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QUANDLE HOMOMORPHISMS OF KNOT QUANDLES TO ALEXANDER QUANDLES

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ABSTRACT

A quandle is a set with a binary operation satisfying some properties. A quandle homomorphism is a map between quandles preserving the structure of their binary operations. A knot determines a quandle called a knot quandle. We show that the number of all quandle homomorphisms of a knot quandle of a knot to an Alexander quandle is completely determined by Alexander polynomials of the knot. Further we show that the set of all quandle homomorphisms of a knot quandle to an Alexander quandle has a module structure.

Keywords: knot, quandle, coloring, Alexander polynomial

1. Introduction

A quandle, introduced in [6, 1], is defined to be a set X with a binary operation $* : X \times X \to X$ satisfying some properties. A typical example of a quandle is a group with the binary operation derived from the conjugation of elements in the group. In this sense the notion of a quandle is obtained from the notion of a group by forgetting the product structure, but not forgetting the structure of conjugation. Another typical example of a quandle is a knot quandle, introduced in [6]; it is a quandle derived from a knot. D. Joyce showed in [6] that the set of knot quandles is a complete invariant of knots, while it is known that a set of knot groups is not a complete invariant of knots.

A quandle homomorphism is defined to be a map of a quandle to another quandle preserving the structure of their binary operations. It is known to be useful for classifying knots to count the number of all homomorphisms of a knot group to a fixed finite group. It might be useful that to count the number of all quandle homomorphisms of a knot quandle to a quandle.

In this paper we show that the number of all quandle homomorphisms of a knot quandle of a knot to an Alexander quandle is completely determined by the series of the Alexander polynomials of the knot (Theorem 1). Further we show that the set of all quandle homomorphisms of a knot quandle to an Alexander quandle has a module structure.

For each quandle, there is a cohomology group of the quandle. Further there is an invariant of links called cocycle invariant associate to each 2-cocycle of the cohomology group (see [1, 2, 3]). The cocycle invariant is called trivial if the value is a positive integer. In particular, it is known that if the 2-cocycle used is a coboundary, then the invariant is trivial. Theorem 1 gives a way of computation of the trivial cocycle invariant of the knot derived from an Alexander quandle.

2. Knot Quandles

In this section, we review definitions of quandles, Alexander quandles and knot quandles.

Definition 1 ([1, 6]). A *quandle* is defined to be a set X with a binary operation $*: X \times X \to X$ satisfying the following properties:

(Q1) For each $x \in X$, $x * x = x$.
(Q2) For each pair of $x, y \in X$, there is a unique element $z \in X$ such that $x = z * y$.
(Q3) For each triple of $x, y, z \in X$, $(x * y) * z = (x * z) * (y * z)$.

The property (Q2) is equivalent to the following property:

(Q2′) For any $x, y \in X$, there is a binary operation $\overline{*} : X \times X \to X$ such that $(x * y) \overline{*} y = x = (x \overline{*} y) * y$.

We remark that $\overline{*}$ is uniquely determined such that the element z described in (Q2) is equal to $x \overline{*} y$.

Let q be a natural number, Λ_q the Laurent polynomial ring in t with coefficients in the cyclic group $\mathbb{Z}_q$ of order q and $J \subset \Lambda_q$ an ideal of Λ_q. The quotient ring Λ_q/J with the binary operation such that $x*y = tx + (1-t)y$ for each pair of $x, y \in \Lambda_q/J$ is called an *Alexander quandle*. We remark that $x \overline{*} y$ is equal to $t^{-1}x + (1-t^{-1})y$.

Definition 2. Let X and Y be quandles. A *quandle homomorphism* of X to Y is defined to be a map $\varphi : X \to Y$ satisfying $\varphi(x*y) = \varphi(x) * \varphi(y)$ for any $x, y \in X$.

Throughout this paper let K be an oriented knot in the 3-sphere S^3 and D_K a regular diagram of K. We put R_{D_K} to be the set $\{a_1, a_2, \cdots, a_n\}$ of all over arcs of D_K. Here the number n is equal to the number of all over arcs of D_K. We regard S^3 as being $\mathbb{R}^3$ together with an extra point at infinity. We may assume that D_K lies in $\mathbb{R} \times \mathbb{R} \times \{0\}$. Let $b_1, b_2, \cdots, b_n$ be arcs in $\mathbb{R} \times \mathbb{R} \times (-\epsilon, 0]$ such that each b_i satisfies a condition illustrated in Fig. 1 at each crossing point. Then the union of D_K and these arcs is isomorphic to K. Therefore we put K again to be this union in the following.

Let N be the union of the closed unit disk $D^2 = \{z \in \mathbb{C} \mid |z| \leq 1\}$ and the interval $\{z \in \mathbb{R} \subset \mathbb{C} \mid 1 \leq z \leq 5\}$. An *elemental map* of K is defined to be a

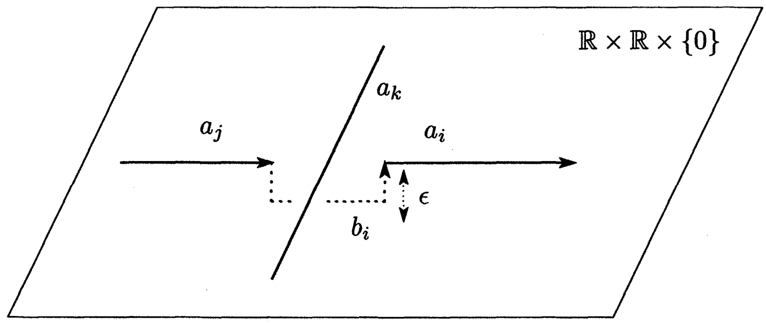


Fig. 1. b_i connects a_j and a_i

continuous map $\mu : (N, 0) \to (\mathbb{R}^3 \cup \{\infty\}, K)$ such that $\mu^{-1}(K)$ is equal to 0, $\mu(5)$ is equal to $(0,0,1)$, $\mu|_{D^2}$ is an embedding and the Gauss linking number of two curves K and $\mu|_{\partial D^2}$ is equal to 1 (See Fig. 2). Let $Q(K)$ be the set of all homotopy classes of elemental maps of K. For each $[\mu], [\nu] \in Q(K)$, We define an element $[\mu] * [\nu]$ of $Q(K)$ as a homotopy class of an elemental map $\mu * \nu$ such that $\mu * \nu(z) = \mu(z)$ for $|z| \leq 1$, $\mu(4z-3)$ for $1 \leq z \leq 2$, $\nu(13-4z)$ for $2 \leq z \leq 3$, $\nu(e^{2(z-3)\pi i})$ for $3 \leq z \leq 4$ and $\nu(4z-15)$ for $4 \leq z \leq 5$ (See Fig. 3). It is known, see [6], that this is well defined and satisfies the defining properties (Q1), (Q2) and (Q3) of a quandle. Therefore $Q(K)$ becomes a quandle with this binary operation $*$. This quandle is called the *knot quandle* of K.

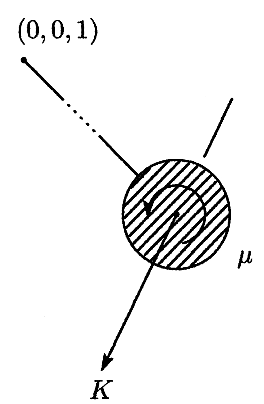


Fig. 2. An elemental map μ

Let $[\mu_i]$ be the homotopy class of an elemental map illustrated in Fig. 4 for each i. We remark that arbitrary element of $Q(K)$ is generated by $[\mu_1], [\mu_2], \cdots, [\mu_n]$. Further there is an equation of generators $[\mu_{k_m}] *^{\epsilon_m} [\mu_{l_m}] = [\mu_m]$ corresponding to

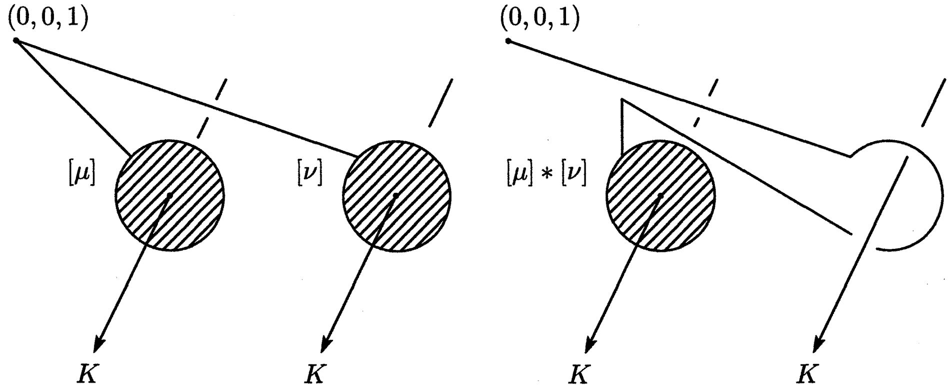


Fig. 3. The rule of the binary operation

each crossing point τ_m (illustrated in Fig. 5). Here $*^{\epsilon}$ denotes a binary operation $*$ if $\epsilon = 1$, $\overline{*}$ if $\epsilon = -1$. Therefore for an arbitrary quandle X, a quandle homomorphism $\varphi : Q(K) \to X$ is determined by deciding $\varphi([\mu_i]) \in X$ for each i satisfying $\varphi([\mu_m]) = \varphi([\mu_{k_m}]) *^{\epsilon_m} \varphi([\mu_{l_m}])$. Therefore the map $c : R_{D_K} \to X$ with $c(a_i) = \varphi([\mu_i])$ determines a quandle homomorphism. This map is called a *coloring* on D_K by X (see [1]).

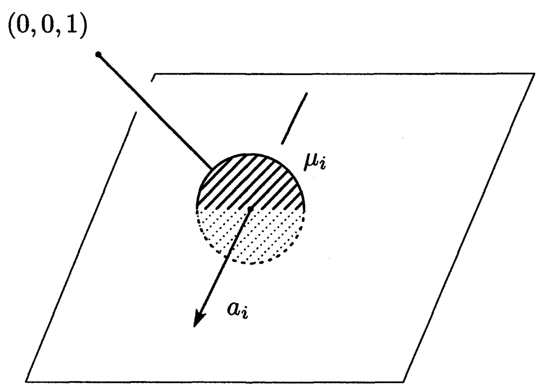
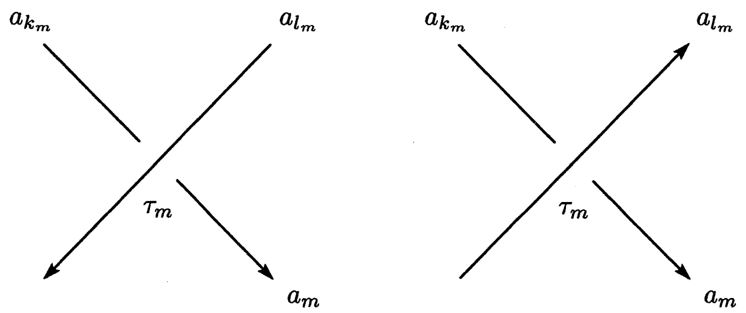


Fig. 4. The generator of the knot quandle

3. Quandle Homomorphisms to Alexander Quandles

In this section, we investigate quandle homomorphisms of a knot quandle to Alexander quandles.

Let K, D_K and R_{D_K} be as in Section 2. Let α_i be an element of the knot



$[\mu_{k_m}] * [\mu_{l_m}] = [\mu_m]$ $(\epsilon_m = 1)$ $\qquad$ $[\mu_{k_m}] \overline{*} [\mu_{l_m}] = [\mu_m]$ $(\epsilon_m = -1)$

Fig. 5. The equation of generators corresponding to a crossing point

group of K represented by a loop going around the over arc a_i (See Fig. 6). The *Wirtinger presentation* derived from D_K is the presentation of the knot group of the knot which given by $\langle \alpha_1, \alpha_2, \cdots, \alpha_n \mid \theta_1, \theta_2, \cdots, \theta_n \rangle$ (see [7]). Here the relation θ_m is made up corresponding to each crossing point of D_K by the rule illustrated in Fig. 7.

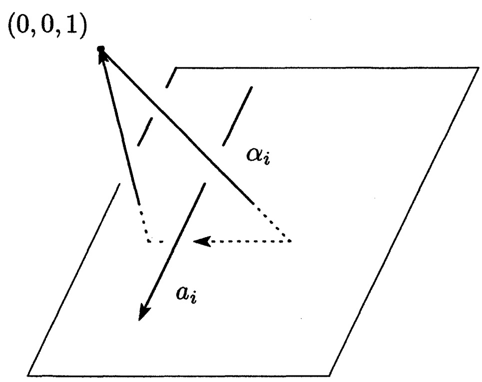


Fig. 6. The generator of the knot group

Let $F_n = \langle \alpha_1, \alpha_2, \cdots, \alpha_n \rangle$ be a free group and $\mathbb{Z}F_n$ its group ring. Let $\partial/\partial\alpha_i : \mathbb{Z}F_n \to \mathbb{Z}F_n$ be the Fox free differential operator in the variable α_i (see [5]) and $a : \mathbb{Z}F_n \to \mathbb{Z}[t, t^{-1}]$ the ring homomorphism defined by putting $a(\alpha_1) = a(\alpha_2) = \cdots = a(\alpha_n) = t$. The *Alexander matrix* A_{D_K} of the regular diagram D_K is defined by $A_{D_K} = (a \circ \partial/\partial\alpha_i(\theta_j))_{i,j \leq n}$.

The *i-th Alexander polynomial* $\Delta_K^{(i)}(t)$ of K is defined to be the greatest common divisor polynomial of all $(n-i-1)$-th minor determinants of A_{D_K}. They are

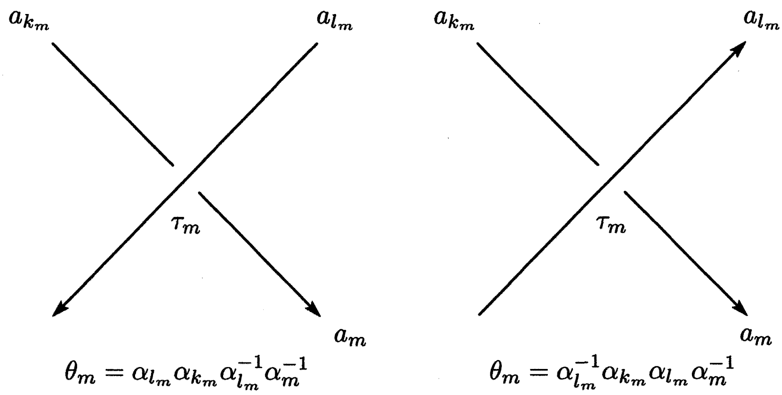


Fig. 7. The relation of the Wirtinger presentation

invariants of knots (see [5, 7]). In particular, the 0-th Alexander polynomial is called the *Alexander polynomial* and simply denoted by $\Delta_K(t)$.

It is known that $\Delta_K^{(i)}(t)$ is divisible by $\Delta_K^{(i+1)}(t)$ for each i.

Let Λ_q/J be an Alexander quandle and y_i an element of Λ_q/J associate to each over arcs a_i of D_K. For each crossing point of D_K, if elements y_{k_m}, y_{l_m} and y_m satisfy the rule illustrated in Fig. 8 then $(a_1, a_2, \cdots, a_n) \mapsto (y_1, y_2, \cdots, y_n)$ induces a quandle homomorphism of $Q(K)$ to Λ_q/J (See [4]).

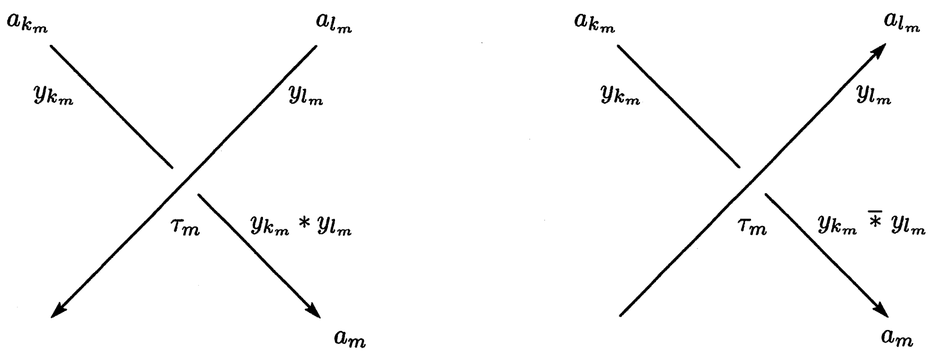


Fig. 8. A map satisfying these rules is a coloring

Let $B_{D_K} = \sum_{m=1}^{n} B_m$ be an $(n \times n)$-matrix where B_m is the $(n \times n)$-matrix corresponding to each crossing point τ_m (illustrated in Fig. 8) such that (k_m, m) entry is t^{ϵ_m}, (l_m, m) entry is $1 - t^{\epsilon_m}$ and otherwise is 0. Here ϵ_m means the sign of crossing point τ_m.

Let y_i be an element of Λ_q/J and $\gamma : Q(K) \to \Lambda_q/J$ a map which takes $[\mu_i]$ to y_i for each i. The map γ is a quandle homomorphism if and only if

$(y_1, y_2, \cdots, y_n)(B_{D_K} - E_n)$ is equal to $(0, 0, \cdots, 0)$. Here E_n denotes the n-dimensional identity matrix.

Lemma 1. *We have* $(B_{D_K} - E_n) = A_{D_K}$.

Proof. We write matrices A_{D_K} and B_{D_K} by (a_{ij}) and (b_{ij}), respectively. Let δ_{ij} be the Kronecker's delta, i.e. $\delta_{ij} = 1$ if $i = j$, 0 otherwise. If the crossing point τ_m (illustrated in Fig. 8) is a positive crossing, then $a_{im} = -1$ for $i = m$, t for $i = k_m$, $1 - t$ for $i = l_m$, 0 otherwise, $b_{im} = 0$ for $i = m$, t for $i = k_m$, $1 - t$ for $i = l_m$, 0 otherwise. If the crossing point τ_m is a negative crossing, then $a_{im} = -1$ for $i = m$, t^{-1} for $i = k_m$, $1 - t^{-1}$ for $i = l_m$, 0 otherwise, $b_{im} = 0$ for $i = m$, t^{-1} for $i = k_m$, $1 - t^{-1}$ for $i = l_m$, 0 otherwise. Therefore, a_{im} is equal to $b_{im} - \delta_{im}$ for all pairs of i, m □.

The above lemma says that the set of all quandle homomorphisms of the knot quandle $Q(K)$ to an Alexander quandle Λ_q/J is determined by the Alexander matrix of A_{D_K}.

In particular, if $q = p$ is a prime number, then:

Theorem 1. *Let p be a prime number, J an ideal of the ring Λ_p and $Q(K)$ a knot quandle. For each $i \geq 0$, we put $e_i(t) = \Delta_K^{(i)}(t)/\Delta_K^{(i+1)}(t)$. Then the number of all quandle homomorphisms of the knot quandle $Q(K)$ to the Alexander quandle Λ_p/J is equal to the cardinality of the module $\Lambda_p/J \oplus \oplus_{i=0}^{n-2}\{\Lambda_p/(e_i(t), J)\}$.*

Proof. Let $\psi : (\Lambda_p/J)^n \to (\Lambda_p/J)^n$ be the map which takes a row vector x to xA_{D_K}.

On one side we show that the number of all quandle homomorphisms is equal to the cardinality of $\ker\psi$. It is clear from the Lemma 1.

On the other hand we show that the cardinality of the module of the theorem is equal to the cardinality of $\operatorname{coker}\psi$. Since p is a prime number, Λ_p/J is a principal ideal domain. Therefore A_{D_K} can be expressed as

$$A_{D_K} = U \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & e_0(t) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & e_{n-3}(t) & 0 \\ 0 & 0 & \cdots & 0 & e_{n-2}(t) \end{pmatrix} V \quad ,$$

with some unimodular matrices U and V. It takes that the cardinality of $\operatorname{coker}\psi$ is equal to the cardinality of the module of the theorem.

Since $(\Lambda_p/J)^n/\ker\psi$ is isomorphic to $\operatorname{Im}\psi$, the cardinality of $\ker\psi$ is equal to the cardinality of $(\Lambda_p/J)^n/\operatorname{Im}\psi$. Therefore the cardinality of $\ker\psi$ is equal to the cardinality of $\operatorname{coker}\psi$ □.

It is shown by definition of a quandle homomorphism that for any quandle X, if each element of $Q(K)$ is mapped to a fixed element $x \in X$ then this map becomes a quandle homomorphism. This homomorphism is called a *trivial coloring* on K. It is clear that for any knot, the number of trivial colorings on the knot is equal to the cardinality of the quandle. Hence if $\Lambda_p/(e_i(t), J) = 0$ for all $i \geq 0$, then there is no quandle homomorphism other than trivial colorings. Therefore, we have following corollaries:

Corollary 1. *For a knot quandle $Q(K)$, if the Alexander polynomial $\Delta_K(t)$ is equal to 1 then for any Alexander quandle Λ_p/J there is no quandle homomorphism other than trivial colorings.*

Corollary 2. *For an Alexander quandle Λ_p/J and a knot quandle $Q(K)$, there is no quandle homomorphism other than trivial colorings if and only if the ideal generated by $\Delta_K(t)$ and J is equal to Λ_p.*

Let φ, φ' be quandle homomorphisms of a knot quandle $Q(K)$ to an Alexander quandle Λ_q/J and $f(t)$ an element of Λ_q/J. Let $[\mu]$ be an element of $Q(K)$. The *sum* of φ and φ' is defined to be the map which takes an element $[\mu]$ to $\varphi([\mu]) + \varphi'([\mu])$. Further the *scalar product* of φ by $f(t)$ is defined to be the map which takes $[\mu]$ to $f(t)\varphi([\mu])$.

Lemma 2. *The sum and the scalar product are quandle homomorphisms.*

The sum and scalar product of quandle homomorphisms correspond to the sum and scalar product of the vector space $\ker\psi$ of Theorem 1, respectively. Moreover we have the following proposition:

Proposition 1. *The set of all quandle homomorphisms of the knot quandle $Q(K)$ to the Alexander quandle Λ_p/J is isomorphic to $\Lambda_p/J \oplus \oplus_{i=0}^{n-2}\{\Lambda_p/(e_i(t), J)\}$.*

Proof. Let $\psi' : (\Lambda_p/J)^n \to (\Lambda_p/J)^n$ be the map which takes a row vector x' to $x'UA_{D_K}V$. Then $\ker\psi$ is isomorphic to $\ker\psi'$. The map $u : \ker\psi \to \ker\psi'$ such that $y \mapsto yU^{-1}$ for all $y \in \ker\psi$, gives such an isomorphism.

Further since the matrix $UA_{D_K}V$ is a diagonal matrix, $\ker\psi'$ is isomorphic to $\ker(f(t) \mapsto 0) \oplus \oplus_{i=0}^{n-2}\ker(f(t) \mapsto e_i(t)f(t))$ for all $f(t) \in \Lambda_p/J$ and $\mathrm{coker}\psi$ is isomorphic to $\Lambda_p/(J,0) \oplus \oplus_{i=0}^{n-2}\Lambda_p/(J, e_i(t))$. Let $g_J(t) \in \Lambda_p/J$ be the generator of the ideal J. Then the map $d : \ker\psi' \to \mathrm{coker}\psi$ such that $(y'_{-1}, y'_0, \cdots, y'_{n-2}) \mapsto (y'_{-1}, y'_0 g_J(t)e_0(t)^{-1}, \cdots, y'_{n-2}g_J(t)e_{n-2}(t)^{-1})$ for all $(y'_{-1}, y'_0, \cdots, y'_{n-2}) \in \ker\psi'$ gives such an isomorphism of $\ker\psi'$ to $\mathrm{coker}\psi$.

Therefore the composite map $d \circ u$ defines an isomorphism of the Proposition 1 □.

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References

[1] J. Carter, D. Jelsovsky, S. Kamada, L. Langford, M. Saito, *Quandle cohomology and state-sum invariants of knotted curves and surfaces*, preprint at http://xxx.lanl.gov/abs/math.GT/9903135

[2] J. Carter, D. Jelsovsky, S. Kamada, M. Saito, *Computations of quandle cocycle invariants of knotted curves and surfaces*, preprint at http://xxx.lanl.gov/abs/math.GT/9906115

[3] J. Carter, D. Jelsovsky, S. Kamada, M. Saito, *Quandle homology groups, their Betti numbers, and virtual knots*, preprint at http://xxx.lanl.gov/abs/math.GT/9909161

[4] R. Fenn, C. Rourke, *Racks and links in codimension two*, Journal of Knot Theory and Its Ramifications Vol. **1** No. 4 (1992), 343–406

[5] R. H. Fox, *Free differential calculus. II. The isomorphism problem of groups*, Ann. of Math. (2) **59** (1954), 196–210

[6] D. Joyce, *A classifying invariant of knots, the knot quandle*, J. Pure Appl. Algebra **23** (1982), 37–65

[7] A. Kawauchi, *A survey of knot theory*, Translated and revised from the 1990 Japanese original by the author, Birkhuser Verlag, Basel, 1996