -0.8, 1, 1.2, -0.8)$	1.25	0.11 -0.19	6 (left)
2	$a = (-0.4, 0.6, 0, -0.5, 0.4, -1, -0.5, 0.3, -0.9, -0.7)$ $b = (-0.2, -0.7, -1.1, -0.2, -0.8, -1.2, -0.1, -0.4, -0.7, -0.9)$	1.45	0.095 -0.16	6 (right)
3	$a = (0, -0.6, -0.6, 0.1, -0.9, 0.3, -0.5, 1, 0.2, 0.1)$ $b = (-0.2, -0.7, 0.4, 0.8, -0.4, -0.4, -0.5, -1.1, 0.9, 0.3)$	1.29	0.049 -0.11	7 (left)
4	$a = (0.2, 0.9, -0.7, -0.2, 1, -0.2, -0.8, -0.4, -1.1, 0.3)$ $b = (-0.6, 0.1, 1.2, 0.3, 0.9,$	1.57	0.078 -0.032	7 (right)

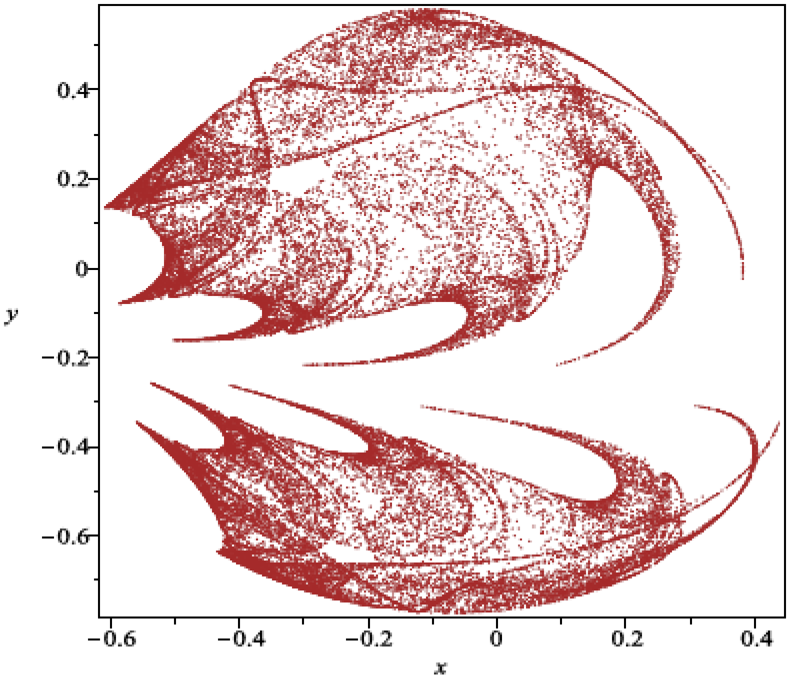
Chaotic attractors of cubic maps





Chaotic attractors of cubic maps





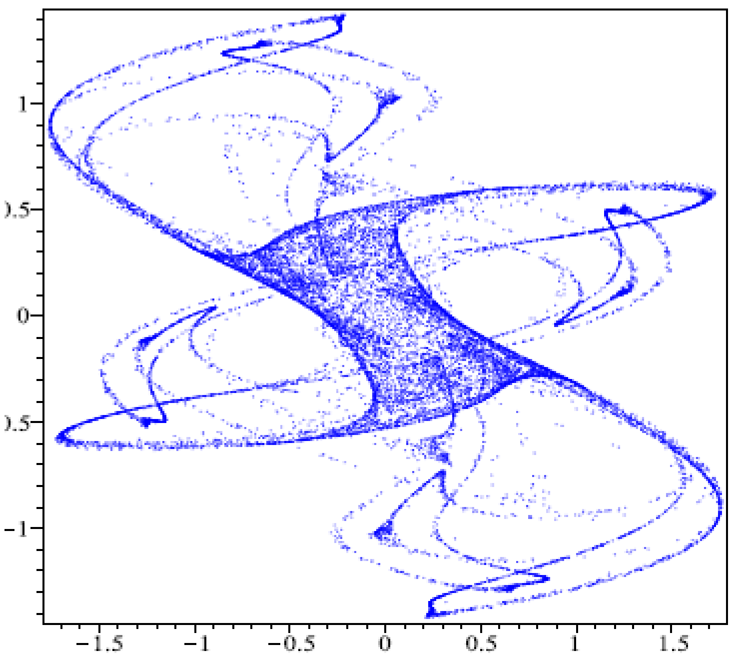
Other fascinating iterated maps

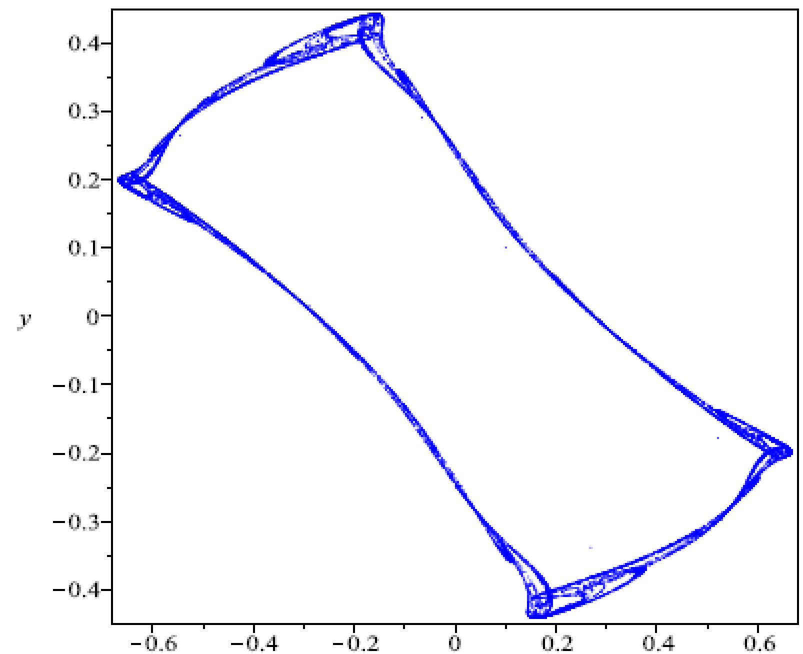
The King's dream. The King's Dream is a simple, yet beautiful fractal. The formulae to produce it were developed by Clifford Pickover, and published in his book "Chaos in Wonderland". The iterated equations for this fractal are:

$$x_{n+1} = \sin(by_n) + c\sin(bx_n)$$
$$y_{n+1} = \sin(ax_n) + d\sin(ay_n), \quad n = 0, 1, 2, \ldots$$

where

$$a = -0.966918, \; b = 2.879879, \; c = 0.765145, \; d = 0.7447228.$$

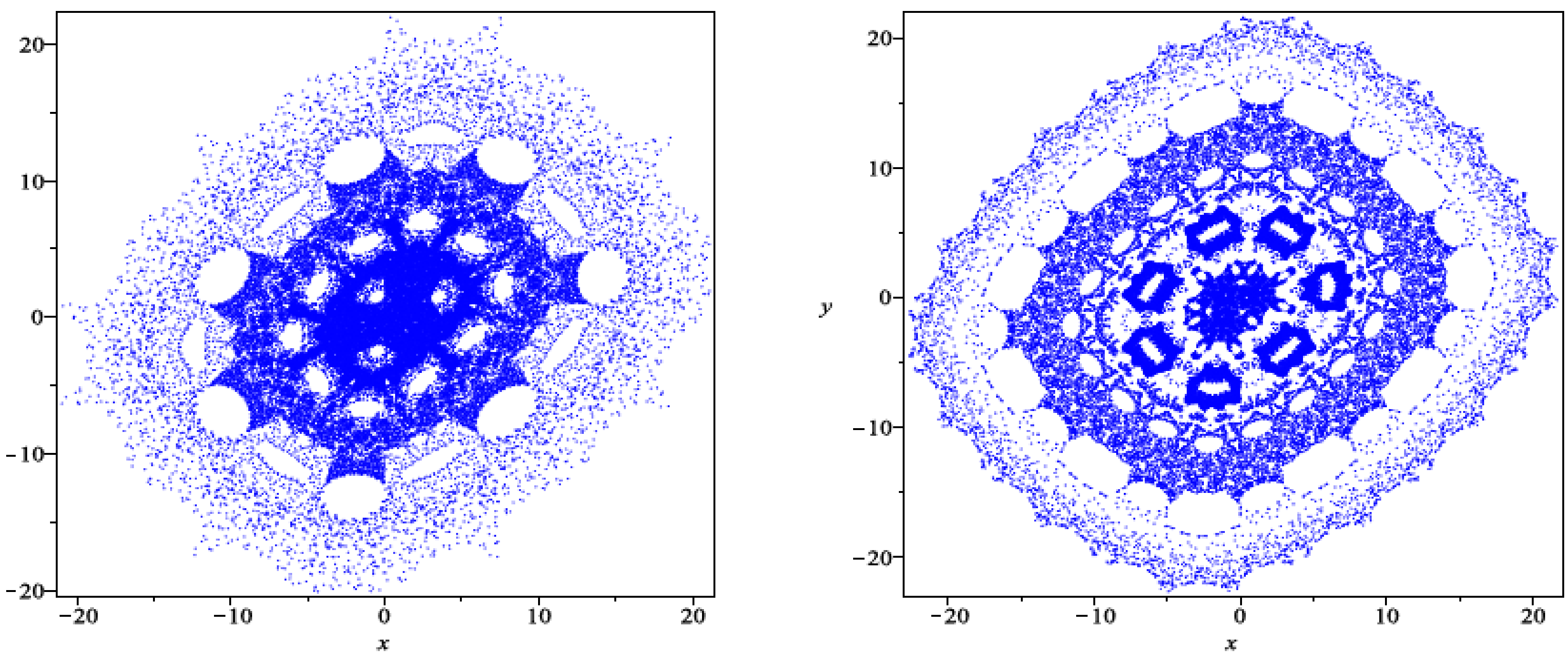




Barry Martin Fractal

$$x_{n+1} = y_n - \mathrm{sign}(x_n)\sqrt{|bx_n - c|}$$
$$y_{n+1} = a - x_n, \quad n = 0, 1, 2, \ldots$$

Three special functions are used in the first equation. The first is the square root. Most computer languages reference this as SQRT. Next comes the absolute value function, $|\ |$. Lastly is the signum function *sign*. The *sign* function returns a value of 1 if the x_n value is positive, and to return a value of -1 if the value is negative. The constants a, b and c may take any values. The fractal, presented on Figure 9 (left) is computed for $a = 1$, $b = 2$, $c = 3$; the right plot is obtained for $a = -1$, $b = 1$, $c = -1$.



Appendix: Maple commands

- Maple commands for computing and visualizing the attractor of (at least one of) the examples in Tables 1 and 2;
- Maple commands for computing the fractal dimension of (at least one of) the examples in Tables 1 and 2;
- Maple commands for computing the Lyapunov exponents of (at least one of) the examples in Tables 1 and 2.