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Algebraic Geometry — *The canonical ring of a* 3*-connected curve*, by Marco Franciosi and Elisa Tenni, communicated on 8 November 2013.

Abstract. — Let C be a projective curve either reduced with planar singularities or contained in a smooth algebraic surface.

We show that the canonical ring $R(C, \omega_C) = \bigoplus_{k \geq 0} H^0(C, {\omega_C}^{\otimes k})$ is generated in degree 1 if C is 3-connected and not (honestly) hyperelliptic; we show moreover that $R(C, L) = \bigoplus_{k \geq 0} H^0(C, L^{\otimes k})$ is generated in degree 1 if C is reduced with planar singularities and L is an invertible sheaf such that $\deg L_{|B} \geq 2p_a(B) + 1$ for every $B \subseteq C$[†].

Key words: Algebraic curve, Noether's theorem, canonical ring.

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1. Introduction

Let C be a projective curve either reduced with planar singularities or contained in a smooth algebraic surface, ω_C be its dualizing sheaf of C and L be an invertible sheaf on C.

The main result of this paper is Theorem 3.3 stating that the canonical ring

$$R(C, \omega_C) = \bigoplus_{k \geq 0} H^0(C, {\omega_C}^{\otimes k})$$

is generated in degree 1 if C is a 3-connected and not honestly hyperelliptic curve (see Definition 2.1 and Definition 2.2). This is a generalization to singular curves of the classical theorem of Noether for smooth curves (see [1, §III.2]) and can be regarded as a first step in a more general analysis of the Koszul groups $\mathcal{K}_{p,q}(C, \omega_C)$ of 3-connected curves (see [11] for the definition and the statement of the so called "*Green's conjecture*"). A detailed explanation of the role that the Koszul groups of smooth and singular curves play in the geometry of various moduli spaces can be found in [2].

Additional motivation for the present work comes from the theory of surface fibrations, as shown by Catanese and Ciliberto in [4] and Reid in [16]. Indeed, given a surface fibration $f : S \to B$ over a smooth curve B, the *relative canonical algebra* $R(f) = \bigoplus_{n \geq 0} f_*(\omega_{S/B}^{\otimes n})$ gives important information on the geometry of

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the surface. It is clear that the behaviour of $R(f)$ depends on the canonical ring of every fibre.

The main result on the canonical ring for singular curves in the literature is the 1-2-3 conjecture, stated by Reid in [16] and proved in [8] and [14], which says that the canonical ring $R(C, \omega_C)$ of a connected Gorenstein curve of arithmetic genus $p_a(C) \geq 3$ is generated in degree 1, 2, 3, with the exception of a small number of cases. More recently in [9] the first author proved that the canonical ring is generated in degree 1 under the strong assumption that C is *even* (i.e., $\deg_B K_C$ is even on every subcurve $B \subseteq C$).

We remark that our result implies in particular that the canonical ring of a regular surface of general type is generated in degree ≤ 3 if there exists a curve $C \in |K_S|$ 3-connected and not honestly hyperelliptic (see [9, Theorem 1.2]). Moreover one can apply the same argument of Konno (see [14, Theorem III]) and see that the relative canonical algebra of a relatively minimal surface fibration is generated in degree 1 if every fibre is 3-connected and not honestly hyperelliptic.

Our second result is Theorem 4.2 stating that the ring

$$R(C, L) = \bigoplus_{k \geq 0} H^0(C, L^{\otimes k})$$

considered as an algebra over $H^0(C, \mathcal{O}_C)$, is generated in degree 1 if C is reduced with planar singularities and L is an invertible sheaf such that $\deg L_{|B} \geq 2p_a(B) + 1$ for every $B \subseteq C$. This is a generalization of a theorem of Castelnuovo (see [15]) on the projective normality of smooth projective curves. This result can be useful when dealing with properties (for instance Brill-Noether properties) of a family of smooth curves which degenerates to C.

In [8], [9], [14] the analysis of the canonical ring is based on the study of the Koszul groups $\mathcal{K}_{p,q}(C, \omega_C)$ (with p, q small) and their vanishing properties, together with some vanishing results for invertible sheaves of low degree.

In this paper we use a completely different approach. Our method is inspired by the arguments developed in a series of papers by Green and Lazarsfeld which appeared in the late '80s (see [11], [12]) and it is based on the generalization to singular curves of Clifford's theorem given by the authors in [10].

Given an invertible sheaf L such that the map $H^0(C, L) \otimes H^0(C, L) \to H^0(C, L^{\otimes 2})$ fails to be surjective, we exhibit a zero-dimensional scheme $S \subset C$ such that the map $H^0(C, L) \otimes H^0(C, L) \to H^0(S, \mathcal{O}_S)$ induced by the restriction also fails to be surjective. Thus its dual map

$$\varphi : \mathrm{Ext}^1(\mathcal{O}_S, \omega_C \otimes L^{-1}) \to \mathrm{Hom}(H^0(C, L), H^1(C, \omega_C \otimes L^{-1}))$$

is not injective. From the analysis of an extension in the Kernel of φ we conclude that the cohomology of $\mathcal{I}_S \cdot L$ must satisfy some numerical conditions. This in turn contradicts Clifford's theorem when $L = \omega_C$, or $\deg L_{|B} \geq 2p_a(B) + 1$ on every $B \subseteq C$.

Finally, we wish to stress the role of numerical connectedness in generalizing Noether's theorem. By the results of [6] (see [6, §2, §3] or Theorem 2.3) and our main result Theorem 3.3 we have the following implications for a connected curve C

$$\begin{aligned} &C \text{ 3-connected, not honestly hyperelliptic} \\ &\quad \Rightarrow R(C, \omega_C) \text{ is generated in degree 1 and } \omega_C \text{ is ample} \\ &\quad \Rightarrow \omega_C \text{ is very ample.} \end{aligned}$$

If C is reduced it is known that the three properties are equivalent (see [5]). However this is false when C is not reduced. To see that the converse of the first implication fails one can take $C = 2F$, where F is a non-hyperelliptic fibre of a surface fibration. In Examples 3.5 and 3.6 we construct 3-disconnected curves with very ample canonical sheaves which fail Noether's Theorem, thus proving that the converse of the second implication is false too. These examples support our belief that 3-connected curves are the most natural generalization of smooth curves when dealing with the properties of the canonical embedding.

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2. NOTATION AND PRELIMINARY RESULTS

We work over an algebraically closed field $\mathbb{K}$ of characteristic ≥ 0.

Throughout this paper a curve C will be a Cohen-Macaulay scheme of pure dimension 1. It will be *projective, either reduced with planar singularities* (i.e. such that for every point $P \in C$ it is $\dim_{\mathbb{K}} \mathcal{M}/\mathcal{M}^2 \leq 2$ where $\mathcal{M}$ is the maximal ideal of $\mathcal{O}_{C,P}$) *or contained in a smooth algebraic surface* X, in which case we allow C to be reducible and non-reduced. Notice that C is Gorenstein.

In both cases we will use the standard notation for curves lying on smooth algebraic surface, writing $C = \sum_{i=1}^{s} n_i \Gamma_i$, where Γ_i are the irreducible components of C and n_i are their multiplicities.

A subcurve $B \subseteq C$ *is a Cohen-Macaulay subscheme of pure dimension* 1; it will be written as $\sum m_i \Gamma_i$, with $0 \leq m_i \leq n_i$ for every i.

Notice that under these assumptions every subcurve $B \subset C$ is Gorenstein too.

ω_C denotes the dualizing sheaf of C (see [13], Chap. III, §7), and $p_a(C)$ the arithmetic genus of C, $p_a(C) = 1 - \chi(\mathcal{O}_C)$. K_C denotes a canonical divisor.

DEFINITION 2.1. *A curve C is* honestly hyperelliptic *if there exists a finite morphism* $\psi : C \to \mathbb{P}^1$ *of degree* 2 (*see* [6, §3] *for a detailed treatment*).

If A, B are subcurves of C such that $A + B = C$, then their product $A \cdot B$ is

$$A \cdot B = \deg_B(K_C) - (2p_a(B) - 2) = \deg_A(K_C) - (2p_a(A) - 2).$$

If C is contained in a smooth algebraic surface X this corresponds to the intersection product of curves as divisors on X.

Definition 2.2. *C is* m-connected *if for every decomposition $C = A + B$ in effective, both non-zero curves, one has $A \cdot B \geq m$. C is* numerically connected *if it is 1-connected.*

First we recall some useful results proved in [5] and [6].

Theorem 2.3 ([6], §2, §3). *Let C be a Gorenstein curve. Then*

(i) *If C is 1-connected then $H^1(C, \omega_C) \cong \mathbb{K}$.*
(ii) *If C is 2-connected and $C \not\cong \mathbb{P}^1$ then $|\omega_C|$ is base point free.*
(iii) *If C is 3-connected and C is not honestly hyperelliptic then ω_C is very ample.*

Proposition 2.4 ([6], Lemma 2.4). *Let C be a projective scheme of pure dimension 1, let $\mathscr{F}$ be a coherent sheaf on C, and $\varphi : \mathscr{F} \to \omega_C$ a non-vanishing map of $\mathcal{O}_C$-modules. Set $\mathscr{J} = \operatorname{Ann} \varphi \subset \mathcal{O}_C$, and write $B \subset C$ for the subscheme defined by $\mathscr{J}$. Then B is Cohen–Macaulay and φ has a canonical factorization of the form*

$$\mathscr{F} \twoheadrightarrow \mathscr{F}_{|B} \hookrightarrow \omega_B = \mathscr{H}om_{\mathcal{O}_C}(\mathcal{O}_B, \omega_C) \subset \omega_C,$$

where $\mathscr{F}_{|B} \hookrightarrow \omega_B$ is generically onto.

Proposition 2.5 [5]. *Let C be a projective scheme of pure dimension 1, let $\mathscr{F}$ be a rank 1 torsion-free sheaf on C.*

(i) *If $\deg(\mathscr{F}_{|B}) \geq 2p_a(B) - 1$ for every subcurve $B \subseteq C$ then $H^1(C, \mathscr{F}) = 0$.*
(ii) *If $\mathscr{F}$ is invertible and $\deg(\mathscr{F}_{|B}) \geq 2p_a(B)$ for every subcurve $B \subseteq C$ then $|\mathscr{F}|$ is base point free.*
(iii) *If $\mathscr{F}$ is invertible and $\deg(\mathscr{F}_{|B}) \geq 2p_a(B) + 1$ for every subcurve $B \subseteq C$ then $\mathscr{F}$ is very ample on C.*

As we mentioned in the introduction, our approach to the analysis of the ring $R(C, L) = \bigoplus_{k \geq 0} H^0(C, L^{\otimes k})$ for a line bundle L builds on the generalization of Clifford's theorem proved by the authors in [10]. In the rest of this section we recall the main results we need from [10], namely, the notion of subcanonical cluster and Clifford's theorem, and we prove some technical lemmas on the cohomology of rank 1 torsion-free sheaves.

Definition 2.6. *A* cluster *S of* degree *r is a zero-dimensional subscheme of C with* length $\mathcal{O}_S = \dim_k \mathcal{O}_S = r$. *A cluster $S \subset C$ is* subcanonical *if the space $H^0(C, \mathscr{I}_S \omega_C)$ contains a generically invertible section, i.e., a section s_0 which does not vanish on any subcurve of C.*

Theorem 2.7 ([10], Theorem A). *Let C be a projective 2-connected curve either reduced with planar singularities or contained in a smooth algebraic surface, and let $S \subset C$ be a subcanonical cluster.*

Assume that S is a Cartier divisor or alternatively that there exists a generically invertible section $H \in H^0(C, \mathscr{I}_S K_C)$ such that $\operatorname{div}(H) \cap \operatorname{Sing}(C_{\mathrm{red}}) = \emptyset$.

Then

$$h^0(C, \mathcal{I}_S K_C) \le p_a(C) - \frac{1}{2}\deg(S).$$

Moreover if equality holds then the pair (S, C) *satisfies one of the following assumptions:*

(i) S *is trivial, i.e., it is either empty or it is a canonical divisor* K_C*;*
(ii) C *is honestly hyperelliptic and* S *is a multiple of the honest* g^1_2*;*
(iii) C *is* 3*-disconnected* (*i.e., there is a decomposition* $C = A + B$ *with* $A \cdot B = 2$).

Remark 2.8. Let C and S be as in Theorem 2.7. Then Riemann-Roch Theorem implies that

$$h^0(C, \mathcal{I}_S K_C) + h^1(C, \mathcal{I}_S K_C) \le p_a(C) + 1$$

and equality holds if one of the three cases listed in Theorem 2.7 is satisfied.

Remark 2.9. If $Z_0 \subset Z$ are clusters, then the natural restriction map $\mathcal{O}_Z \twoheadrightarrow \mathcal{O}_{Z_0}$ induces an inclusion $\mathrm{Ext}^1(\mathcal{O}_{Z_0}, \mathcal{O}_C) \hookrightarrow \mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}_C)$.

Lemma 2.10. *Let* C *be a Gorenstein curve and* Z *a cluster. Assume that there exists an extension* $\xi \in \mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}_C)$ *such that* $\xi \notin \mathrm{Ext}^1(\mathcal{O}_{Z_0}, \mathcal{O}_C)$ *for every proper subcluster* $Z_0 \subsetneq Z$*. Then the corresponding extension of sheaves can be written as*

$$0 \to \mathcal{O}_C \to \mathcal{H}om(\mathcal{I}_Z, \mathcal{O}_C) \to \mathcal{O}_Z \to 0.$$

Proof. Consider an extension corresponding to ξ

$$0 \to \mathcal{O}_C \to E_\xi \to \mathcal{O}_Z \to 0. \tag{1}$$

We prove first that E_ξ is torsion free. Indeed, if $\mathrm{Tor}(E_\xi) \ne 0$ then there exists a subcluster $Z_0 \subset Z$ and a sheaf $E_0 \cong E_\xi/\mathrm{Tor}(E_\xi)$ which fits in the following commutative diagram:

$$\begin{tikzcd}
 & & \mathrm{Tor}(E_\xi) \arrow[r, "\cong"] \arrow[d] & \mathrm{Tor}(E_\xi) \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_C \arrow[r] \arrow[d] & E_\xi \arrow[r] \arrow[d] & \mathcal{O}_Z \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_C \arrow[r] \arrow[d] & E_0 \arrow[r] \arrow[d] & \mathcal{O}_{Z_0} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

In particular there exists a proper subcluster Z_0 such that the extension corresponding to E_0 in $\mathrm{Ext}^1(\mathcal{O}_{Z_0}, \mathcal{O}_C)$ corresponds to ξ, which is impossible.

Since E_ξ is a rank 1 torsion-free sheaf it is reflexive, i.e., there is a natural isomorphism $\mathcal{H}om(\mathcal{H}om(E_\xi, \mathcal{O}_C), \mathcal{O}_C) \cong E_\xi$. Dualizing sequence (1) we see that $\mathcal{H}om(E_\xi, \mathcal{O}_C) \cong \mathcal{I}_Z$ since $\mathcal{E}xt^1(\mathcal{O}_Z, \mathcal{O}_C) \cong \mathcal{O}_Z$ and $\mathcal{E}xt^1(E_\xi, \mathcal{O}_C) = 0$, hence $E_\xi \cong \mathcal{H}om(\mathcal{I}_Z, \mathcal{O}_C)$. □

Remark 2.11. For every cluster $S \subset C$ and every invertible sheaf L on C, considering the embedding $\imath : S \hookrightarrow C$, with abuse of notation, we will write L instead of $\imath^*(L)$. Moreover, throughout the paper we will repeatedly use the isomorphisms $\mathcal{O}_S \cong \mathcal{O}_S \cdot L$ and $H^0(S, \mathcal{O}_S)^* \cong \mathrm{Ext}^1(\mathcal{O}_S, \omega_C \otimes L^{-1}) \cong \mathrm{Ext}^1(\mathcal{O}_S \cdot L, \omega_C \otimes L^{-1})$.

A useful tool in the analysis of the multiplication map $H^0(C, L)^{\otimes 2} \to H^0(C, L^{\otimes 2})$ is the restriction to a suitable cluster S. Indeed the composition of the multiplication map $H^0(C, L)^{\otimes 2} \to H^0(C, L^{\otimes 2})$ with the evaluation map $H^0(C, L^{\otimes 2}) \to H^0(S, \mathcal{O}_S)$ yields a map $H^0(C, L) \otimes H^0(C, L) \to H^0(S, \mathcal{O}_S)$.

Lemma 2.12. *Let C be a Gorenstein curve and L an effective line bundle on C. Let S be a cluster such that the restriction map*

$$H^0(C, L) \otimes H^0(C, L) \to H^0(S, \mathcal{O}_S)$$

is not surjective. Then there exists a nonempty subcluster $S_0 \subseteq S$ such that

$$h^0(C, L) + h^1(C, L) \leq h^0(C, \mathcal{I}_{S_0} L) + h^1(C, \mathcal{I}_{S_0} L).$$

Proof. Let S be a cluster such that the restriction map

$$H^0(C, L) \otimes H^0(C, L) \to H^0(S, \mathcal{O}_S)$$

is not surjective. By Serre duality the dual map

$$\varphi : \mathrm{Ext}^1(\mathcal{O}_S, \omega_C \otimes L^{-1}) \to \mathrm{Hom}(H^0(C, L), H^1(C, \omega_C \otimes L^{-1}))$$

is not injective. The dual map φ is given as follows: consider an element $\xi \in \mathrm{Ext}^1(\mathcal{O}_S, \omega_C \otimes L^{-1})$ and its corresponding extension

$$0 \to \omega_C \otimes L^{-1} \to E_\xi \to \mathcal{O}_S \to 0.$$

Let $c_\xi : H^0(S, \mathcal{O}_S) \to H^1(C, \omega_C \otimes L^{-1})$ be the connecting homomorphism induced by the extension. Then the restriction map $r : H^0(C, L) \to H^0(S, \mathcal{O}_S)$ induces a map $\varphi_\xi = c_\xi \circ r : H^0(C, L) \to H^1(C, \omega_C \otimes L^{-1}))$ given as follows

$$\begin{tikzcd}
 & & & H^0(C, L) \arrow[d, "r"] \arrow[dr, "\varphi_\xi"] & \\
0 \arrow[r] & H^0(C, \omega_C \otimes L^{-1}) \arrow[r] & H^0(C, E_\xi) \arrow[r, "f_\xi"] & H^0(S, \mathcal{O}_S) \arrow[r, "c_\xi"] & H^1(C, \omega_C \otimes L^{-1})
\end{tikzcd}$$

The map φ_ξ is precisely $\varphi(\xi) \in \mathrm{Hom}(H^0(C,L), H^1(C, \omega_C \otimes L^{-1}))$. By definition $\varphi(\xi) = 0$ if and only if $\mathrm{Im}(r) \subset \mathrm{Im}(f_\xi)$. In particular if $\varphi(\xi) = 0$ then we have $\dim \mathrm{Im}(r) \leq \dim \mathrm{Im}(f_\xi)$ which implies that

$$h^0(C,L) + h^1(C,L) \leq h^0(C, E_\xi) + h^0(C, \mathscr{I}_S L). \tag{2}$$

In order to prove the lemma let S_0 be minimal (with respect to the inclusion) among the subclusters of S for which the restriction $\mathrm{Sym}^2 H^0(C,L) \to H^0(S_0, \mathscr{O}_{S_0})$ fails to be surjective. This implies that if $Z \subsetneq S_0$ is any proper subcluster then the map

$$\varphi_0 : \mathrm{Ext}^1(\mathscr{O}_Z, \omega_C \otimes L^{-1}) \to \mathrm{Hom}(H^0(C,L), H^1(C, \omega_C \otimes L^{-1}))$$

is injective. Note that φ_0 factors through φ:

$$\begin{tikzcd}
\mathrm{Ext}^1(\mathscr{O}_Z, \omega_C \otimes L^{-1}) \arrow[d] \arrow[dr, "\varphi_0"] & \\
\mathrm{Ext}^1(\mathscr{O}_{S_0}, \omega_C \otimes L^{-1}) \arrow[r, "\varphi"] & \mathrm{Hom}(H^0(C,L), H^1(C, \omega_C \otimes L^{-1}))
\end{tikzcd}$$

By the minimality of S_0 if $\xi \in \mathrm{Ext}^1(\mathscr{O}_{S_0}, \omega_C \otimes L^{-1})$ is in the kernel of φ, then ξ must not belong to the image of $\mathrm{Ext}^1(\mathscr{O}_Z, \omega_C \otimes L^{-1})$ for every $Z \subsetneq S_0$. The corresponding extension E_ξ is isomorphic to $\mathscr{H}om(\mathscr{I}_{S_0}, \mathscr{O}_C) \otimes \omega_C \otimes L^{-1} \cong \mathscr{H}om(\mathscr{I}_{S_0} L, \omega_C)$ thanks to Lemma 2.10. Thus $h^0(C, E_\xi) = h^1(C, \mathscr{I}_{S_0} L)$ by Serre duality. Inequality (2) becomes

$$h^0(C,L) + h^1(C,L) \leq h^0(C, \mathscr{I}_{S_0} L) + h^1(C, \mathscr{I}_{S_0} L). \qquad \square$$

3. Noether's theorem for singular curves

The aim of this section is to prove Noether's theorem for singular curves. For the proof we use two main ingredients: a generalization of the *free pencil trick* (see Lemma 3.2), and the surjectivity of the restriction map $H^0(C, \omega_C)^{\otimes 2} \to H^0(S, \mathscr{O}_S)$ for a suitable cluster S.

Lemma 3.1. *Let C be a projective curve which is either reduced with planar singularities or contained in a smooth algebraic surface. Assume that C is 2-connected and $p_a(C) \geq 2$. Let $H \in H^0(C, \omega_C)$ be a generic section.*

Then there exists a cluster S contained in $\operatorname{div} H$ *such that the following hold:*

1. $h^0(C, \mathscr{I}_S \omega_C) = 2$
2. *the evaluation map $H^0(C, \mathscr{I}_S K_C) \otimes \mathscr{O}_C \to \mathscr{I}_S \omega_C$ is surjective.*

Proof. Since $|K_C|$ is base point free thanks to Theorem 2.3 we may assume that H is generically invertible and $\operatorname{div} H$ is a length $2p_a(C) - 2$ cluster. Thus for every integer $v \in \{1, \ldots, p_a(C)\}$ there exists at least one cluster $S_v \subseteq \operatorname{div} H$ such that $h^0(C, \mathscr{I}_{S_v}\omega_C) = v$. In particular we may take a cluster S such that $h^0(C, \mathscr{I}_S\omega_C) = 2$ and S is maximal up to inclusion among the clusters contained in $\operatorname{div} H$ with this property. S is the desired cluster. Indeed, if it were $S_0 \supsetneq S$ such that the image of the evaluation map $H^0(C, \mathscr{I}_S K_C) \otimes \mathscr{O}_C \to \mathscr{I}_S\omega_C$ was $\mathscr{I}_{S_0}\omega_C \subsetneq \mathscr{I}_S\omega_C$ then we would have $h^0(C, \mathscr{I}_{S_0}K_C) = 2$, contradicting the maximality of S. □

Even though the sheaf $\mathscr{I}_S\omega_C$ defined in the above lemma is not usually a line bundle by abuse of notation we will call it a *free pencil*.

LEMMA 3.2. *Let the pair (C, S) be as in the previous lemma. Then the map*

$$H^0(C, \mathscr{I}_S\omega_C) \otimes H^0(C, \omega_C) \xrightarrow{m} H^0(C, \mathscr{I}_S\omega_C^{\otimes 2}) \tag{3}$$

is surjective.

PROOF. Consider the evaluation map $H^0(C, \mathscr{I}_S\omega_C) \otimes \omega_C \xrightarrow{ev} \mathscr{I}_S\omega_C^{\otimes 2}$ and its kernel $\mathscr{K}$:

$$0 \to \mathscr{K} \to H^0(C, \mathscr{I}_S\omega_C) \otimes \omega_C \to \mathscr{I}_S\omega_C^{\otimes 2} \to 0. \tag{4}$$

The map (3) is surjective if and only if $h^1(C, \mathscr{K}) = 2$ since $h^1(C, \mathscr{I}_S\omega_C^{\otimes 2}) = 0$ by Proposition 2.5. In the rest of the proof we establish $h^1(C, \mathscr{K}) = 2$.

We have

$$\mathscr{K} \cong \mathscr{H}om(\mathscr{I}_S\omega_C, \omega_C).$$

Indeed consider a basis $\{x_0, x_1\}$ for $H^0(C, \mathscr{I}_S\omega_C)$ and define the map

$$\begin{aligned} \imath : \mathscr{H}om(\mathscr{I}_S\omega_C, \omega_C) &\to H^0(C, \mathscr{I}_S\omega_C) \otimes \omega_C \\ \varphi &\mapsto x_0 \otimes \varphi(x_1) - x_1 \otimes \varphi(x_0). \end{aligned}$$

Our aim is to check that $\imath$ is injective and $\imath(\mathscr{H}om(\mathscr{I}_S\omega_C, \omega_C))$ is precisely $\mathscr{K}$. It is clear that $\operatorname{Im}(\imath) \subset \mathscr{K}$. Moreover $\imath$ is injective since the sheaf $\mathscr{I}_S\omega_C$ is generated by its sections x_0 and x_1. It is straightforward to check that over the points $P \in C$ not belonging to S (where both the sheaves $\mathscr{H}om(\mathscr{I}_S\omega_C, \omega_C)$ and $\mathscr{K}$ are invertible), $\imath$ induces an isomorphism. Moreover computing the Euler characteristic we have

$$\chi(\mathscr{H}om(\mathscr{I}_S\omega_C, \omega_C)) = \chi(\mathscr{K}) = \deg S - (p_a(C) - 1)$$

hence the map $\imath$ induces an isomorphism between $\mathscr{H}om(\mathscr{I}_S\omega_C, \omega_C)$ and $\mathscr{K}$.

We know that

$$H^1(C, \mathscr{K})^* = \operatorname{Hom}(\mathscr{K}, \omega_C) = H^0(C, \mathscr{H}om(\mathscr{K}, \omega_C))$$

It is easy to check that $\mathscr{I}_S\omega_C$ is reflexive, i.e.,

$$\mathscr{H}om(\mathscr{H}om(\mathscr{I}_S\omega_C, \omega_C), \omega_C) \cong \mathscr{I}_S\omega_C$$

thus $h^1(C, \mathscr{K}) = h^0(C, \mathscr{I}_S\omega_C) = 2$. □

We may now prove our main theorem.

THEOREM 3.3. *Let C be a projective curve either reduced with planar singularities or contained in a smooth algebraic surface. Assume that C is 3-connected, not honestly hyperelliptic and $p_a(C) \geq 3$. Then the map*

$$\mathrm{Sym}^n H^0(C, \omega_C) \to H^0(C, \omega_C^{\otimes n})$$

is surjective for every $n \geq 0$.

PROOF. It is already known that the canonical ring $R(C, \omega_C) = \bigoplus_{n\geq 0} H^0(C, \omega_C^{\otimes n})$ is generated in degree at most 2: see Konno [14, Proposition 1.3.3] or Franciosi [8, Theorem C]. Notice that, even though both papers deal with the case of divisors on smooth surfaces, their proofs go through without changes to reduced curves with planar singularities. Thus to prove the theorem it is sufficient to show that the map in degree 2 is surjective:

$$H^0(C, \omega_C) \otimes H^0(C, \omega_C) \to H^0(C, \omega_C^{\otimes 2}).$$

We consider a free pencil $\mathscr{I}_S\omega_C$ (as in Lemma 3.1) and study the following commutative diagram:

$$\begin{tikzcd}
H^0(C, \mathscr{I}_S\omega_C) \otimes H^0(C, \omega_C) \arrow[r, hook] \arrow[d, "m"] & H^0(C, \omega_C)^{\otimes 2} \arrow[r] \arrow[d, "r"] \arrow[dr, "p"] & H^0(S, \mathscr{O}_S) \otimes H^0(C, \omega_C) \arrow[d] \\
H^0(C, \mathscr{I}_S\omega_C^{\otimes 2}) \arrow[r, hook] & H^0(C, \omega_C^{\otimes 2}) \arrow[r] & H^0(S, \mathscr{O}_S)
\end{tikzcd}$$

A simple diagram chase shows that if both the maps m and p are surjective, then the product map r is surjective too, proving the theorem.

Lemma 3.2 states precisely that the map m is surjective.

The map p must be surjective too: if not, we could apply Lemma 2.12 and conclude that there exists a nonempty subcanonical cluster $S_0 \subseteq S$, contained in a generic section in $H^0(C, \omega_C)$, such that

$$h^0(C, \mathscr{I}_{S_0}\omega_C) + h^1(C, \mathscr{I}_{S_0}\omega_C) \geq p_a(C) + 1. \tag{5}$$

By Theorem 2.7 and Remark 2.8 we know that this cannot happen if C is 3-connected and not honestly hyperelliptic, and $S_0 \neq \emptyset$, K_C. □

Remark 3.4. The assumptions of Theorem 3.3 are sharp. One can find examples of 3-disconnected curves, with very ample canonical sheaf, such that the map $\mathrm{Sym}^2 H^0(C, \omega_C) \to H^0(C, \omega_C^{\otimes 2})$ is not surjective, see Examples 3.5 and 3.6.

Moreover, one expects the surjectivity of the map to fail for 3-disconnected curves thanks to a simple observation on reduced curves. Indeed, consider a connected *reduced* curve, 2-connected but 3-disconnected (hence with at least two components). In this case there exists a subcurve $B \subset C$ such that $\omega_{C|B} \cong \omega_B \otimes \mathcal{O}_B(S)$, where S is a length 2 cluster and by [6, §3] we conclude that its canonical sheaf ω_C is not very ample. In particular if C does not contain rational curves then ω_C turns out to be ample but not very ample. Therefore $R(C, \omega_C)$ cannot be generated in degree 1.

Example 3.5. Let B be a smooth genus b curve with $b \geq 4$ and let D be a general effective divisor on B of degree $b+3$. The linear system $|D|$ is very ample and induces an embedding of B in $\mathbb{P}^3$ (see [1, Ex. V.B.1]).

Define the ruled surface $X = \mathbb{P}_B(\mathcal{O}_B \oplus \mathcal{O}_B(D - K_B))$: the map $f : X \to B$ has a section Γ with selfintersection $(-b+5)$ (see [13, §V.2] for the main numerical properties). Consider the curve $C = 2\Gamma$: we have that $p_a(C) = b+4$ and C is 2-disconnected (numerically disconnected if $b \geq 5$). By adjunction we have

$$K_C = (K_X + C)_{|C} = f^*(D)$$

and it is easy to check that it is very ample on C by analyzing the standard decomposition

$$0 \to \omega_\Gamma \to \omega_C \to \omega_{C|\Gamma} \to 0.$$

Since Γ is a section of $f : X \to B$, f induces an isomorphism $(B, \mathcal{O}_B(D)) \cong (\Gamma, \omega_{C|\Gamma})$. Therefore it is immediately seen that $|\omega_C|$ separates length 2 clusters.

The map

$$\mathrm{Sym}^2 H^0(C, \omega_C) \xrightarrow{q_0} H^0(C, \omega_C^{\otimes 2})$$

is not surjective, as one could see from the following diagram:

$$\begin{tikzcd}
H^0(\Gamma, \omega_\Gamma) \otimes H^0(C, \omega_C) \arrow[r] \arrow[d] & H^0(C, \omega_C)^{\otimes 2} \arrow[r] \arrow[d, "q"] & H^0(\Gamma, \omega_{C|\Gamma}) \otimes H^0(\Gamma, \omega_{C|\Gamma}) \arrow[d, "p"] \\
H^0(\Gamma, \omega_\Gamma \otimes \omega_C) \arrow[r] & H^0(C, \omega_C^{\otimes 2}) \arrow[r, two heads] & H^0(\Gamma, \omega_{C|\Gamma}^{\otimes 2})
\end{tikzcd}$$

Indeed if the map q_0 were surjective (hence q), then the map p would be surjective as well. Since $\omega_{C|\Gamma} \cong \mathcal{O}_B(D)$ the image of the map p is the same as the image of the map

$$p_0 : \mathrm{Sym}^2 H^0(B, \mathcal{O}_B(D)) \to H^0(B, \mathcal{O}_B(2D))$$

which is not surjective, as one can easily check by computing the dimension of the two spaces.

EXAMPLE 3.6. We now show an example of a 2-connected but 3-disconnected curve with very ample canonical sheaf which does not satisfy Theorem 3.3.

Let B be a smooth hyperelliptic curve of genus $b \geq 3$ contained in a smooth algebraic surface. Assume $B^2 = 2$ and consider the curve $C = 2B$. For example, run the construction of the previous example with $b = 3$.

If the degree 2 line bundle $\omega_{C|B} \otimes \omega_B^{-1}$ is not effective then ω_C turns out to be very ample. Furthermore, consider the commutative diagram

$$\begin{tikzcd}
H^0(B,\omega_B) \otimes H^0(C,\omega_C) \arrow[r] \arrow[d] & H^0(C,\omega_C)^{\otimes 2} \arrow[r] \arrow[d, "q"] & H^0(B,\omega_{C|B}) \otimes H^0(\Gamma,\omega_{C|B}). \arrow[d, "p"] \\
H^0(B,\omega_B \otimes \omega_C) \arrow[r] & H^0(C,\omega_C^{\otimes 2}) \arrow[r, two heads] & H^0(B,\omega_{C|B}^{\otimes 2})
\end{tikzcd}$$

As in the previous example, we know that if the map q is surjective, then the map p is surjective as well, but this contradicts the assumption of B being hyperelliptic by [12, Corollary 1.4].

4. CASTELNUOVO'S THEOREM FOR REDUCED CURVES

In this section we prove a generalization of Castelnuovo's theorem for reduced curves.

In the proof we will apply Lemma 2.12 and the following Proposition, which is a Clifford-type result for line bundles of high degree.

PROPOSITION 4.1. *Let C be a projective reduced curve with planar singularities and L a line bundle on C such that*

$$\deg L_{|B} \geq 2p_a(B) + 1 \quad \textit{for every } B \subset C.$$

If S is a cluster contained in a generic section $H \in H^0(C,L)$ then

$$h^0(C,\mathcal{I}_S L) + h^1(C,\mathcal{I}_S L) < h^0(C,L).$$

PROOF. Notice at first that $H^1(C,L) = 0$ and $|L|$ is very ample by Proposition 2.5. Therefore a generic hyperplane section consists of $\deg L$ smooth points. Moreover by Riemann-Roch theorem we have

$$\begin{aligned}
h^0(C,\mathcal{I}_S L) + h^1(C,\mathcal{I}_S L) < h^0(C,L) &\Leftrightarrow h^0(C,\mathcal{I}_S L) < h^0(C,L) - \frac{1}{2}\deg S \\
&\Leftrightarrow h^1(C,\mathcal{I}_S L) < \frac{1}{2}\deg S
\end{aligned}$$

We argue by induction on the number of irreducible components of C. Suppose that C is irreducible or that the statement holds for every reduced curve with fewer components of C. If C is disconnected the statement is trivial, hence we may assume that C is connected. If $h^0(C, \mathscr{I}_S L) = 0$ or $h^1(C, \mathscr{I}_S L) = 0$ the result is trivial too, thus we may assume $h^0(C, \mathscr{I}_S L) > 0$ and $h^1(C, \mathscr{I}_S L) > 0$.

Suppose at first that there exists a proper subcurve $B \subset C$ such that

$$H^1(C, \mathscr{I}_S L) \cong H^1(B, \mathscr{I}_S L_{|B}).$$

By induction we have that $h^1(C, \mathscr{I}_S L) < \frac{1}{2} \deg S_{|B} \leq \frac{1}{2} \deg S$ and we may conclude.

If there is no subcurve $B \subset C$ as above (e.g. when C is irreducible) we can easily deduce from Proposition 2.4 that there exists a generically surjective map $\mathscr{I}_S L \hookrightarrow \omega_C$, hence we may assume that there exists a subcanonical cluster Z such that

$$\mathscr{I}_S L \cong \mathscr{I}_Z \omega_C.$$

Since S is contained in a generic section of $H^0(C, L)$ it is a Cartier divisor, hence Z is a Cartier divisor too.

If C is 2-connected we apply Theorem 2.7 and we conclude since

$$\begin{aligned} h^0(C, \mathscr{I}_S L) = h^0(C, \mathscr{I}_Z \omega_C) &\leq p_a(C) - \frac{1}{2} \deg Z \\ &= \frac{1}{2} \deg L + 1 - \frac{1}{2} \deg S < h^0(C, L) - \frac{1}{2} \deg S. \end{aligned}$$

If C is 2-disconnected, we can find a decomposition $C = C_1 + C_2$, such that $C_1 \cdot C_2 = 1$ and C_1 is 2-connected (see [7, Lemma A.4]). Thus we consider the following exact sequence:

$$0 \to \mathscr{I}_{Z|C_1} \omega_{C_1} \to \mathscr{I}_Z \omega_C \to (\mathscr{I}_Z \omega_C)_{|C_2} \to 0.$$

We know by induction that

$$\begin{aligned} h^0(C_2, (\mathscr{I}_Z \omega_C)_{|C_2}) = h^0(C, (\mathscr{I}_S L)_{|C_2}) &< h^0(C_2, L_{|C_2}) - \frac{1}{2} \deg S_{|C_2} \\ &= -p_a(C_2) + 1 + \deg L_{|C_2} - \frac{1}{2} \deg S_{|C_2}. \end{aligned}$$

We apply Theorem 2.7 to $\mathscr{I}_{Z|C_1} \omega_{C_1}$ since the single point $C_1 \cap C_2$ is a base point for $|\omega_C|$, hence the space $H^0(C_1, \mathscr{I}_{Z|C_1} \omega_{C_1}) \cong H^0(C_1, \mathscr{I}_{Z|C_1} \omega_{C|C_1})$ contains an invertible section, that is $Z_{|C_1}$ is a subcanonical cluster. Thus

$$h^0(C_1, \mathscr{I}_{Z|C_1} \omega_{C_1}) \leq p_a(C_1) - \frac{1}{2} \deg Z_{|C_1} = \frac{1}{2} \deg L_{|C_1} - \frac{1}{2} \deg S_{|C_1} + \frac{1}{2}$$

and we conclude since

$$\begin{aligned} h^0(C, \mathcal{I}_S L) &= h^0(C, \mathcal{I}_Z \omega_C) \leq h^0(C_2, (\mathcal{I}_Z \omega_C)_{|C_2}) + h^0(C_1, \mathcal{I}_{Z|C_1} \omega_{C_1}) \\ &< -p_a(C_2) + 1 + \deg L_{|C_2} - \frac{1}{2} \deg S_{|C_2} + \frac{1}{2} \deg L_{|C_1} - \frac{1}{2} \deg S_{|C_1} + \frac{1}{2} \\ &\leq h^0(C, L) - \frac{1}{2} \deg S. \end{aligned}$$ □

THEOREM 4.2. *Let C be a projective reduced curve with planar singularities and let L be a line bundle on C such that*

$$\deg L_{|B} \geq 2p_a(B) + 1 \quad \textit{for every } B \subset C.$$

Then the product map

$$\mathrm{Sym}^n H^0(C, L) \to H^0(C, L^{\otimes n})$$

is surjective for every $n \geq 1$.

PROOF. Notice at first that $H^1(C, L) = 0$ and L is very ample by Proposition 2.5.
If $n \geq 2$ the map

$$H^0(C, L^{\otimes n}) \otimes H^0(C, L) \to H^0(C, L^{\otimes (n+1)})$$

is surjective by [8, Proposition 1.5] since $H^1(C, L^{\otimes n} \otimes L^{-1}) = 0$. In order to prove the theorem we check that the map in degree 2 is surjective:

$$\mathrm{Sym}^2 H^0(C, L) \to H^0(C, L^{\otimes 2}).$$

To this aim we consider a generic hyperplane section $S = \mathrm{div}\, L$ and the following commutative diagram

$$\begin{tikzcd} H^0(C, \mathcal{I}_S L) \otimes H^0(C, L) \arrow[r, hook] \arrow[d, two heads] & H^0(C, L) \otimes H^0(C, L) \arrow[r] \arrow[d, "r"] \arrow[dr, "p"] & H^0(S, \mathcal{O}_S) \otimes H^0(C, L) \arrow[d] \\ H^0(C, \mathcal{I}_S L^{\otimes 2}) \arrow[r, hook] & H^0(C, L^{\otimes 2}) \arrow[r, two heads] & H^0(S, \mathcal{O}_S) \end{tikzcd}$$

Notice that the first column is surjective since $\mathcal{I}_S L \cong \mathcal{O}_C$ while the second row is exact since $H^1(C, \mathcal{I}_S L^{\otimes 2}) \cong H^1(C, L) = 0$. A simple diagram chase shows that the map r is surjective if and only if the map p is surjective.

It is $h^0(C, \mathcal{I}_{S_0} L) + h^1(C, \mathcal{I}_{S_0} L) < h^0(C, L)$ for every subcluster $S_0 \subseteq S$ by Proposition 4.1, hence the map p must be surjective by Lemma 2.12. □

REMARK 4.3. If C is numerically connected our result implies that the embedded curve $\varphi_L(C) \subset \mathbb{P}H^0(C, L)^*$ is arithmetically Cohen-Macaulay.

REMARK 4.4. At present, we do not know if the result holds for non-reduced curves with similar assumptions on the multidegree of the line bundle L. We have partial results under some assumptions on the connectedness of the curve and the self-intersection of its components. Notice that in [8] it has been shown to be true for adjoint divisor.

Since one of the main applications of this result concerns the analysis of moduli space of curves (see [3]) we have preferred to keep a simple statement under this strong assumption.

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