Corso di laurea in Ingegneria Gestionale/ Chimica
Esame di ALGEBRA LINEARE - anno accademico 2013/2014

**Prova scritta del 20/12/2013**
TEMPO A DISPOSIZIONE: 120 minuti

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## PRIMA PARTE

PUNTEGGIO : risposta mancante = 0 ; risposta esatta = +1 risposta sbagliata = -1
calcoli e spiegazioni non sono richiesti

- $z = -\sqrt{3} - i \implies z^3 = \boxed{-8i}$ ; $z^{-1} = \boxed{-\frac{\sqrt{3}}{4} + i\frac{1}{4}}$

- Dati $W$ e $Z$ i seguenti sottospazi di $\mathbb{R}^3$ :

$$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \ : \ x_1 + 2x_2 - x_3 = 0 \right\},\ Z = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \right\rangle.$$

Allora $\mathbb{R}^3 = W \bigoplus Z$ vero [✓ falso]

Determinare una base di $W \cap Z$:

$$\boxed{\left\{ \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \right\}}$$

- $A = \begin{pmatrix} 3 & 1 & 1 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 3 \end{pmatrix} \implies \dim(Ker(l_A)) = \boxed{4}$ $\quad \mathrm{rg}(A) = \boxed{2}$

- Le soluzioni del sistema $\begin{pmatrix} 3 & 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ costituiscono uno spazio affine di dimensione = $\boxed{3}$

- $\det \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 2 & 3 & -1 \\ 0 & 3 & 3 & -1 \\ 1 & 0 & 0 & 1 \end{pmatrix} = \boxed{3}$

- $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \implies A$ è diagonalizzabile [✓ vero] falso

- $A = \begin{pmatrix} 1 & 5 \\ 2 & 8 \end{pmatrix} \implies A^{-1} = \begin{pmatrix} -4 & \frac{5}{2} \\ 1 & -\frac{1}{2} \end{pmatrix}$

compito 20-12-2013

Traccia sol.

① $$\begin{cases} (z-3i)^3 = 2\,(3i+\overline{z}) \\ |z-6i| \leq |z| \end{cases}$$

(i) $w = z - 3i$

$\overline{w} = \overline{z} + 3i \longrightarrow$ (i) $\Leftrightarrow$ $w^3 = 2\overline{w}$

$w = \rho \cdot e^{i\vartheta}$

(i) $\Leftrightarrow$ $w^3 = \rho^3 \cdot e^{i3\vartheta}$

$\overline{w} = 2 \cdot \rho \cdot e^{-i\vartheta}$

(i) $\Leftrightarrow$ $$\begin{cases} \rho^3 = 2\rho \quad , \rho \in \mathbb{R}^+ \\ 3\vartheta = -\vartheta + 2k\pi, \quad k \in \mathbb{Z} \end{cases}$$

sol.: $\rho = 0$ $\Updownarrow$ $w=0$

$$\begin{cases} \rho = \sqrt{2} \\ \vartheta = \dfrac{2k\pi}{4} \end{cases} \quad k = 0, \dots, 3$$

$$W_i = \begin{cases} \sqrt{2} \\ \sqrt{2}i \\ -\sqrt{2} \\ -\sqrt{2}i \\ 0 \end{cases}$$

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$$\longrightarrow \quad z_0 = \sqrt{2}+3i$$

$$z_1 = i(\sqrt{2}+3)$$

$$z_2 = -\sqrt{2}+3i$$

$$z_3 = -\sqrt{2}+3i$$

$$z_4 = 3i$$

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ii) $|z-6i| \le |z| \iff$

$$\sqrt{x^2+(y-6)^2} \le \sqrt{x^2+y^2} \iff$$

$\dots \iff$ $y \geq 3$
x qualsiasi

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CONCLUSIONE:

soluzioni sistema $z_0$ $z_1$ $z_2$ $z_4$

②

$$A_t = \begin{pmatrix} -1 & 1 & t \\ t & 2 & -2 \\ -1 & 1 & 3 \end{pmatrix}$$

$$\det(A_t) = t^2 - t - 6$$

$$\det(A_t) \neq 0 \iff t \neq -2, 3$$

$t = -2$: $$A = \begin{pmatrix} -1 & 1 & -2 \\ -2 & 2 & -2 \\ -1 & 1 & 3 \end{pmatrix} \qquad rg = 2$$

$t = 3$ $$A = \begin{pmatrix} -1 & 1 & 3 \\ 3 & 2 & -2 \\ -1 & 1 & 3 \end{pmatrix} \qquad rg = 2$$

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$$t \neq 2, 3 \quad \begin{cases} rg = 3 \\ \dim(\ker) = 0 \end{cases}$$

$$t = 2, 3 \quad \begin{cases} rg = 2 \\ \dim(\ker) = 3 - 2 = 1 \end{cases}$$

$$A_t \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ -3 \end{pmatrix}$$

Se $t \neq -2, 3$ $\quad rg(A_t) = 3$

$(A_t : b)$ $\quad 3 \times 4 \Rightarrow rg \leq 3$

$\Rightarrow \quad t \neq -2, 3 \quad 3 \geq rg(A_t : b) \geq rg(A_t) = 3$

$\Rightarrow \quad \exists !$ soluzione

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casi particolari

$t = 3$

$$(A : b) = \begin{pmatrix} -1 & 1 & 3 & -3 \\ 3 & 2 & -2 & 2 \\ -1 & 1 & 3 & -3 \end{pmatrix}$$

$b = -$ III col.

$\Rightarrow \quad rg(A : b) = rg(A) = 2$

Per $t = 3$ $\quad \exists$ soluzione

& $\dim\{\text{SOLUZIONI}\} = 1$

$t = -2$

$$(A\vdots b) = \left(\begin{array}{cccc} -1 & 1 & -2 & -3 \\ -2 & 2 & -2 & 2 \\ -1 & 1 & 3 & 3 \\ & \multicolumn{3}{c}{\underbrace{\hphantom{1 \quad -2 \quad 3}}_{M}} \end{array}\right)$$

$\det(M) \neq 0 \Rightarrow rg(A \vdots b) = 3$

CIOE' per $t = -2$

$$rg(A \vdots b) = 3 > 2 = rg(A)$$

$\Rightarrow$ non $\exists$ SOLUZIONE.

(iii) $W = \langle \begin{pmatrix}1\\7\\1\end{pmatrix} \rangle$ $\quad\dim(W)=1$

$$\mathbb{R}^3 = W \oplus \mathrm{Im}(\ell_{A_t})$$

$$\Longleftrightarrow \begin{cases} \dim(W) + \dim(\mathrm{Im}\,\ell_{A_t}) = 3 \\ \left\{\begin{pmatrix}1\\7\\1\end{pmatrix}, v_1, v_2\right\} \text{ BASE di } \mathbb{R}^3 \end{cases}$$

dove $\{v_1, v_2\}$ BASE di $\mathrm{Im}(\ell_{A_t})$

→ Unici casi possibili $t=-2,\ 3$

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$t=-2$: BASE $\mathrm{Im}(\ell_{A_t}) = \left\{ \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}-2\\-2\\3\end{pmatrix} \right\}$

$$\det \begin{pmatrix} 1 & 1 & -2 \\ 7 & 2 & -2 \\ 1 & 1 & 3 \end{pmatrix} \neq 0 \Rightarrow \left\{ \begin{pmatrix}1\\7\\1\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}-2\\-2\\3\end{pmatrix} \right\}$$

è base di $\mathbb{R}^3$

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$t=3$ Base $\mathrm{Im}(\ell_{A_t}) = \left\{ \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}3\\-2\\3\end{pmatrix} \right\}$

$$\det \begin{pmatrix} 1 & 1 & 3 \\ 7 & 2 & -2 \\ 1 & 1 & 3 \end{pmatrix} = 0 \Rightarrow \left\{ \begin{pmatrix}1\\7\\1\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}-2\\-2\\3\end{pmatrix} \right\}$$

non è base

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RISPOSTA: $t=-2$

③ $\mathrm{Im}(f) = \left\langle \begin{pmatrix} -1 \\ 3 \end{pmatrix} \right\rangle$

$\mathrm{Ker}(f) = \mathrm{Im}(f) = \left\langle \begin{pmatrix} -1 \\ 3 \end{pmatrix} \right\rangle$

$A \Leftrightarrow f$

$rg(A) = 1$

Possiamo scegliere $A = \begin{pmatrix} -1 & -\alpha \\ 3 & 3\alpha \end{pmatrix}$

$\mathrm{Ker}(f) = \left\langle \begin{pmatrix} -1 \\ 3 \end{pmatrix} \right\rangle \Leftrightarrow \quad A \cdot \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\Leftrightarrow \begin{cases} 1 - 3\alpha = 0 \\ -3 + 9\alpha = 0 \end{cases} \qquad \alpha = \frac{1}{3}$$

RISPOSTA : $A = \begin{pmatrix} -1 & -\frac{1}{3} \\ 3 & 1 \end{pmatrix}$

④

$$A = \begin{pmatrix} 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{pmatrix}$$

$$P_A(\lambda) = \det(A - \lambda\, Id) = \dots = \lambda^4$$

Autovalori : 0 $\quad$ m.a. = 4

$$rg(A) = 2 \qquad \begin{pmatrix} \text{I col} = -\text{II col} \\ \text{III col} = 0 \\ \text{I, IV lin. ind.} \end{pmatrix}$$

$$\Rightarrow \quad m.g.(0) = 4 - 2 = 2$$

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ii) Autovettori :

$$A \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \quad \left\{ t \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} : \quad (s,t) \neq (0,0) \right\}$$

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iii)

$$Ker(f) \Leftrightarrow \begin{cases} x_1 - x_2 \qquad - x_4 = 0 \\ \qquad\qquad\qquad x_4 = 0 \end{cases}$$

Poichè $rg(A) = 2$

(sono sufficienti 2 equazioni ind.)

$$\mathrm{Im}(f) = \left\langle \begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}-1\\0\\1\\-1\end{pmatrix} \right\rangle = \langle \text{I col.}, \text{IV col} \rangle$$

$$\mathrm{Im}(f) = \alpha \cdot \begin{pmatrix}1\\0\\0\\1\end{pmatrix} + \beta \begin{pmatrix}-1\\0\\1\\-1\end{pmatrix}$$

$$\mathrm{Im}(f): \begin{cases} x_1 = \alpha - \beta \\ x_2 = 0 \\ x_3 = \beta \\ x_4 = \alpha - \beta \end{cases}$$

$\mathrm{Ker}(f) \cap \mathrm{Im}(f)$ : sostituiamo nel sistema di eq. del ker:

$$\begin{cases} (\alpha - \beta) \qquad - (\alpha - \beta) = 0 \\ \alpha - \beta = 0 \end{cases}$$

Sol. $\alpha = \beta$

CONCLUSIONE: $\mathrm{Im}(f) \cap \mathrm{Ker}(f) =$

$$\alpha \cdot \begin{pmatrix}1\\0\\0\\1\end{pmatrix} + \alpha \begin{pmatrix}-1\\0\\1\\-1\end{pmatrix} = \alpha \cdot \begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \left\langle \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \right\rangle$$