Models for real subspace arrangements and stratified manifolds

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October 12, 2004

Abstract

Let us consider a central subspace and half-space arrangement $\mathcal{A}$ in an euclidean vector space V and let $\mathcal{M}(\mathcal{A})$ be its complement. We construct some compactifications for the C^∞ manifold $\mathcal{M}(\mathcal{A})/\mathbb{R}^+$. They turn out to be C^∞ manifolds with corners whose boundary is determined by simple combinatorial data. This generalizes a construction described by Kontsevich in his paper on deformation quantization of Poisson manifolds (see [7]).

Then we extend the construction to more general objects, i.e. stratified real manifolds whose stratification locally "looks like" the one induced by an arrangement of linear (half-)spaces. This is a real version of MacPherson and Procesi paper [10]; the models we obtain are again C^∞ manifolds with corners equipped with a nice combinatorial description of the boundary.

1 Introduction

Let us consider a central subspace arrangement $\mathcal{A}$ in an euclidean vector space V of dimension n and let $\mathcal{M}(\mathcal{A})$ be its complement. The first goal of this paper is to construct some compactifications for the C^∞ manifold $\mathcal{M}(\mathcal{A})/\mathbb{R}^+$. These will turn out to be C^∞ manifolds with corners and we will give a detailed description of their boundary.

The combinatorics involved in this description is due to the work [1] of De Concini and Procesi, in which models for complex subspace arrangements are constructed from the point of view of algebraic geometry.

As a first generalization we can consider a mixed subspace and half-space arrangement: what we obtain is again a C^∞ manifold with corners. In the particular case of the mixed complex and real configuration spaces considered by Kontsevich in [7], this provides a different construction of the associated compactifications.

The main ingredient in the definition of our compactifications is the embedding of the complement of the arrangement into a product of spheres. Then we take the closure of this embedding.

Another way to describe the same construction is via an explicit sequence of real blow-ups. This different strategy allows us to generalize our point of view of subspace arrangements and to focus on real stratified manifolds X whose stratification locally "looks like" the stratification induced by a system of subspaces and half-spaces (in this case we say that the stratification is "conical" or that X is "conically stratified", see Section 7).

For "real blow-up" in this setting we mean the "balls, beams and plates" construction which in [8] is described in the more general context of cone-like Whitney stratifications. Its essential step is the following: let R be a minimal stratum in a conically stratified riemannian manifold X (possibly with corners); then we replace each q in the closure $\overline{R}$ by the set of rays in T_qX which are normal to the tangent cone $T_q\overline{R}$ (more details in Section 9, of course riemannian assumption is not necessary).

In general we can associate to a conically stratified manifold X many distinct sets ("building sets") of combinatorial data (see Section 8). Given a building set $\mathcal{G}$, we will show how a series of real blow-ups construct a model $\tilde{X}_{\mathcal{G}}$ of X. Here we are providing a real version of MacPherson and Procesi paper [10]: we mean that $\tilde{X}_{\mathcal{G}}$ is a C^∞ manifold with corners such that

1. except for the open dense stratum, all the strata of $\tilde{X}_{\mathcal{G}}$ lie in the boundary;
2. the codimension 1 strata are in a natural bijective correspondence, via the blow-up map, with the elements of $\mathcal{G}$;
3. combinatorial data encoded by $\mathcal{G}$ allow us to control intersections of closures of strata.

The interest of $\tilde{X}_{\mathcal{G}}$ essentially lies in property 3 above: we can fully predict the combinatorics of the boundary (this can be useful for instance when one applies Stokes' theorem).

Among these models we find all the compactifications of subspace arrangements we mentioned above, as well as models for any real configuration spaces and models of spaces of matrices.

The content of this paper is divided into two main parts. The first one includes sections from 2 to 6, which are devoted to the compactifications of complements of arrangements: after recalling some combinatorial notions from [1] in Section 2 (nested sets and building sets), the compactifications are defined in Section 3, and the essential point is to prove that they are C^∞ manifold with corners (which is done in Section 4) and that their boundary can be described in terms of the combinatorics of nested sets (Section 5). In Section 6 we generalize the construction to half-spaces and, as an application, we reobtain Kontsevich's manifolds.

The second part is made by sections from 7 to 11 and deals with models of stratified manifolds. Even if these models include the compactifications we described above, we discuss the two cases separately, since in the linear case we can give particularly concrete proofs: everything follows from the explicit construction of an atlas of charts which cover the compactifications (see Section

4). When we deal with the general case our proofs are essentially based on the local behaviour of real blow-ups. The main definitions concerning stratified manifolds are provided by Sections 7 and 8, while in Section 9 we focus on the blow-up of a minimal stratum, which is the basic step in our construction. Section 10 deals with series of blow-ups, determining when a series of blow-ups construct a model and when different series give, up to diffeomorphism, the same result (see Theorem 10.2).

In Section 11 we focus on the combinatorial structure of the boundary of a model, in terms of the (transversal) intersections of the closures of codimension 1 strata (see Theorem 11.1). Here nested sets play a crucial role as well as building sets (we are referring to a natural generalization of objects defined in Section 2), and this actually singles out a combinatorial property of the compactifications of subspace arrangements that extends to models of stratified manifolds: from a purely combinatorial point of view, they (as well as models in [1] and [10]) are examples of the "resolutions" of meet-semilattices studied in [3].

Acknowledgments. The author wishes to express his gratitude to Corrado De Concini for suggesting the idea of the construction of real models and for many stimulating conversations.

2 Some combinatorics

Let us recall some definitions from [1]. According to the notation used in the Introduction, let $\mathcal{A}$ be a central subspace arrangement. We denote by $\mathcal{A}^\perp$ the arrangement formed by the subspaces orthogonal to the subspaces of $\mathcal{A}$:

$$\mathcal{A}^\perp = \{B^\perp \mid B \in \mathcal{A}\}$$

Then we denote by $\mathcal{C}_\mathcal{A}^\perp$ the closure, under the sum, of $\mathcal{A}^\perp$, that is to say, the set of subspaces in V which are sums of subspaces in $\mathcal{A}^\perp$.

Definition 2.1 *Given a subspace $U \in \mathcal{C}_\mathcal{A}^\perp$, a decomposition of U is a collection of non zero subspaces $U_1, U_2, \ldots, U_k \in \mathcal{C}_\mathcal{A}^\perp$ $(k > 1)$ which satisfy the following properties:*

1. *$U = U_1 \oplus U_2 \oplus \ldots \oplus U_k$*
2. *for every subspace $A \subset U$ in $\mathcal{C}_\mathcal{A}^\perp$, we have that $A \cap U_1,\ A \cap U_2, \ldots, A \cap U_k$ lie in $\mathcal{C}_\mathcal{A}^\perp$ and $A = (A \cap U_1) \oplus (A \cap U_2) \oplus \ldots \oplus (A \cap U_k)$*

Definition 2.2 *If a subspace in $\mathcal{C}_\mathcal{A}^\perp$ does not admit a decomposition, it is called "irreducible". The set of all irreducible subspaces is denoted by $\mathcal{F}_\mathcal{A}^\perp$.*

The following proposition essentially says that irreducible subspaces give a decomposition which has the expected "good" properties:

Proposition 2.1 *Every subspace $U \in \mathcal{C}_{\mathcal{A}}^{\perp}$ has a unique decomposition $U = \oplus_{i=1}^{k} U_i$ into irreducible subspaces. This is called "the irreducible decomposition" of U. If $A \subset U$ is irreducible, then $A \subset U_i$ for exactly one i.*

In the sequel building sets of subspaces will play a crucial role.

Definition 2.3 *The collection of subspaces $\mathcal{A}^{\perp} \subset V$ is called "building set" if every element C of $\mathcal{C}_{\mathcal{A}}^{\perp}$ is the direct sum $C = G_1 \oplus G_2 \oplus \ldots \oplus G_k$ of the set of the maximal elements $G_1, G_2, \ldots, G_k$ of $\mathcal{A}^{\perp}$ contained in C. We say in this case that $\{G_1, \ldots, G_k\}$ is "the building decomposition of C in $\mathcal{A}^{\perp}$".*

Remarks.

1) One can easily see that the "building decomposition of C in $\mathcal{A}^{\perp}$" is a decomposition in the previous sense.

2) The sets $\mathcal{C}_{\mathcal{A}}^{\perp}$ and $\mathcal{F}_{\mathcal{A}}^{\perp}$ defined above are building sets. Furthermore, for every building set $\mathcal{A}^{\perp}$, we have $\mathcal{F}_{\mathcal{A}}^{\perp} \subset \mathcal{A}^{\perp} \subset \mathcal{C}_{\mathcal{A}}^{\perp}$. Let in fact $A^{\perp} \in \mathcal{C}_{\mathcal{A}}^{\perp}$ be irreducible. Now $A^{\perp}$ can be decomposed in $\mathcal{A}^{\perp}$, but then $A^{\perp} \in \mathcal{A}^{\perp}$ since $A^{\perp}$ is irreducible. This proves the first inclusion, the second being trivial.

Let now $\mathcal{B}^{\perp}$ be a building set such that $\mathcal{C}_{\mathcal{B}}^{\perp} = \mathcal{C}_{\mathcal{A}}^{\perp}$. This implies that

$$\mathcal{F}_{\mathcal{A}}^{\perp} = \mathcal{F}_{\mathcal{B}}^{\perp} \subset \mathcal{B} \subset \mathcal{C}_{\mathcal{B}}^{\perp} = \mathcal{C}_{\mathcal{A}}^{\perp}$$

Therefore in the family of building sets that have the same closure under the sum we can always find a minimum and a maximum element with respect to inclusion.

We can now introduce the notion of "nested set" (see [1]) by means of the following definitions. This notion generalizes the one introduced by Fulton and MacPherson in their paper [4] on models of configuration spaces.

Definition 2.4 *A set $\mathcal{S}$ of subspaces in V is called nested if, given any of its subset $\{U_1, \ldots, U_k\}$, $k \geq 2$, of pairwise non comparable elements, one has $U = U_1 \oplus \cdots \oplus U_k$ and $U \notin \mathcal{S}$.*

Definition 2.5 *Let $\mathcal{K}$ be a building set of subspaces in V. A subset $\mathcal{S} \subset \mathcal{K}$ is called "nested relative to $\mathcal{K}$", or $\mathcal{K}$-nested, if*

1. *$\mathcal{S}$ is nested*
2. *given a subset $\{U_1, \ldots, U_h\}$ of pairwise non comparable elements in $\mathcal{S}$, then $C = U_1 \oplus \cdots \oplus U_h$ is the decomposition of C in $\mathcal{K}$.*

3 Construction of the compactification

Let $\mathcal{A}$ be a subspace arrangement. We are ready to define a compactification of the C^{∞} manifold $\widehat{\mathcal{M}}(\mathcal{A}) = \mathcal{M}(\mathcal{A})/\mathbb{R}^{+}$. Let us denote by $S(V)$ the $n-1$-th dimensional unit sphere in V, and, for every subspace $A \subset V$, let $S(A) =$

$A \cap S(V)$. Then we can consider the compact manifold

$$K = S(V) \times \prod_{A \in \mathcal{A}} S(A^{\perp})$$

and notice that there is an open embedding

$$\phi : \widehat{\mathcal{M}}(\mathcal{A}) \longrightarrow K$$

This is obtained as a composition of the section $s : \widehat{\mathcal{M}}(\mathcal{A}) \mapsto \mathcal{M}(\mathcal{A})$ provided by

$$s([p]) = \frac{p}{|p|} \in S(V) \cap \mathcal{M}(\mathcal{A})$$

with the map

$$\mathcal{M}(\mathcal{A}) \mapsto S(V) \times \prod_{A \in \mathcal{A}} S(A^{\perp})$$

where on each factor we have a well defined projection.

Definition 3.1 *We denote by $Y_{\mathcal{A}}$ the closure in K of $\phi(\widehat{\mathcal{M}}(\mathcal{A}))$.*

In the next section we will prove that, when $\mathcal{A}^{\perp}$ is a building set, $Y_{\mathcal{A}}$ is a smooth manifold with corners. We recall that an n-th dimensional manifold with corners is a manifold that can be covered by a C^{∞} atlas of charts which are diffeomorphic to open subsets of $(\mathbb{R}^{\geq 0})^n$.

Remark.

We notice that, given a subspace arrangement $\mathcal{B}$ there are in general several building sets $\mathcal{A}^{\perp}$ such that $\mathcal{C}_{\mathcal{B}}^{\perp} = \mathcal{C}_{\mathcal{A}}^{\perp}$, which also implies that $\widehat{\mathcal{M}}(\mathcal{A}) = \widehat{\mathcal{M}}(\mathcal{B})$. Therefore we will have several compactifications $Y_{\mathcal{A}}$ of $\widehat{\mathcal{M}}(\mathcal{B})$, with obvious surjective projections $Y_{\mathcal{A}_1} \mapsto Y_{\mathcal{A}_2}$ if $\mathcal{A}_2 \subset \mathcal{A}_1$. Remark 2 of the preceding section assures us that all these manifolds are projections of a maximum manifold ($Y_{\mathcal{C}_{\mathcal{B}}^{\perp}}$) and project onto a minimum manifold ($Y_{\mathcal{F}_{\mathcal{B}}^{\perp}}$).

4 The open charts

Let $\mathcal{A}^{\perp}$ be building; notice that also $\mathcal{A}_0^{\perp} = \{B^{\perp} \mid B \in \mathcal{A}\} \cup \{0^{\perp}\}$ is building. We are going to construct an open covering of $\phi(\widehat{\mathcal{M}}(\mathcal{A}))$ by some charts which are associated to the $\mathcal{A}_0^{\perp}$-nested sets of $\mathcal{A}_0^{\perp}$ which contain $\{0^{\perp}\} = V$.

Remark.

From now on “nested set” will mean “$\mathcal{A}_0^{\perp}$-nested set which contains V”.

Definition 4.1 *Given a subspace $C \subset V$, we define the following two (possibly empty) subspace arrangements.*

1. $\mathcal{A}_C = \{H \in \mathcal{A} \mid C \subset H\}$

2. $\mathcal{A}^C = \{B \cap C \mid B \in \mathcal{A} - \mathcal{A}_C\}$

Furthermore, given two subspaces $H, C \subset V$, we will denote by $\mathcal{A}_H^C$ the subspace arrangement $\mathcal{A}_H^C = \{B \cap C \mid B \in \mathcal{A}_H - (\mathcal{A}_C \cap \mathcal{A}_H)\}$.

Let us now give a graduation to the elements of a nested set $\mathcal{S}$ in $\mathcal{A}_0^{\perp}$. Recall that a nested set can be represented by a graph, which is an oriented tree, in the following way. The vertices of the tree are labeled by the elements of $\mathcal{S}$, and the root is V; let then $A^{\perp}$ and $B^{\perp}$ be two elements of $\mathcal{S}$ such that $A^{\perp}$ is maximal (with respect to inclusion) among the elements of $\mathcal{S}$ strictly included in $B^{\perp}$: then we draw an edge which joins the vertices $A^{\perp}$ and $B^{\perp}$ and is oriented from $B^{\perp}$ to $A^{\perp}$. We say that an element $X^{\perp}$ of $\mathcal{S}$ has degree n if it is connected to the root by a n-edges oriented path.

Given a vertex $A^{\perp}$ of degree n in the graph associated to $\mathcal{S}$, we denote by $\mathcal{S}^{A^{\perp}}$ the (possibly empty) set of the elements of degree $n+1$ which are connected by an edge with $A^{\perp}$. Furthermore, we denote by $\mathcal{S}_{\cap}^{A^{\perp}}$ the common intersection of $A^{\perp}$ and of all the subspaces which are orthogonal to the subspaces in $\mathcal{S}^{A^{\perp}}$ (we put $\mathcal{S}_{\cap}^{A^{\perp}} = A^{\perp}$ if $\mathcal{S}^{A^{\perp}}$ is empty).

We can now associate to (the graph of) $\mathcal{S}$ an open set $\widehat{U}_{\mathcal{S}}$. It is constructed as a product of open sets, according to the following algorithm. We provide an open manifold in correspondence with every element in $\mathcal{S}$. The open manifold which corresponds to the root V is :

$$N_V = \mathcal{M}_{\mathcal{S}_{\cap}^V}(\mathcal{A}^{\mathcal{S}_{\cap}^V}) \cap S(V).$$

Here and from now on we use the following notation: if $\mathcal{A}$ is a subspace arrangement whose elements are contained in a subspace F of V, $\mathcal{M}_F(\mathcal{A})$ will denote the complement of $\mathcal{A}$ in F. Notice that, if $\mathcal{S} = \{V\}$, we have $N_V = \mathcal{M}(\mathcal{A}) \cap S(V)$. Now, given an element $A^{\perp} \in \mathcal{S}$ we construct

$$N_{A^{\perp}} = \mathcal{M}_{\mathcal{S}_{\cap}^{A^{\perp}}}(\mathcal{A}_A^{\mathcal{S}_{\cap}^{A^{\perp}}}) \cap S(V)$$

Then, for any $A^{\perp}$ in $\mathcal{S}$, we consider a "small" positive real number $\varepsilon_{A^{\perp}}$, and we can define $\widehat{U}_{\mathcal{S}}$ as

$$\widehat{U}_{\mathcal{S}} = N_V \times \prod_{A^{\perp} \in \mathcal{S} - \{V\}} N_{A^{\perp}} \times (0, \varepsilon_{A^{\perp}})$$

Choosing in every space N_V or $N_{A^{\perp}}$ a ball $\rho(N_V)$ or $\rho(N_{A^{\perp}})$ we obtain an open subset $\widehat{U}_{\mathcal{S}}(\rho)$ of $\widehat{U}_{\mathcal{S}}$. We can embed $\widehat{U}_{\mathcal{S}}(\rho)$ in $\widehat{\mathcal{M}}(\mathcal{A}) \subset K$ as a chart using the following map $\widehat{\tau}$:

$$(p_V, \ldots, p_{A^{\perp}}, t_{A^{\perp}}, \ldots) \overset{\widehat{\tau}}{\mapsto} [p_V + \text{the point in } (\mathcal{S}_{\cap}^V)^{\perp} \text{ such that}, \forall\, A^{\perp} \in \mathcal{S} - \{V\},$$

$$\text{its orthogonal projection to } \mathcal{S}_{\cap}^{A^{\perp}} \text{ is } t_{T_1^{\perp}} t_{T_2^{\perp}} \cdots t_{A^{\perp}} p_{A^{\perp}}]$$

where $T_1^\perp, T_2^\perp \ldots$ are all the internal vertices in the path which connects V to $A^\perp$.

The map $\widehat{\tau}$ is a well defined embedding provided that the balls $\rho(N_V)$, $\rho(N_{A^\perp})$ and the numbers $\varepsilon_{A^\perp}$ are sufficiently small. Therefore we have an open atlas $\widehat{\mathcal{U}} = \bigcup_{\mathcal{S}} \widehat{U}_{\mathcal{S}}(\rho)$ which covers $\widehat{\mathcal{M}}(\mathcal{A})$ (we remark that $\mathcal{S}$ ranges over all the nested sets in $\mathcal{A}_0^\perp$ which contain V and ρ over all possible suitable collections of balls $\rho(N_V)$, $\rho(N_{A^\perp})$).

If we allow the real numbers $\varepsilon_{A^\perp}$ to be 0, we have the corresponding new space

$$U_{\mathcal{S}}(\rho) = \rho(N_V) \times \prod_{A^\perp \in \mathcal{S} - \{V\}} \rho(N_{A^\perp}) \times [0, \varepsilon_{A^\perp})$$

which is diffeomorphic to an open set of a simplicial cone $\mathbb{R}^d_{\geq 0}$.

Remark. In the sequel we will often write $\widehat{U}_{\mathcal{S}}$ and $U_{\mathcal{S}}$ instead of $\widehat{U}_{\mathcal{S}}(\rho)$ and $U_{\mathcal{S}}(\rho)$, the choice of a collection of balls $\rho(N_V)$, $\rho(N_{A^\perp})$ being implicit.

Proposition 4.1 *The open embedding $\widehat{\tau} : \widehat{U}_{\mathcal{S}} \mapsto \widehat{\mathcal{M}}(\mathcal{A})$ can be extended by continuity to an injective map $\tau : U_{\mathcal{S}} \mapsto K$ such that the boundary of $U_{\mathcal{S}}$ maps into $K - \widehat{\mathcal{M}}(\mathcal{A})$.*

Proof.

The embedding can be obviously extended by continuity to $U_{\mathcal{S}}$. Let us check the injectivity. Suppose that

$$\tau((p_V, \ldots, p_{A^\perp}, t_{A^\perp}, \ldots)) = \tau((q_V, \ldots, q_{A^\perp}, r_{A^\perp}, \ldots)).$$

Let us consider a branch in the graph of $\mathcal{S}$, and rename its vertices denoting each of them by its degree: $V = 0, \ldots, s$. Let j be the first index such that $t_j = 0$ (here we allow j to be equal to $s+1$, which means that all the t_j's are different from 0). Since the p_i's (the $q_i's$) are a set of orthonormal vectors, the projection of $\tau((p_V, \ldots, p_{A^\perp}, t_{A^\perp}, \ldots))$ to the first factor of K, i.e. $S(V)$, reveals that we have $r_j = 0$ and, for $i < j$, $p_i = q_i$ and $t_i = r_i$. Notice that this also proves the assertion concerning the boundary. Let now j_1 be the second index such that $t_{j_1} = 0$. The projection to $S(j)$ reveals that $r_{j_1} = 0$ and that, for $i < j_1$, $p_i = q_i$ and $t_i = r_i$. Applying the same reasoning to every branch of the graph of $\mathcal{S}$ we prove the injectivity of τ. ∎

Identifying $U_{\mathcal{S}}$ with its image in $\widehat{\mathcal{M}}(\mathcal{A})$ via τ, we have that $\widehat{\mathcal{M}}(\mathcal{A}) \subset \mathcal{U} = \bigcup_{\mathcal{S}} U_{\mathcal{S}} \subset Y_{\mathcal{A}}$.

Theorem 4.2 *We have that $\mathcal{U} = \bigcup_{\mathcal{S}} U_{\mathcal{S}} = Y_{\mathcal{A}}$ and this is a C^∞ atlas which gives $Y_{\mathcal{A}}$ the structure of a C^∞ manifold with corners.*

Proof.

Let us view a point p in the boundary of $\widehat{\mathcal{M}}(\mathcal{A}) \subset K$ as the limit of a path $\delta = \delta(t) : [0, 1) \mapsto \widehat{\mathcal{M}}(\mathcal{A})$. We will construct (the graph of) a nested set $\mathcal{S}$ such that $U_{\mathcal{S}}$ contains p. If we look at this path in $S(V)$, we can choose the minimal subspace B in the intersection lattice of $\mathcal{A}$ such that δ converges to a point in B. Let $B^\perp = B_1^\perp \oplus \cdots \oplus B_\varrho^\perp$ be the direct sum of $B^\perp$ in terms of the maximal elements of $\mathcal{A}^\perp$ which are included in $B^\perp$ (this is possible since $\mathcal{A}_0^\perp$ is building). Then $B = B_1 \cap \cdots \cap B_\varrho$ and $B_1^\perp, \ldots, B_\varrho^\perp$ are the elements of degree 1 of $\mathcal{S}$. Now, for every $1 \leq i \leq \varrho$ let us consider the projection $\delta_{B_i^\perp}$ of δ to $S(B_i^\perp)$ (it is well defined since $\delta \subset \widehat{\mathcal{M}}(\mathcal{A})$). Let v_i be the limit of the vector $\delta_{B_i^\perp}(t)$ as $t \to 1$. If v_i does not lie in any subspace of the intersection lattice of $\mathcal{A}_{B_i}$, we will not add any outgoing edge from $B_i^\perp$ to the graph of $\mathcal{S}$. Otherwise, let C_i be the minimal subspace in the intersection lattice of $\mathcal{A}_{B_i}$ such that v_i belongs to C_i. Then we can decompose $C_i^\perp$ as $C_i^\perp = C_{i1}^\perp \oplus \cdots \oplus C_{i\mu_i}^\perp$ (notice that $\mathcal{A}_{B_i}^\perp$ is building), and $\{B_1^\perp, \ldots, B_\varrho^\perp, C_{i1}^\perp, \ldots, C_{i\mu_i}^\perp\}$ is a nested set. Therefore our second step in the construction of $\mathcal{S}$ is to put $\mathcal{S} = \{B_1^\perp, \ldots, B_\varrho^\perp, \ldots, C_{i1}^\perp, \ldots, C_{i\mu_i}^\perp, \ldots\}$. We can now project $\delta_{B_i^\perp}$ to $C_{ij}^\perp$ for every i and j and continue. In this way we have proved that $p \in U_\mathcal{S}(\rho)$ with $\mathcal{S}$ as above and obvious ρ; therefore

$$\mathcal{U} = \bigcup_{\mathcal{S},\rho} U_\mathcal{S}(\rho) = Y_\mathcal{A}.$$

It remains to show that the transition maps are C^∞. This is immediate in the case of two charts $U_\mathcal{S}(\rho)$ and $U_\mathcal{S}(\rho')$. When two different nested sets $\mathcal{S}$ and $\mathcal{T}$ are involved, it suffices to restrict to the case when $\mathcal{S}$ and $\mathcal{T}$ differ only in their elements of top degree s. Namely we have that:

1. $\mathcal{S}$ and $\mathcal{T}$ have the same elements of degree $< s$
2. the elements of degree s of $\mathcal{T}$ are obtained from the elements of degree s of $\mathcal{S}$ by deleting one element
3. s is the top degree for $\mathcal{T}$

Let us denote by $J^\perp$ the element of degree s which is in $\mathcal{S}$ but not in $\mathcal{T}$, and by $K^\perp$ (resp. $L^\perp$) the element of degree $s-1$ (resp. $s-2$) which includes $J^\perp$. Furthermore, let $J_1^\perp, \ldots, J_l^\perp$ be the other elements of degree s which are included in $K^\perp$. We will consider the case when $l \geq 1$ (the other case is similar and easier). Let $p = (p_V, \ldots, p_{A^\perp}, t_{A^\perp}, \ldots)$ be a point in $U_\mathcal{S}$ and $q = (q_V, \ldots, q_{A^\perp}, r_{A^\perp}, \ldots)$ be a point in $U_\mathcal{T}$ and let us put $p = q \in U_\mathcal{S} \cap U_\mathcal{T}$.

The following relations are immediate:

$$p_{A^\perp} = q_{A^\perp}$$

for every $A^\perp \in (\mathcal{S} \cap \mathcal{T} - \{K^\perp\})$,

$$t_{A^\perp} = r_{A^\perp}$$

for every $A^\perp \in (\mathcal{S} \cap \mathcal{T} - \{K^\perp, J_1^\perp, \ldots, J_l^\perp\})$. Then we want to express $t_{K^\perp}, t_{J_i^\perp}, t_{J^\perp}, p_{K^\perp}, p_{J^\perp}$ in terms of $r_{K^\perp}, r_{J_i^\perp}, q_{K^\perp}, q_{J_i^\perp}$.

Definition 4.2 *Given a point* $(p_V, \ldots, p_{A^\perp}, t_{A^\perp}, \ldots) \in U_{\mathcal{S}}$ *and an element* $B^\perp \in \mathcal{S}$*, we denote by* $\phi_{\mathcal{S},B^\perp}$ *the function*

$$\phi_{\mathcal{S},B^\perp} \, : \, U_{\mathcal{S}} \mapsto B^\perp$$

which assigns to a point $p = (p_V, \ldots, p_{A^\perp}, t_{A^\perp}, \ldots) \in U_{\mathcal{S}}$ *the point* $\phi_{\mathcal{S},B^\perp}(p)$ *in* $(\mathcal{S}_\cap^{B^\perp})^\perp \cap B^\perp = \bigoplus_{C^\perp \in \mathcal{S}^{B^\perp}} C^\perp$ *which satisfies the following property: for every* $D^\perp \in \mathcal{S}$ *,* $D^\perp \subsetneq B^\perp$*, the orthogonal projection of* $\phi_{\mathcal{S},B^\perp}(p)$ *to* $\mathcal{S}_\cap^{D^\perp}$ *is* $t_{G_1^\perp} t_{G_2^\perp} \cdots t_{D^\perp} p_{D^\perp}$*, where the* $G_i^\perp$*'s are all the internal vertices in the path which connects* $B^\perp$ *to* $D^\perp$*.*

The projection of $p = q$ to $S(L^\perp)$ gives

$$\frac{p_{L^\perp} + \phi_{\mathcal{S},L^\perp}(p)}{|p_{L^\perp} + \phi_{\mathcal{S},L^\perp}(p)|} = \frac{q_{L^\perp} + \phi_{\mathcal{T},L^\perp}(q)}{|q_{L^\perp} + \phi_{\mathcal{T},L^\perp}(q)|} \tag{1}$$

Projecting to $\mathcal{S}_\cap^{L^\perp} = \mathcal{T}_\cap^{L^\perp}$ we have

$$\frac{p_{L^\perp}}{|p_{L^\perp} + \phi_{\mathcal{S},L^\perp}(p)|} = \frac{q_{L^\perp}}{|q_{L^\perp} + \phi_{\mathcal{T},L^\perp}(q)|}$$

Passing to the norms

$$|p_{L^\perp} + \phi_{\mathcal{S},L^\perp}(p)| = |q_{L^\perp} + \phi_{\mathcal{T},L^\perp}(q)|$$

which allows us to deduce from (1)

$$\phi_{\mathcal{S},L^\perp}(p) = \phi_{\mathcal{T},L^\perp}(q)$$

Projecting to $K^\perp$ we obtain

$$t_{K^\perp}(p_{K^\perp} + \phi_{\mathcal{S},K^\perp}(p)) = r_{K^\perp}(q_{K^\perp} + \phi_{\mathcal{T},K^\perp}(q)) \tag{2}$$

Let us denote by π_C the orthogonal projection onto a subspace C in V. We have that

$$r_{K^\perp} \pi_{\mathcal{S}_\cap^{K^\perp}}(q_{K^\perp}) = t_{K^\perp} p_{K^\perp}$$

In norm this implies

$$r_{K^\perp} |\pi_{\mathcal{S}_\cap^{K^\perp}}(q_{K^\perp})| = t_{K^\perp} \tag{3}$$

which is a C^∞ expression for $t_{K^\perp}$ in terms of $r_{K^\perp}$ and $q_{K^\perp}$. In fact the projection of $p = q$ to $S(K^\perp)$ shows that $\pi_{\mathcal{S}_\cap^{K^\perp}}(q_{K^\perp})$ is always different from 0 (being equal to a non zero multiple of $p_{K^\perp}$). Then the expression for $p_{K^\perp}$ is

$$p_{K^\perp} = \frac{\pi_{\mathcal{S}_\cap^{K^\perp}}(q_{K^\perp})}{|\pi_{\mathcal{S}_\cap^{K^\perp}}(q_{K^\perp})|}$$

Now equations (2) and (3) imply

$$t_{K^\perp} t_{J_i^\perp} = r_{K^\perp} r_{J_i^\perp}$$

from which we deduce

$$t_{J_i^\perp} = \frac{r_{J_i^\perp}}{|\pi_{\mathcal{S}_\cap^{K^\perp}}(q_{K^\perp})|}$$

Furthermore, from (2) and (3) we obtain

$$t_{J^\perp} = \frac{|\pi_{J^\perp}(\phi_{\mathcal{T},K^\perp}(q) + q_{K^\perp})|}{|\pi_{\mathcal{S}_\cap^{K^\perp}}(q_{K^\perp})|}$$

which is a C^∞ expression for $t_{J^\perp}$ in terms of $r_{J_i^\perp}, q_{K^\perp}, q_{J_i^\perp}$.

In fact $\pi_{J^\perp}(\phi_{\mathcal{T},K^\perp}(q)+q_{K^\perp}) \neq 0$ otherwise $\phi_{\mathcal{T},K^\perp}(q)+q_{K^\perp} \in J \cap K^\perp$ which contradicts the construction of $U_{\mathcal{T}}$.

We have thus shown that the transition map from $U_{\mathcal{T}}$ to $U_{\mathcal{S}}$ is C^∞. In the other direction we proceed in the same way and we obtain

$$r_{K^\perp} = t_{K^\perp}|p_{K^\perp} + \pi_{\mathcal{T}_\cap^{K^\perp}}(\phi_{\mathcal{S},K^\perp}(p))|$$

$$q_{K^\perp} = \frac{p_{K^\perp} + \pi_{\mathcal{T}_\cap^{K^\perp}}(\phi_{\mathcal{S},K^\perp}(p))}{|p_{K^\perp} + \pi_{\mathcal{T}_\cap^{K^\perp}}(\phi_{\mathcal{S},K^\perp}(p))|}$$

$$r_{J_i^\perp} = |p_{K^\perp} + \pi_{\mathcal{T}_\cap^{K^\perp}}(\phi_{\mathcal{S},K^\perp}(p))|t_{J_i^\perp}$$

Notice that $|p_{K^\perp} + \pi_{\mathcal{T}_\cap^{K^\perp}}(\phi_{\mathcal{S},K^\perp}(p))| \neq 0$ since $|\pi_{\mathcal{T}_\cap^{K^\perp}}(\phi_{\mathcal{S},K^\perp}(p))| << |p_{K^\perp}|$. ■

5 The boundary of the compactification

In this section we will focus on the boundary $\mathcal{D}$ of $\widehat{\mathcal{M}}(\mathcal{A})$ in $Y_{\mathcal{A}}$. We will show that $\mathcal{D}$ is the union $\mathcal{D} = \bigcup_{A \in \mathcal{A}} \mathcal{D}_A$ of manifolds with corners $\mathcal{D}_A$ of codimension 1.

Furthermore we will characterize all the non empty intersections of these $\mathcal{D}_A$'s and show that they are manifolds of the same type as $Y_{\mathcal{A}}$, i.e. compactifications of suitable real subspace arrangements.

Let us denote by

$$p : K \supset Y_{\mathcal{A}} \mapsto S(V)$$

the projection onto the first factor of K. Then we have $\mathcal{D} = p^{-1}(\bigcup_{A \in \mathcal{A}} A)$ and define

$$\mathcal{D}_A = p^{-1}\overline{(A - \bigcup_{B \in \mathcal{A}^A} B)}$$

Theorem 5.1 *We have that $\mathcal{D}_A$ is equal to the closure in K of*

$$\bigcup_{\rho} (U_{\{V,A^\perp\}}(\rho) \cap \{t_{A^\perp} = 0\}).$$

Moreover

$$\mathcal{D}_A \cong Y_{\mathcal{A}_A^{A^\perp}} \times Y_{\mathcal{A}^A}$$

Proof.

Let $\{y_n\}$ be a succession of points in $\widehat{\mathcal{M}}(\mathcal{A}) \subset K$ which converges to a point $q \in p^{-1}(A - \bigcup_{B \in \mathcal{A}^A} B)$. Notice that the points $\{y_n\}$ definitely belong to $\widehat{U}_{\mathcal{S}}(\rho)$ where $\mathcal{S} = \{V, A^\perp\}$ and ρ is opportunely chosen. If we read these points in $S(V)$, they can be written (up to normalization)

$$y_n = q_n + t_n a_n$$

where $q_n \in N_V \subset A$, $a_n \in N_{A^\perp} \subset A^\perp$, $t_n \in (0, \varepsilon_{A^\perp})$. Then it must be $t_n \to 0$ and therefore the successions $\{(q_n, a_n, t_n)\}$ and $\{(q_n, a_n, 0)\}$ in $U_{\{V, A^\perp(\rho)\}}$ have the same limit $q \in Y_{\mathcal{A}}$. This shows that q belongs to the closure of $\bigcup_\rho (U_{\{V,A^\perp\}}(\rho) \cap \{t_{A^\perp} = 0\})$. Thus

$$p^{-1}(A - \bigcup_{B \in \mathcal{A}^A} B) \subset \overline{\bigcup_\rho (U_{\{V,A^\perp\}}(\rho) \cap \{t_{A^\perp} = 0\})}$$

which implies

$$\overline{p^{-1}(A - \bigcup_{B \in \mathcal{A}^A} B)} \subset \overline{\bigcup_\rho (U_{\{V,A^\perp\}}(\rho) \cap \{t_{A^\perp} = 0\})}$$

Since the other inclusion is trivial, it remains to prove that

$$\overline{\bigcup_\rho (U_{\{V,A^\perp\}}(\rho) \cap \{t_{A^\perp} = 0\})} \cong Y_{\mathcal{A}^A} \times Y_{\mathcal{A}_A^{A^\perp}}$$

There is the obvious diffeomorphism

$$\theta \, : \, N_V \times N_{A^\perp} \xrightarrow{\cong} U_{\{V,A^\perp\}} \cap \{t_{A^\perp} = 0\}$$

Notice that N_V can be embedded in $K_V = S(A) \times \prod_{B \in \mathcal{A}^A} S(B^\perp)$ and $N_{A^\perp}$ in $K_{A^\perp} = S(A^\perp) \times \prod_{C \in \mathcal{A}_A^{A^\perp}} S(C^\perp)$; the closure of $N_V \times N_{A^\perp}$ in $K_V \times K_{A^\perp}$ is precisely $Y_{\mathcal{A}^A} \times Y_{\mathcal{A}_A^{A^\perp}}$ (notice that $(\mathcal{A}^A)^\perp$ and $(\mathcal{A}_A^{A^\perp})^\perp$ are building).

We can then construct a C^∞ imbedding i

$$i \, : \, K_V \times K_{A^\perp} \mapsto K$$

given by the product of the following imbeddings:

$$S(A) \mapsto S(V)$$

$$S(B^\perp) \xrightarrow{\pi_{\check{B}^\perp}} \check{B}^\perp \mapsto S(\check{B}^\perp) \text{ if } B \in \mathcal{A}^A,\ \check{B} \in \mathcal{A},\ \check{B} \cap A = B$$

$$S(C^\perp) \mapsto S(\check{C}^\perp) \text{ if } C \in \mathcal{A}_A^{A^\perp},\ \check{C} \in \mathcal{A}_A,\ \check{C} \cap A^\perp = C$$

$$S(A^\perp) \mapsto S(A^\perp)$$

We observe that i coincides with θ when restricted to $N_V \times N_{A^\perp}$. Now

$$i(Y_{\mathcal{A}^A} \times Y_{\mathcal{A}_A^{A^\perp}})$$

is closed since it is the image of a compact space. Therefore $i(Y_{\mathcal{A}^A} \times Y_{\mathcal{A}_A^{A^\perp}})$ contains $\overline{\bigcup_\rho (U_{\{V,A^\perp\}}(\rho) \cap \{t_{A^\perp} = 0\})}$. But $Y_{\mathcal{A}^A} \times Y_{\mathcal{A}_A^{A^\perp}}$ is the closure of $N_V \times N_{A^\perp}$ which is diffeomorphic to $\bigcup_\rho (U_{\{V,A^\perp\}}(\rho) \cap \{t_{A^\perp} = 0\})$: it follows that

$$i(Y_{\mathcal{A}^A} \times Y_{\mathcal{A}_A^{A^\perp}}) \subset \overline{\bigcup_\rho (U_{\{V,A^\perp\}}(\rho) \cap \{t_{A^\perp} = 0\})}$$

Since i restricted to $Y_{\mathcal{A}^A} \times Y_{\mathcal{A}_A^{A^\perp}}$ is a homeomorphism with its image, the claim follows. ■

Let us now focus on the intersections of the varieties $\mathcal{D}_A$ in the boundary. It suffices to study these intersections in the open charts. Let us consider the projection $p(q) \in S(V)$ of a point q which belongs to $U_\mathcal{S}$ ($\mathcal{S}$ nested in $\mathcal{A}_0^\perp$, $V \in \mathcal{S}$).

By construction of $U_\mathcal{S}$, $\overline{p^{-1}(C - \bigcup_{D \in \mathcal{A}^C} D)} \cap U_\mathcal{S} = \emptyset$ if $C^\perp$ does not belong to $\mathcal{S}$, while $\overline{p^{-1}(C - \bigcup_{D \in \mathcal{A}^C} D)} \cap U_\mathcal{S}$ can be expressed by the equation $t_{C^\perp} = 0$ in $U_\mathcal{S}$ if $C^\perp \in \mathcal{S}$.

We can then conclude that $\mathcal{D}_C \cap U_\mathcal{S} \neq \emptyset$ if and only if $C^\perp \in \mathcal{S}$. This proves the following:

Theorem 5.2 *Let $\mathcal{T}$ be a subset of $\mathcal{A}_0^\perp$ which includes V; then $\mathcal{D}_\mathcal{T} = \bigcap_{B^\perp \in \mathcal{T}} \mathcal{D}_B$ is non empty if and only if $\mathcal{T}$ is nested in $\mathcal{A}_0^\perp$.*

In this case a proof similar to the one of Theorem 5.1 gives:

Theorem 5.3 *Let $\mathcal{T}$ be a subset of $\mathcal{A}_0^\perp$ which includes V and is nested in $\mathcal{A}_0^\perp$. Then, for $A^\perp \in \mathcal{T}$ the sets $(\mathcal{A}_A^{\mathcal{T}_\cap^{A^\perp}})^\perp$ are building and*

$$\mathcal{D}_\mathcal{T} \cong \prod_{A^\perp \in \mathcal{T}} Y_{\mathcal{A}_A^{\mathcal{T}_\cap^{A^\perp}}}$$

6 Mixed subspace and half-space arrangements

6.1 Compactifications of mixed subspace and half-space arrangements

In this section we extend the definition of the compactifications to the case of a mixed subspace and half-space arrangement. We will obtain again a family of compact manifolds with corners which includes the case of the compactifications of configuration spaces introduced by Kontsevich in [7].

Definition 6.1 *Let us consider a subspace arrangement $\mathcal{A}$ in V and an hyperplane arrangement $\mathcal{H}$ in V with the property that every hyperplane L in $\mathcal{H}$ is equipped with an orientation (represented by the choice of a unitary vector v_L orthogonal to L). Then $\mathcal{AH} = \mathcal{A} \cup \mathcal{H}$ is a mixed subspace and half-space arrangement and we will denote its complement by*

$$\mathcal{M}(\mathcal{AH}) = V - \left(\bigcup_{A \in \mathcal{A}} A \cup \bigcup_{L \in \mathcal{H}} \{x \in V \mid (x, v_L) \leq 0\} \right)$$

Let us now denote by $Adm(\mathcal{AH})$ the collection of all the building sets $\mathcal{G}^\perp$ such that $\mathcal{C}_\mathcal{G}^\perp = \mathcal{C}_{\mathcal{A}\cup\mathcal{H}}^\perp$.

As we know, in $Adm(\mathcal{AH})$ there are a minimum and a maximum element with respect to inclusion, i.e. $\mathcal{F}_{\mathcal{A}\cup\mathcal{H}}^\perp$ and $\mathcal{C}_{\mathcal{A}\cup\mathcal{H}}^\perp$. It is important to notice that every element of $Adm(\mathcal{AH})$ includes $H^\perp$ for every $H \in \mathcal{H}$ (the orthogonal complement of a hyperplane is a line and the lines are irreducible).

Let us now consider a building set $\mathcal{G}^\perp$ in $Adm(\mathcal{AH})$. We can construct the compactification $Y_\mathcal{G}$ which by definition is included in a product of spheres; in particular among these spheres there are the copies of S^0 associated to the hyperplanes of $\mathcal{H}$. The two points of $S^0 = S(H^\perp)$ $(H \in \mathcal{H})$ represent in this case the two half-spaces determined by H. Let us denote by p_H the point representing the half-space $\{x \in V \mid (x, v_H) > 0\}$.

Definition 6.2 *We will denote by $Y_\mathcal{G}^\mathcal{H}$ the intersection of $Y_\mathcal{G}$ with the closed sets C_H $(H \in \mathcal{H}$ $)$ defined in this way: the projection of C_H onto $S(H^\perp)$ is p_H.*

The varieties $Y_\mathcal{G}^\mathcal{H}$ are C^∞ compact manifolds with corners, being union of connected components of compact manifolds with corners.

6.2 Kontsevich's construction

Let us recall a construction of Kontsevich's compactifications of configuration spaces which appear in [7]. Given two non negative integers n, m satisfying $2n + m \geq 2$ we can consider the quotient space

$$C_{n,m} = \{\, (p_1, \ldots, p_n; q_1, \ldots, q_m) \mid p_i \in \mathcal{H},\ q_j \in \mathbb{R},\ p_i \neq p_j \ \forall\, i \neq j,\ q_s \neq q_t \ \forall\, s \neq t \,\} / G_1$$

where G_1 is the real Lie group of holomorphic transformations which preserve the half-plane and the point ∞:

$$G_1 = \{az + b \mid a \in \mathbb{R},\, a > 0,\ b \in \mathbb{R}\}$$

and $\mathcal{H}$ is the Lobacevsky plane. Notice that $C_{n,m}$ is a C^∞ manifold of dimension $2n+m-2$. Analogously, given $n \geq 2$, we have the C^∞ manifold

$$C_n = \{\, (p_1, \ldots, p_n) \in \mathbb{C}^n \mid p_i \neq p_j \;\forall\, i \neq j \,\} / G_2$$

where G_2 is the real Lie group of dimension 3:

$$G_2 = \{az + b \mid a \in \mathbb{R},\, a > 0,\, b \in \mathbb{C}\}\,.$$

Let us now consider the map

$$\phi_{n,m} \;:\; C_{n,m} \mapsto \mathcal{L}_{n,m} = (S^1)^{n(n-1)+nm} \times (\mathbb{P}_{\mathbb{C}})^{\left(\begin{array}{c} n+m \\ 3 \end{array}\right)^3} \times (\mathbb{P}_{\mathbb{C}})^{n(n+m-1)}$$

defined by

$$\phi_{n,m}([(p_1, \ldots, p_n; q_1, \ldots, q_m)]) =$$

$$= \left(Arg(p_i - p_j),\, Arg(p_i - \overline{p}_j),\, Arg(p_r - q_k),\, \frac{\tau_s - \tau_l}{\tau_t - \tau_l},\, \frac{\gamma_s - p_l}{\gamma_s - \overline{p}_l} \right)$$

where in the formula $i > j$, $s > t$, τ_s, τ_l, τ_t are three distinct points among $\tau_1 = p_1, \ldots, \tau_n = p_n, \tau_{n+1} = q_1, \ldots, \tau_{n+m} = q_m$ and γ_s is a point among $p_1, \ldots, \widehat{p_l}, \ldots, p_n, q_1, \ldots, q_m$. Of course, in the cases when n, m are small, in the above definition we use only the coordinates which are well defined (for example, if $n \leq 2$ the quotients $\frac{p_s - p_l}{p_t - p_l}$ do not appear). In particular when $n = 1$, $m = 0$ the target space is a point. Analogously, we define the map

$$\phi_n \;:\; C_n \mapsto \mathcal{L}_n = (S^1)^{\left(\begin{array}{c} n \\ 2 \end{array}\right)} \times (\mathbb{P}_{\mathbb{C}})^{\left(\begin{array}{c} n \\ 3 \end{array}\right)^3}$$

$$\phi_n([(p_1, \ldots, p_n)]) = \left(Arg(p_r - p_k),\, \frac{p_k - p_i}{p_j - p_i} \right)$$

where $r > k$ and $k > j$.

The maps $\phi_{n,m}$ and ϕ_n turn out to be embeddings (see [5] for further details).

Remark.

The definition of $\phi_{n,m}$ and ϕ_n given in [7] is slightly different and doesn't assure injectivity, even if one deduces from the description of the compactifications that $\phi_{n,m}$ and ϕ_n actually are embeddings. The definitions provided above settle this problem. This difficulty is completely overcome by the new approach proposed in the next subsection.

Now one can obtain the compactifications:

Definition 6.3 *The space $\overline{C}_{n,m}$ (resp. $\overline{C}_n$) is the closure of the image of $\phi_{n,m}$ (resp. ϕ_n) in the target space.*

Following [7], we will show how to give to $\overline{C}_n$ and $\overline{C}_{n,m}$ the structure of smooth manifolds with corners.

Let us first describe a continuous section s^{cont} of the natural projection map

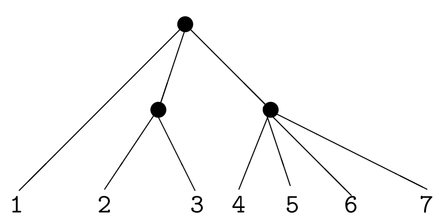
$$Conf_n = \{ (p_1, \dots, p_n) \in \mathbb{C}^n \mid p_i \neq p_j \ \forall\, i \neq j \} \mapsto C_n$$

Given a point $p = [(p_1, \dots, p_n)] \in C_n$ we put $s^{cont}(p) = (q_1, \dots, q_n)$, where $(q_1, \dots, q_n)$ is in the fiber of p and

1. the diameter of the set $\{q_1, \dots, q_n\}$ is equal to 1
2. the center of the minimal circle in $\mathbb{C}$ containing $\{q_1, \dots, q_n\}$ is 0.

We will say that $\{q_1, \dots, q_n\}$ is a configuration of points "in standard position". In every G_2-orbit in $Conf_n$ there is one and only one point which gives rise to a configuration in standard position.

Let us now introduce a family of open charts in C_n which are parametrized by a family of rooted oriented trees. The trees we are dealing with are all the rooted oriented trees with n leaves labeled with the numbers from 1 to n and such that the number of edges which stem from each vertex (which is not a leaf) is greater than or equal to two. For instance:



Let us denote by T such a tree and by $Star(v)$ the set of edges which start from a given vertex v. We can then parametrize an open set U_T in C_n in the following way:

1. for every vertex v of T (except leaves) we provide a configuration c_v of points in standard position labeled by the set $Star(v)$: if u is a vertex of T which is adjacent to v and follows v in the orientation, we denote by vu the corresponding edge in $Star(v)$ and we have the point p_{vu} in c_v;
2. for every vertex v except leaves and the root of the tree, there is a vertex w which precedes v in the orientation and such that $wu \in Star(w)$; then we provide the scale $s_v > 0$ with which we should put in the configuration c_w a copy of c_v centered at the point p_{wv} (which is deleted).

Then we have a continuous atlas $\mathcal{U} = \bigcup_T U_T$ which covers C_n. The compactification $\overline{C}_n$ is achieved by formally allowing some of the scales s_v to be equal to 0. Then $\overline{C}_n$ turns out to be a topological manifold with corners, with strata C_T labeled by the admissible trees T. According to the construction, C_T is isomorphic to the product $\prod_v C_{Star(v)}$, where v ranges over all the vertices of T except leaves.

In order to introduce a smooth structure on $\overline{C}_n$ it is now sufficient to choose, for every $m \leq n$, a smooth section s^{smooth} of the projection $Conf_m \mapsto C_m$ instead of s^{cont}. Then the coordinates near a point in a stratum C_T are given by the scales $s_v \in \mathbb{R}_{\geq 0}$ close to 0 and by the local coordinates in the manifolds $C_{Star(v)}$ (what we obtain turns out to be a compatible family of C^∞ open charts which cover $\overline{C}_n$). The case of the manifold $C_{n,m}$ can be treated in a similar way. In [7] the following appropriate new definition of "standard position" for the points belonging to a finite subset S of $\mathcal{H} \cup \mathbb{R}$ is given.

Definition 6.4 *Let S be as above. Then the elements of S are said to be in "standard position" if*

1. *the projection of the convex hull of S to the horizontal line $\mathbb{R}$ is either 0 or an interval with center 0,*
2. *the maximum of the diameter of S and of the distance from S to $\mathbb{R}$ is equal to 1.*

Notice that any configuration of n points in $\mathcal{H}$ and m points in $\mathbb{R}$ can be put uniquely in standard position using the group G_1. Then we can cover $\overline{C}_{n,m}$ using open charts which are constructed in a similar way as in the case of $\overline{C}_n$. Every chart is associated to a rooted oriented tree which has two different types of leaves (n "complex' leaves corresponding to points in $\mathcal{H}$ and m "real" leaves corresponding to points in $\mathbb{R}$). The number of edges that stem from every vertex of the tree is ≥ 2 and also internal vertices are of "real" or "complex" type: a "complex" vertex corresponds to a point in $\mathcal{H}$ (i.e. to a cluster of points belonging to $\mathcal{H}$ that converge to a point in $\mathcal{H}$), while a "real" vertex represents a point in the boundary $\mathbb{R}$ of $\mathcal{H}$ (i.e. a cluster of points, belonging to $\mathcal{H}$, or to $\mathbb{R}$, or some to $\mathcal{H}$, some to $\mathbb{R}$, that converge to a point in $\mathbb{R}$). As a consequence, the resulting strata are isomorphic to the product of manifolds of type C_j and $C_{r,s}$. Then $C_{n,m}$ is given the structure of $\mathbb{C}^\infty$ manifold in the same way explained for C_n.

6.3 Kontsevich's spaces viewed as models of arrangements

Now we want to show that Kontsevich's compactifications are special cases of our compactifications of subspace and half-space arrangements. Let us focus on the manifolds $C_{n,m}$ and $\overline{C}_{n,m}$ (the case of C_n and $\overline{C}_n$ is similar and easier since it does not involve half-spaces). We choose the subspace arrangement $\mathcal{A}_{n,m} \subset \mathbb{R}^{2n+m} = \{(x_1, y_1, \ldots, x_n, y_n, w_1, \ldots, w_m)\}$ made by the subspaces $\{0\}$, $H_{i_1,\ldots,i_r} : \{x_{i_1} = \cdots = x_{i_r}\} \cap \{y_{i_1} = \cdots = y_{i_r}\}$ $(1 \leq i_1 < \cdots < i_r \leq n$, $r \geq 2)$, $W_{j_1,\ldots,j_s} : \{w_{j_1} = \cdots = w_{j_s}\}$ $(1 \leq j_1 < \cdots < j_r \leq m$, $s \geq 2)$ and $H^0_{i_1,\ldots,i_r,j_1,\ldots,j_p} : \{x_{i_1} = \cdots = x_{i_r} = w_{j_1} = \cdots = w_{j_p}\} \cap \{y_{i_1} = \cdots = y_{i_r} = 0\}$ for $1 \leq i_1 < \cdots < i_r \leq n$, $1 \leq j_1 < \cdots < j_p \leq m$, $p + r \geq 2$. Notice that $H^0_{j_1,\ldots,j_p} =$

$W_{j_1,\ldots,j_p}$. Then we add the hyperplane arrangement $U = \{U_i : \{y_i = 0\}\}$ whose hyperplanes U_i are equipped with orthogonal vectors $(0,\ldots,0,1,0,\ldots,0)$ (1 in the $2i$-th position).

If we let the group G_1 to act on the complement of this configuration we obtain the space $C_{n,m}$ (here and in the sequel we "forget" that in the original definition this space is equipped with hyperbolic geometry).

Now in order to construct the compactification we choose in $Adm(\mathcal{A}_{n,m}U)$ the building set of irreducibles: it turns out that this is $(\mathcal{A}_{n,m}U)^{\perp}$. In the sequel we will often write $\mathcal{F} = \mathcal{A}_{n,m}U$ for brevity.

Then we can take into account the translations by reducing to the mixed arrangement in $\mathbb{R}^{2n+m-1} \cong \{(0, y_1, \ldots, x_n, y_n, w_1, \ldots, w_m)\}$ obtained by putting $x_1 = 0$ in the defining relations of $\mathcal{A}_{n,m}U$ (we still denote by $\mathcal{A}_{n,m}, U, \mathcal{F} = \mathcal{A}_{n,m}U$ the new arrangements). Let us now consider the complement of the mixed subspace and half-space arrangement $\mathcal{F}$: the quotient $\widehat{\mathcal{M}}(\mathcal{F}) = \mathcal{M}(\mathcal{F})/\mathbb{R}^+$ coincides with $C_{n,m}$. What we want to prove is

Theorem 6.1 *The manifold with corners $Y_{\mathcal{F}}^U$ is diffeomorphic to $\overline{C}_{n,m}$.*

Proof.
First we will show that there is a homeomorphism

$$\theta \,:\, Y_{\mathcal{F}}^U \mapsto \overline{C}_{n,m},$$

then we will look at the respective local charts and notice that θ restricts to local diffeomorphisms. The definition of θ is the natural one: as a first step θ identifies the points of the open components $\widehat{\mathcal{M}}(\mathcal{A}_{n,m}\mathcal{U})$ and $C_{n,m}$; then it can be extended by continuity: given a succession of points in $\widehat{\mathcal{M}}(\mathcal{A}_{n,m}\mathcal{U})$ which converges to a point $p \in Y_{\mathcal{F}}^U$, we can identify via θ this succession with a succession of points in $C_{n,m}$ which still converges (in $\overline{C}_{n,m}$) and define the limit to be $\theta(p)$. We only need to check injectivity; then by standard topological arguments it follows that θ is a homeomorphism. Let us suppose for instance that there exist two distinct points $x, y \in Y_{\mathcal{F}}^U$ such that $\theta(x) = \theta(y)$ and the projections of x and y to $S(H^{\perp}_{i_1,\ldots,i_r})$ differ. Let us now consider the product $\Pi_{i_1,\ldots,i_r}$ of the factors S^1 and $\mathbb{P}_{\mathbb{C}}$ which in the definition of $\overline{C}_{n,m}$ involve only the indices $i_1, \ldots, i_r$ and let $p_{i_1,\ldots,i_r}$ be the projection of $\overline{C}_{n,m}$ onto $\Pi_{i_1,\ldots,i_r}$. Now we notice that the following diagram is commutative:

$$\begin{array}{ccc} Y_{\mathcal{F}}^U & \xrightarrow{\theta} & \overline{C}_{n,m} \\ \downarrow & & \downarrow{\scriptstyle p_{i_1,\ldots,i_r}} \\ S(H^{\perp}_{i_1,\ldots,i_r}) & \xrightarrow[\gamma]{} & \Pi_{i_1,\ldots,i_r} \end{array}$$

where the map γ is injective since it is analogue to the embedding used in the definition of $\overline{C}_r$ (commutativity is clear on the open part and passes to the boundary by continuity). This implies that $p_{i_1,\ldots,i_r}(\theta(x)) \neq p_{i_1,\ldots,i_r}(\theta(y))$, which is a contradiction. Repeating a similar argument for all the elements in $\mathcal{F}$ we can conclude the proof of the injectivity of θ.

Now we observe that the opens charts for $Y_{\mathcal{F}}^{U}$ associated to the nested sets and the charts for $\overline{C}_{n,m}$ provided by Kontsevich are in bijective correspondence. In fact the trees with "real" and "complex" vertices described in the preceding subsection are a realization of the trees associated to the nested sets according to the rules given in Section 4. The correspondence is the following. Let us consider of a tree T with real and complex vertices. Let v be a real vertex such that the subtree which stems from it contains the complex leaves $j_1, \ldots, j_p$ and the real leaves $w_1, \ldots, w_s$. The corresponding element in the nested set is $(H^0_{j_1,\ldots,j_p,w_1,\ldots,w_s})^{\perp}$. If v is a complex vertex such that the subtree which stems from it contains the complex leaves $j_1, \ldots, j_p$, the corresponding element in the nested set is $(H_{j_1,\ldots,j_p})^{\perp}$.

Then, given a nested set $\mathcal{S}$ and its corresponding tree $\mathcal{S}^{tree}$, let us consider the charts $U_{\mathcal{S}} \subset Y_{\mathcal{F}}^{U}$ and $U_{\mathcal{S}^{tree}} \subset \overline{C}_{n,m}$: they are products of open balls in the manifolds $N_{A^{\perp}}$ (resp. $C_{r,k}$ or C_j) and of certain small intervals. One can easily check that θ (up to the choice of the same "small" ε's and of the radii) identifies $U_{\mathcal{S}}$ and $U_{\mathcal{S}^{tree}}$ sending each $N_{A^{\perp}}$ to its corresponding $C_{r,k}$ or C_j in a diffeomorphic way. In fact:

1. if $A = H_{j_1,\ldots,j_p}$, $N_{A^{\perp}}$ is seen as the manifold inside $Conf_p$ of elements of norm 1 and such that if we sum their coordinates we obtain $(0,0) \in \mathbb{C}$.

2. if $A = H^0_{j_1,\ldots,j_p,w_1,\ldots,w_s}$, $(p \geq 1)$, $N_{A^{\perp}}$ is seen as the manifold inside $Conf_{p,s}$ of elements of norm 1 and such that if we sum their complex coordinates we obtain a complex number of the form ai $(a \in \mathbb{R})$.

3. if $A = W_{j_1,\ldots,j_p}$, $N_{A^{\perp}}$ is seen as the manifold inside $Conf_{0,p}$ of elements of norm 1 and such that if we sum their coordinates we obtain $0 \in \mathbb{R}$.

■

Remark. The subspace arrangement $\mathcal{A}_n \subset \mathbb{R}^{2n-2} = \{(0, 0, x_2, y_2, \ldots, x_n, y_n)\}$, made by the subspaces $\{0\}$, $H_{i_1,\ldots,i_r} = \{x_{i_1} = \cdots = x_{i_r}\} \cap \{y_{i_1} = \cdots = y_{i_r}\}$ (where $1 \leq i_1 < \cdots < i_r \leq n$ and we adopt the convention that $x_1 = y_1 = 0$), provides a real presentation of C_n, given that $\widehat{\mathcal{M}}(\mathcal{A}_n) = C_n$. The dual $\mathcal{A}_n^{\perp}$ is building and a proof similar to the previous one shows that

Theorem 6.2 *The manifold with corners $Y_{\mathcal{A}_n}$ is diffeomorphic to $\overline{C}_n$.*

Example: the Eye

Let us consider the mixed subspace and half-space arrangement in $\mathbb{R}^4 = \{(x_1, y_1, x_2, y_2)\}$ provided by the subspaces $\{0\}$, $\{x_1 = x_2\} \cap \{y_1 = y_2\}$ and by the hyperplanes $\{y_1 = 0\}$, $\{y_2 = 0\}$ with respective orthogonal vectors $(0, 1, 0, 0)$ and $(0, 0, 0, 1)$. The complement describes the following situation: two distinct points p_1, p_2 in $\mathbb{C}$ with $Im(p_1)$, $Im(p_2) > 0$. Proceeding according to Kontsevich's construction we let the group of transformations of the complex plane $\{az+b | a \in \mathbb{R}^+, b \in \mathbb{R}\}$ act on this configuration and obtain the space $C_{2,0}$ (we are "forgetting" hyperbolic geometry). Turning to the real presentation, we can take into account the translations by reducing to the mixed arrangement

$\mathcal{A}_{2,0}\mathcal{U}$ in $\mathbb{R}^3 = \{(0, y_1, x_2, y_2)\}$ provided by the subspaces $\{0\}$, $\{x_2 = 0\} \cap \{y_1 = y_2\}$ and by the hyperplanes $\{y_1 = 0\}$, $\{y_2 = 0\}$ with respective orthogonal vectors $(0, 1, 0, 0)$ and $(0, 0, 0, 1)$. Notice that in this case $\mathcal{F}$ coincides with $\mathcal{A}_{2,0}\mathcal{U}$. What remains is the complement $\widehat{\mathcal{M}}(\mathcal{AU}) = \mathcal{M}(\mathcal{A}_{2,0}\mathcal{U})/\mathbb{R}^+$ which can be embedded into the two-dimensional compact manifold $Y_{\mathcal{A}_{2,0}\mathcal{U}}$.

A picture can immediately show that $Y^U_{\mathcal{A}_{2,0}\mathcal{U}}$ is diffeomorphic to $\overline{C}_{2,0}$. In fact $Y^U_{\mathcal{A}_{2,0}\mathcal{U}}$ is the closed section of the two-dimensional sphere delimited by the half-spaces $y_1 > 0$, $y_2 > 0$ with the further property that the point $\{y_1 = y_2\} \cap S(\mathbb{R}^3)$ is substituted by a copy of S^1 (this can be seen using the local chart $U_{\{\mathbb{R}^3, \{y_1 = -y_2\}\}}$). Therefore $Y^U_{\mathcal{A}_{2,0}\mathcal{U}}$ is diffeomorphic to the following space "The Eye", which in its turn is diffeomorphic to $\overline{C}_{2,0}$ (see[7]):

7 Conical stratifications of C^∞ manifolds with corners

The compactifications we described until now have been constructed by closing the image of an immersion into a product of spheres. Another way to describe the same construction is via an explicit sequence of real blow-ups. In fact, roughly speaking, if a point p lies in a subspace L, the local normal disk to L in V (centered in p) projects to the component $S(L^\perp)$ in such a way that p in the compactification is replaced by the set of rays in the normal disk. This is actually the local picture of a real blow-up. Looking at things from this point of view, it turns out that our construction is essentially local. Thus we can use our tools to define models for a wider family of manifolds, the so called "conically stratified" manifolds.

Let us give their definition, which is inspired by the one which MacPherson and Procesi used in the complex case (see [10]).

Let X be a C^∞ real manifold with corners of dimension n. We can assume, without loss of generality, that X is embedded in $\mathbb{R}^N$.

Definition 7.1 *A stratification of X is a decomposition $X = \bigcup_\alpha S_\alpha$, where $\{S_\alpha\}$ is a locally finite family of locally closed disjoint submanifolds of X called the strata, which satisfy the following three properties:*
i) there is a unique open dense stratum;
ii) the closure $\overline{S}_\alpha$ of each stratum is a union of strata;
iii) for every $r = 1, \ldots, n-1$ the r-dimensional component of the boundary is a union of strata.

Remarks.
1) Condition ii) of the above definition implies that the set of strata forms a poset: $S_\alpha \geq S_\beta$ if and only if S_β lies in the closure of S_α.
2) We are not assuming that the strata are connected. The assumption that there is a open dense stratum is useful since it simplifies the notation, but it is not essential. Every connected component of the open dense stratum of X "generates" a connected component in the final model.

3) If $Y \mapsto X$ is a submersion, a stratification of X determines an "induced stratification" of Y whose strata are inverse images of strata.
4) If X and Y are both stratified, then there is a product stratification whose strata are products of strata of X with strata of Y.
5) The "trivial stratification" of X is the stratification made by the single stratum X itself.

In the sequel, we will be interested in stratifications whose strata are locally $\mathbb{R}^+$-stable, that is to say, the strata are cones in the local pictures. This leads to the definition of conical stratification.

Definition 7.2 *A n-dimensional "spherical slice" is the intersection of an open ball centered at 0 in $\mathbb{R}^n$ with $\mathbb{R}^a \times (\mathbb{R}^{\geq 0})^{n-a}$. A "spherical slice stratified as a cone" is a stratified spherical slice whose stratification is $\mathbb{R}^+$ stable and includes the stratum $\{0\}$.*

Notice that, according to condition iii) of Definition 7.1, in a spherical slice stratified as a cone the r-th dimensional component of the boundary is a union of strata.

Definition 7.3 *A stratification of X is "conical" if, given any point $x \in X$, there exists an open neighborhood A of x and a diffeomorphism of A with a product of a " tangent" disk D_T, and a spherical slice D_N ("normal slice") such that x corresponds to $(0,0)$ and the stratification induced in A is the product of the trivial stratification of D_T with a stratification of D_N as a cone.*

Definition 7.4 *A stratification of X is "bounded" if the strata are numerable and there exists a positive number M such that the cardinality of every chain of strata $S_{i_1} < S_{i_2} < \cdots < S_{i_j} < \cdots$ is less than M.*

Given a manifold X equipped with a bounded conical stratification, we want to construct models for it. A model is a stratified C^∞ manifold with corners $\tilde{X}$ which is obtained from X by a series of blow-ups along some strata. There is a natural correspondence between the stratifications of X and $\tilde{X}$ and, except for the open dense stratum, the strata of $\tilde{X}$ lie in the boundary.

The construction of these models will be completed in Section 10. In Section 11 we will give a nice combinatorial description of the boundary of the models, in terms of the (transversal) intersections of the closures of of codimension 1 strata (see Theorem 11.1).

In general, there are several models associated to a given X. More precisely, there are as many models as there are building sets associated to the stratification of X. Here we are talking about a generalization of the definition of building sets (see Section 2) which will be explained in the next section.

8 Building sets of strata

First we notice that, given a stratum S of a conical stratification and two points x, y in S, the splittings $D_T \times D_N$ of the neighbourhoods of x and y are diffeomorphic (up to restriction to smaller neighbourhoods), i.e., that each stratum determines a splitting.

This can be proven by noticing that there is a canonical stratification (independent from local systems) of the normal cone bundle $T_S X$, which the local splittings can be deduced from.

The stratification can be constructed by a process of taking differences as follows. Let Z be a closed union of strata in X, then we can consider all the smooth curves from an open set $U \subset \mathbb{R}^{\geq 0}$ (containing the origin) to X which send 0 to S and U to Z. The derivatives in 0 of all such curves project to vectors in the normal cone to S along Z. These vectors provide by definition a closed union of strata in $T_S X$. Then taking differences we can obtain a stratification of $T_S X$ which, if restricted to a local splitting $D_T \times D_N$, gives the local stratification. This forces the neighbourhoods of two points in S to be diffeomorphic as stratified manifolds (of course up to restriction to smaller neighbourhoods).

Definition 8.1 *Let W be a set of strata $\geq$ of a certain stratum S. We say that S is factored by W if for any (and hence every) point $x \in S$ there is a neighborhood A of x with a decomposition*

$$D = D_T \times D_N^1 \times \cdots \times D_N^m$$

where D_T is a disk trivially stratified, D_N^i (for every $i = 1, \ldots, m$) is a spherical slice stratified as a cone and the strata of W intersected with A are the products of 0 in one normal slice with the open dense stratum in the other normal slices. Additionally, a stratum is factored by the set consisting of itself.

Definition 8.2 *Building sets of strata.*
Given a conical stratification $\mathcal{R}$, a set $\mathcal{G}$ of strata is a building set if any stratum $S \in \mathcal{R}$ is factored by the minimal elements in $\mathcal{G}$ which are $\geq S$. Such elements are called "the $\mathcal{G}$-factors" of S.

We notice that the set $\mathcal{R}$ of all strata is building. It is the largest one with respect to inclusion. There is also a minimum building set in $\mathcal{R}$ which is the building set of irreducible strata. Its definition is a generalization of the corresponding one in Section 2.

Definition 8.3 *Given a conical stratification $\mathcal{R}$, a stratum S is called "reducible" if for any (and hence every) point $x \in S$ there is a neighbourhood A of x with the following property: A, equipped with the induced stratification, admits a splitting*

$$A = D_T \times D_N^1 \times D_N^2$$

where D_N^1 and D_N^2 are spherical slices of positive dimension stratified as cones and D_T is trivially stratified. Otherwise S is called "irreducible".

9 Blowing up strata

In this section we will describe how to blow up the closure of a minimal stratum in a building set $\mathcal{G}$ (i.e. a stratum S in $\mathcal{G}$ such that every stratum $< S$ does not belong to $\mathcal{G}$). This will be the main ingredient of our construction.

Let us first study the closure $\overline{S}$ of S.

Proposition 9.1 *Let S be a minimal stratum in the building set $\mathcal{G}$. Then every point $y \in \overline{S}$ has a neighbourhood D with a product stratification $D = D_{\overline{S}} \times D_N$ where D_N is a normal slice stratified as a cone, while $D_{\overline{S}} \times 0 = \overline{S} \cap D$.*

Remark.
The splitting $D = D_{\overline{S}} \times D_N$ of the above claim is not a splitting of the same kind of the ones which appear in the Definition 7.3 of conical stratifications. In fact $D_{\overline{S}}$ does not need to be a trivially stratified disk.

Proof.
We have to prove the proposition only for points in $\overline{S} - S$. Let y be such a point. Then y belongs to some stratum L which is not in $\mathcal{G}$. This stratum has a splitting in terms of its factors: they are the minimal elements in $\mathcal{G}$ which are $\geq L$ and therefore S, by minimality, is one of these factors. Then there is a neighbourhood D of y which has this factorization:

$$D = D_T \times D_N^1 \times \cdots \times D_N^m$$

and we know that $S \cap D$ is the product of 0 in one normal slice (say D_N^i) with the open dense stratum in the other normal slices.

Then we can take D_N to be D_N^i and $D_{\overline{S}}$ to be the product of all the other factors.

■

The construction that allows us to obtain the "real blow-up" of X along $\overline{S}$ is well known and sometimes it is called "the balls, beams and plates" construction. In [8] it is described in the more general context of Whitney stratifications. Focussing on our picture, let us discuss a local real blow-up, namely, the blow-up of D_N in 0.

If D_N is a m-dimensional spherical slice, let $S(D_N)$ denote the unit sphere in $\mathbb{R}^m$. Then the blow-up $BL_0(D_N)$ of D_N in 0 consists in embedding $D_N - \{0\}$ in $D_N \times S(D_N)$ and taking the closure.

Since our goal is to obtain a stratified manifold we have to define a stratification of $BL_0(D_N)$. This can be done in the following way: $BL_0(D_N) - \pi^{-1}(0) = D_N - \{0\}$ (here π is the projection to D_N) is stratified in the same way $D_N - \{0\}$ is, while $\pi^{-1}(0)$ is stratified by the projection to $S(D_N)$ of the stratification of $D_N - \{0\}$ (this is well defined since the stratification of $D_N - \{0\}$ is $\mathbb{R}^+$-stable).

Notice that the obtained stratification is still conical. Now, looking at the proof of Proposition 9.1 (and keeping the same notation), we notice that the fiber of the normal cone bundle to $\overline{S}$ in X is diffeomorphic to D_N^i. Therefore the real blow-up $BL_{\overline{S}}X$ can be constructed by glueing together local blow-ups of D_N^i in 0. This makes it clear that it is a C^∞ manifold with corners.

Proposition 9.2 *The blow-up $BL_{\overline{S}}X$ has an induced conical stratification characterized by the property that, over a neighbourhood $D = D_{\overline{S}} \times D_N$ of a point $\overline{y} \in \overline{S}$ it coincides with the product stratification $BL_{\overline{S}\cap D}D = D_{\overline{S}} \times BL_0 D_N$.*

Proof. If we define on $BL_{\overline{S}}X$ a stratification which has the claimed local property, this turns out to be conical since it is locally conical (the product of two conical stratifications is conical).

There is no problem concerning the definition of the stratification in the complement of the exceptional divisor. It suffices to take the same stratification as $X - \overline{S}$.

Now, the exceptional divisor can be viewed as the unit sphere bundle $UT_{\overline{S}}X$ associated to the normal cone bundle $T_{\overline{S}}X$ (here we are referring to the metric of the ambient $\mathbb{R}^N$, but this involves no loss of generality).

Then, on one side we can equip $T_{\overline{S}}X$ with a stratification in the same way as $T_S X$ in Section 8. On the other side we can equip $T_{\overline{S}}X$ with the stratification given by taking inverse images of strata of $\overline{S}$. The common refinement of these two stratifications is again a conical stratification of $T_{\overline{S}}X$ and we choose for $UT_{\overline{S}}X$ the induced stratification. It is easy to check that this stratification locally coincides with the one given by the local splittings. ∎

10 A series of blow-ups

Our aim now is to find in the new conically stratified manifold $BL_{\overline{S}}X$ a building set of strata $\mathcal{G}'$ which allow us to continue our blowing up process. This is provided by

Proposition 10.1 *The following set of strata $\mathcal{G}'$ in $BL_{\overline{S}}X$ is building: in the complement of the exceptional divisor we take all the strata of $\mathcal{G} - \{S\}$; then we add the open dense stratum S' in the exceptional divisor.*

Notice that the stratum S' is a codimension 1 stratum in the boundary. Then we can go on and blow up a minimal stratum in $BL_{\overline{S}}X$. The idea of the construction of the model, which we will denote by $X_{\mathcal{G}}$, is to continue until we obtain a model equipped with a building set whose strata are all the codimension 1 strata in the boundary. This can be done in a finite number of steps if the number of strata of X is finite (which is always the case when X is compact); nevertheless, this construction works also in the more general case of a bounded conical stratification.

It turns out that $X_{\mathcal{G}}$ is independent of the choices involved.

Proposition 10.2 *Suppose that $X^1_{\mathcal{G}}$ and $X^2_{\mathcal{G}}$ are two models of X obtained starting from the building set $\mathcal{G}$ and blowing up minimal elements in two different orders. Then $X^1_{\mathcal{G}}$ and $X^2_{\mathcal{G}}$ are diffeomorphic.*

Proof.

We will proof the claim when both $X^1_{\mathcal{G}}$ and $X^2_{\mathcal{G}}$ are constructed by blowing up at every step a stratum which is of minimal dimension among the minimal strata in the building set (the proof in the general case is similar but needs a more complicated notation).

The open dense strata in X, $X^1_{\mathcal{G}}$ and $X^2_{\mathcal{G}}$ can be identified by construction.

Let us now look at what happens locally. Let $x \in X$ and take a factorization (according to Definition 8.1) of a neighbourhood D of x:

$$D = D_T \times D^1_N \times \cdots \times D^m_N.$$

Now, let S_1, S_2, $\ldots S_j$ be all the minimal elements in $\mathcal{G}$ of minimal dimension and suppose that the first i of them intersect D. We can assume that, for every $h = 1, \ldots, i$,

$$S_h \cap D = D_T \times op(D^1_N) \times \cdots \times op(D^{h-1}_N) \times 0 \times op(D^{h+1}_N) \times \cdots \times op(D^m_N)$$

where $op()$ denotes the open dense stratum. Then, after the first j blow-ups of the construction of both $X^1_{\mathcal{G}}$ and $X^2_{\mathcal{G}}$, the part which is over D is

$$D_T \times Bl_0(D^1_N) \times \cdots \times Bl_0(D^i_N) \times D^{i+1}_N \times \cdots \times D^m_N$$

and this is independent on the ordering among the S_j's.

This shows that the map which identifies the open dense strata (that is to say, the internal parts) of $X^1_{\mathcal{G}}$ and $X^2_{\mathcal{G}}$ can be extended to a diffeomorphism between the two spaces. ■

11 The boundary of the model

Let us consider a manifold X equipped with a bounded conical stratification and a model $X_{\mathcal{G}}$ obtained by blowing up the strata of a building set $\mathcal{G}$. In this section we will describe the structure of the boundary of $X_{\mathcal{G}}$. The main combinatorial objects involved in this description are the $\mathcal{G}$-nested sets: the following definition generalizes the one given in the linear case (see Section 2).

Definition 11.1 *$\mathcal{G}$-nested sets of strata.*

A set $\mathcal{T} \subset \mathcal{G}$ of strata is called $\mathcal{G}$-nested if it satisfies the following property: let $A_1, \ldots, A_k$ be the minimal elements of $\mathcal{T}$ and let $\mathcal{T}_i$ be the set of elements in $\mathcal{T}$ that are $> A_i$. Then $A_1, \ldots, A_k$ are all the $\mathcal{G}$-factors of some single stratum of X and $\mathcal{T}_i$ is nested, as defined by induction.

Now we can characterize all the strata of $X_{\mathcal{G}}$ in a nice combinatorial way.

Theorem 11.1 *1) The codimension 1 strata in the boundary of $X_{\mathcal{G}}$ are in bijective correspondence with the elements of $\mathcal{G}$. The stratum D_G which corresponds to $G \in \mathcal{G}$ is the inverse image of G via the blow-up map $X_{\mathcal{G}} \mapsto X$.*
2) Let us consider in $X_{\mathcal{G}}$ the family of strata indexed by some subset $\mathcal{T}$ of $\mathcal{G}$. The intersection of the closures of these strata is non empty if and only if $\mathcal{T}$ is $\mathcal{G}$-nested.
3) The strata of $X_{\mathcal{G}}$ can be indexed by the $\mathcal{G}$-nested sets.

Remark.
If the intersection of point 2) is not empty, then it is transversal, being the intersection of closures of boundary strata in a manifold with corners.

Proof. If S is a codimension 1 stratum in the boundary of X then it belongs to $\mathcal{G}$ and coincides with D_S. The other codimension 1 strata in the boundary of $X_{\mathcal{G}}$ are obtained by construction by blowing up the (proper transforms of the) closures of the remaining elements in $\mathcal{G}$.

Let now $\mathcal{T}$ be a subset of $\mathcal{G}$. If $\mathcal{T}$ is nested, then we can take its minimal elements $A_1, \ldots, A_k$ which are all the $\mathcal{G}$-factors of a certain stratum K. We have a local factorization of K

$$D_T \times D_N^1 \times \cdots D_N^k$$

which after blowing up along (the proper transforms of) the closures of $A_1, \ldots, A_k$ becomes

$$D_T \times Bl_0(D_N^1) \times \cdots Bl_0(D_N^k)$$

This means that the intersection of the closures of $D_{A_1}, \ldots, D_{A_k}$ is nonempty. Proceeding further we can blow up the proper transforms of the minimal strata in $\mathcal{T}_i$ for every i (where, according to definition of nested sets, $\mathcal{T}_i$ is the set of elements in $\mathcal{T}$ that are $> A_i$). We can repeat at this step the same reasoning and obtain that the common intersection of the closures of all the involved codimension 1 strata in the boundary is non empty. And so on...

If instead $\mathcal{T}$ is not nested, this means that at a certain step of this process we find the following situation. There is a manifold X', a building set $\mathcal{G}'$ (whose are elements are in bijective correspondence with the elements of $\mathcal{G}$), and elements $B_1, \ldots, B_k$ belonging to $\mathcal{T} \subset \mathcal{G}'$ (we are identifying elements of $\mathcal{G}$ and $\mathcal{G}'$ according to the bijective correspondence) such that the following condition holds:
for every stratum B in X', it is not true that all the factors of B are $B_1, \ldots, B_k$.

In this case we will proof that $\overline{D}_{B_1} \cap \cdots \cap \overline{D}_{B_k}$ is empty. Let us suppose the contrary, and let $p \in \overline{D}_{B_1} \cap \cdots \cap \overline{D}_{B_k}$.

Now, if we consider the factorization associated to a stratum B in X':

$$D = D_T \times D_N^1 \times \cdots \times D_N^j$$

there are two possibilities:
1) for a certain i, $\overline{B}_i \cap D$ is empty;
2) condition 1) does not occur and we can find a spherical slice (say D_N^i) and a subset $B_{i_1}, \ldots B_{i_l}$ $(l \geq 2)$ of $\{B_1, \ldots, B_k\}$ such that $B_{i_h} \cap D_N^i \neq 0$ for $h = 1, \ldots, l$ and $\overline{B}_{i_1} \cap \cdots \cap \overline{B}_{i_l} \cap D_N^i = 0$.

Let us denote by $\pi : X_{\mathcal{G}} \mapsto X'$ the blow-up map: then we have that, if condition 1) holds, $\pi(p)$ does not belong to D.

If we proof that $\pi(p)$ does not belong to D also if condition 2) holds, we find a contradiction, since open sets like D cover X'.

So let us assume condition 2) and denote by K the stratum which locally appears as

$$D_T \times op(D_N^1) \times op(D_N^{i-1}) \times 0 \times op(D_N^{i+1}) \times \cdots \times op(D_N^j)$$

Since we are interested in the local picture, we can assume that in X' K is a minimal element of $\mathcal{G}'$ and that the next step in the blow-up process is $BL_{\overline{K}}X'$.

Let us denote by $\pi' : X_{\mathcal{G}} \mapsto BL_{\overline{K}}X'$ the blow-up map. If $\pi(p) \in D$, then $\pi'(p)$ must be a point in the exceptional divisor.

Our aim is to cover the exceptional divisor by some charts and show that $\pi'(p)$ cannot belong to any one of these. A family of open sets in $BL_{\overline{K}}X'$ which cover the exceptional divisor can be obtained by repeating the following construction. We choose a point v in $\overline{K}$ in X' and form a factorization

$$D_T \times D_N^1 \times \cdots \times D_N^r$$

of a neighbourhood of v. Since K is a minimal stratum in the building set it must appear as a factor (say $D_T \times 0 \times op(D_N^2) \times \cdots \times op(D_N^r)$). Then the open set we are searching for is

$$D_T \times BL_0(D_N^1) \times \cdots \times D_N^r$$

Now we notice that in this open set the the closures of the proper transforms of $B_{i_1}, \ldots B_{i_l}$ have an empty common intersection. This follows from $B_{i_h} \cap D_N^1 \neq 0$ for $h = 1, \ldots, l$ and $\overline{B}_{i_1} \cap \cdots \cap \overline{B}_{i_l} \cap D_N^1 = 0$. Therefore the point $\pi'(p)$ cannot belong to the exceptional divisor. This completes the contradiction. ■

Examples and remarks.

1) As mentioned in the Introduction, one can easily check that the real compactifications defined in Section 3 are examples of this more general blow-up construction. In that case we dealt with strata which are subsets of linear subspaces in $\mathbb{R}^n$, and we could show an explicit embedding of the open dense stratum in a product of spheres. From the point of view of Sections 7-11, we recover this embedding step by step, since the blow-up locally appears as the closure of an embedding $D_N - \{0\} \mapsto D_N \times S(D_N)$.

Kontsevich's configuration spaces in [7] are among the above examples, but, more generally, real and complex configuration spaces are in a natural way conically stratified manifolds.

Several further examples of conically stratified manifolds are provided by manifolds of matrices: for instance, real or complex $n \times m$ matrices stratified by rank, or real symmetric matrices stratified by their indices and rank.

2) We notice that there is a rather standard way to obtain manifolds with corners as "models" of stratified spaces, when the set of strata is a poset. In fact, given a stratum Y, the complement of the tubular neighbourhoods of all strata smaller than Y, when intersected with Y is a manifold with corners. It is embedded inside Y but its interior is in fact diffeomorphic to Y.

This corresponds, in our setting, to choosing the inclusion maximum building set associated to the stratification and construct the maximum model: in this case we provide a general description of the combinatorics of the boundary.

It often happens that smaller models are more interesting than the maximum one: we already showed the example of Kontsevich's configuration spaces which are obtained using the minimum building set.

As another example, we can consider the hyperplane arrangement in $\mathbb{R}^3$ made by the hyperplanes orthogonal to the roots of a root system of type A_3. One can realize its picture by embedding $\mathbb{R}^3$ into $\mathbb{R}^4$ as the subspace $\{x_1+x_2+x_3+x_4=0\}$ and considering the arrangement induced by the hyperplanes $x_i - x_j = 0$ $(4 \geq i > j \geq 1)$. Every compactification has 24 connected components and we can easily embed it inside the 2-sphere S^2 in such a way that each component lies in a Weyl chamber.

Let $\mathcal{F}_4^\perp$ be the minimum building set; it is formed by the subspaces A_I where I ranges over all subsets of $\{1,2,3,4\}$ of cardinality at least two and $A_I = \{\sum_{s,\, r\in I} \mathbb{R}(x_r - x_s)\}$.

Let now $Y_{\mathcal{F}_4}$ be the minimum compactification and let us choose a Weyl chamber W (for instance the one such that $x_1 > x_2 > x_3 > x_4$). We notice that the connected component $Y_{\mathcal{F}_4}(W)$ of $Y_{\mathcal{F}_4}$ inside W is a convex body which realizes the Stasheff associahedron (see [9]) for 4 letters. In fact it is a spherical pentagon: its vertices are in bijective correspondence with the maximal nested sets in $\mathcal{F}_4^\perp$ that can be formed by the subspaces A_I with I consisting of consecutive numbers (for instance $A_{\{2,3,4\}}$). These, in their turn, are in bijective correspondence with all distinct complete bracketings of 4 letters. Three edges of the penthagon are diffeomorphic copies of the open walls of the Weyl chamber, while the other two edges are the blow-ups of the two points $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}, 0)$ and $(1,0,0,0)$ in $\overline{W} \cap S^2$ which are intersection of three hyperplanes.

The Weyl group in this case is the symmetric group S_4; by means of elementary linear algebra we can show that there is a linearization P of $Y_{\mathcal{F}_4}$ such that if we let S_4 act on P and take the convex hull of the orbit $S_4 \cdot P$ we obtain a realization of Kapranov's permutoassociahedron KP_4 (see [6]). We conjecture that this construction of Stasheff's and Kapranov's polytopes extends to every dimension and can be generalized to all root systems.

References

[1] C. De Concini, C. Procesi, Wonderful models of subspace arrangements. Selecta Mathematica, New series, 1 (1995), 459-494.

[2] C. De Concini, C. Procesi, Hyperplane arrangements and holonomy equations. Ibid, 495-536.

[3] E. Feichtner, D.Kozlov, Incidence combinatorics of resolutions, preprint 2000.

[4] W. Fulton, R. MacPherson, A compactification of configuration space. Annals of Math., 139 (1994), 183-225.

[5] G.Gaiffi, Compactifications of Configuration Spaces, in Seminari di Geometria Algebrica 1998-1999, Scuola Normale Superiore di Pisa, 1999.

[6] M.M. Kapranov, The permutoassociahedron, Mac Lane's coherence theorem and asymptotic zones for the KZ equation., J. of Pure and App. Algebra 85 (1993) 119-142.

[7] M. Kontsevich, Deformation quantization of Poisson manifolds, I. q-alg/9709040.

[8] G. Kuperberg, D. Thurston, Perturbative 3-manifolds invariants by cut-and-paste topology, math.GT/9912167.

[9] J.D. Stasheff, Homotopy associativity of H spaces I, Trans. Amer. Math. Soc. 108 (1963), 275-292.

[10] R. MacPherson, C. Procesi, Making conical compactifications wonderful. Selecta Mathematica, New series, 4 (1998), 125-139.