Hausdorff Ultrafilters[*]

Mauro Di Nasso

Dipart. di Matematica "L. Tonelli"

Università di Pisa, Italy.

`dinasso@dm.unipi.it`

Marco Forti

Dipart. di Matem. Applicata "U. Dini"

Università di Pisa, Italy.

`forti@dma.unipi.it`

Abstract

We name *Hausdorff* those ultrafilters that provide ultrapowers whose natural topology (S-topology) is Hausdorff. E.g. selective ultrafilters are Hausdorff. Here we give necessary and sufficient conditions for product ultrafilters to be Hausdorff. Moreover we show that no regular ultrafilter over the "small" uncountable cardinal $\mathfrak{u}$ can be Hausdorff. ($\mathfrak{u}$ is the least size of an ultrafilter basis on ω.) We focus on countably incomplete ultrafilters, but our main results hold also for κ-complete ultrafilters.

2000 Mathematics Subject Classification 03E05; 03H05; 54D80.

Introduction

Let $\mathcal{U}$ be an ultrafilter over the set I. Given any set X, the corresponding ultrapower $X^I/\mathcal{U}$ can be given a natural topology by taking the family of all subsets of the form $A^I/\mathcal{U}$ for $A \subseteq X$ as a (cl)open basis. (This topology is

named S-topology (*standard topology*) in nonstandard analysis.) It turns out that the resulting topological space is Hausdorff if and only if the ultrafilter $\mathcal{U}$ satisfies a simple algebraic property, namely:

(H) for all $f, g : I \to I$ ($f(\mathcal{U}) = g(\mathcal{U}) \implies f \equiv_{\mathcal{U}} g$).[1]

To the authors' knowledge, this property was first isolated for ultrafilters over ω in [8], under the name "property (C)". There it is proved that (C) holds for all 3-*arrow* ultrafilters, as introduced in [2]. In particular, weakly Ramsey ultrafilters, and also products of pairs of non-isomorphic selective ultrafilters have property (C). Uniform Hausdorff ultrafilters over I are called *separating* in [13], where it is shown that they cannot be *regular* if $|I| \geq 2^{\aleph_0}$. Several large cardinal consequences are then derived from the existence of separating ultrafilters on uncountable sets. A corresponding property $S(\mathcal{I})$ for (possibly non-maximal) ideals was studied in [16]. Focusing on uncountable cardinals κ, a combinatorial condition is given there in order to let $S(\mathcal{I})$ *fail* for suitable κ-complete ideals.

The Hausdorff ultrafilters came to the authors' attention in their attempts [3, 9] to isolate a few topological properties that could characterize the non-standard models. It is a well-known fact in nonstandard analysis that every nonstandard model X^* can be canonically mapped into the Stone-Čech compactification βX of the discrete space X (see e.g. [17]). Starting from this fact, it is proved that a nonstandard model X^* is Hausdorff (when equipped with the S-topology) if and only if it is isomorphic to an *invariant* (i.e. Rudin-Keisler downward closed) subspace of βX where all points (ultrafilters) satisfy the property (H). The name *Hausdorff ultrafilters* was first used in [9], to stress the "topological" significance of that special class of ultrafilters. Also from a model-theoretic point of view, Hausdorff ultrafilters could be of interest. In fact their existence is equivalent to that of nonstandard extensions of the complete structure on the natural numbers where *different elements have different types*.

The consistency strength of Hausdorff ultrafilters seems to be an interesting problem. The countable case remained unsettled till very recently.[2] On the one hand, it is easily seen that all *selective* ultrafilters are Hausdorff, but the two notions are different. For instance, under the Continuum Hypothesis CH (or even Martin's Axiom MA), there are Hausdorff ultrafilters that are

---

[1] See Section 1 for definitions and notation.

[2] The question has been posted by the authors to the "Logic café" of Logic Colloquium '03, Helsinki.

not P-points (hence not selective) and there are (necessarily non-selective) P-points that are not Hausdorff. However the most interesting question as to whether the existence of Hausdorff ultrafilters is provable in ZFC alone has been solved in the negative by T. Bartoszynski and S. Shelah [1], when this paper was almost completed.

Here we prove a general criterion for a product ultrafilter to be Hausdorff. In particular, under a natural incompatibility condition, Hausdorff ultrafilters are stable under products by Hausdorff P-points. Thus, provided many non-isomorphic selective ultrafilters are at hand, one can produce a large class of Hausdorff ultrafilters by means of iterated products. As first pointed out in [13], Hausdorff ultrafilters on uncountable sets are highly problematic, connected as they are to *irregular* ultrafilters. Here we show that Hausdorff uniform ultrafilters on the *small* uncountable cardinal $\mathfrak{u}$ cannot be *regular*. ($\mathfrak{u}$ is the least size of an ultrafilter basis on ω.) However, regular uniform ultrafilters on ω_1 can be constructed under Martin's axiom plus $2^{\aleph_0} = \aleph_2$. The consistency of regular uniform Hausdorff ultrafilters on cardinals greater than ω_1 is an open problem.

The paper is organized as follows. In Section 1, we characterize those products of ultrafilters that are Hausdorff. In Section 2 we give an upper bound for the size of sets admitting regular uniform Hausdorff ultrafilters. In the final Section 3, we face the problem of delimiting the consistency strength of Hausdorff ultrafilters. We briefly discuss also κ-complete ultrafilters over an uncountable measurable cardinal κ. We conclude with a list of open questions and suggestions for further research.

1 Products of Hausdorff ultrafilters

While fixing notation, we recall some basic definitions and facts. Throughout this paper, $\mathcal{U}$ and $\mathcal{V}$ will denote ultrafilters on I and J respectively. We are mainly interested in the countable case, but all results of this section are independent of the sizes of the sets I and J.

The *additivity* of a nonprincipal ultrafilter $\mathcal{U}$ is the least amount of sets outside $\mathcal{U}$ having union in $\mathcal{U}$. $\mathcal{U}$ is κ-*complete* if its additivity is at least κ,

i.e. $\mathcal{U}$ is (nonprincipal and) closed under intersections of size less than κ. It is well known that the additivity of any ultrafilter is a measurable cardinal.[3] Ultrafilters of additivity ω are usually called *countably incomplete*.

Two functions f and g with domain I are *equivalent modulo $\mathcal{U}$* (or $\mathcal{U}$-equivalent) iff $\{i \in I \mid f(i) = g(i)\} \in \mathcal{U}$. In this case we write $f \equiv_\mathcal{U} g$. If $f : I \to J$, the family $f(\mathcal{U}) = \{B \subseteq J \mid f^{-1}(B) \in \mathcal{U}\}$ is an ultrafilter on J.[4] The *Rudin-Keisler preordering* on ultrafilters is defined by setting $\mathcal{V} \leq_{RK} \mathcal{U}$ whenever $\mathcal{V} = f(\mathcal{U})$ for some f. In this case we say that $\mathcal{V}$ is *RK-below* $\mathcal{U}$ (or $\mathcal{U}$ is *RK-above* $\mathcal{V}$). It is a basic fact in the theory of ultrafilters that $f(\mathcal{U}) = \mathcal{U} \Leftrightarrow f \equiv_\mathcal{U} id$, where id denotes the identity map. As a consequence, both $\mathcal{U} \leq_{RK} \mathcal{V}$ and $\mathcal{V} \leq_{RK} \mathcal{U}$ hold if and only if $f(\mathcal{U}) = \mathcal{V}$ for some $f : I \to J$ which is 1-1 when restricted to a suitable $U \in \mathcal{U}$. In this case $\mathcal{U}$ and $\mathcal{V}$ are *isomorphic*, and we write $\mathcal{U} \cong \mathcal{V}$. Thus $\leq_{RK}$ becomes a *partial order* on the equivalence classes of isomorphic ultrafilters. We say that $\mathcal{U}, \mathcal{V}$ are *incomparable* if $\mathcal{U} \not\leq_{RK} \mathcal{V}$ and $\mathcal{V} \not\leq_{RK} \mathcal{U}$; they are *incompatible* if there is no non-principal ultrafilter $\mathcal{W} \leq_{RK} \mathcal{U}, \mathcal{V}$. A non-principal ultrafilter $\mathcal{U}$ is *selective* (or *Ramsey*) if every $f : I \to I$ is $\mathcal{U}$-equivalent either to a constant or to a 1-1 function. Selective ultrafilters are exactly the minimal elements of the RK-preordering, i.e. those lying immediately above the principal ultrafilters. An ultrafilter is *uniform* if all its elements have the same cardinality. Every nonprincipal ultrafilter is isomorphic to a uniform ultrafilter over κ, where κ is the least size of a set in $\mathcal{U}$. However in the following we shall not restrict ourselves to uniform ultrafilters.

The *product filter* $\mathcal{U} \times \mathcal{V}$ is the filter on $I \times J$ generated by all products $U \times V$ where $U \in \mathcal{U}$ and $V \in \mathcal{V}$. Notice that, in general, $\mathcal{U} \times \mathcal{V}$ is not *ultra*. The *product ultrafilter* $\mathcal{U} \otimes \mathcal{V}$ (sometimes called *tensor product*) is defined by setting:

$$X \in \mathcal{U} \otimes \mathcal{V} \iff \{i \in I \mid \{j \in J \mid (i,j) \in X\} \in \mathcal{V}\} \in \mathcal{U}.$$

Let $\{\mathcal{V}_i \mid i \in I\}$ be a family of ultrafilters on J. The *$\mathcal{U}$-limit* $\lim_\mathcal{U} \mathcal{V}_i$ of the family is the ultrafilter on J such that:

$$X \in \lim_\mathcal{U} \mathcal{V}_i \iff \{i \in I \mid X \in \mathcal{V}_i\} \in \mathcal{U}.$$

---

[3] A cardinal κ is measurable if there is a nonprincipal κ-complete ultrafilter on κ. (Notice that we include ω among the measurable cardinals.)

[4] The notation $f_*(\mathcal{U})$ is frequently used in the literature. We adhere to the simpler notation $f(\mathcal{U})$ since no ambiguity is likely to arise.

Clearly, $\lim_{\mathcal{U}} \mathcal{V}_i = \lim_{\mathcal{U}} \mathcal{W}_i$ whenever $\{i \in I \mid \mathcal{V}_i = \mathcal{W}_i\} \in \mathcal{U}$.

If not explicitly mentioned otherwise, we refer to [10] for all set-theoretic notions and facts used in this paper, and to [7] for definitions and facts concerning ultrapowers, ultrafilters, and nonstandard models.

Let us repeat our basic definition of Hausdorff ultrafilter:

Definition 1.1 An ultrafilter $\mathcal{U}$ is *Hausdorff* if it satisfies the condition:

(H) for all $f, g : I \to I$ ($f(\mathcal{U}) = g(\mathcal{U}) \implies f \equiv_{\mathcal{U}} g$).

Notice that the above condition (H) is automatically satisfied whenever any of the functions f and g is 1-1 or constant when restricted to some set $A \in \mathcal{U}$. In particular, all selective ultrafilters are Hausdorff. It is easily seen that Hausdorff ultrafilters are downward closed with respect to the Rudin-Keisler preordering.

We begin with a simple characterization of Hausdorff ultrafilters.

Lemma 1.2 *The ultrafilter $\mathcal{U}$ is Hausdorff if and only if whenever an ultrafilter $\mathcal{W} \leq_{RK} \mathcal{U}$ includes the square filter $\mathcal{V} \times \mathcal{V}$ of some nonprincipal $\mathcal{V}$, then $\mathcal{W}$ includes also the diagonal $\Delta = \{(j, j) \mid j \in J\}$.*

Proof.   Assume that $f : I \to J \times J$ is a function such that $\mathcal{V} \times \mathcal{V} \subseteq f(\mathcal{U}) = \mathcal{W}$. Let $\pi_1, \pi_2 : J \times J \to J$ be the canonical projections. Then $\pi_1(\mathcal{W}) = \pi_2(\mathcal{W}) = \mathcal{V}$, while $\pi_1 \equiv_{\mathcal{W}} \pi_2$ if and only if $\mathcal{W}$ includes the diagonal $\Delta \subseteq J \times J$. Since Hausdorff ultrafilters are Rudin-Keisler downward closed, we obtain the direct implication.

Conversely, let $f, g : I \to J$ be such that $f(\mathcal{U}) = g(\mathcal{U}) = \mathcal{V}$ is nonprincipal. Define $h : I \to J \times J$ by $h(i) = (f(i), g(i))$. Then clearly $\mathcal{W} = h(\mathcal{U}) \supseteq \mathcal{V} \times \mathcal{V}$, and so $\mathcal{W}$ includes the diagonal. But then f and g are equivalent modulo $\mathcal{U}$, since they agree on $h^{-1}(\Delta) \in \mathcal{U}$. Therefore $\mathcal{U}$ is Hausdorff. $\square$

Corollary 1.3 *If the product $\mathcal{U} \otimes \mathcal{V}$ is Hausdorff, then the ultrafilters $\mathcal{U}$ and $\mathcal{V}$ are Hausdorff and incompatible.*

Proof.   If $\mathcal{W} = \mathcal{U} \otimes \mathcal{V}$ is Hausdorff, then both $\mathcal{U}$ and $\mathcal{V}$ are $\leq_{RK} \mathcal{W}$, hence Hausdorff. If $f : I \to K$, $g : J \to K$ are such that $f(\mathcal{U}) = g(\mathcal{V}) = \mathcal{D}$, define $h : I \times J \to K \times K$ by $h(i, j) = (f(i), g(j))$. Then the ultrafilter $h(\mathcal{W})$ includes $\mathcal{D} \times \mathcal{D}$, hence also the diagonal, according to Lemma 1.2. Now

$$h^{-1}(\Delta) = \{(i, j) \mid f(i) = g(j)\} \in \mathcal{U} \otimes \mathcal{V} \iff \{i \mid \{j \mid f(i) = g(j)\} \in \mathcal{V}\} \in \mathcal{U}.$$

5

But this implies that both f and g are constant, modulo $\mathcal{U}$ and $\mathcal{V}$ respectively, and so $\mathcal{D}$ is principal. This proves that $\mathcal{U}$ and $\mathcal{V}$ are incompatible.

$\square$

Our criterion to test whether a product of Hausdorff ultrafilters is itself Hausdorff is obtained by adding a further condition, namely:

Theorem 1.4 *The product $\mathcal{U} \otimes \mathcal{V}$ is Hausdorff if and only if the following conditions are satisfied:*
(i) $\mathcal{U}$ and $\mathcal{V}$ are Hausdorff and incompatible;
(ii) for $i \in I$, let $\mathcal{V}_i' \neq \mathcal{V}_i''$ be (possibly principal) ultrafilters on J that are $\leq_{RK} \mathcal{V}$. Then $\lim_\mathcal{U} \mathcal{V}_i' \neq \lim_\mathcal{U} \mathcal{V}_i''$.

Proof. Assume first that the product $\mathcal{U} \otimes \mathcal{V}$ is Hausdorff. The condition (i) holds by Corollary 1.3. Proceeding by contradiction, let the ultrafilters $\mathcal{V}_i' = f_i(\mathcal{V})$, $\mathcal{V}_i'' = g_i(\mathcal{V})$, and $\mathcal{U}$ witness that condition (ii) fails. Put $f(i,j) = f_i(j)$ and $g(i,j) = g_i(j)$. A straightforward application of the definitions proves the equalities

$$f(\mathcal{U} \otimes \mathcal{V}) = \lim_\mathcal{U} \mathcal{V}_i' = \lim_\mathcal{U} \mathcal{V}_i'' = g(\mathcal{U} \otimes \mathcal{V}).$$

Now $f \equiv_{\mathcal{U} \otimes \mathcal{V}} g$ if and only if $\{i \mid f_i \equiv_\mathcal{V} g_i\} \in \mathcal{U}$. But the latter condition is not realized, because $f_i(\mathcal{V}) = \mathcal{V}_i' \neq \mathcal{V}_i'' = g_i(\mathcal{V})$, and so $f_i \not\equiv_\mathcal{V} g_i$, for all i. Thus we obtain the contradiction that $\mathcal{U} \otimes \mathcal{V}$ is not Hausdorff.

Vice versa, assume that $\mathcal{U}$ and $\mathcal{V}$ satisfy conditions (i) and (ii). Given $f, g : I \times J \to K$, define the functions $f_i, g_i : J \to K$ by $f_i(j) = f(i,j)$ and $g_i(j) = g(i,j)$. Since $\mathcal{V}$ is Hausdorff, $\{i \mid f_i(\mathcal{V}) \neq g_i(\mathcal{V})\} = \{i \mid f_i \not\equiv_\mathcal{V} g_i\} \in \mathcal{U}$ if and only if $f \not\equiv_{\mathcal{U} \otimes \mathcal{V}} g$. In this case, take two sequences of ultrafilters $\mathcal{V}_i', \mathcal{V}_i''$ such that $\mathcal{V}_i' \neq \mathcal{V}_i''$ for all i, and $\mathcal{V}_i' = f_i(\mathcal{V})$, $\mathcal{V}_i'' = g_i(\mathcal{V})$ whenever $f_i \not\equiv_\mathcal{V} g_i$. By condition (ii), we conclude that

$$f(\mathcal{U} \otimes \mathcal{V}) = \lim_\mathcal{U} f_i(\mathcal{V}) = \lim_\mathcal{U} \mathcal{V}_i' \neq \lim_\mathcal{U} \mathcal{V}_i'' = \lim_\mathcal{U} g_i(\mathcal{V}) = g(\mathcal{U} \otimes \mathcal{V}).$$

$\square$

Notice that also *principal* ultrafilters are allowed in condition (ii). As a consequence, that condition already implies that the ultrafilters $\mathcal{U}$ and $\mathcal{V}$ are incompatible. In fact, if $f(\mathcal{U}) = g(\mathcal{V})$ is nonprincipal, let $\mathcal{V}_i'$ be the principal ultrafilter generated by $\{f(i)\}$ and let $\mathcal{V}_i'' = g(\mathcal{V})$ for all $i \in I$. Then clearly $\lim_\mathcal{U} \mathcal{V}_i' = \lim_\mathcal{U} \mathcal{V}_i''$, contradicting condition (ii). In order to simplify the above criterion, the following notions are useful:

Definition 1.5 A family $\{\mathcal{V}_i \mid i \in I\}$ of ultrafilters over J is *discrete* if there exists a family $\{A_i \mid i \in I\}$ of subsets of J such that $A_i \in \mathcal{V}_i$ and $A_i \cap A_j = \emptyset$ whenever $\mathcal{V}_i \neq \mathcal{V}_j$. The family $\{\mathcal{V}_i \mid i \in I\}$ is *$\mathcal{U}$-compatible* if there exists a nonprincipal ultrafilter $\mathcal{W}$ such that $\{i \in I \mid \mathcal{W} \leq_{RK} \mathcal{V}_i\} \in \mathcal{U}$.

Theorem 1.6 *Let $\mathcal{U}, \mathcal{V}$ be incompatible Hausdorff ultrafilters, and let κ be the least size of a set in $\mathcal{U}$. The product $\mathcal{U} \otimes \mathcal{V}$ is Hausdorff if the following conditions are satisfied for every family $\{\mathcal{V}_i \mid i \in I\}$ of ultrafilters $\leq \mathcal{V}$:*

(a) every subfamily of size κ is discrete;

(b) if $\lim_{\mathcal{U}} \mathcal{V}_i \leq_{RK} \mathcal{U}$, then the family $\{\mathcal{V}_i \mid i \in I\}$ is $\mathcal{U}$-compatible.

    Proof. Let $\{\mathcal{V}_i' \mid i \in I\}$ and $\{\mathcal{V}_i'' \mid i \in I\}$ be families of ultrafilters satisfying the hypotheses of condition (ii) of Theorem 1.4. Put $A' = \{i \in I \mid \mathcal{V}_i' \text{ principal}\}$ and $A'' = \{i \in I \mid \mathcal{V}_i'' \text{ principal}\}$. We show that (a) and (b) together imply (ii). We distinguish three cases.

Case 1: $A' \in \mathcal{U}$ and $A'' \in \mathcal{U}$. Pick functions $f, g : I \to J$ such that $\{f(i)\} \in \mathcal{V}_i'$ and $\{g(i)\} \in \mathcal{V}_i''$ for all $i \in A' \cap A''$. As $f \not\equiv_{\mathcal{U}} g$ and $\mathcal{U}$ is Hausdorff, we have

$$\lim_{\mathcal{U}} \mathcal{V}_i' = f(\mathcal{U}) \neq g(\mathcal{U}) = \lim_{\mathcal{U}} \mathcal{V}_i''.$$

So condition (ii) of Theorem 1.4 is satisfied.

Case 2: $A' \notin \mathcal{U}$ and $A'' \notin \mathcal{U}$. Assume w.l.o.g. that all $\mathcal{V}_i', \mathcal{V}_i''$ are nonprincipal. Enumerate

$$\{\mathcal{V}_i' \mid i \in I\} \cup \{\mathcal{V}_i'' \mid i \in I\} = \{\mathcal{W}_h \mid h \in H\}$$

in such a way that $\mathcal{W}_h \neq \mathcal{W}_k$ for $h \neq k$. Let $\sigma, \tau : I \to H$ be the functions such that $\mathcal{V}_i' = \mathcal{W}_{\sigma(i)}$ and $\mathcal{V}_i'' = \mathcal{W}_{\tau(i)}$. Notice that $\lim_{\mathcal{U}} \mathcal{V}_i' = \lim_{\sigma(\mathcal{U})} \mathcal{W}_h$ and $\lim_{\mathcal{U}} \mathcal{V}_i'' = \lim_{\tau(\mathcal{U})} \mathcal{W}_h$. Since $\sigma \not\equiv_{\mathcal{U}} \tau$ and $\mathcal{U}$ is Hausdorff, it follows that $\sigma(\mathcal{U}) \neq \tau(\mathcal{U})$. So we can find a set $A \in \mathcal{U}$ of size κ such that $\sigma(A) \cap \tau(A) = \emptyset$. The family $\{\mathcal{W}_h \mid h \in \sigma(A) \cup \tau(A)\}$ is discrete, by condition (a). Choose a family $\{A_h \mid h \in \sigma(A) \cup \tau(A)\}$ of subsets of J such that $A_h \in \mathcal{W}_h$, and $A_h \cap A_k = \emptyset$ for $h \neq k$. Put $X = \bigcup_{h \in \sigma(A)} A_h$ and $Y = \bigcup_{h \in \tau(A)} A_h$. Then $X \in \lim_{\sigma(\mathcal{U})} \mathcal{W}_h$, $Y \in \lim_{\tau(\mathcal{U})} \mathcal{W}_h$, and $X \cap Y = \emptyset$. Hence the limits are different, and condition (ii) of Theorem 1.4 is fulfilled also in this case.

Case 3: $A' \notin \mathcal{U}$ and $A'' \in \mathcal{U}$ (the case $A' \in \mathcal{U}$ and $A'' \notin \mathcal{U}$ is symmetric). Assume w.l.o.g. that all $\mathcal{V}_i'$ are nonprincipal. As in case 1, pick a function

7

$g : I \to J$ such that $\lim_{\mathcal{U}} \mathcal{V}''_i = g(\mathcal{U})$. If $\lim_{\mathcal{U}} \mathcal{V}'_i = \lim_{\mathcal{U}} \mathcal{V}''_i \leq_{RK} \mathcal{U}$, then pick a nonprincipal $\mathcal{W}$ (on K, say) witnessing that condition (b) is fulfilled. Let $A \in \mathcal{U}$ be a set of size κ such that $\mathcal{W} \leq_{RK} \mathcal{V}'_i$ for all $i \in A$. Choose functions $f_i : J \to K$ such that $f_i(\mathcal{V}'_i) = \mathcal{W}$ and subsets $B_i \subseteq J$ witnessing that the family $\{\mathcal{V}'_i \mid i \in A\}$ is discrete. Pick a function $f : J \to K$ that agrees with f_i on B_i, for every $i \in A$. Then

$$\mathcal{W} = \lim_{\mathcal{U}} f_i(\mathcal{V}'_i) = \lim_{\mathcal{U}} f(\mathcal{V}'_i) = f\left(\lim_{\mathcal{U}} \mathcal{V}'_i\right) \leq_{RK} \lim_{\mathcal{U}} \mathcal{V}'_i \leq_{RK} \mathcal{U}.$$

But obviously $\mathcal{W} \leq_{RK} \mathcal{V}$, contradicting the incompatibility of $\mathcal{U}$ and $\mathcal{V}$.

$\square$

Call κ-*P-point* a κ-complete ultrafilter $\mathcal{U}$ on I such that every function $f : I \to I$ is $\mathcal{U}$-equivalent either to a constant or to a "$<\kappa$-to-1" function. (We say that f is $<\kappa$-*to*-1 if the fibers of f are "small", i.e. $|f^{-1}(i)| < \kappa$ for all $i \in I$.) Obviously every κ-complete selective ultrafilter is a κ-P-point. In particular, $\mathcal{U}$ is a uniform selective ultrafilter on the uncountable set I if and only if $\kappa = |I|$ is measurable, and $\mathcal{U}$ is isomorphic to a *normal κ-ultrafilter* (see e.g. [11]).

When $\mathcal{V}$ is a κ-P-point, the necessary condition (i) of Thorem 1.4 becomes sufficient:

Corollary 1.7 *Let κ be the least size of a set in $\mathcal{U}$ and let $\mathcal{V}$ be a κ-P-point. Then $\mathcal{U} \otimes \mathcal{V}$ is Hausdorff if and only if $\mathcal{U}$ and $\mathcal{V}$ are Hausdorff and incompatible.*

Proof. It is well known that a family of size not exceeding κ of κ-P-points is discrete (see e.g. [11]). Moreover a κ-P-point is selective whenever its additivity exceedes κ. On the other hand a κ-P-point of additivity κ is isomorphic to a κ-ultrafilter, i.e. a uniform κ-complete ultrafilter over κ. So, if the additivity of $\mathcal{V}$ is uncountable, there is a unique normal ultrafilter lying RK-below $\mathcal{V}$ (see e.g. [14]), which is, up to isomorphism, the least nonprincipal ultrafilter $\leq_{RK} \mathcal{V}$. On the other hand, if $\mathcal{V}$ is countably incomplete, and so $\kappa = \aleph_0$, it follows from Theorem 5 of [4] that any sequence of nonprincipal ultrafilters $\leq_{RK} \mathcal{V}$ has a nonprincipal RK-lower bound $\mathcal{W} \leq_{RK} \mathcal{V}$. Therefore the conditions (a)-(b) of Theorem 1.6 are always met.

$\square$

Similarly, if $\mathcal{U}$ is a κ-P-point, then the condition (b) of Theorem 1.6 is trivial, because only constant κ-sequences can converge to κ-P-points,

and so (a) alone suffices for $\mathcal{U} \otimes \mathcal{V}$ to be Hausdorff. More generally, provided sufficiently many incompatible Hausdorff κ-P-points are available, one can produce other Hausdorff ultrafilters that are (Rudin-Keisler) greater and greater. Namely, by recalling that a P-point lies RK-below a product if and only if it lies RK-below a factor (see e.g. [6]), one can prove by induction:

Corollary 1.8 *Let $\{\mathcal{U}_h \mid h \in H\}$ be a family of pairwise incompatible Hausdorff κ-P-points over κ. Then the product $\mathcal{U}_{h_1} \otimes \ldots \otimes \mathcal{U}_{h_n}$ is Hausdorff if and only if the indices $h_1, \ldots, h_n$ are all different.*

2   Regular Hausdorff ultrafilters

Recall that the ultrafilter $\mathcal{U}$ on I is (κ, ν)-*regular* if there exists a family $\mathcal{F} \subseteq \mathcal{U}$ of size ν such that the intersection of any subfamily of size κ of $\mathcal{F}$ is empty. $\mathcal{U}$ is *regular* if it is $(\omega, |I|)$-regular, i.e. if there is a family $\{A_i \mid i \in I\} \subset \mathcal{U}$ such that every infinite subfamily has empty intersection. Every regular ultrafilter is countably incomplete, and when I is countable the two notions are equivalent (but in general regularity is a *proper* strengthening of countable incompleteness).

According to [13], a uniform ultrafilter $\mathcal{U}$ on I is λ-*separating* if, whenever the functions $f, g : I \to \lambda$ are such that $f(\mathcal{U}) = g(\mathcal{U})$ is uniform, then $f \equiv_\mathcal{U} g$. It is easily seen that $\mathcal{U}$ is Hausdorff if and only if it is λ-separating for all $\lambda \leq |I|$. For cardinals $\kappa \leq \lambda$ put

$\mathfrak{u}_{\kappa\lambda} =$ the *least size of a κ-subbasis of a uniform κ-complete ultrafilter* on λ.

($\mathcal{B}$ is a κ-subbasis of the filter $\mathcal{F}$ if the intersections of less than κ elements of $\mathcal{B}$ generate $\mathcal{F}$.) Notice that the cardinal $\mathfrak{u}_{\kappa\lambda}$ may be undefined for some (all) λ, when κ is uncountable. Clearly $\mathfrak{u} = \mathfrak{u}_{\omega\omega}$ is the least size of an ultrafilter basis on any set I. Since obviously $\mathfrak{u}_{\kappa\lambda} \leq 2^\lambda$, the following theorem generalizes and improves Theorem 2.5 of [13]:

Theorem 2.1   *There are no $(\kappa, \mathfrak{u}_{\kappa\lambda})$-regular λ-separating κ-complete ultrafilters. In particular, if $|I| \geq \mathfrak{u}$, no regular ultrafilter on I is Hausdorff.*

Proof.   Let $\mathcal{U}$ be a $(\kappa, \mathfrak{u}_{\kappa\lambda})$-regular κ-complete ultrafilter on I. Let $\{A_\alpha \mid \alpha < \mathfrak{u}_{\kappa\lambda}\}$ be a family of $\mathfrak{u}_{\kappa\lambda}$-many sets in $\mathcal{U}$ such that every subfamily of size κ has empty intersection. For $i \in I$ put $F_i = \{\alpha < \mathfrak{u}_{\kappa\lambda} \mid i \in U_\alpha\}$.

Then F_i has size less than κ, and we can assume w.l.o.g. that it is nonempty. Let $\mathcal{B} = \{B_\alpha \mid \alpha < \mathfrak{u}_{\kappa\lambda}\}$ be a κ-subbasis of a uniform κ-complete ultrafilter $\mathcal{D}$ on λ. Choose functions $f, g : I \to \lambda$ such that $f(i), g(i) \in \bigcap_{\alpha \in F_i} B_\alpha$ and $f(i) \neq g(i)$ for all $i \in I$. Then clearly every $B_\alpha \in \mathcal{B}$ belongs both to $f(\mathcal{U})$ and $g(\mathcal{U})$. Hence $f(\mathcal{U}) = g(\mathcal{U}) = \mathcal{D}$, but $f \not\equiv_{\mathcal{U}} g$, and $\mathcal{U}$ cannot be λ-separating. $\square$

The cardinal $\mathfrak{u}$ is one of several cardinal invariants of the continuum considered in the literature (see [19, 5]). As usual for cardinals that are the sizes of distinguished subsets of $\mathcal{P}(\omega)$, all what is provable in ZFC about $\mathfrak{u}$ is that $\aleph_1 \leq \mathfrak{u} \leq 2^{\aleph_0}$ (see [5]). So, under CH, there are no (ω, ω_1)-regular Hausdorff ultrafilters. In fact, uniform Hausdorff ultrafilters on the continuum require very strong set theoretic hypotheses. In [13] one derives from the existence of such ultrafilters that, under CH, there are inner models with measurable cardinals. Even when the continuum is larger than $\aleph_1$, one could expect that similar results be derivable from Theorem 2.1. E.g. if $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$ and $\mathfrak{u} = \aleph_1$, then an argument similar to those of [13] yields the same implication. On the other hand, in [12] it is proved that if $\diamondsuit$ holds and there is a normal ω_1-dense ideal over ω_1,[5] then for any nonprincipal ultrafilter $\mathcal{D}$ on ω there exists a uniform ultrafilter $\mathcal{U}$ on $\mathbb{R}$ such that $\mathcal{V} <_{RK} \mathcal{U}$ implies $\mathcal{V} \leq_{RK} \mathcal{D}$. Since $\diamondsuit$ implies CH, one can choose $\mathcal{D}$ to be Hausdorff and so $\mathcal{U}$ is a uniform Hausdorff ultrafilter over $\mathbb{R}$. On the contrary, when the continuum is "large" and $\mathfrak{u} = 2^{\aleph_0}$, the existence of Hausdorff uniform ultrafilters over "large" sets should be obtained by very weak assumptions. E.g., Martin's Axiom together with $2^{\aleph_0} = \aleph_2$ yields the existence of regular uniform Hausdorff ultrafilters over ω_1.[6]

3   Final remarks and open questions

3.1   Sums of Hausdorff ultrafilters

Let $\{\mathcal{V}_i \mid i \in I\}$ be a family of ultrafilters on J. The $\mathcal{U}$-sum ultrafilter $\sum_{\mathcal{U}} \mathcal{V}_i$ is the ultrafilter on $I \times J$ defined as follows:

$$X \in \sum_{\mathcal{U}} \mathcal{V}_i \iff \{i \in I \mid \{j \in J \mid (i,j) \in X\} \in \mathcal{V}_i\} \in \mathcal{U}.$$

---

[5] The consistency of this hypothesis has been proved by Woodin by assuming $\mathsf{AD}_{\mathbb{R}} + $ "Θ is regular".

[6] Personal communication by K. Hrbàček.

Notice that when all $\mathcal{V}_i = \mathcal{V}$, then $\sum_{\mathcal{U}} \mathcal{V}_i = \mathcal{U} \otimes \mathcal{V}$.

The results of Section 1 can be easily adapted to ultrafilter sums.

Theorem 3.1 *The sum $\sum_{\mathcal{U}} \mathcal{V}_i$ is Hausdorff if and only if the following conditions are satisfied:*
(i) $\mathcal{U}$ is Hausdorff and $\{i \in I \mid \mathcal{V}_i$ is Hausdorff $\} \in \mathcal{U}$;
(ii) let $\mathcal{V}_i' \neq \mathcal{V}_i''$ be (possibly principal) ultrafilters on J such that $\mathcal{V}_i', \mathcal{V}_i'' \leq_{RK} \mathcal{V}_i$ for all $i \in I$. Then $\lim_{\mathcal{U}} \mathcal{V}_i' \neq \lim_{\mathcal{U}} \mathcal{V}_i''$.

Recall that a family $\{\mathcal{V}_i \mid i \in I\}$ of ultrafilters is $\mathcal{U}$-*incompatible* if it is not $\mathcal{U}$-incompatible, i.e. if $\{i \in I \mid \mathcal{W} \leq_{RK} \mathcal{V}_i\} \in \mathcal{U}$ only holds for principal $\mathcal{W}$.

Theorem 3.2 *Let $\{\mathcal{V}_i \mid i \in I\}$ be a $\mathcal{U}$-incompatible family of Hausdorff ultrafilters, and let κ be the least size of a set in $\mathcal{U}$. Then the sum $\sum_{\mathcal{U}} \mathcal{V}_i$ is Hausdorff if the following conditions are fulfilled for every choice of nonprincipal ultrafilters $\mathcal{V}_i' \leq_{RK} \mathcal{V}_i$:*
(a) every subfamily of size κ is discrete;
(b) $\lim_{\mathcal{U}} \mathcal{V}_i' \not\leq_{RK} \mathcal{U}$.

Corollary 3.3 *Let $\mathcal{U}$ be a κ-P-point and let $\{\mathcal{V}_i \mid i \in I\}$ be a $\mathcal{U}$-incompatible family of κ-P-points. Then the sum $\sum_{\mathcal{U}} \mathcal{V}_i$ is Hausdorff.*

We omit the proofs, which are straightforward modifications of the corresponding proofs of Section 1. Having so treated those Hausdorff ultrafilters that are Rudin-Frolik composite, the next natural step would be to investigate Hausdorff P-points. We cannot enter here into this topic, that we aim to consider in a separate paper.

3.2 Uncountably complete Hausdorff ultrafilters

We deal now briefly with κ-ultrafilters, i.e. κ-complete ultrafilters on sets of measurable size $\kappa > \omega$. Ultrapowers modulo such ultrafilters preserve well-foundedness, and this fact has several important consequences. In particular the natural ordering on the ultrapower $\kappa^\kappa/\mathcal{U}$ is a well ordering, and this is the main tool used in the investigation of κ-complete ultrafilters (see [15, 14, 11]). Moreover also the Rudin-Keisler preordering on κ-ultrafilters is well-founded, and so every RK-descending chain is finite, and ends with a selective (normal) one. If $\mathcal{U}$ is a κ-P-point, then there is *exactly one* normal κ-ultrafilter $\mathcal{D} \leq_{RK}$

$\mathcal{U}$, namely $\mathcal{D} = f(\mathcal{U})$, where f is the least nonconstant function in $\kappa^\kappa/\mathcal{U}$. In particular the existence of ω_1-complete Hausdorff ultrafilters is equivalent to that of uncountable measurable cardinals. Moreover, if $\mathcal{U}$ is a normal κ-ultrafilter, then in $L[\mathcal{U}]$ any κ-ultrafilter is a power of $\mathcal{U}$ (see [15]), and so $\mathcal{U}$ is simultaneously the *unique κ-P*-point, the *unique κ-Q*-point, and the *unique* Hausdorff κ-ultrafilter.

The above results on products and sums also hold for κ-ultrafilters. Besides, due to the mentioned properties, some of them admit simplified formulations. E.g.

Corollary 3.4 *Let $\mathcal{U}$ be a Hausdorff κ-ultrafilter and let $\mathcal{V}$ be a Hausdorff κ-P-point RK-above the normal κ-ultrafilter $\mathcal{D}$. Then $\mathcal{U} \otimes \mathcal{V}$ is Hausdorff if and only if $\mathcal{D} \not\leq_{RK} \mathcal{U}$.*

3.3 The consistency strength of Hausdorff ultrafilters

It is well known that, under CH or MA, there are $2^{2^{\aleph_0}}$ nonisomorphic selective ultrafilters over ω. So, under any of these hypotheses, Corollary 1.8 provides equally many Hausdorff ultrafilters that are not P-points. Another interesting consequence is that there are strictly RK-increasing sequences of Hausdorff ultrafilters (which are necessarily not 3-arrow, see [2]). Taking direct limits of such sequences, one produces *invariant analytic coherent non-principal subspaces* of $\beta\omega$ in the sense of [9], thus answering a question posed there. Moreover, under the same hypotheses, one can inductively define both Hausdorff and non-Hausdorff nonselective P-points. (Under CH this fact follows from results of [8].) On the other hand there are models of ZFC with only non-selective P-points over ω, models with a sole (necessarily selective) P-point (up to isomorphism), and also models without P-points (see [18]). The relationships between Hausdorff ultrafilters and other special classes (P- and Q-points, arrow ultrafilters, etc.) will be investigated in a forthcoming paper.

Be it as it may, as far as we do not abide ZFC as our foundational theory, we cannot prove that Hausdorff ultrafilters exist at all. In fact the forcing models recently defined by Bartoszynski and Shelah in [1] have no Hausdorff ultrafilters, even in the weaker sense that considers only $<\omega$-to-1 functions in the condition (H).

3.4 Some open questions

The results of the preceding sections suggest several questions. We itemize below a few of those that seem most interesting to us.

1. Is it consistent that a product of Hausdorff incompatible ultrafilters be non-Hausdorff? and that $\mathcal{U} \otimes \mathcal{V}$ is Hausdorff whereas $\mathcal{V} \otimes \mathcal{U}$ is not?

2. Is the existence of Hausdorff ultrafilters derivable from set-theoretic hypotheses weaker than those providing selective ultrafilters? (E.g. from $\mathfrak{x} = \mathfrak{c}$, for suitable cardinal invariant $\mathfrak{x}$ of the continuum?)

3. Is it consistent with ZFC that there are regular Hausdorff ultrafilters on sets of size greater than $\aleph_1$?

4. Assuming some large cardinal axiom, is it consistent with ZFC + ¬CH that there are uniform Hausdorff ultrafilters on the continuum?

5. What about non-normal Hausdorff κ-ultrafilters?

References

[1] A. Bartoszynski, S. Shelah - *There may be no Hausdorff ultrafilters*, manuscript (2003).

[2] J.E. Baumgartner, A.D. Taylor - Partitions theorems and ultrafilters, *Trans. Amer. Math. Soc.* **241** (1978), 283–309.

[3] V. Benci, M. Di Nasso, M. Forti - Hausdorff nonstandard extensions, *Bol. Soc. Parana. Mat.* (3) **20** (2002), 9–20.

[4] A. Blass - The Rudin-Keisler ordering of P-points, *Trans. Amer. Math. Soc.* **179** (1973), 79–90.

[5] A. Blass - Combinatorial cardinal characteristics of the continuum, to appear in *Handbook of Set Theory* (M. Foreman, M. Magidor, A. Kanamori, eds.).

[6] A. Blass, G. Moche - Finite preimages under the natural map from $\beta(\mathbb{N} \times \mathbb{N})$ to $\beta\mathbb{N} \times \beta\mathbb{N}$, *Topology Proceedings* (to appear).

[7] C.C. Chang, H.J. Keisler - *Model Theory* (3rd edition), North-Holland, Amsterdam 1990.

[8] M. Daguenet-Teissier - Ultrafiltres à la façon de Ramsey, *Trans. Amer. Math. Soc.* **250** (1979), 91–120.

[9] M. Di Nasso, M. Forti, Topological and nonstandard extensions, *Monatsh. Math.* (accepted).

[10] T. JECH - *Set Theory* (3rd edition), Springer, Berlin 2002.

[11] A. KANAMORI - Ultrafilters over a measurable cardinal, *Ann. Math. Logic* **11** (1977), 315–332.

[12] A. KANAMORI - Finest partitions for ultrafilters, *J. Symb. Logic* **511** (1986), 327–356.

[13] A. KANAMORI, A.D. TAYLOR - Separating ultrafilters on uncountable cardinals, *Israel J. Math.* **47** (1984), 131–138.

[14] J. KETONEN - Ultrafilters over measurable cardinals, *Fund. Math.* **77** (1973), 257–269.

[15] K. KUNEN - Some applications of iterated ultrapowers in set theory, *Ann. Math. Logic* **1** (1970), 179–227.

[16] A.H. MEKLER, D.H. PELLETIER, A.D. TAYLOR - A note on a lemma of Shelah concerning stationary sets, *Proc. Amer. Math. Soc.* **83** (1981), 764–768.

[17] S. NG, H. RENDER - The Puritz order and its relationship to the Rudin-Keisler order, in *Reuniting the Antipodes - Constructive and Nonstandard Views of the Continuum* (P. Schuster, U. Berger and H. Osswald, eds.), *Synthèse Library* **306**, Kluwer A. P., New York 2001, 157–166.

[18] S. SHELAH - *Proper and Improper Forcing* (2nd edition), Springer, Berlin 1998.

[19] E.K. VAN DOUWEN - The integers and topology, in *Handbook of Set Theoretic Topology* (K. Kunen, J.E. Vaughan, eds.), North-Holland, Amsterdam 1984, 111–168.