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Combinatorics of Numbers

. 4
. 9
13
19
25
29
33
39
43
49
53
61
67
68

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Contents

Introduction 4
§1. Filters and ultrafilters 9
§2. Ultrafilters on topological spaces 13
§3. Space of ultrafilters 19
§4. Semigroup of ultrafilters 25
§5. Ramsey Theorem 29
§6. Hindman Theorem 33
§7. Van der Waerden Theorem 39
§8. Hales-Jewett Theorem 43
§9. Rado Theorem 49
§10. Furstenberg-Weiss Theorem 53
§11. Partition of groups and rings 61
References 67
Index 68

INTRODUCTION

Schur proved the following theorem in 1916: *For every finite coloring of the set of natural numbers there exists a monochromatic solution of the equation* $x+y=z$. In other word, given a partition $\mathbb{N} = A_1 \cup \dots \cup A_m$ of the set of natural numbers, at least one of the sets A_i contains elements x, y such that $x+y \in A_i$. As a consequence of this theorem, Schur proved that for every natural m the congruence $x^m + y^m = z^m \pmod p$ has a nontrivial solution for every sufficiently large prime number p.

In 1927, van der Waerden proved the famous theorem on arithmetic progressions: *If the set of natural numbers is partitioned into a finite number of subsets, then at least one of this subsets contains arithmetic progressions of arbitrary finite length.*

In 1930, Ramsey, in his paper on mathematical logic, proved the following theorem. *Let k, m be arbitrary natural numbers. If the family of all k-element subsets of the set of natural numbers $\mathbb{N}$ is colored into m colors, then there is an infinite subset $A \subseteq \mathbb{N}$ such that all its k-element subsets have the same color.*

In his cycle of papers published in 1933, Rado generalized the Schur theorem onto systems of linear homogeneous Diophantine equations. Here we formulate his result for a single homogeneous equation. A linear homogeneous equation $a_1x_1 + \dots + a_1x_1 = 0$ with nonzero entire coefficients is called regular if for every finite coloring of the set of natural numbers there exists a monochromatic solution of this equation. By the Rado theorem, an equation is regular iff some nonempty sum of its coefficients is equal to zero. For example, the equations $x+y-z=0$, $x+y-2z=0$ are regular, while the equation $x+y-3z=0$ is not.

In one of the papers from the mentioned cycle Rado exposed Gallai's proof of a multidimensional counterpart of the van der Waerden theorem. Let k, m, l be arbitrary natural numbers and $\mathbb{N}^k$

the set of k-dimensional vectors with positive integer coordinates. For every partition $\mathbb{N}^k = A_1 \cup \cdots \cup A_m$ there are a subset A_i and subsets

$$B_1 = \{a_1, a_1 + d, \dots, a_1 + nd\},$$

$$\dots\dots\dots\dots$$

$$B_k = \{a_k, a_k + d, \dots, a_k + nd\}, \quad a_1, \dots, a_k, d \in \mathbb{N},$$

such that $B_1 \times \cdots \times B_k \subseteq A_i$.

In 1963, Hales and Jewett, investigating a higher dimensional counterpart of tic-tac-toe, proved an unexpected theorem on partitions of semigroups: *Suppose the free semigroup S with generators $a_1, \dots, a_n$ is partitioned into a finite number of subsets, $S = A_1 \cup \cdots \cup A_m$. There exists a containing the letter x semigroup word $f(x)$ in the alphabet $\{a_1, \dots, a_n, x\}$ and a subset A_i such that $f(a_1), \dots, f(a_n) \in A_i$.*

The Gallai Theorem and, in particular, the van der Waerden Theorem are easy consequences of the Hales-Jewett Theorem. In 1972, Graham, Leeb, and Rothchild deduced the following new result from the Hales-Jewett Theorem: *Suppose an infinite-dimensional vector space over a finite field is partitioned into a finite number of subsets. Then at least one of subsets of the partition contains affine subspaces of arbitrary finite dimension.*

At the end of 60s, Rado, Folkman, Sanders, Rothschild, and Graham independently proved the following generalization of the Schur Theorem: *Denote by $FS(A)$ the set of all finite sums of distinct elements from the set $A \subseteq \mathbb{N}$. For every partition $\mathbb{N} = A_1 \cup \cdots \cup A_m$ of the set of natural numbers there exist a subset A_i and an arbitrarily large finite subset A such that $FS(A) \subseteq A_i$.*

In 1974, Hindman essentially extended this theorem by proving existence of infinite subset A with the property $FS(A) \subseteq A_i$. Because of its extremal complexity, Hindman's proof did not suit the specialists (including Hindman himself). Baumgartner slightly simplified Hindman's arguments, but his proof was still too complicated. Attempts of finding a simple proof of the Hindman Theorem soon led to a success and initiated development of topological and algebraic methods in the combinatorics of numbers.

In 1975, Glaser developed Galvin's idea and deduced the Hindman Theorem from the following known result: *Suppose S is a compact space with a semigroup operation which is continuous with respect to the second argument. Then S contains an idempotent, i.e., an element s such that $s^2 = s$.* This statement should be applied to the Stone-Čech compactification $\beta\mathbb{N}$ of the discrete space $\mathbb{N}$ of positive integers. The elements of the compact space $\beta\mathbb{N}$ are exactly the ultrafilters — special families of subsets of $\mathbb{N}$. The following is one of the main properties of ultrafilters: *for any partition of $\mathbb{N}$ into a finite number of subsets one of the set of this partition is an element of the ultrafilter.*

The operation $+$ on the set of natural numbers can be naturally extended to a semigroup operation on $\beta\mathbb{N}$, which is continuous with respect to the second argument. The following property of idempotents in the semigroup $\beta\mathbb{N}$ is crucial in Glaser's proof of the Hindman Theorem: *If a subset $B \subseteq \mathbb{N}$ is an element of an idempotent ultrafilter, then B contains an infinite subset A such that $FS(A) \subseteq B$.*

Since the late 70s, the semigroup $\beta\mathbb{N}$ becomes one of the most popular objects of attention for experts in the combinatorics of numbers and topological algebra. Here is an incomplete list of investigators of this semigroup: Hindman, van Douwen, Pym, Blass, Bergelson, Strauss. Creating the method of ultrafilters in the combinatorics of numbers is the most important result of investgations of the semigroup $\beta\mathbb{N}$.

The aim of this book is a systematic and self-contained exposition of the method of ultrafilters and demonstration of its applications. The structure of the book is very simple. The initial four sections are devoted to development of the technique of ultrafilters. Further, proofs of the mentioned above main results of the combinatorics of numbers and of some theorems that were first proved by the method of ultrafilters are presented.

In 1979, Hindman proved that for each finite partition $\mathbb{N} = A_1 \cup \cdots \cup A_m$ there exists a subset A_i and infinite subsets A, B with the property $FS(A) \subseteq A_i$, $FP(B) \subseteq A_i$. Here $FP(B)$ denotes the set of all finite products of distict elements of the subset B.

In 1990, Bergelson and Hindman made a synthesis of the additive and multiplicative versions of the van der Waerden Theorem: *For each finite partition* $\mathbb{N} = A_1 \cup \dots \cup A_m$ *there exists an element* A_i *of this partition which contains arbitrary long arithmetic and geometric progressions.*

In 1993, the author proved the following two theorems on partitions of groups and rings.

Let G *be a group. For each finite partition* $G = A_1 \cup \dots \cup A_m$ *of* G *there exists* $A = A_i$ *such that the following statements hold:*

1) $G = A^{-1}AK$ *for some finite subset* $K \subseteq G$*;*

2) $(A^{-1}A)^n$ *is a finite index subgroup of* G *for some natural number* n*;*

3) $f(A) \cap A^{-1}AA^{-1}A \neq \varnothing$ *for every homomorphism* $f\colon G \to G$*.*

Let R *be an infinite associative ring with division. If* R *is partitioned into a finite number of subsets* $R = A_1 \cup \dots \cup A_n$*, then there exists* m *such that for the subset* $A = A_m \setminus \{0\}$ *the equalities hold:*

$$R = A^{-1}A - A^{-1}A + A^{-1}A - A^{-1}A = A^{-1}AA^{-1}A - A^{-1}AA^{-1}A.$$

Finally, note that, besides the method of ultrafilters, another topology-algebraic method was created in the combinatorics of numbers. This is the method of symbolic dynamics developed by Furstenberg and his collaborators. We mention only one brilliant result — the Furstenberg-Weiss Theorem on joint recurrency that implies van der Waerden Theorem.

Let $T_1, \dots, T_k$ *be commuting continuous mappings of a compact metric space* X*. Then there exist a point* $x \in X$ *and an increasing sequence* $\langle n_i \rangle$ *of natural numbers such that*

$$T_1^{n_i}(x) \underset{i\to\infty}{\longrightarrow} x, \dots, T_k^{n_i}(x) \underset{i\to\infty}{\longrightarrow} x.$$

Acknowledgements. This text was written (in Russian) in 1991 as lecture notes of the course given at the Faculty of Mechanics and Mathematics, Kyiv University.

Being a specialist in topological algebra, I'd like to note that my interest to the combinatorics of numbers was stimulated by the excellent paper [6] of Neal Hindman. Eughen Zelenyuk was one of the first listeners of the course as well as the first reader of this text; later he obtained outstanding results in the β-theory (see [15]). Igor Guran and Taras Banakh used the manuscript as a material for their courses and seminars. An idea of this edition appeared at the Lviv University; this idea was explicitly formulated by Michael Zarichnyi. He and Taras Banakh translated the text in English. Oleg Gutik and Andrii Teleiko made the TeX file of the book. Volodya Dmyterko organized a rapid materialization of the project.

My sincere thanks to everybody who made the publication of this book possible.

§1. Filters and ultrafilters.

A family $\mathcal{F}$ of subsets of a set X is called *centered* if the intersection of any finite number of its elements is nonempty.

A family $\mathcal{F}$ of subsets of a set X is called a *filter* if the following conditions hold:

F1) $\varnothing \notin \mathcal{F}$;

F2) $F_1, \ldots, F_n \in \mathcal{F} \Rightarrow F_1 \cap \cdots \cap F_n \in \mathcal{F}$;

F3) $F \in \mathcal{F},\ F \subseteq F' \Rightarrow F' \in \mathcal{F}$.

Clearly, every filter is a centered family of subsets and every centered family is contained in some filter. In order to construct such a filter, first consider all the finite intersections of elements of a centered family and then take all their supersets.

1.1. Example. Let $x \in X$ and $\mathcal{F}_x = \{F \subseteq X : x \in F\}$. The filter $\mathcal{F}_x$ is called a *principal* filter corresponding to the element x.

1.2. Example. Let X be a topological space and $x \in X$. The family of all neighborhoods of x forms a filter.

1.3. Example. Let X be an infinite set, $\mathcal{F}_x = \{F \subseteq X : X \setminus F \text{ is finite}\}$. The filter $\mathcal{F}$ is called a *Fréchet* filter.

The family of all filters on a fixed set X is partially ordered by the relation $\mathcal{F}_1 \subseteq \mathcal{F}_2$. A filter which is maximal with respect to this partial order is called an *ultrafilter*. Any principal filter is an ultrafilter. Nonprincipal ultrafilters are called *free*. Obviously, an ultrafilter $\mathcal{F}$ is free iff $\bigcap \mathcal{F} = \varnothing$. Thus, free ultrafilters can exist on infinite sets only. We use the following classical statement which is equivalent to the axiom of choice for constructing free ultrafilters.

1.4. Lemma (Kuratowski-Zorn). *If every chain (i.e. linearly ordered set) in a partially ordered set has an upper bound, then the set has a maximal element.*

1.5. Ultrafilter Theorem. *Every filter on a set is contained in an ultrafilter.*

Proof. Let $\mathcal{F}'$ be a filter on a set X. Consider a collection α of all filters $\mathcal{F}$ on X which contain $\mathcal{F}'$. Let γ be a chain in α. It follows immediately from the definition of filter that $\{F \in \mathcal{F} : \mathcal{F} \in \gamma\}$ is a filter on X which is an upper bound for γ. By Lemma 1.4 the family α has a maximal element, i.e. an ultrafilter which contains $\mathcal{F}'$. □

1.6. Corollary. *If X is an infinite set, then there exists a free ultrafilter on X.*

Proof. Consider an ultrafilter on X which contains the Fréchet filter on X. □

1.7. Ultrafilter Criterion. *A filter $\mathcal{F}$ on a set X is an ultrafilter iff either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$, for every subset A of X.*

Proof. Necessity. Let $\mathcal{F}$ be an ultrafilter on X, $A \subseteq X$, and $A \notin \mathcal{F}$. Condition F3) implies $F \setminus A \neq \varnothing$ for every $F \in \mathcal{F}$. If $F_1, \ldots, F_n \in \mathcal{F}$, then

$$(F_1 \setminus A) \cap \cdots \cap (F_n \setminus A) = (F_1 \cap \cdots \cap F_n) \setminus A \neq \varnothing,$$

and therefore the family $\{F \setminus A : F \in \mathcal{F}\}$ is centered. By Theorem 1.5, there exists an ultrafilter $\mathcal{F}'$ which contains this family. Obviously, $\mathcal{F} \subseteq \mathcal{F}'$ and hence $\mathcal{F} = \mathcal{F}'$. Since $X \setminus A \in \mathcal{F}'$, we have $X \setminus A \in \mathcal{F}$.

Sufficiency. Suppose, a filter $\mathcal{F}$ satisfies the indicated property and $\mathcal{F}$ is contained in an ultrafilter $\mathcal{F}'$. If $\mathcal{F} \neq \mathcal{F}'$, there exists a subset $A \in \mathcal{F}'$ such that $A \notin \mathcal{F}$. By the condition, $X \setminus A \in \mathcal{F}$ and hence $X \setminus A \in \mathcal{F}'$. However, $A \cap (X \setminus A) = \varnothing$ and this contradicts the definition of filter. Thus, $\mathcal{F} = \mathcal{F}'$ and $\mathcal{F}$ is an ultrafilter. □

1.8. Corollary. *If $\mathcal{F}$ is an ultrafilter on a set X, $F \in \mathcal{F}$, and $F = F_1 \cup \cdots \cup F_m$, then there exists i such that $F_i \in \mathcal{F}$.*

Proof. Suppose the contrary: $F_1 \notin \mathcal{F}, \ldots, F_m \notin \mathcal{F}$. By Ultrafilter Criterion, $X \setminus F_1 \in \mathcal{F}, \ldots, X \setminus F_m \in \mathcal{F}$. Since

$$(X \setminus F_1) \cap \cdots \cap (X \setminus F_m) = X \setminus (F_1 \cup \cdots \cup F_m),$$
$$(X \setminus F_1) \cap \cdots \cap (X \setminus F_m) \in \mathcal{F},$$

we have $X \setminus F \in \mathcal{F}$. Since $F \in \mathcal{F}$ and $\mathcal{F}$ is a filter, we obtain a contradiction. □

The property of ultrafilters indicated in Corollary 1.8 lies in background of proofs of most theorems on partitions of infinite sets. Suppose we have constructed on a set X an ultrafilter such that each its element satisfies a certain property (defined on subsets of X). Then, for each partition $X = A_1 \cup \cdots \cup A_m$ at least one of the sets A_i satisfies this property. For example, in order to prove the van der Waerden Theorem it is sufficient to construct on the set of natural numbers an ultrafilter such that each its element contains arithmetic progressions of arbitrary length.

1.9. Ultrafilter Restriction Theorem. *If $\mathcal{F}$ is an ultrafilter on a set X and $F \in \mathcal{F}$, then $\mathcal{F}' = \{F' : F' \subseteq F\}$ is an ultrafilter on F.*

Proof. We can immediately check conditions F1)–F3) thus showing that $\mathcal{F}'$ is a filter on F. Let $F_1 \subseteq F$ and $F_1 \notin \mathcal{F}$. Then $F_1 \notin \mathcal{F}$ and, by Ultrafilter Criterion, $X \setminus F_1 \in \mathcal{F}$. Since $F \in \mathcal{F}$ and $X \setminus F_1 \in \mathcal{F}$, we have $(X \setminus F_1) \cap F = F \setminus F_1 \in \mathcal{F}$. Thus, $F \setminus F_1 \in \mathcal{F}'$ and, by Criterion 1.7, $\mathcal{F}'$ is an ultrafilter. □

Suppose a map $f: X \to Y$ is given and $\mathcal{F}$ is an ultrafilter on X. Note that

$$\bar{f}(\mathcal{F}) = \{A \subseteq Y : f^{-1}(A) \in \mathcal{F}\}$$

is a filter on the set Y.

1.10. Ultrafilter Image Theorem. *Let $\mathcal{F}$ be an ultrafilter on a set X. For every map $f: X \to Y$ the filter $\bar{f}(\mathcal{F})$ is an ultrafilter on the set Y.*

Proof. If $Y = Y_1 \cup Y_2$, then $X = f^{-1}(Y_1) \cup f^{-1}(Y_2)$. By Ultrafilter Criterion, either $f^{-1}(Y_1) \in \mathcal{F}$ or $f^{-1}(Y_2) \in \mathcal{F}$. To be definitive, suppose that $F = f^{-1}(Y_1)$ and $F \in \mathcal{F}$. Then $f(F) = Y_1$ and $f(F) \in \bar{f}(\mathcal{F})$. Thus, $Y_1 \in \bar{f}(\mathcal{F})$ and we have only to use Ultrafilter Criterion. □

Ultrafilters were introduced by Riesz in 1909. They were widely used only after a paper of Ulam (1929). Riesz and Ulam proved ex-

istence of free ultrafilters on countable sets in the presence of the axiom of choice. The following question is natural: in the Zermelo-Fraenkel set theory (ZF) without the axiom of choice, are there free ultrafilters on countable sets? In 1970 Solovay proved that the following statement is consistent with ZF: every subset of the real line is Lebesgue measurable? But already in 1938 Sierpiński proved that the existence of a free ultrafilter on a coutable set implies existence of a subset of the real line which is not Lebesgue measurable. Thus, the non-existence of free ultrafilters on a countable set is consistent with ZF. In this book we accept the axiom of choice C and work in the standard set theory ZFC.

EXERCISES.

1. A filter $\mathcal{F}$ on an infinite set X is called *uniform* if $|F| = |X|$ for every subset $F \in \mathcal{F}$. Prove that every uniform filter is contained in some uniform ultrafilter.
2. Let $\mathcal{F}$ be an ultrafilter on an infinite set of cardinality α. Prove that $|\mathcal{F}| = 2^{\alpha}$.
3. Let $\mathcal{F}$ be a filter on a set X which is not an ultrafilter. Prove that $\mathcal{F}$ is contained in at least two distinct ultrafilters.
4. Prove that the cardinality of the family of all ultrafilters on an infinite set of cardinality α is $2^{2^{\alpha}}$.
5. Let X be an infinite set, $\operatorname{Exp} X$ a family of all subsets of X. A map $\mu: \operatorname{Exp} X \to \{0, 1\}$ is called two-valued finitely-additive measure if the following conditions are satisfied:
   1) $\mu(X) = 1$, $\mu(\{x\}) = 0$ for each $x \in X$;
   2) if $A_1, \ldots, A_n \subset X$ and $A_i \cap A_j = \varnothing$ for $i \neq j$, then $\mu(A_1 \cup \cdots \cup A_n) = \mu(A_1) + \cdots + \mu(A_n)$.

   Prove that for every infinite set X there exists a two-valued finitely-additive measure defined on $\operatorname{Exp} X$.
6. Prove that every infinite family of subsets of a set X contains a subfamily $\mathfrak{A}$ of the same cardinality such that either $\mathfrak{A}$ is centered or $\mathfrak{A}' = \{X \setminus A : A \in \mathfrak{A}\}$ is centered.

§2. ULTRAFILTERS ON TOPOLOGICAL SPACES.

Suppose that to each point x of a set X a collection $\mathcal{B}(x)$ of subsets of X, which are called *neighborhoods* of x, is assigned so that the following conditions are satisfied:

(B1) $x \in U$ for every neighborhood $U \in \mathcal{B}(x)$;
(B2) if $U \subseteq V$, $U \in \mathcal{B}(x)$, then $V \in \mathcal{B}(x)$;
(B3) if $U_1, \ldots, U_n \in \mathcal{B}(x)$, then $U_1 \cap \cdots \cap U_n \in \mathcal{B}(x)$;
(B4) if $U \in \mathcal{B}(x)$, then there is a neighborhood $V \in \mathcal{B}(x)$ such that $U \in \mathcal{B}(y)$ for every $y \in V$.

A subset $A \subseteq X$ is defined to be *open*, if A is a neighborhood of each its point, i.e. $A \in \mathcal{B}(x)$ for every $x \in A$. Evidently, open sets satisfy the following properties:

(O1) X, $\varnothing$ are open sets;
(O2) if $U_1, \ldots, U_n$ are open sets, then $U_1 \cap \cdots \cap U_n$ is an open set;
(O3) if U_α, $\alpha \in J$, is a collection of open sets, then $\bigcup\{U_\alpha : \alpha \in J\}$ is an open set.

The family τ of all open subsets is called the *topology* on the set X, and the pair (X, τ) is called a *topological space*. Remark, that we have defined neighborhoods of points firstly, and then, with their help — open subsets. But we could make otherwise. Suppose we are given a family τ of subsets of a set X, which satisfies the conditions (O1)–(O3). A subset $W \subseteq X$ is called a *neighborhood* of a point x, if there is an open set $U \in \tau$ such that $x \in U \subseteq W$. Then the family $\mathcal{B}(x)$, $x \in X$, satisfies the conditions (B1)–(B4).

A point $x \in X$ is called *a cluster point* of a set $A \subseteq X$, if $W \cap A \neq \varnothing$ for every neighborhood W of x. The set A of all cluster points of the set A is called the *closure* of A. A subset which coincides with its closure is called *closed*. It is easy to see that a subset A is closed iff its complement $X \setminus A$ is open. This yields that closed subsets possess the following properties:

(C1) $X, \varnothing$ are closed subsets;
(C2) if $F_1, \ldots, F_n$ are closed subsets, then $F_1 \cup \cdots \cup F_n$ is a closed subset;

(C3) if F_α, $\alpha \in J$, are closed subsets, then $\bigcap\{F_\alpha : \alpha \in J\}$ is a closed subset.

Now we employ filters and ultrafilters to characterize certain basic properties of topological spaces and their maps. By definition, a filter $\mathcal{F}$ on a topological space (X, τ) converges to a point x if $W \in \mathcal{F}$ for every neighborhood W of x. In this case the point x is called a *limit of the filter* $\mathcal{F}$. Observe that a subset $A \subseteq X$ is closed iff a limit of any filter containing A belongs to the set A.

A topological space is called *Hausdorff* if any two distinct points of this space have disjoint neighborhoods.

2.1. Theorem. *A topological space (X, τ) is Hausdorff iff every filter on X has at most one limit.*

Proof. Suppose the space (X, τ) is Hausdorff, $\mathcal{F}$ a filter on X converging to two distinct points x, y. Choose two disjoint neighborhoods $U, V \in \mathcal{F}$ of x, y respectively. Then we have $U \cap V = \varnothing$, a contradiction with the definition of filter.

Now suppose that each filter on X has at most one limit but the space (X, τ) is not Hausdorff. Choose two distinct points $x, y \in X$ such that $U \cap V \neq \varnothing$ for any neighborhoods U and V of x and y. The family $\{U \cap V : U \in \mathcal{B}(x), V \in \mathcal{B}(y)\}$ is centered. It can be completed to a filter $\mathcal{F}$. Evidently, $\mathcal{B}(x) \subseteq \mathcal{F}$, $\mathcal{B}(y) \subseteq \mathcal{F}$, the filter $\mathcal{F}$ converges to two distinct points x and y, a contradiction with the hypothesis. □

Let $(X_1, \tau_1), (X_2, \tau_2)$ be two topological spaces. A map $f\colon X_1 \to X_2$ is called *continuous at a point* $x \in X_1$, if $f^{-1}(V)$ is a neighborhood of x for every neighborhood V of the point $f(x)$. A map which is continuous at every point of X is called *continuous*.

2.2. Theorem. *A map $f\colon X_1 \to X_2$ is continuous at a point $x \in X_1$ iff for every ultrafilter $\mathcal{F}$ on X_1 convergent to x, the ultrafilter $\bar{f}(\mathcal{F})$ converges to $f(x)$.*

Proof. Suppose f is continuous at x but the ultrafilter $\bar{f}(\mathcal{F})$ does not converge to $f(x)$ for some ultrafilter $\mathcal{F}$ convergent to x. Select a neighborhood V of $f(x)$ such that $V \notin \bar{f}(\mathcal{F})$. Let $U = X_2 \setminus V$. By the Ultrafilter Criterium, $U \in \bar{f}(\mathcal{F})$ and thus $f^{-1}(U) \in \mathcal{F}$. On the other hand, by the continuity of f, $f^{-1}(V)$ is a neighborhood of x. Since $\mathcal{F}$ converges to x, $f^{-1}(V) \in \mathcal{F}$. But $f^{-1}(U) \cap f^{-1}(V) = \varnothing$, a contradiction with the fact that $\mathcal{F}$ is a filter.

Suppose the ultrafilter $\bar{f}(\mathcal{F})$ converges to $f(x)$ for every ultrafilter $\mathcal{F}$ convergent to x. Suppose the map f is not continuous at x. Select such a neighborhood V of point $f(x)$ that $f^{-1}(V)$ is not a neighborhood of x. The centered collection $\{U \setminus f^{-1}(V) : U \in \mathcal{B}(x)\}$ can be completed to an ultrafilter $\mathcal{F}$. Evidently, $\mathcal{F}$ converges to x. Since, by our assumption, the ultrafilter $\bar{f}(\mathcal{F})$ converges to $f(x)$, we have $f^{-1}(V) \in \mathcal{F}$. Then we obtain $X_1 \setminus f^{-1}(V) \in \mathcal{F}$, $f^{-1}(V) \in \mathcal{F}$, a contradiction with the fact that $\mathcal{F}$ is a filter. □

A topological space is called *compact* if every open cover of the space has a finite subcover.

2.3. Theorem. *A topological space (X, τ) is compact iff every ultrafilter on X is convergent.*

Proof. Suppose a space X is compact but there is an ultrafilter $\mathcal{F}$ on X which converges to no point of X. Then for every point $x \in X$ there is an open neighborhood U_x of x such that $U_x \notin \mathcal{F}$. From the cover of X by open subsets U_x, $x \in X$, select a finite subcover $X = U_{x_1} \cup \cdots \cup U_{x_n}$. By Corollary 1.8, at least one of the subsets $U_{x_1}, \ldots, U_{x_n}$ belongs to $\mathcal{F}$ contradicting to the choice of these sets.

Suppose every ultrafilter on X converges, but the space (X, τ) is not compact. Then there is a cover U_α, $\alpha \in J$, of (X, τ) which has no finite subcover. For every finite subset $F \subseteq J$ let $\mathcal{U}(F) = \bigcup\{\mathcal{U}_\alpha : \alpha \in F\}$. By our hypothesis, $X \setminus \mathcal{U}(F) \neq \varnothing$. The centered family $\{X \setminus \mathcal{U}(F) : F$ is finite subset from $J\}$ can be completed to an ultrafilter $\mathcal{F}$. By the hypothesis $\mathcal{F}$ converges to some $x \in X$. Let $\mathcal{U}_\alpha$ be an element of the cover containing x. Clearly $\mathcal{U}_\alpha \in \mathcal{F}$. Let $F = \{\alpha\}$. By the construction of $\mathcal{F}$, $X \setminus \mathcal{U}(F) \in \mathcal{F}$, a contradiction with $\mathcal{U}(F) \in \mathcal{F}$. □

Let (X_α, τ_α), $\alpha \in J$, be a collection of topological spaces, and $X = \prod\{X_\alpha : \alpha \in J\}$ the Cartesian product of X_α. The elements of X are the functions $f: J \to \bigcup\{X_\alpha : \alpha \in J\}$ satisfying $f(\alpha) \in X_\alpha$

for all $\alpha \in J$. Fix a finite subset $K = \{\alpha_1, \ldots, \alpha_n\} \subseteq J$ and for every $\alpha \in K$ fix a neighborhood $\mathcal{U}_\alpha$ of f_α in (X_α, τ_α). Such a choice determines the standard neighborhood $W(\mathcal{U}_{\alpha_1}, \ldots, \mathcal{U}_{\alpha_n}) = \{g \in X : g(\alpha_1) \in \mathcal{U}_{\alpha_1}, \ldots, g(\alpha_n) \in \mathcal{U}_{\alpha_n}\}$ of $f \in X$. Under a neighborhood of f we understand any subset of X containing a standard neighborhood of f. It is easy to see that the family $\mathcal{B}(f)$, $f \in X$ of all neighborhoods satisfies (B1)–(B4). The so defined topology on X is called *Tychonov* and X equipped with this topology is called the *Tychonov product* of the family (X_α, τ_α), $\alpha \in J$.

The Tychonov topology can be characterized as the weakest topology on the Cartesian product X such that all projections $\mathrm{pr}_\alpha: X \to X_\alpha$, where $\mathrm{pr}_\alpha(f) = f(\alpha)$, are continuous.

2.4. Tychonov Theorem. *The Tychonov product of any collection of compact topological spaces is a compact space.*

Proof. Consider an arbitrary ultrafilter $\mathcal{F}$ on X. By the Ultrafilters Image theorem, $\overline{\mathrm{pr}}_\alpha(\mathcal{F})$ is an ultrafilter on the compact space (X_α, τ_α). By Theorem 2.3 the ultrafilter $\overline{\mathrm{pr}}_\alpha(\mathcal{F})$ converges to some point $x_\alpha \in X_\alpha$. Let $f \in X$ be such that $f(\alpha) = x_\alpha$ for every $\alpha \in J$. By definition of the Tychonov topology the ultrafilter $\mathcal{F}$ converges to the point $f \in X$. Now Theorem 2.3. completes the proof. □

Let X be a set and m a positive integer. A collection $\mathcal{A}$ of subsets of X is called *m-regular* with respect to a subset $Y \subseteq X$ if for every partition $Y = Y_1 \cup \cdots \cup Y_m$ there is k and a subset $A \in \mathcal{A}$ such that $A \subseteq Y_k$. If a collection $\mathcal{A}$ is not m-regular with respect to Y, then there is a partition $Y = Y_1 \cup \cdots \cup Y_m$ such that $A \not\subseteq Y_i$ for every $A \in \mathcal{A}$ and for every $i = 1, \ldots, m$. Such a partition is called *non-regular*.

2.5. Compactness Theorem for partitions. *If a collection $\mathcal{A}$ of subsets of a set X is m-regular with respect to X and every element of $\mathcal{A}$ is a finite subset, then there is a finite subset $Y \subseteq X$ such that $\mathcal{A}$ is m-regular with respect to Y.*

Proof. Let $M = \{1, \ldots, m\}$, M_x be a copy of M for every $x \in X$. Consider the Tychonov product $M^X = \prod\{M_x : x \in X\}$, where each factor is endowed with the discrete topology. Suppose our theorem is not valid. Then for every finite subset $Y \subseteq X$ there is a non-regular partition $Y = Y_1 \cup \dots \cup Y_m$. Define the characteristic function h of this partition letting $h(x) = i$ iff $x \in Y_i$. Extend the function h to a map $f_Y \in M^X$. Let $F_Y = \{f_K : Y \subseteq K, K \text{ is a finite subset of } X\}$. The centered family $\{F_Y : Y \text{ is a finite subset of } X\}$ can be completed to an ultrafilter $\mathcal{F}$. By Tychonov Theorem, the ultrafilter $\mathcal{F}$ converges to a point $f \in M^X$. Consider $X = X_1 \cup \dots \cup X_m$, where $X_i = \{x \in X : f(x) = i\}$. By the conditions of the theorem there is k and $A \in \mathcal{A}$ such that $A \subseteq X_k$. Consequently, $f(x) = k$ for all $x \in A$. Since $\mathcal{F}$ converges to f, there is a finite subset $Y \subseteq X$ such that $A \subseteq Y$ and $f_Y(x) = f(x)$ for all $x \in X$. The restriction of f_Y to the subset Y determines a nonregular partition $Y = Y_1 \cup \dots \cup Y_m$. Since $f_Y(x) = k$ for all $x \in A$, we have $A \subseteq Y_k$, contradicting to the non-regularity of this partition. □

2.6. Ultrafilters and amenability. Let G be a group, $\operatorname{Exp} G$ the family of all subsets of G, $[0,1]$ the unit interval. A map $\mu\colon \operatorname{Exp} G \to [0,1]$ is called *left-invariant measure* if:

1) $\mu(G) = 1$;
2) if $A_1, \dots, A_m$ are pairwise disjoint subsets of G, then

$$\mu(A_1 \cup \dots \cup A_m) = \mu(A_1) + \dots + \mu(A_m);$$

3) $\mu(gA) = \mu(A)$ for every subset $A \subseteq G$ and every $g \in G$, where $gA = \{ga : a \in A\}$.

If a left-invariant measure $\mu\colon \operatorname{Exp} G \to [0,1]$ exists, then the group G is called *amenable*. It is known that given a left-invariant measure on a group G, one can construct a two side invariant measure, which is called a *Banach measure*.

Using the technique of ultrafilters, let us prove the amenability of the group $\mathbb{Z}$ of integers. For any subset $A \subseteq \mathbb{Z}$ and natural number n let

$$\mu_n(A) = \frac{|A \cap [-n,n]|}{2n+1}.$$

Define the map $\mu_A\colon \mathbb{N} \to [0,1]$ letting $\mu_A(n) = \mu_n(A)$. Fix a free ultrafilter $\mathcal{F}$ on the set of natural numbers. Since $[0,1]$ is compact

and Hausdorff, the ultrafilter $\overline{\mu_A}(\mathcal{F})$ converges to a unique limit $\mu(A)$. Therefore, we have defined a map $\mu\colon \operatorname{Exp} G \to [0,1]$. The required properties of μ follow from the following results:

1′) $\mu_n(\mathbb{Z}) = 1$ for every natural n;
2′) if $A_1, \dots, A_m$ are pairwise disjoint subsets of $\mathbb{Z}$, then

$$\mu_n(A_1 \cup \dots \cup A_m) = \mu_n(A_1) + \dots + \mu_n(A_m)$$

for every natural n;

3′) $|\mu_n(A+a) - \mu_n(A)| \leqslant \frac{|a|}{2n+1}$ for every integer a and natural n.

We refer the interested reader to the book [2] for the details on the theory of amenable groups.

EXERCISES.

1. A group G is defined to satisfy the *Følner condition* if for every $\varepsilon > 0$ and every finite subset $K \subseteq G$ there is a nonempty finite subset $U \subseteq G$ such that for every $g \in K$ we have

$$\frac{|(gU \setminus U) \cup (U \setminus gU)|}{|U|} < \varepsilon.$$

Prove that each group satisfying the Følner condition is amenable.

2. Prove that each abelian group is amenable.
3. Let F be the free group with two generators a, b. For each $i \in \mathbb{Z}$ let H_i be the set of all words of F that can be written as $a^i b^j$ for some $j \in \mathbb{Z}$. The partition $F = \bigcup\{H_i : i \in \mathbb{Z}\}$ is called the *von Neumann partition*. Using this partition prove that the group F is not amenable.

§3. Space of ultrafilters.

Let X be a set, βX the set of all ultrafilters on X. For a subset $A \subseteq X$ put $\bar{A} = \{p \in \beta X : A \in p\}$. A subset $W \subseteq \beta X$ is called a neighborhood of an ultrafilter $p \in \beta X$ if there exists $A \in p$ such that $\bar{A} \subseteq W$. Conditions (B1)–(B4) are easy to check. Thus, we have introduced a topology on the set of ultrafilters by means of system of neighborhoods at each point.

Let us show that for every subset $A \subseteq X$ the set $\bar{A}$ is open and closed in the space βX. If $p \in \bar{A}$, then $A \in p$ and, consequently, $\bar{A}$ is a neighborhood of each its point, i.e., $\bar{A}$ is open. Since $\beta X \setminus \bar{A} = \overline{X \setminus A}$, we see that the subset $\bar{A}$ is closed as the complement of an open subset.

Let $p, q \in \beta X$. Choose a subset $A \in p$ such that $A \notin q$. Then $X \setminus A \in q$ and $\bar{A}, \overline{X \setminus A}$ are disjoint neighborhoods of the points p, q, i.e., the space βX is Hausdorff.

3.1. Theorem. *The space βX is compact.*

Proof. Let $\mathfrak{A}$ be an open cover of the space βX. Since each open subset on βX is the union of subsets of the form $\bar{A}$, without loss of generality, we may suppose that $\mathfrak{A} = \{\bar{A} : A \in \mathcal{F}\}$, where $\mathcal{F}$ is a family of subsets of X. Let

$$\mathcal{F}' = \{X \setminus A : A \in \mathcal{F}\}.$$

Suppose $\mathcal{F}'$ is centered, then $\mathcal{F}'$ is contained in some ultrafilter p. Since $\mathfrak{A}$ is a cover of βX, there exists a subset $A \in \mathcal{F}$ such that $p \in \bar{A}$. On the other hand, $X \setminus A \in p$. Since $A \in p$, we have $X \setminus A \in p$, thus obtaining a contradiction with the fact that p is a filter. Therefore, the family $\mathcal{F}'$ is not centered. Choose subsets $A_1, \ldots, A_n \in \mathcal{F}$ such that

$$(X \setminus A_1) \cap \cdots \cap (X \setminus A_n) = \varnothing.$$

Then $A_1 \cup \cdots \cup A_n = X$ and $\bar{A}_1 \cup \cdots \cup \bar{A}_n = \beta X$. Thus, $\{\bar{A}_1, \ldots, \bar{A}_n\}$ is a finite subcover of $\mathfrak{A}$. □

Let φ be a filter on a set X and $\overline{\varphi} = \{p \in \beta X : \varphi \subseteq p\}$. Since $\overline{\varphi} = \bigcap\{\bar{A} : A \in \varphi\}$, we see that $\overline{\varphi}$ is a nonempty closed subset of βX.

3.2. Theorem. *For every nonempty closed subset $H \subseteq \beta X$ there exists a filter φ on the set X such that $H = \overline{\varphi}$.*

Proof. Let $\varphi = \{F \subseteq X : F \in p$ for every ultrafilter $p \in H\}$. Obviously, φ is a filter and $H \subseteq \overline{\varphi}$. Suppose $H \neq \overline{\varphi}$ and choose an ultrafilter $q \in \overline{\varphi} \setminus H$. Since H is closed in βX, there exists a subset $B \in q$ such that $\overline{B} \cap H = \varnothing$. Hence, $X \setminus B \in p$ for every ultrafilter $p \in H$. By definition of the filter φ, we have $X \setminus B \in \varphi$. Since $q \in \overline{\varphi}$, we see that $X \setminus B \in q$ which contradicts to the fact that q is a filter. □

Identify the set X with the subset of all principal ultrafilters in βX. If $x \in X$, then $\overline{\{x\}}$ is a neighborhood of the principal ultrafilter x and $\overline{\{x\}} = \{x\}$. Thus, each point of X is isolated in βX, i.e., X is a discrete subspace.

Show that the subspace X is dense in βX. Let $p \in \beta X$ and $A \in p$. Choose an arbitrary element $a \in A$. Then the principal ultrafilter a is contained in the neighborhood $\bar{A}$ of the ultrafilter p.

3.3. Theorem. *Each map $f: X \to Y$ of the subspace $X \subseteq \beta X$ into a compact Hausdorff space Y can be extended to a continuous map $\bar{f}: \beta X \to Y$.*

Proof. Fix an arbitrary ultrafilter $p \in \beta G$. By Theorem 1.10, $\bar{f}(p)$ is an ultrafilter on Y. Since Y is a compact Hausdorff space, by Theorem 2.3, the ultrafilter $\bar{f}(p)$ converges to a unique point y. Put $\bar{f}(p) = y$. Clearly, $\bar{f}$ is an extension of the map f onto βX. Show that $\bar{f}$ is continuous. Let V be an arbitrary closed neighborhood of the point $y = \bar{f}(p)$. Then $f^{-1}(V) \in p$ and $\bar{f}(\bar{A}) \subseteq V$, where $A = f^{-1}(V)$. We have only to note that $\bar{A}$ is a neighborhood of the ultrafilter p. □

A centered family $\mathcal{F}$ of subsets of a set X is called a *base* of an ultrafilter p if for every subset $A \in p$ there exists a subset $F \in \mathcal{F}$

such that $F \subseteq A$. Note that $\mathcal{F} \subseteq p$ and p is a unique ultrafilter with this property.

Any map $f: X \to \beta X$ can be extended to a continuous map $\bar{f}: \beta X \to \beta X$. Fix an ultrafilter $p \in \beta X$, a subset $A \in p$ and for every element $a \in A$ choose an arbitrary set $F_a \in f(a)$. Show that the centered family of subsets of the form $\bigcup\{F_a : a \in A, F_a \in f(a)\}$ is a base of the ultrafilter $\bar{f}(p)$. Choose an arbitrary subset $B \in f(a)$. Since $\overline{B}$ is a neighborhood of the ultrafilter $\bar{f}(p)$ and the map $\bar{f}$ is continuous, there exists a subset $A \in p$ such that $\bar{f}(\bar{A}) \subseteq \overline{B}$. Then $\bar{f}(a) \in \overline{B}$ for every element $a \in A$. Since $\bar{f}$ is an extension of the map f, we have $f(a) \in \overline{B}$, i.e., $B \in f(a)$ for every $a \in A$. Put $F_a = B$, $a \in A$. Then $\bigcup\{f_a : a \in A\} \subseteq B$ and the proof is complete.

The constructed base of the ultrafilter $\bar{f}(p)$ will be called *canonical*.

The space of ultrafilters is a particular case of a general construction, the Čech-Stone compactification of a Tychonov space. Further, all spaces considered are assumed to be Hausdorff.

A topological space X is called *Tychonov* (or *completely regular*) if for every closed subset $F \subseteq X$ and every point $a \notin F$ there exists a continuous function $f: X \to [0,1]$ such that $f(F) = 0$, $f(a) = 1$. In other words, there exists sufficiently many functions from X to $[0,1]$ in order to separate points and closed subsets. Obviously, every discrete space is Tychonov.

In 1937 Čech and Stone independently proved the following theorem. For every Tychonov space X there exists a compact space βX satisfying the following properties:

1) X is a dense subset of the space βX;
2) every continuous map f from X into a compact space Y extends to a continuous map $\bar{f}: \beta X \to Y$.

Clearly, the space βX satisfying the conditions of this theorem is unique up to homeomorphisms which are the identity on X. The space βX is called the *Čech-Stone compactification* of X. Thus, the considered above space βX of ultrafilters on the set X is essentially the Čech-Stone compactification of the discrete space X.

The minimal cardinality of dense subsets of a topological space

X is called the *density* of X.

Note that if the space X is Hausdorff, Y is a dense subset in X, and $|Y| = \alpha$, then $|X| \leqslant 2^{2^\alpha}$. Indeed, to each point $x \in X$ assign the family of subsets $\{U \cap Y : U \text{ is a neighborhood of } x\}$. Since X is Hausdorff, we thus obtain an injective map of X into $\operatorname{Exp}\operatorname{Exp} Y$.

3.4. Lemma. *If X is an infinite discrete space, $|X| = \alpha$, then $d(X^{(2^\alpha)}) = \alpha$.*

Proof. Let $\mathcal{D}^\alpha$ be the Tychonov power of the discrete two-point space $\mathcal{D}$. Since $|\mathcal{D}^\alpha| = 2^\alpha$, the family of indexes of the Tychonov power $X^{(2^\alpha)}$ can be identified with $\mathcal{D}^\alpha$. Consider the family $\mathcal{B}$ of all open and closed subsets of the compact space $\mathcal{D}^\alpha$. Since every open and closed subset of $\mathcal{D}^\alpha$ is the union of a finite family of standard neighborhoods of points, we have $|\mathcal{B}| = \alpha$. For every finite subset $\{f_k : k = 1, \dots, n\} \subseteq \mathcal{D}^\alpha$ there exists a partition $\mathfrak{A} = \{A_k : k = 1, \dots, n\}$ of the space $\mathcal{D}^\alpha$ by subsets from $\mathcal{B}$ such that $f_k \in A_k$. Further, for any such partition $\mathfrak{A}$ and any function $\varphi: \{1, \dots, n\} \to X$ we define a map $\Phi_{\mathfrak{A},\varphi}: \mathcal{D}^\alpha \to X$ by $\Phi_{\mathfrak{A},\varphi}(f) = \varphi(k)$ whenever $f \in A_k$. Put $Y = \{\Phi_{\mathfrak{A},\varphi}(f) : \varphi \in X^{|\mathfrak{A}|}\}$. By construction, the set Y is dense in $X^{(2^\alpha)}$ and

$$|Y| \leqslant \sum\{\alpha^n |\mathcal{B}|^n : n = 1, 2, \dots\} = \alpha.$$

Thus, $d(X^{(2^\alpha)}) \leqslant \alpha$. The reverse inequality is a consequence of the following fact: the image of each dense subset in a Tychonov product under the projection map onto a factor is dense in this factor. □

3.5. Theorem (Hewitt-Marczewski-Pondiczery). *Let $\{X_i : i \in I\}$ be a family of topological spaces, α an infinite cardinal number, and $d(X_i) \leqslant \alpha$ for each $i \in I$. If $|I| \leqslant 2^\alpha$, then $d(\prod\{X_i : i \in I\}) \leqslant \alpha$.*

Proof. Consider a discrete space X of cardinality α and for each $i \in I$ fix a map f_i of the space X onto a dense subspace Y_i of X_i. Define the map $f: X^I \to \prod\{X_i : i \in I\}$ by the condition: $\operatorname{pr}_i f(p) = f_i(\operatorname{pr}_i p)$. By Lemma 3.4, there exists a dense in X^I subspace Y of cardinality $\leqslant \alpha$. Then $f(Y)$ is dense in $\prod\{Y_i : i \in I\}$ and, consequently, $d(\prod\{X_i : i \in I\}) \leqslant \alpha$. □

3.6. Corollary. *If X is an infinite discrete space of cardinality α, then $|\beta X| = 2^{2^\alpha}$.*

Proof. Since the Hausdorff space βX contains a dense subset of cardinality α, we have $|\beta X| \leqslant 2^{2^\alpha}$. Prove the reverse inequality. By Theorem 3.5, the compact space $\mathcal{D}^{(2^\alpha)}$ contains a dense subspace Y of cardinality α. Extend a bijection $f\colon X \to Y$ to a continuous map $\bar{f}\colon \beta X \to \mathcal{D}^{(2^\alpha)}$. Since $\bar{f}(\beta X)$, being the continuous image of a compactum, is closed in $\mathcal{D}^{(2^\alpha)}$ and $Y \subseteq \bar{f}(\beta X)$, we have $\bar{f}(\beta X) = \mathcal{D}^{(2^\alpha)}$. Consequently, $|\beta X| \geqslant 2^{2^\alpha}$. □

3.7. Independent families of subsets. Let $\mathfrak{A}$ be a family of subsets of an infinite set X with $|X| = \alpha$. A map $f\colon \mathfrak{A} \to \operatorname{Exp} X$ is called a *function of choice* of either $f(A) = A$ or $f(A) = X \setminus A$ for every subset $A \in \mathfrak{A}$. A family of subsets $\mathfrak{A}$ is called *independent* provided $\bigcap\{f(A) : A \in \mathcal{F}\} \neq \varnothing$ for every finite subfamily $\mathcal{F} \subseteq \mathfrak{A}$ and arbitrary function of choice f.

Prove that there exists an independent family $\mathfrak{A}$ of subsets of the set X of cardinality 2^α. Let $Y = \{y_k : x \in X\}$ be a dense subset of the space $\mathcal{D}^J$, $|J| = 2^\alpha$. Define the family $\mathfrak{A} = \{F(\eta) : \eta \in J\}$ of subsets of the set X by the condition

$$x \in F(\eta) \Leftrightarrow \operatorname{pr}_\eta y_x = 1.$$

Let $\mathcal{F} = \{F(\eta_1), \ldots, F(\eta_k)\}$ be a finite subset of $\mathfrak{A}$ and f a function of choice. Choose an element y_x so that

$$\operatorname{pr}_{\eta_i} y_x = \begin{cases} 1, & \text{if } f\big(F(\eta_i)\big) = F(\eta_i); \\ 0, & \text{if } f\big(F(\eta_i)\big) = X \setminus F(\eta_i). \end{cases}$$

Then $x \in \bigcap\{f(A) : A \in \mathcal{F}\}$ and hence $\mathfrak{A}$ is independent.

EXERCISES.

1. Let X be an infinite family of cardinality α. Prove that the cardinality of the set of all uniform ultrafilters on X is equal $|\beta X|$.
2. Prove that the cardinality of every disjoint family of open subsets in the space βX does not exceed the cardinality of X.

3. Let X be a countable set. Construct a disjoint family of open sets in the space $\beta X \setminus X$ whose cardinality is continuum.
4. Prove that the closure of every open subset in the space βX is open.
5. Let p be an arbitrary element of the space βX. Prove that each countable cover of the space $\beta X \setminus \{p\}$ contains a finite subcover.

§4. Semigroup of ultrafilters.

Let S be a semigroup endowed with the discrete topology. We are going to expose a construction of extending the multiplication operation from S onto βS.

For each element $a \in S$ define the map $R_a\colon S \to S$ by the rule: $R_a(x) = xa$ for every $x \in S$. Since $S \subset \beta S$, the map R_a can be extended to a continuous map $\overline{R}_a\colon \beta S \to \beta S$. Clearly, for every ultrafilter $p \in \beta S$ the filter $\overline{R}_a(p)$ is an ultrafilter with a canonical base $\{Pa : P \in p\}$. Thus, we have defined the product $pa = \overline{R}_a(p)$ of the ultrafilter $p \in \beta S$ and the element $a \in S$.

Further, for each ultrafilter $p \in \beta S$ we can consider the map $L_p\colon S \to S$ defined by the formula $L_p(x) = px$ for every $x \in S$. Extend the map L_p to a continuous map $\overline{L}_p\colon \beta S \to \beta S$. If $q \in \beta S$, the ultrafilter $\overline{L}_p(q)$ is called the *product* of the ultrafilters p and q and is denoted by pq.

Find explicitly a canonical base of the ultrafilter pq as the image of the ultrafilter q under the map $\overline{L}_p$. Take an arbitrary subset $Q \in q$ and for each element $x \in Q$ choose a subset $P_x \in p$. Actually, the subset $\bigcup\{P_x x : x \in Q\}$ is, by the definition from §3, an element of a canonical base of the ultrafilter $\overline{L}_p(q)$.

It follows from continuity of the map $\overline{L}_p$ that the multiplication operation in βS is continuous with respect to the second argument with the first argument fixed. Therefore, for each subset $A \in pq$ there exists a subset $Q \in q$ such that $p\overline{Q} \subseteq \bar{A}$. It follows from continuity of the map $\overline{R}_a$, $a \in S$ that the multiplication operation is continuous with respect to the first argument provided the fixed second argument is a principal ultrafilter. Thus, for every subset $A \in pa$ there exists a subset $P \in p$ such that $\overline{P}_a \subseteq \bar{A}$.

Using these properties of the multiplication, we are able to prove the associativity of the multiplication operation. Let $p, q, r \in \beta S$. Since $(pq)r$ and $p(qr)$ are ultrafilters, it is sufficient to show that $(pq)r \subseteq p(qr)$. Fix an arbitrary subset $A \in (pq)r$ and choose a subset $R \in r$ such that $(pq)\overline{R} \subseteq \bar{A}$. Consequently, $(pq)x \in \bar{A}$ for

every element $x \in R$. Since x is a principal ultrafilter, there exists a subset $F_x \subseteq pq$ such that $\overline{F}_x x \subseteq \bar{A}$. Further, choose a subset $Q_x \in q$ such that $q\overline{Q}_x \subseteq F_x$. Thus, $(p\overline{Q}_x)x \subseteq \bar{A}$ for every $x \in R$. Choose an arbitrary element $y \in Q_x$. Since $(py)x = p(yx)$, we have $p(yx) \in \bar{A}$ for every $y \in Q_x$. Put $B = \bigcup\{Q_x x : x \in R\}$ and note that B is an element of a canonical base of the ultrafilter qr and, in particular, $qr \in \overline{B}$. Since $p\overline{B} \subseteq \bar{A}$, we have $p(qr) \in \bar{A}$, i.e. $A \in p(qr)$. By arbitrarity of $A \in (pq)r$ we conclude that $(pq)r \subseteq p(qr)$.

Suppose now that S is a right cancellative semigroup, i.e., the condition $xa = ya$ implies $x = y$. Prove that the closed subset $\beta S \setminus S \subseteq \beta S$ is a subsemigroup of the semigroup βS. Indeed, let $p, q \in \beta S \setminus S$ and $A \in pq$. Choose subsets $Q \in q$ and $P_x \in p$, $x \in Q$ such that $\bigcup\{P_x x : x \in Q\} \subseteq A$. Since $p \in \beta S \setminus S$, every subset P_x is infinite. The right cancellativity implies that the subsets $P_x x$ are infinite. Consequently, the subset A is infinite and $pq \in \beta S \setminus S$.

A semigroup endowed with a Hausdorff topology is called *left-topological* if the multiplication operation is continuous with respect to the second argument with the first argument fixed, i.e., the left shift by each element of the semigroup is a continuous map. We have proved above that βS is a compact left-topological semigroup.

Recall that an element s of a semigroup is called an *idempotent* if $s^2 = s$. A subset A is called a *left (right) ideal* of a semigroup S if $xA \subseteq A$ (respectively $Ax \subseteq A$) for every $x \in S$. An *ideal* of a semigroup is a subset that is both a left and a right ideal.

4.1. Theorem. *Every compact left-topological semigroup contains an idempotent.*

Proof. Consider any chain $\mathfrak{A}$ of closed subsemigroups of the semigroup S. By compactness of the semigroup, the set $\bigcap\mathfrak{A}$ is nonempty and, consequently, is a closed subsemigroup. Therefore, every chain of closed subsemigroups has a lower bound. By the Kuratowski-Zorn Lemma, there exists a minimal closed subsemigroup H of S. Fix an arbitrary element $e \in H$. Since $(eH)(eH) \subseteq eH$, the subset eH is a subsemigroup. Continuity of the left shift by the element e, compactness, and Hausdorffness of the semigroup S

imply that eH is a closed subsemigroup. Since H is a minimal closed subsemigroup and $eH \subseteq H$, we have $eH = H$. Consider the set $E = \{x \in H : ex = e\}$. Since $e \in H$, we have $E \neq \varnothing$. Obviously, E is a subsemigroup. It follows from continuity of the left shift by e that E is a closed subsemigroup. Since $E \subseteq H$, we have $E = H$. Thus, $e \in E$ and $e^2 = e$. □

4.2. Theorem. *Let S be a compact left-topological semigroup. Every minimal right ideal R of the semigroup S is closed. Every right ideal H of the semigroup S contains a minimal right ideal.*

Proof. Take an arbitrary element $a \in R$. Since R is a minimal right ideal, we have $aS = R$. Continuity of the left shift by the element a, compactness, and Hausdorffness of S imply closedness of the subset aS. Prove the second statement. Take an arbitrary element $h \in H$. Since $hS \subseteq H$, we see that H contains a closed right ideal of the semigroup S. Consider an arbitrary chain $\mathfrak{A}$ of contained in H closed right ideals of the semigroup S. Since S is compact, the subset $\bigcap \mathfrak{A}$ is nonempty and, consequently, is a closed right ideal. By the Kuratowski-Zorn Lemma, there exists a minimal closed right ideal of the semigroup S contained in H. Take an arbitrary element $a \in K$. Since $aS \subseteq K$ and aS is a closed right ideal, we have $aS = K$. Therefore, K is a minimal right ideal of the semigroup S. □

A systematic exposition of the semigroup theory with different continuity properties of operations is given in the book [5]. Solutions of problems 1–3 below can be found in Chapter 2 of the book [3].

EXERCISES.

1. Let S be an arbitrary semigroup and R a minimal right ideal of S. Prove the following statements:
   1) aR is a minimal right ideal of the semigroup S for every element $a \in S$;
   2) $\bigcup\{aR : a \in S\}$ is a minimal ideal of S;
   3) R contains no proper right ideal of the semigroup of R.
2. Let S be a semigroup without proper right ideals, $E(S)$ the set of idempotents of the semigroup S and $E(S) \neq \varnothing$. Prove the following statements:

   1) $es = e$ for every $e \in E(S)$ and $s \in S$;
   2) Se is a group with the unit e;
   3) the semigroup S is isomorphic to the direct product $Se \times E(S)$.
3. Let e be an idempotent of a semigroup S and eS a minimal right ideal of S. Prove the following statements:
   1) Se is a minimal left ideal of the semigroup S for every element $a \in S$;
   2) SeS is a minimal ideal of the semigroup S;
   3) eSe is a subgroup of the semigroup of S.
4. Prove that the closure of a left ideal of any left-topological semigroup is a left ideal.
5. Suppose a left-topological semigroup S contains a dense subset A such that $ax = xa$ for every $a \in A$, $x \in S$. Prove that every closed left ideal of the semigroup S is an ideal.

§5. Ramsey Theorem.

5.1. Ramsey Theorem (infinite version). *Let k, m be natural numbers and $[\mathbb{N}]^k$ denote the family of all k-element subsets of the set of natural numbers. For each coloring $\chi\colon [\mathbb{N}]^k \to \{1, \ldots, m\}$ of the set $[\mathbb{N}]^k$ into m colors there exists an infinite subset $A \subseteq \mathbb{N}$ such that all its k-element subsets have the same color.*

Proof. The case $k = 1$ is obvious and is an infinite version of the Dirichlet principle.

Consider the case $k = 2$. We identify elements of the set $[\mathbb{N}]^2$ with edges of complete graph with $\mathbb{N}$ vertices. Let $X_0 = \mathbb{N}$ and fix any point $x_0 \in X_0$. Among edges connecting the point x_0 with points of the set $X_0 \setminus \{x\}$ infinitely many have the same color, say, c_0. Let

$$X_1 = \{i \in X_0 \setminus \{x_0\} : \chi(x_0, i) = c_0\}.$$

Fix any point $x_1 \in X_1$ and consider edges connecting x_1 with other points of the set X_1. Among such edges there are infinitely many having the same color, say, c_2. Let

$$X_2 = \{i \in X_1 \setminus \{x_1\} : \chi(x_1, i) = c_2\}.$$

Fix any point $x_2 \in X_2$, $x_2 > x_1$ and select in $X_2 \setminus \{x_2\}$ an infinite subset X_3 such that all edges connecting the point x_2 with points of the set X_3 have the same color, say, c_3.

Continuing in this way, we will construct a sequence $T = \{x_0, x_1, \ldots\}$ such that for each edge $\{t, t'\}$ connecting points of T the color of $\{t, t'\}$ depends only on $\min\{t, t'\}$. Using this fact we may define a new coloring χ^* letting $\chi^*(t) = \chi(\{t, t'\})$, where $t' \in T$ is any point with $t' > t$. By the Dirichlet principle, some infinite subset A is monochromatic with respect to χ^*, i.e., $\chi^*(a) = c$ for every $a \in A$. By the definition of χ^*, this precisely means that all two-element subsets of the set $A \subseteq T$ have the same color with respect to χ.

Consider the case $k = 3$. Let $X_0 = \mathbb{N}$ and fix a point $x_0 \in X_0$. Any coloring $\chi\colon [\mathbb{N}]^3 \to \{1, \ldots, m\}$ induces a coloring χ_0 of pairs from $X_0 \setminus \{x_0\}$ by the rule $\chi_0(i, j) = \chi(x_0, i, j)$. By the Ramsey Theorem for $k = 2$, the set $X_0 \setminus \{x_0\}$ contains an infinite subset X_1 such that $\chi(\{i, j\}) = c_1$ for all distinct $i, j \in X_1$. Fix any $x_1 \in X_1$ with $x_1 > x_0$. The coloring χ induces a coloring χ_1 of pairs from $X_1 \setminus \{x_1\}$ by the rule $\chi_1(\{i, j\})$. By the Ramsey Theorem for $k = 2$ the set $X_1 \setminus \{x_1\}$ contains an infinite subset X_2 such that $\chi_1(\{i, j\}) = c_2$ for all distinct $i, j \in X_2$. Continuing in this way, we may construct a set $T = \{x_0, x_1, \ldots\}$ such that the color of any three-element subset $\{t, t', t''\}$ depends only on $\min\{t, t', t''\}$. Define a new coloring χ^* of the set T letting $\chi^*(t) = \chi(\{t, t', t''\})$ for any t', t'', $t < t' < t''$. By the Dirichlet principle, there exists an infinite subset $A \subseteq T$ monochromatic with respect to χ^*. By the definition of χ^*, all elements of $[A]^3$ are colored by one color.

We can assure readers that the same arguments work for any k. □

5.2. Ramsey Theorem (finite version). *For every natural numbers k, l, m, $k \leqslant l$, there exists a natural number $n(k, l, m)$ such that for every $n \geqslant n(k, l, m)$ and arbitrary coloring $\psi\colon [1, \ldots, n]^k \to \{1, \ldots, m\}$ of the family $[1, \ldots, n]^k$ of all k-subsets of the set $\{1, \ldots, n\}$ into m colors there exists an l-subset of the set $\{1, \ldots, n\}$ such that all its k-subsets have the same color.*

Proof. Put $X = [\mathbb{N}]^k$ and $\mathfrak{A} = \{[B]^k : B \subset \mathbb{N}, |B| = l\}$. Consider an arbitrary partition $X = A_1 \cup \cdots \cup A_m$. By the infinite version of Ramsey Theorem, there exists an infinite subset $A \subset \mathbb{N}$ such that $[A]^k \subseteq A_i$. Choose an arbitrary l-subset $B \subset A$. Then $[B]^k \subseteq A_i$. Thus, the family $\mathfrak{A}$ of subsets of the set X is m-regular with respect to X. By the definition, every subset from the family $\mathfrak{A}$ is finite. By the compactness theorem for partitions (Theorem 2.5), there exists a finite subset $Y \subseteq [\mathbb{N}]^k$ such that the family $\mathfrak{A}$ is m-regular with respect to Y. There exists a natural number $n(k, l, m)$ such that $Y \subseteq [1, \ldots, n(k, l, m)]^k$. Then the subset $[1, \ldots, n]^k$ is m-regular with respect to the family $\mathfrak{A}$ for all $n \geqslant n(k, l, m)$. This completes the proof. □

The minimal natural number $n(k, l, m)$ satisfying the finite version of Ramsey Theorem is called the *Ramsey number* and is denoted by $R(k,l,m)$.

5.3. Schur Theorem (infinite version). *For every coloring $\chi\colon \mathbb{N} \to \{1,\dots,m\}$ of the set of natural numbers into m colors the equation $x+y=z$ has a monochromatic solution.*

Proof. Define the coloring $\chi^*\colon [\mathbb{N}]^2 \to \{1,\dots,m\}$ by the rule

$$\chi^*\{i,j\} = \chi(|i-j|).$$

By the infinite version of Ramsey Theorem, there exists an infinite subset $A \subseteq \mathbb{N}$ such that $\chi^*\{i,j\} = \text{const}$ for every distinct elements $i,j \in A$. Choose three elements $i,j,k \in A$, $i<j<k$. Then

$$\chi^*\{i,j\} = \chi^*\{j,k\} = \chi^*\{i,k\}.$$

Sinse $(j-i)+(k-j) = k-i$, we obtain a monochromatic solution $j-i$, $k-j$, $k-i$. □

5.4. Schur Theorem (finite version). *For every natural number m there exists a natural number $S(m)$ such that for every $n \geqslant S(m)$ and an arbitrary coloring $\chi\colon \{1,\dots,n\} \to \{1,\dots,m\}$ the equation $x+y=z$ has a monochromatic solution in the set $\{1,\dots,n\}$.*

Proof. Consider the family $\mathfrak{A}$ of the triples of distinct natural numbers $\{i,j,k\}$ satisfying the condition $i+j=k$. By the infinite version of Schur Theorem, the family $\mathfrak{A}$ is m-regular with respect to $\mathbb{N}$. By the compactness theorem for partitions, the family $\mathfrak{A}$ is m-regular with respect to some finite subset $Y \subseteq \mathbb{N}$. We can take as $S(m)$ any natural number such that $Y \subseteq \{1,\dots,S(m)\}$. □

5.5. Corollary. *For every natural number m the congruence $x^m + y^m \equiv z^m \pmod p$ has nontrivial solutions for every sufficiently large prime number p.*

Proof. It is well-known that the multiplicative group $\mathbb{Z}_p^* = \{1,\dots,p\}$ of the field $\mathbb{Z}_p^*$ is cyclic. Take an arbitrary generator g of

the group $\mathbb{Z}_p^*$. For each $k \in \mathbb{Z}_p^*$ there exists a unique t, $0 \leqslant t < p-1$, such that $k = g^t \pmod p$. Write $t = i + mj$, $0 \leqslant i < m$, and define the coloring $\chi\colon \mathbb{Z}_p^* \to \{1,\dots,m-1\}$ by $\chi(k) = i$. By Theorem 5.4, for $p = S(m)$ there exist numbers $a,b,c \in \mathbb{Z}_p^*$ such that $a+b=c$ and a,b,c have the same color, say, i. Hence,

$$a \equiv g^{i+mj(a)} \pmod p,\ \ b \equiv g^{i+mj(b)} \pmod p,\ \ c \equiv g^{i+mj(c)} \pmod p,$$
$$g^{i+mj(a)} + g^{i+mj(b)} \equiv g^{i+mj(c)} \pmod p.$$

Multiplying the above relation by g^{-1} in the group $\mathbb{Z}_p^*$ we obtain the required solution:

$$x = g^{j(a)}, \quad y = g^{j(b)}, \quad z = g^{j(c)}.$$ □

Further information on the Ramsey Theorem, its applications and generalizations can be found in the book [1].

EXERCISES.

1. Prove that $R(2,3,2) = 6$.
2. Let $\langle a_n \rangle$ be an infinite sequence of distinct elements of the group G. Prove that for every partition $G = A_1 \cup \dots \cup A_m$ there exists a subset A_k and subsequence $\langle b_n \rangle$ of the sequence $\langle a_n \rangle$ such that $b_i b_j^{-1} \in A_k$ for every $i > j$.
3. Suppose each antichain in a partially ordered set X is finite. Prove that X contains an infinite chain. (Recall that an *antichain* is a subset consisting of mutually incomparable elements).
4. Prove that for every natural number m there exists a natural number $f(m)$ such that every subset of the plane consisting of $f(m)$ points in general position (i.e. each line contains at most two points of the set) contains m points which are vertices of a convex m-polygon.
5. A subset A of a metric space (X,d) is called *uniformly discrete* provided there exists $\epsilon > 0$ such that $d(x,y) > \epsilon$ for every distinct elements $x,y \in A$. Suppose an infinite subset $Y \subseteq X$ contains no nontrivial Cauchy sequence. Prove that there exists an infinite uniformly discrete subset $A \subseteq Y$.
6. A free ultrafilter p on the set of natural numbers is called a *Ramsey ultrafilter* provided for each finite partition $[\mathbb{N}]^2 = A_1 \cup \dots \cup A_n$ there exists a subset A_i and a subset $A \in p$ such that $[A]^2 \subseteq A_i$. Using the Ramsey Theorem and the Continuum Hypothesis consruct a Ramsey ultrafilter. The Continuum Hypothesis can be applied in the following form of a counting principle: every set of the continuum cardinality can be totoally ordered so that every its initial segment is countable.

§6. Hindman Theorem.

Let $\langle a_n \rangle$ be an infinite sequence of elements of a semigroup S. Denote by $\mathrm{FP}\langle a_n \rangle$ the collection of all elements of the semigroup having the form

$$a_{n_k} a_{n_{k-1}} \dots a_{n_1}, \qquad \text{where } n_1 < n_2 < \dots < n_k.$$

A subset A of the semigroup S is called an FP-*set* provided there exists an infinite sequence $\langle a_n \rangle$ of distinct elements from A such that $\mathrm{FP}\langle a_n \rangle \subseteq A$. If the semigroup S is commutative and the semigroup operation is denoted by "+", we use the symbol FS instead of FP.

6.1. Lemma. *If a free ultrafilter p on a semigroup S is an idempotent of the semigroup βS, then any subset $A \in p$ is an* FP-*set.*

Proof. Put $A_0 = A$. Since $pp = p$ and $p \in \bar{A}_0$, by continuity of the multiplication with respect to the second argument, there exists a subset $B \in p$, $B \subseteq A_0$, such that $p\overline{B} \subseteq \bar{A}_0$. Fix an arbitrary element $a_1 \in B$. Since $pa_1 \in \bar{A}_0$ and $a_1 \in S$, by continuity of multiplication with respect to the first argument, there exists a subset $A_1 \in p$ such that

$$\bar{A}_1 a_1 \subseteq \bar{A}_0, \quad A_1 \subseteq A_0, \quad a_1 \notin A_1.$$

Consequently,

$$A_1 a_1 \subseteq A_0, \quad A_1 \subseteq A_0, \quad a_1 \in A_0 \setminus A_1.$$

Analogously, for the subset $A_1 \in p$ we choose an element a_2 and a subset $A_2 \in p$ such that

$$A_2 a_2 \subseteq A_1, \quad A_2 \subseteq A_1, \quad a_2 \in A_1 \setminus A_2.$$

Continuing the process we construct a sequence $\langle a_n \rangle$ and a decreasing chain of subsets

$$A_0 \supseteq A_1 \supseteq \dots \supseteq A_n \supseteq \dots$$

such that

$$A_n \in p, \quad A_n a_n \subseteq A_{n-1}, \quad a_n \in A_{n-1} \setminus A_n,$$

for each natural number n. Obviously, all elements of the sequence $\langle a_n \rangle$ are distinct and $\mathrm{FP}\langle a_n \rangle \subseteq A$. □

6.2. Hindman Theorem for semigroups. *Suppose an infinite semigroup S is either a semigroup without idempotents or a right cancellative semigroup. Then for every finite partition $S = A_1 \cup \dots \cup A_m$ at least one of elements of the partition is an* FP-*set.*

Proof. By Lemma 6.1, it is sufficient to prove existence of a free ultrafilter on S which is an idempotent of the semigroup βS.

First suppose S contains no idempotent. By Theorem 4.1, there exists an idempotent element p of the semigroup βS. By the condition, the ultrafilter p cannot be principal and, consequently, $p \in \beta S \setminus S$.

Let S be a right cancellativee semigroup. In §4 we have already proved that $\beta S \setminus S$ is a closed subsemigroup of the semigroup βS. Applying Theorem 4.1 to the semigroup $\beta S \setminus S$ we find an idempotent $p \in \beta S \setminus S$. □

6.3. Remark. Theorem 6.2 cannot be extended onto arbitrary semigroups. Here is a trivial counterexample. Consider an infinite set X, fix an element $a \in X$ and define the multiplication in S by the formula: $xy = a$ for every $x, y \in X$. The required partition is $X = X_1 \cup X_2$, where $X_1 = \{a\}$, $X_2 = X \setminus X_1$.

The set of natural numbers considered with the addition (multiplication) operation is a cancellative semigroup. It immediately follows from Theorem 6.2 that for every partition $\mathbb{N} = A_1 \cup \dots \cup A_m$ there exists an FS-set A_i and an FP-set A_j. The following theorem shows that the indexes i, j can be choosed equal.

6.4. Hindman Theorem for natural numbers. *For every partition $\mathbb{N} = A_1 \cup \dots \cup A_m$ of the set of natural numbers there exists a subset A_i which is an* FS-*set as well as an* FP-*set.*

Proof. Consider the family I of all ultrafilters on $\mathbb{N}$ such that every their element is an FS-set. By Lemma 6.1, I contains all idempotents of the semigroup $\beta(\mathbb{N},+)$, in particular, $I \neq \varnothing$. Note also that I is a closed subset in $\beta\mathbb{N}$.

Show that I is a right ideal of the semigroup $\beta(\mathbb{N},\cdot)$. Let $p \in I$ and $q \in \beta\mathbb{N}$. Fix an arbitrary subset $A \in pq$ and choose a subset $Q \in q$ such that $p\overline{Q} \subseteq \bar{A}$. Take an arbitrary element $a \in Q$. Since $pa \in \bar{A}$, there exists a subset $P \in p$ such that $P_a \subseteq A$. Since P is an FS-set, there exists an infinite sequence $\langle a_n \rangle$ of distinct elements from P such that $\mathrm{FS}\langle a_n \rangle \subseteq P$. Obvious inclusions

$$PS\langle a_n a \rangle \subseteq Pa \subseteq A$$

imply that A is an FS-set. By arbitrarity of $A \in pq$, we conclude that $pq \in I$.

Thus, the closed subset I is a right ideal and, in particular, a subsemigroup of the semigroup $\beta(\mathbb{N},\cdot)$. By Theorem 4.1, I contains an idempotent p of the semigroup $\beta(\mathbb{N},\cdot)$. By definition, the subset I consists of free ultrafilters. It immediately follows from Lemma 6.1 that every subset $A \in p$ is an FP-set. Consequently, every subset $A \in p$ is both FP-set and FS-set. Now we have only to choose a set of the partition $\mathbb{N} = A_1 \cup \cdots \cup A_m$ that is an element of the ultrafilter p. □

Consider an application of the Hindman Theorem to the ring theory. Let R be an arbitrary (not necessarily associative) ring. A subset $A \subseteq R$ is called *algebraic* provided there exists a finite collection $f_1(x), \ldots, f_m(x)$ of ring polynomials such that

$$A = \bigcup \{a \in R : f_i(a) = 0\}.$$

A point $b \in R$ is called *algebraically isolated* if $R \setminus \{b\}$ is an algebraic subset of the ring R. Note that if a ring has at least one isolated point, then all points of this ring are algebraically isolated.

6.5. Arnautov Theorem. *No infinite ring has algebraically isolated points.*

First we formulate two simple lemmas.

6.6. Lemma. *Let R be a ring, $f(x) \in R[x]$, $\deg f(x) = n$. There exists a polynomial $g(x) \in R[x]$ such that $\deg g(x) < n$ and*

$$f(x+a) = f(x) + f(a) + g(x)$$

for all elements $a \in R$.

Proof is obtained by removing parentheses.

6.7. Lemma. *Let $f(x) \in R[x]$, $\deg f(x) = n$, $n \in \mathbb{N}$, and*

$$A = \{a_1, \ldots, a_{n+1}\} \subset R.$$

If $f(b) = 0$ for every $b \in \mathrm{FS}(A)$, then $f(0) = 0$.

Proof. Induction by n. If $n = 1$, then

$$f(a_1) = f(a_2) = f(a_1 + a_2) = 0.$$

By Lemma 6.6,

$$f(a_1 + a_2) = f(a_1) + f(a_2) + g(a_2).$$

Consequently, $g(a_2) = 0$. Since $\deg g(x) = 0$, we have $g(0) = 0$. The relationship

$$(a_1) = f(a_1) + f(0) + g(0)$$

implies $f(0) = 0$.

Suppose $\deg f(x) = n$. Then for every $c \in \mathrm{FS}\{a_2, \ldots, a_{n+1}\}$ the following relation holds

$$f(a_1 + c) = f(a_1) + f(c) + g(c).$$

Since

$$f(a_1 + c) = f(a_1) = f(c) = 0,$$

we see that $g(c) = 0$ for every $c \in \mathrm{FS}\{a_2, \ldots, a_{n+1}\}$. By induction, $f(0) = 0$.

The relationship

$$f(a_1 + c) = f(a_1) + f(0) + g(0)$$

implies $f(0) = 0$. □

6.8. Proof of Theorem 6.5. Suppose the contrary. Then the zero of some infinite ring R is an algebraically isolated point. Choose polynomials $f_1(x), \dots, f_m(x)$ such that

$$R \setminus \{0\} = A_1 \cup \dots \cup A_m,$$

where

$$A_i = \{a \in R : f_i(a) = 0\}.$$

By the Hindman Theorem applied to the additive group of the ring R, in one of subsets of the partition, say A_i, there exists an infinite sequence $\langle a_n \rangle \subseteq A_i$ with $\mathrm{FS}\langle a_n \rangle \subseteq A_i$. Suppose $\deg f_i(x) = k$. Set $A = \{a_1, \dots, a_{k+1}\}$. By Lemma 6.7, $f_i(0) = 0$, a contradiction. □

6.9. Remark. A ring R is called *topologizable* provided there exists a nondiscrete Hausdorff topology on R such that the ring operations of addition, multiplication, and distraction are continuous. By the Markov Theorem, any countable ring is topologizable if and only if the zero is not algebraically izolated point. Thus, the Markov and Arnautov theorems together imply topologizability of every countable ring.

A history of the proof of the Hindman Theorem as well as its generalizations and an information on the semigroup $\beta\mathbb{N}$ are given in the survey paper [6]. Our exposition of the proof of the Arnautov Theorem follows the paper [13].

EXERCISES.

1. Prove the following generalization of the Hindman Theorem. Suppose an infinite semigroup S either contains no idempotent or is a right-cancellative semigroup. If every FP-set from P is partioned onto a finite number of subsets, then at least one of subsets from the partition is an FP-set.
2. An idempotent p of the semigroup βS, for some semigroup S, is called a *strong idempotent* provided p has a base of subsets of the form $\mathrm{FP}\langle a_n \rangle$,

where $\langle a_n \rangle$ is a sequence of different elements of the semigroup S. Suppose an infinite semigroup S either contains no idempotent or is a right-cancellative semigroup. Using the Continuum Hypothesis, construct a strong idempotent of the semigroup βS.

3. Construct a partition $\mathbb{N} = A_1 \cup A_2$ of the set of natural numbers such that $A + A \not\subseteq A_i$ for every nonempty subset $A \subseteq A_i$.

§7. Van der Waerden Theorem.

Let m be a natural number, S the Tychonov product of m copies of the semigroup $\beta(\mathbb{N},+)$. The elements of the space S will be represented as vectors $\vec{p} = (p_1,\dots,p_m)$. Note that the set S is a compact left topological semigroup with respect to coordinate-wise addition of vectors. Put

$$E^* = \{(a, a+d, \dots, a+(m-1)d) : a \in \mathbb{N}, d \in \mathbb{N} \cup \{0\}\},$$
$$I^* = \{(a, a+d, \dots, a+(m-1)d) : a \in \mathbb{N}, d \in \mathbb{N}\}.$$

Denote by E and I the closures in the semigroup S of the subsets E^* and I^*, respectively.

7.1. Lemma. *E is a subsemigroup of the semigroup S and I is an ideal of the semigroup E.*

Proof. Let $\vec{p} = (p_1,\dots,p_m)$, $\vec{q} = (q_1,\dots,q_m)$, $\vec{p},\vec{q} \in E$. Take an arbitrary neighborhood $V_1 \times \dots \times V_m$ of the element $\vec{p}+\vec{q}$. Using the continuity of the addition with respect to the second argument we can choose a neighborhood $U_1 \times \dots \times U_m$ of the element $\vec{q}$ such that

$$\vec{p} + (U_1 \times \dots \times U_m) \subseteq V_1 \times \dots \times V_m.$$

Choose elements $a \in \mathbb{N}$, $d \in \mathbb{N} \cup \{0\}$ ($d \in \mathbb{N}$ if $q \in I$) such that

$$(a, a+d, \dots, a+(m-1)d) = \vec{x},$$

$\vec{x} \in U_1 \times \dots \times U_m$. Since $\vec{p}+\vec{x} \in V_1 \times \dots \times V_m$ and $a, a+d, \dots, a+(m-1)d \in \mathbb{N}$, there exists a neighborhood $W_1 \times \dots \times W_m$ of the element $\vec{p}$ such that

$$(W_1 \times \dots \times W_m) + \vec{x} \subseteq V_1 \times \dots \times V_m.$$

Choose elements $b \in \mathbb{N}$, $e \in \mathbb{N} \cup \{0\}$ ($e \in \mathbb{N}$ if $p \in I$) such that

$$(b, b+e, \dots, b+(m-1)e) = \vec{y},$$

$\vec{y} \in W_1 \times \dots \times W_m$. Then $\vec{y}+\vec{x} \in V_1 \times \dots \times V_m$,

$$\vec{y}+\vec{x} = (a+b, a+b+d+e, \dots, a+b+(m-1)(d+e)).$$

Hence, $\vec{y}+\vec{x} \in E^*$ and, if either $\vec{p} \in I$ or $\vec{q} \in I$, then $\vec{y}+\vec{x} \in I^*$. □

7.2. Lemma. *If $p \in \beta\mathbb{N}$ and $\vec{p} = (p,\dots,p)$, then $\vec{p} \in E$.*

Proof. Let $U_1 \times \dots \times U_m$ be an arbitrary neighborhood of the element $\vec{p}$. Then $U = U_1 \cap \dots \cap U_m$ is a neighborhood of the element p. Choose an arbitrary element $a \in \mathbb{N} \cap U$. Then

$$(a,\dots,a) \in (U_1 \times \dots \times U_m) \cap E^*. \qquad \square$$

7.3. Lemma. *If R is a minimal right ideal of $\beta(\mathbb{N},+)$, $p \in R$, and $\vec{p} = (p,\dots,p)$, then $\vec{p} \in I$.*

Proof. By Theorem 4.2, there exists a minimal right ideal F in the right ideal $\vec{p}+E$ of the semigroup E. Since F, by Theorem 4.2, is a closed subsemigroup, Theorem 4.2 guarantees existence of an idempotent $\vec{q} \in F$. Since $\vec{q} \in \vec{p}+E$, we have $\vec{q} = \vec{p}+\vec{r}$ for some element $\vec{r} \in E$. Let $\vec{q} = (q_1,\dots,q_m)$, $\vec{r} = (r_1,\dots,r_m)$. Then $q_i = p + r_i \in p + \beta\mathbb{N}$. Since R is a minimal right ideal of the semigroup $\beta(\mathbb{N},+)$ and $p \in R$, we have $R = p + \beta\mathbb{N} = q_i + \beta\mathbb{N}$. Choose an element $t_i \in \beta\mathbb{N}$ such that $q_i + t_i = p$. Then $q_i + q_i + t_i = q_i + t_i = p$. Consequently, $\vec{q}+\vec{p} = \vec{p}$ and $\vec{p} \in F$.

It remains to prove that $F \subseteq I$. Since $FI \subseteq F$ and $FI \subseteq I$, we have $FI \subseteq F \cap I$. Consequently, $F \cap I \neq \varnothing$. Since I is an ideal and F a right ideal, we see that $F \cap I$ is a right ideal of the semigroup E and, by minimality of F we obtain $F \subseteq I$. □

7.4. Theorem. *If R is a minimal right ideal of the semigroup $\beta(\mathbb{N},+)$, $p \in R$, then every subset $A \Subset p$ contains arbitrarily long arithmetic progressions.*

Proof. Fix a natural number m and consider a neighborhood $\bar{A} \times \dots \times \bar{A}$ of the element $\vec{p} = (p,\dots,p)$. By Lemma 7.3 there exists $x \in I^* \cap (\bar{A} \times \dots \times \bar{A})$. Then $\vec{x} = (a, a+d, \dots, a+(m-1)d)$, $a, d \in \mathbb{N}$ while $a, a+d, \dots, a+(m-1)d \in A$. □

7.5. Van der Waerden Theorem (infinite version). *If the set of natural numbers is partitioned into a finite number of subsets, then at least one of subsets of the partition contains arbitrarily long arithmetic progressions.*

Proof. By Theorem 4.2 the semigroup $\beta(\mathbb{N},+)$ contains minimal right ideals. Suppose R is one of them. Choose an ultrafilter $p \in R$

and a subset of the partition being an element of the ultrafilter p. We have only to apply Theorem 7.4. □

7.6. Van der Waerden Theorem (finite version). *For each natural numbers k, m there exists a number $W(k,m)$ satisfying the condition: if $n \geqslant W(k,m)$, $\{1,\dots,n\} = A_1 \cup \dots \cup A_m$, then one of the subsets A_i contains an arithmetic progression of lenght k.*

Proof. Consider the family $\mathfrak{A}$ of all k-subsets of $\mathbb{N}$ that are arithmetic progressions (with respect to the natural ordering). By the infinite version of van der Waerden Theorem the family $\mathfrak{A}$ is m-regular with respect to $\mathbb{N}$. By the Compactness Theorem for partitions, the family $\mathfrak{A}$ is m-regular with respect to some finite subset $Y \subset \mathbb{N}$. Then the required natural number (k,m) can be determined by the condition $Y \subseteq \{1,\dots,W(k,m)\}$. □

7.7. Remark. Considering the semigroup $(\mathbb{N},\cdot)$ instead of the semigroup $(\mathbb{N},+)$ and geometric progressions instead of arithmetic ones in the above constructions it is easy to prove multiplicative counterparts of Propositions 7.1–7.6. In particular, if R is a minimal right ideal of the semigroup $\beta(\mathbb{N},\cdot)$, $p \in R$, then every subset $A \in p$ contains arbitrarily long geometric progressions. Hence, for every partition $\mathbb{N} = A_1 \cup \dots \cup A_m$ there exist subsets A_i and A_j such that A_i contains arbitrarily long arithmetic progressions and A_j contains arbitrarily long geometric progressions. The following theorem shows that we may suppose $i = j$.

7.8. Bergelson-Hindman Theorem. *For every partition of the set of natural numbers $\mathbb{N} = A_1 \cup \dots \cup A_m$ there exists a subset A_i that contains both arbitrarily long arithmetic and arbitrarily long geometric progressions.*

Proof. Consider the family W of all ultrafilters p onto $\mathbb{N}$ with the following property: each subset $P \in p$ contains arbitrarily long arithmetic progressions. By Theorem 7.4, W is nonempty. Show that W is a right ideal of the semigroup $\beta(\mathbb{N},\cdot)$. Let $p \in W$, $q \in \beta\mathbb{N}$. Fix an arbitrary subset $A \in pq$ and choose a subset $Q \in q$ such that $p\overline{Q} \subseteq \bar{A}$. Choose an arbitrary element $a \in Q$. Then $Pa \subseteq A$ for some subset $P \in p$. By the definition of W, the subset P contains arbitrarily long arithmetic progressions. But then the set Pa also contains arbitrarily long arithmetic progressions, and consequently, so does A. By arbitrarity of $A \in pq$, we conclude that $pq \in W$.

Hence, W is a right ideal of the semigroup $\beta(\mathbb{N},\cdot)$. By Theorem 4.2, there exists a minimal right ideal R of the semigroup $\beta(\mathbb{N},\cdot)$ contained in W. Consider an arbitrary ultrafilter $p \in R$ and choose a subset A_i of the partition such that A_i is an element of the ultrafilter p. Since $p \in W$, the set A_i contains arbitrarily long arithmetic progressions. Since $p \in W$, the set A_i contains arbitrarily long geometric progressions. □

Finally note that our exposition of the results of this section follows the paper [7].

EXERCISES.

1. A filter φ on the set of the natural numbers is called *additive* provided for every subset $A \in \varphi$ there exists a subset $B \in \varphi$ such that $B + B \subseteq A$. A subset $A \subseteq \mathbb{N}$ is called *additive* if A is an element of some additive filter. Prove that any subset A is additive iff there exists a decreasing sequence of subsets $A \supseteq A_1 \supseteq \dots \supseteq A_n \supseteq \dots$ such that $A_1 + \dots + A_n \subseteq A$ for every natural number n.
2. Let φ be an additive filter, $\overline{\varphi} = \{p \in \beta\mathbb{N} : \varphi \subseteq P\}$. Prove the following statements:
   (1) $\overline{\varphi}$ is a subsemigroup of the semigroup $\beta(\mathbb{N},+)$;
   (2) if R is a minimal right ideal of the semigroup $\overline{\varphi}$, $p \in R$, then every subset $A \in p$ contains arbitrarily long arithmetic progresions.
3. Prove that for any partition of an additive subset $A = A_1 \cup \dots \cup A_m$ there exists a subset A_i such that A_i is both FS-set and FP-set, and contains arbitrarily long arithmetic and geometric progressions.

§8. HALES-JEWETT THEOREM.

Let m be a natural number, S the free semigroup with generators $a_1, \dots, a_m$. Denote by H the Tychonov product of m copies of the semigroup βS. The elements of the space H will be denoted as vectors $\vec{p} = (p_1, \dots, p_m)$. Note that the space H is a compact left topological semigroup with respect to coordinatewise multiplication.

Consider an independent variable x and denote by F the set of all words $f(x)$ in the alphabet $\{a_1, \dots, a_m, x\}$. Denote by $F' \subseteq F$ the subset of all words $f(x)$ containing the element x. Set

$$X^* = \{(f(a_1), \dots, f(a_m)) : f(x) \in F\},$$
$$Y^* = \{(f(a_1), \dots, f(a_m)) : f(x) \in F'\}.$$

Let X and Y denote the closures in the semigroup H of the subsets X^* and Y^* respectively. The following statements 8.1–8.5 can be proved anagously to the statements 7.1–7.5.

8.1. Lemma. *X is a subsemigroup of the semigroup H, Y is an ideal of the subsemigroup X.*

8.2. Lemma. *If $p \in \beta S$ and $\vec{p} = (p, \dots, p)$, then $\vec{p} \in X$.*

8.3. Lemma. *If R is a minimal right ideal of the semigroup βS, $p \in R$, and $\vec{p} = (p, \dots, p)$, then $\vec{p} \in R$.*

8.4. Theorem. *If R is a minimal right ideal of the semigroup βS, $p \in R$, then for every subset $A \in p$ there exists a word $f(x) \in F'$ such that $f(a_1), \dots, f(a_m) \in A$.*

8.5. Hales–Jewett Theorem. *Suppose the free semigroup with generators $a_1, \dots, a_m$ is partitioned into a finite number of subsets $S = S_1 \cup \dots \cup S_n$. There exists a semigroup word $f(x)$ in the alphabet $\{a_1, \dots, a_m, x\}$ containing the element x and a subset S_i such that $f(a_1), \dots, f(a_n) \in S_i$.*

An original proof of Theorem 8.5 is sketched in Exercises 4–7 at the end of this section. We can derive some corollaries from the Hales–Jewett Theorem.

8.6. Theorem (a strengthened version of the van der Waerden Theorem). *Let m be a natural number, $\langle y_k \rangle$ an infinite sequence of natural numbers. For each partition $\mathbb{N} = B_1 \cup \dots \cup B_n$ at least one of the subsets from the partition contains an arithmetic progression $b, b+d, \dots, b+md$, where $d \in \mathrm{FS}\langle y_k \rangle$.*

Proof. Put $\Omega = \{0, 1, \dots, m\}$ and consider the free semigroup S with the set of free generators Ω. Define the map $\varphi: S \to \mathbb{N}$ by the following manner. Arbitrary element $s \in S$ can be written in the form $s_1 \dots s_t$, $s_i \in \Omega$. Put

$$\varphi(s) = 1 + \sum \{s_i y_i : i = 1, \dots, t\}$$

and let

$$S_i = \{s \in S : \varphi(s) \in B_i\}.$$

By the Hales–Jewett Theorem, there exists a word $f(x)$ in the alphabet $\Omega \cup \{x\}$ containing the element x and an index j such that $f(a) \in S_j$ for every $a \in \Omega$. Suppose $f(x) = a_1 \cdot \ldots \cdot a_t$, where $a_i \in \Omega \cup \{x\}$ and for at least one i we have $a_i = x$. Consider the partition of the set $\{1, \dots, t\}$ into two subsets $F = \{i : a_i = x\}$ and $G = \{1, \dots, t\} \setminus F$. Note that $F \neq \varnothing$. Put $d = \sum\{y_i : i \in F\}$, $b = 1 + \sum\{a_i y_i : i \in G\}$. Then $b + ad = \varphi(f(a))$ for every $a \in \Omega$. Consequently,

$$b, b+d, \dots, b+md \in B_j, \; d \in \mathrm{FS}\langle y_k \rangle. \qquad \square$$

8.7. Gallai Theorem. *Let k, m be natural numbers and $\mathbb{N}^k$ the set of k-dimensional vectors with positive integer coordinates. For every partition $\mathbb{N}^k = B_1 \cup \dots \cup B_n$ there exists an index i and subsets*

$$A_1 = \{b_1, b_1 + d, \dots, b_1 + md\}, \dots, A_k = \{b_k, b_k + d, \dots, b_k + md\}$$

such that $A_1 \times \dots \times A_k \subseteq B_j$.

Proof. For the sake of simplicity we consider only the case $k = 2$. Suppose S is the free semigroup with the set of generators

$$\Omega = \{(i, j) : i, j = 0, \dots, m\}.$$

Define the map $\varphi: S \to \mathbb{N}^2$ by the following manner. Each element $s \in S$ can be represented in the form $s_1 \dots s_t \in \Omega$ and we put

$$\varphi(s) = (1,1) + \sum \{s_i : i = 1, \dots, t\}.$$

Let $S_i = \{s \in S : \varphi(s) \in B_i\}$. By the Hales-Jewett Theorem, there exists a containing x word $f(x)$ in the alphabet $\Omega \cup \{x\}$ and an index j such that $f(a) \in S_j$ for every $a \in \Omega$. Suppose $f(x) = a_1 \dots a_t$, where $a_i \in \Omega \cup \{x\}$ and for at least one i we have $x_i = x$. Decompose the set $\{1, \dots, t\}$ into two subsets $F = \{i : a_i = x\}$ and $G = \{1, \dots, t\} \setminus F$. Note that $F \neq \varnothing$. Set $d = |F|$ and

$$(b_1, b_2) = (1,1) + \sum \{a_i : i \in G\}.$$

Since $\varphi(f(a)) \in B_j$ for every $a \in \Omega$, we have $A_1 \times A_2 \subseteq B_j$, where

$$A_1 = \{b_1, b_1 + d, \dots, b_1 + md\},$$
$$A_2 = \{b_2, b_2 + d, \dots, b_2 + md\}.$$

The theorem is proved. □

8.8. Graham-Leeb-Rothschild Theorem. *If an infinite-dimensional vector space V over a finite field F is partitioned into a finite number of subsets, then at least one of the subsets of this partition contains affine subspaces of arbitrary finite dimension.*

Proof. Consider the case $F = \mathbb{Z}_3 = \{0, 1, 2\}$ and find a two-dimensional affine subspace in one of the subsets of a partition $V = V_1 \cup \dots \cup V_k$. Actually, this case demonstrates all essential features of a general proof. Suppose B is a base of the space V and $\langle \vec{y}_n \rangle$, $\langle \vec{z}_n \rangle$ are two disjoint infinite sequences of distinct elements of B. Put

$$\Omega = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,1), (2,2)\}.$$

Denote by $\mathrm{pr}_i(a)$ the ith coordinate of $a \in \Omega$, $i = 1, 2$. Consider the free semigroup S with the set of generators Ω and define

the map $\varphi: S \to V$ by the following manner. Any element $s \in S$ can be represented in the form $s_1 \dots s_t$, $s_i \in \Omega$. Put

$$\varphi(s) = \sum \{\mathrm{pr}_1(s_i)\vec{y}_i + \mathrm{pr}_2(s_i)\vec{z}_i : i = 1, \dots, t\}.$$

Let $S_i = \{s \in S : \varphi(s) \in V_i\}$. By the Hales-Jewett Theorem, there exist an index j and a containing x semigroup word $f(x) = a_1 \dots a_t$ in the alphabet $\Omega \cup \{x\}$ such that $f(a) \in S_j$ for every $a \in \Omega$. Consider the partition of the set $\{1, \dots, t\}$ into two subsets $H = \{i : a_i = x\}$ and $G = \{1, \dots, t\} \setminus H$. Note that $H \neq \varnothing$. Let

$$\vec{u} = \sum \{\vec{y}_i : i \in H\}, \quad \vec{v} = \sum \{\vec{z}_i : i \in H\}.$$

Since $H \neq \varnothing$, by the choice of the sequences $\langle \vec{y}_n \rangle$ and $\langle \vec{z}_n \rangle$ the span W of the vectors $\vec{u}$ and $\vec{v}$ is two-dimensional. Put

$$\vec{b} = \{\mathrm{pr}_1(a_i)\vec{y}_i + \mathrm{pr}_2(a_i)\vec{z}_i : i \in G\}.$$

Then

$$\vec{b} + W = \{\varphi(f(a)) : a \in \Omega\} \subseteq V_j$$

and the theorem is proved. □

Fix natural numbers k, n and consider the set

$$V(k,n) = \{1, \dots, k\}^n$$

of n-dimensional vectors with coordinates from the set $\{1, \dots, k\}$. Suppose $I \subseteq \{1, \dots, n\}$ is a nonempty subset and for each $i \in \{1, \dots, n\} \setminus I$ choose an element $a_i \in \{1, \dots, k\}$. A *combinatorical line* is a set of vectors $(x_1, \dots, x_k) \in V(k,n)$ satisfying the conditions:

(1) $x_i = x_j$ for every $i, j \in I$;
(2) $x_i = a_i$ for every $i \in \{1, \dots, n\} \setminus I$.

Note that, since $I \neq \varnothing$, every combinatorical line in $V(k,n)$ contains exactly n points.

8.9. Theorem. *For every natural numbers k, m there exists a natural number $n(k,m)$ such that if $n \geqslant n(k,m)$, then for every coloring of the set $V(k,n)$ into m colors there exists a monochromatic combinatorical line.*

Proof. Consider the free semigroup with set of free variables $\Omega = \{1, \ldots, k\}$. Consider a containing an element x word $f(a)$ in the alphabet $\Omega \cup \{x\}$ and let $A_f = \{f(a) : a \in \Omega\}$. Denote by $\mathfrak{A}$ the family of all such A_f. By the Hales-Jewett Theorem, the family $\mathfrak{A}$ is m-regular with respect to S. By the compactness theorem for partitions, the family $\mathfrak{A}$ is m-regular with respect to some finite subset $Y \subset S$. Denote by $n(k,m)$ the maximal length of the words from Y. If $n \geqslant n(k,m)$, then the subset S_n consisting of the words of length $\leqslant n$ from S contains the subset Y. Therefore, the family $\mathfrak{A}$ is m-regular with respect to S_n. Construct a map $\varphi \colon S_n \to V(k,n)$. Consider an arbitrary element $s = s_1 \ldots s_n \in S_n$ of length $p \leqslant n$ and put $\varphi(s) = s_1 \ldots s_p s_{p+1} \ldots s_n$, where $s_{p+1} = \cdots = s_n = 1$. Suppose $V(k,n) = V_1 \cup \cdots \cup V_m$ is an arbitrary partition. Since

$$S_n = \varphi^{-1}(V_1) \cup \cdots \cup \varphi^{-1}(V_m)$$

and the family $\mathfrak{A}$ is m-regular with respect to S_n, there exists an index j and a containing x word $f(x)$ in the alphabet $\Omega \cup \{x\}$ such that $A_f \subseteq \varphi^{-1}(V_j)$. It is clear that $\varphi(A_f)$ is a combinatorical line in $V(k,n)$ and $\varphi(A_f) \subseteq V_j$. □

Consider the following higher-dimensional generalization of the tic-tac-toe game. The game $G(k,n)$ has the set $V(k,n)$ as its field. The first player chooses an arbitrary element from $V(k,n)$ and marks it by the symbol "×". The second player chooses one of the nonmarked elements from $V(k,n)$ and marks it by the symbol "o". A winner is a player who first marked a combinatorical line in $V(k,n)$ his/her symbols. It may happen that there is no monochromatic combinatorical line although all the elements of $V(k,n)$ are marked (a drawn game). Standard arguments of the theory of positional games show that either the first player has a winning strategy or the second player has a drawn strategy in $G(k,n)$. It immediately follows from Theorem 8.9 for $m = 2$ that for every natural

number k there exists a natural number n such that the first player always wins in $G(k,n)$.

The Hales-Jewett Theorem is proved in [10]. Our exposition follows the paper [8].

EXERCISES.

1. Suppose S is a finite commutative semigroup. Prove that there exists a natural number n and an element $s \in S$ such that the subset $\{s^n a : s \in S\}$ is a singleton.
2. The Euclidean plane is divided into unit squares by the lines of the form $x = a$, $y = b$, where a, b are arbitrary integers. Some integer number is placed in each unit square. Prove that for every natural number k there exist integers a, b, d, $d \neq 0$, such that the sum of numbers placed in the square

$$\{(x,y) : a \leqslant x \leqslant a+d, b \leqslant y \leqslant b+d\}$$

is divisible by k.
3. Formulate and prove a finite version of the Graham-Leeb-Rothschild Theorem.
4. Let n, m be natural numbers, X, Y sets, $|X| = m$. Suppose a family $\mathfrak{A}$ of subsets from X is n-regular with respect to X and a family $\mathcal{F}$ of subsets from Y is n^m-regular with respect to Y. Prove that the family of subsets

$$\mathfrak{A} \times \mathcal{F} = \{A \times F : A \in \mathfrak{A}, F \in \mathcal{F}\}$$

is n-regular with respect to $X \times Y$.
5. Suppose a family $\mathfrak{A}$ of finite subsets of a semigroup S is n-regular with respect to S for every natural number n. Prove that for every natural numbers k, n the family of subsets

$$\mathfrak{A}_k = \{A_1 \cdot \ldots \cdot A_k : A_i \in \mathfrak{A}\}$$

is n-regular with respect to S.
7. Let S be the free semigroup with a countable family of generators Ω. For a finite subset $A \subset \Omega$ denote by $\mathfrak{A}(A)$ the family of all subsets of the form $f(A)$, where f is a containing an x word in the alphabet $A \cup \{x\}$. Let $I(n,k)$ denote the following statement. For every n-element subset $A \subset \mathfrak{A}$ the family $\mathfrak{A}(A)$ is k-regular with respect to the semigroup generated by the subset A. Using induction prove statements $I(n,k)$ for all natural numbers n, k.

§9. Rado Theorem.

To prove the Rado Theorem, we need the following strengthened version of the van der Waerden Theorem.

9.1. Theorem. *For every natural numbers k, m, s there exists a natural number $M(k,m,s)$ satisfying the condition:*
if

$$n \geqslant M(k,m,s),$$

then for every coloring of the set $\{1,\dots,m\}$ in m colors there exist natural numbers k, d such that the numbers $d+a, a+kd, sd$ have the same color.

Proof. Induction by m. For $m = 1$, we can take $M(k,1,s) = \max\{k+1, s\}$, $d = a = 1$. For $m \geqslant 2$ fixed, suppose $M(k,m-1,s)$ exists for each k, s. For natural numbers x, y, denote by $W(x,y)$ a number determined by the finite version of the van der Waerden Theorem (Theorem 7.6).

Show that $sW\big(kM(k,m-1,s),m\big)$ can be taken as $M(k,m,s)$. Consider an arbitrary coloring

$$\chi\colon \{1,\dots,M\} \to \{1,\dots,m\}.$$

By definition of $W(x,y)$, there exist natural numbers a, d' such that

$$\{a+id' : 1 \leqslant i \leqslant kM(k,m-1,s)\} \subseteq \{1,\dots,W(k,m-1,s),m)\}$$

and all elements $a+id'$ of the progression have the same color, say, red.

There are two possibilities.
1) The number $sd'j$ is red for some $j \leqslant M(k,m-1,s)$. Put $d = jd'$. Then the numbers $a+d,\dots,a+kd, sd$ are red.
2) The number $sd'j$ is not red for every $j \leqslant M(k,m-1,s)$. Hence, the elements of the progression $\{sd'j : j = 1,\dots,M(k,m-1,s)\}$ are colored in $m-1$ colors.

For $j = 1,\dots,M(k,m-1,s)$ let $\chi^*(j) = \chi(cd'j)$. By definition of $M(k,m-1,s)$, there exist numbers $A+D,\dots,A+kD, sD$ from the set $\{1,\dots,M(k,m-1,s)\}$ which have the same color with respect to χ^*. Hence, the numbers

$$sd'A + sd'D, \dots, sd'A + ksDd', s(sd'D)$$

have the same color with respect to the initial coloring χ. □

9.2. Corollary. *For every natural numbers k, s, m and arbitrary coloring of the set of natural numbers into m colors there exists natural numbers a, d such that the numbers*

$$\{a + \lambda d : \lambda = 0, \pm 1, \dots, \pm k\}$$

and sd have the same color.

Proof. By Theorem 9.1 there exist numbers

$$b + d, \dots, b + (2k+1)d, sd$$

which have the same color. Set $a = b + (k+1)d$. □

Recall that a linear equation $c_1x_1 + \dots + c_nx_n = 0$ with nonzero entire coefficients is called *regular* provided for every finite coloring of the set of natural numbers there exists a monochromatic solution of this equation.

9.3. Rado Theorem. *A linear Diophantine equation $c_1x_1 + \dots + c_nx_n = 0$ with nonzero entire coefficients is regular if and only if some nonempty sum of its coefficients is zero.*

Proof. Sufficiency. Suppose the sum of some coefficients is zero. Renumerate the coefficients so that $c_1 + \dots + c_k = 0$. Let χ be a finite coloring of the set $\mathbb{N}$. If $k = n$, put $x_1 = \dots = x_n = 1$. Obviously, this gives a monochromatic solution of the equation. Thus, in the sequel we suppose that $k < n$. Put $B = c_{k+1} + \dots + c_n$. If $B = 0$, then $c_1 + \dots + c_n = 0$, and, once again, $x_1 = \dots = x_n = 1$ is a monochromatic solution. Consequently, we may suppose that $B \neq 0$.

Let

$$A = \text{G.C.D.}(c_1, \ldots, c_k), \quad s = A/\text{G.C.D.}(A, B).$$

There exists an integer t such that $At + Bs = 0$. Choose integers $\lambda_1, \ldots, \lambda_k$ such that $c_1\lambda_1 + \cdots + c_k\lambda_k = At$. We claim now that for every natural numbers a, d the equation has a parametric solution of the form

$$x_i = \begin{cases} a + \lambda_i d, & i = 1, \ldots, k; \\ sd, & i = k+1, \ldots, n. \end{cases}$$

Indeed,

$$\sum_{i=1}^{n} c_i x_i = \sum_{i=1}^{k} c_i x_i + \sum_{i=k+1}^{n} c_i x_i = \sum_{i=1}^{k} c_i(a + \lambda_i d) + \sum_{i=k+1}^{n} c_i cd =$$
$$= a\sum_{i=1}^{k} c_i + d\sum_{i=1}^{k} c_i\lambda_i + sd\sum_{i=k+1}^{n} c_i = a \cdot 0 + dAt + dsB = 0.$$

Finally, choose $k_0 > \max\{|\lambda_i| : i = 1, \ldots, k\}$ and s as above. According to Corollary 9.2 there exist natural numbers a, d such that the numbers $\{a + \lambda d : \lambda = 0, \pm 1, \ldots, \pm k_0\}$ and sd have the same color. However, by definition of k_0, this set contains a solution of the equation.

Necessity. Suppose the contrary: there exists no nonempty zero sum of coefficients of the equation. For prime p, define a coloring χ_p of $\mathbb{N}$ by the following manner. Each natural number x can be represented in the form $x = p^{\alpha}(pt + k)$, where $1 \leqslant k \leqslant p - 1$; put $\chi_p(x) = k$. Thus, χ_p is a $(p-1)$-coloring of $\mathbb{N}$.

Suppose the equation has a monochromatic solution, say, of color k. Let $x_i = p^{\alpha - i}(pt_i + k)$ be such a solution. Without loss of generality, we may suppose that

$$\alpha_1 = \cdots = \alpha_m < \alpha_{m+1} \leqslant \ldots \leqslant \alpha_n$$

(we do not exclude the case $m = n$). Then

$$\sum_{i=1}^{n} c_i p^{\alpha_i}(pt_i + k) = 0.$$

Cancelling this relation by p^{α_1} and passing to residues mod p we obtain

$$\sum_{i=1}^{n} c_i(pt_i + k) \equiv 0 (\operatorname{mod} p),$$
$$k\sum_{i=1}^{n} c_i \equiv 0 (\operatorname{mod} p).$$

Since $1 \leqslant k \leqslant p - 1$ and p is prime, we conclude that

$$\sum_{i=1}^{n} c_i \equiv 0 (\operatorname{mod} p).$$

However, if p is sufficiently large, then, necessarily, $\sum_{i=1}^{n} c_i = 0$, a contradiction. □

The above proof of the Rado Theorem is taken from [1].

EXERCISE.

1. Find a finite coloring of the set $\mathbb{N}$ such that the equation $x + y - 3z = 0$ has no monochromatic solution.

§10. Furstenberg-Weiss Theorem.

A pair (S, X), where X is a compact Hausdorff space and S a semigroup of continuous selfmaps of X, is called a *topological dynamics*.

A nonempty closed subset $A \subseteq X$ is called *invariant* provided $s(A) \subseteq A$ for every map $s \in S$. An invariant set M is called *minimal* if M contains no proper invariant subset.

The set $\mathrm{Orb}(x) = \{s(x) : s \in S\}$ is called the *orbit* of $x \in X$. The closure $\overline{\mathrm{Orb}}(x)$ of the set $\mathrm{Orb}(x)$ is called the *topological orbit* of $x \in X$.

10.1. Lemma. *For every $x \in X$ the topological orbit $\overline{\mathrm{Orb}}(x)$ is an invariant subset.*

Proof. Suppose the contrary and choose a point $y \in \overline{\mathrm{Orb}}(x)$ and a map $s \in S$ such that $s(y) \notin \overline{\mathrm{Orb}}(x)$. Put $V = X \setminus \overline{\mathrm{Orb}}(x)$. Since V is a neighborhood of $s(y)$ and s is continuous, there exists a neighborhood U of y such that $s(U) \subset V$. Since $y \in \overline{\mathrm{Orb}}(x)$, there exists a map $f \in S$ such that $f(x) \in U$. However, $sf(x) \in V$ and $sf(x) \in \mathrm{Orb}(x)$, a contradiction with $V \cap \overline{\mathrm{Orb}}(x) = \varnothing$. □

10.2. Lemma. *A subset $M \subseteq X$ is minimal if and only if $M = \overline{\mathrm{Orb}}(x)$ for every $x \in M$.*

Proof. Suppose M is a minimal subset and $x \in M$. Then $\mathrm{Orb}(x) \subseteq M$, by invariantness of M. Since M is closed, $\overline{\mathrm{Orb}}(x) \subseteq M$. By Lemma 10.1, the subset $\overline{\mathrm{Orb}}(x)$ is invariant and minimality of M implies that $M = \overline{\mathrm{Orb}}(x)$.

To prove the converse implication, suppose $M = \overline{\mathrm{Orb}}(x)$ for every $x \in M$. Given an invariant subset $A \subseteq M$ and $y \in A$, we have $M = \overline{\mathrm{Orb}}(y) \subseteq A$, hence, $M = A$. □

10.3. Lemma. *Every invariant subset $A \subseteq X$ contains a minimal subset.*

Proof. Consider an arbitrary chain $\mathfrak{A}$ of contained in A invariant subsets. Since X is compact and all the elements of $\mathfrak{A}$ are closed,

the set $F = \bigcap \mathfrak{A}$ is nonempty. Obviously, F is a lower bound of the chain $\mathfrak{A}$. By the Kuratowski-Zorn Lemma, there exists a minimal subset contained in $\mathfrak{A}$. □

A point $x \in X$ is called *recurrent* provided $x \in \overline{\mathrm{Orb}}(x)$. In other words, x is recurrent if for every neighborhood U of x there exists a map $s \in S$ such that $s(x) \in U$.

A point $x \in X$ is called *uniformly recurrent* if $x \in \overline{\mathrm{Orb}}(x)$ and the set $\overline{\mathrm{Orb}}(x)$ is minimal. By Lemma 10.2 and 10.3, for every topological dynamics (S, X) there exists a uniformly recurrent point $x \in X$. This statement is known as the Birkhoff Theorem.

To characterize uniformly recurrent points, introduce the following definition. A subset H of a semigroup S is called *syndetic* provided there exist elements $s_1, \ldots, s_n \in S$ satisfying the condition: given any $s \in S$ there exists i such that $s_i s \in H$. In other words, a subset H is syndetic if there exists a finite subset $F \subseteq S$ such that $Fs \cap H \neq \varnothing$ for every $s \in S$.

10.4. Theorem. *Let $x \in X$, $M = \overline{\mathrm{Orb}}(x)$, and $x \in M$. The point x is uniformly recurrent if and only if for every point $y \in M$ and intersecting M open subset U the subset*

$$H = \{s \in S : s(y) \in U\}$$

is syndetic.

Proof. Necessity. Put

$$s^{-1}(U) = \{z \in X : s(z) \in U\}, \quad W = \bigcup \{s^{-1}(U) : s \in S\}.$$

Since all maps from S are continuous, the subset W is open and, consequently, $M \setminus W$ is closed. Suppose that $M \setminus W \neq \varnothing$. Let $z \in M \setminus W$ and f is a fixed element of the semigroup S. By invariantness of M, we have $f(z) \in M$. If $f(z) \in W$, then $sf(y) \in U$ for some map $s \in S$. But then $z \in W$, a contradiction with the choice of z. Therefore, $f(z) \notin W$ and the subset $M \setminus W$ is invariant. Since $U \cap M \neq \varnothing$, we have $M \setminus W \neq M$ thus contradicting minimality of M. Consequently, $M \setminus W = \varnothing$.

The cover $\{s^{-1}(U) : s \in S\}$ of the compact subset M contains a finite subcover $\{s_1^{-1}(U), \dots, s_n^{-1}(U)\}$. Consider an arbitrary element $s \in S$. Since $s(y) \in M$, there exists i such that $s(y) \in s^{-1}(U)$. Hence, $s_i s(y) \in U$ and $s_i s \in H$.

Sufficiency. Suppose x is not uniformly recurrent. Then there exists a proper invariant subset $A \subset M$. Put $U = X \setminus A$ and choose an arbitrary element $y \in A$. Then $U \cap M \neq \varnothing$, $y \in M$, and the subset

$$H = \{s \in S : s(y) \in U\},$$

being empty, cannot be syndetic. □

10.5. Corollary. *Suppose T is a continuous selfmap of a compact Hausdorff space X and $S = \{T^n : n \in \mathbb{N}\}$. A point $x \in X$ is uniformly recurrent if and only if for every its neighborhood U there exists a natural number m such that for every natural number n at least one of the points $T^{n+1}(x), \dots, T^{n+m}(x)$ belongs to U.*

Proof. Suppose x is a uniformly recurrent point and U is a neighborhood of x. By Lemma 10.4, the subset

$$H = \{s \in S : s(x) \in U\}$$

is syndetic. Thus, there exists a finite subset $F \subset S$ such that $Fs \cap H \neq \varnothing$ for every $s \in S$. Without loss of generality we may suppose that $F = \{T, \dots, T^m\}$. Hence, for $s \in T$ there exists a map $T^k \in F$ such that $T^{n+k}(x) \in U$.

Conversely, suppose for every neighborhood U of x there exists a corresponding number m. Suppose that the subset $\overline{\mathrm{Orb}}(x)$ is not minimal and find a proper invariant subset $A \subset \overline{\mathrm{Orb}}(x)$. Since $x \notin A$, there exist open subsets U, V such that $x \in U$, $A \subseteq V$, and $U \cap V \neq \varnothing$. Find a natural number m satisfying the property: for every natural number n at least one of the points $T^{n+1}(x), \dots, T^{n+m}(x)$ is in U. Fix an arbitrary point $a \in A$ and choose a neighborhood W of a such that

$$T(W) \subset U, \dots, T^m(W) \subset U.$$

Since $a \in \overline{\mathrm{Orb}}(x)$, we see that $T^k(a) \in W$, for some natural number k. But then

$$T^{k+1}(x) \in V, \dots, T^{k+m}(x) \in V,$$

a contradiction. □

The following lemma plays the crucial role in the our proof of the Furstenberg-Weiss Theorem on joint recurrence.

10.6. Lemma. *$T_1, \dots, T_k$ be pairwise commuting continuous selfmaps of a compact Hausdorff space X and S the generated by maps $T_1, \dots, T_k$ semigroup such that (S, X) is a minimal topological dynamics (i.e., the space X contains no proper minimal subsets). For every open subset U there exist an open subset $V \subseteq U$ and a natural number n such that*

$$T_1^n(V) \subseteq U, \dots, T_k^n(V) \subseteq U.$$

Proof. Let $x \in U$. Since the point x is uniformly recurrent, by Theorem 10.4, there exists a finite subset $F \subseteq S$ such that $Fs \cap H \neq \varnothing$ for every $s \in S$, where

$$H = \{s \in S : s(x) \in U\}.$$

For every map $T \in F$ put

$$S(T) = \{(n_1, \dots, n_k) \in \mathbb{N}^k : TT_1^{n_1} \dots T_k^{n_k} \in H\}.$$

Since $Fs \cap H \neq \varnothing$ for every $s \in S$, we have

$$\mathbb{N}^k = \bigcup \{S(T) : T \in F\}.$$

By the Gallai Theorem (Theorem 8.7), there exist $T \in F$ and natural numbers $a_1, \dots, a_k, n$ such that

$$TT_1^{a_1+ne_1} \dots T_k^{a_k+ne_k} \in H$$

for every $e_1, \ldots, e_k \in \{0,1\}$.

Put

$$T_0 = TT_1^{a_1} \ldots T_k^{a_k}.$$

Then $T_0(x) \in U$ and, by commutativity of the semigroup S, we have $T_i^n(T_0(x)) \in U$ for every $i = 1, \ldots, k$. Using continuity of the maps $T_1, \ldots, T_k$, choose an open neighborhood V of the point $T(x_0)$ such that $V \subseteq U$ and

$$T_1(V) \subseteq U, \ldots, T_k(V) \subseteq U. \qquad \square$$

10.7. Furstenberg-Weiss Theorem. *Let $T_1, \ldots, T_k$ be pairwise commuting continuous selfmaps of a compact metric space X. There exist a point $x \in X$ and an increasing sequence $\langle n_i \rangle$ of natural numbers such that*

$$T_1^{n_i}(x) \xrightarrow[i\to\infty]{} x, \ldots, T_k^{n_i}(x) \xrightarrow[i\to\infty]{} x.$$

Proof. Using Lemma 10.6 and metrizability of the space X, choose a sequence of open subsets $\langle U_i \rangle$ and a sequence of natural numbers $\langle n_i \rangle$ such that

$$\overline{U}_{i+1} \subset U_i, \quad T_j^{n_i}(\overline{U}_{i+1}) \subseteq U_i, \quad j = 1, \ldots, k, \operatorname{diam} U_1 < 1/i$$

for all natural numbers i. Here $\overline{U}_i$ denotes the closure of U_i and $\operatorname{diam} U_i$ the diameter of U_i, i.e., the supremum of distances between pairs of points of U_i. By compactness of X and the choice of the sequence $\langle U_i \rangle$, we have

$$\bigcap\{\overline{U}_i : i \in \mathbb{N}\} = \{x\}.$$

Obviously, the point x satisfies the statement of the theorem. $\square$

A point x satisfying the statement of Theorem 10.7 is called a *joint recurrent* point of the maps $T_1, \ldots, T_k$.

10.8. Symbolic dynamics. As usual, $\mathbb{Z}$ denotes the set of integers. Suppose K is a finite set and $K^{\mathbb{Z}}$ the Cartesian product

of $\mathbb{Z}$ copies of K. The elements of the set Y will be represented by the vectors of the form

$$(\ldots, y(-2), y(-1), y(0), y(1), y(2), \ldots).$$

We endow Y with the metric d defined as follows: $d(x,y) = 1$ whenever $x(0) \neq y(0)$ and $d(x,y) = \frac{1}{k+1}$, where k is the least natural number such that

$$x(-k) = y(-k), \ldots, x(0) = y(0), \ldots, x(k) = y(k),$$

whenever $x(0) = y(0)$. Note that this metric generates the Tychonov topology on Y (the set K is considered as a discrete topological space).

Define the left shift map $T\colon Y \to Y$ by $T(y) = y'$, where $y'(k) = y(k+1)$ for every $k \in \mathbb{Z}$. Note that T is continuous and put $S = \{T^n : n \in \mathbb{N}\}$.

A topological dynamics (S, X), where X is an invariant subspace of Y, is called a *symbolic dynamics*. Applying a suitable symbolic dynamics, derive the van der Waerden Theorem from the Furstenberg-Weiss Theorem. Similarly, the Gallai Theorem can be derived.

Consider an arbitrary partition $\mathbb{Z} = A_1 \cup \cdots \cup A_k$ and fix a natural number m. Our aim is to find an arithmetic progression of length m in one of the subsets of the partition.

Define the characteristic function $x(t)$ of the partition by the condition: $x(z) = i$ if and only if if $z \in A_i$. The function $x(t)$ can be naturally identified with an element $x \in K^{\mathbb{Z}}$, where $K = \{1, \ldots, k\}$. Put $X = \overline{\operatorname{Orb}}(x)$. Apply the Furstenberg-Weiss Theorem to the family $T, T^2, \ldots, T^m$ of pairwise commuting continuous selfmaps of the space X. Suppose $y \in X$ is a joint recurrent point of the maps $T, T^2, \ldots, T^m$, $y(0) = j$. Consider the neighborhood $U = \{u \in X : u(0) = j\}$ of the point y and choose a natural number n such that

$$T^n(y) \in U, \quad T^{2n}(y) \in U, \ldots, T^{mn}(y) \in U.$$

Using continuity of the map T find a neighborhood V of the point y such that

$$T^n(V) \subseteq U, \quad T^{2n}(V) \subseteq U, \dots, T^{mn}(V) \subseteq U.$$

Since $y \in \overline{\mathrm{Orb}}(x)$ and V is a neighborhood of the point y, there exists a natural number a such that $T^a(x) \in V$. Hence

$$T^n(T^a(x)) \in U, \quad T^{2n}(T^a(x)) \in U, \dots, T^{mn}(T^a(x)) \in U.$$

By the definition of U, we obtain that

$$x(a+n) = x(a+2n) = \cdots = x(a+mn) = j.$$

Thus,

$$\{a+n, a+2n, \dots, a+mn\} \subseteq A_j$$

and we are done.

The original topological proof of the Furstenberg-Weiss Theorem is contained in the survey [11]. Our exposition follows the paper [12].

EXERCISES.

1. Derive the Gallai Theorem from the Furstenberg-Weiss Theorem.
2. Prove the following extension of the Furstenberg-Weiss Theorem: Let $\{T_m : m \in \mathbb{N}\}$ be a countable set of pairwise commuting continuous selfmaps of a compact metric space X. There exists a point $x \in X$ satisfying the condition: for every neighborhood U of x and every natural number m there exists a natural number n such that

$$T_1^n(x) \in U, \dots, T_m^n(x) \in U.$$

3. Show that the metrizability of the compact space X in the Furstenberg-Weiss Theorem is essential. For this, use the following example. Let

$$X = \{re^{2\pi i\theta} : 1 \leqslant r \leqslant 2, \theta \in \mathbb{R}\}$$

be an annulus on the complex plane. Define a new topology on X.

Suppose $x = re^{2\pi i\theta}$. If $1 < t < 2$, the base neighborhood U_ε, $1 < \varepsilon < \min\{r-1, 2-r\}$ is:

$$U_\varepsilon = \{se^{2\pi i\theta} : 0 < |s-r| < \varepsilon\}.$$

If $r = 1$, the base neighborhood L_ε, $0 < \varepsilon < 1$ of x is:

$$L_\varepsilon = \{se^{2\pi i\theta} : 1 \leqslant s \leqslant 1+\varepsilon\} \cup \{se^{2\pi ip} : 1 \leqslant s \leqslant 2, 0 < \theta - p < \varepsilon\}.$$

If $r = 2$, the base neighborhood R_ε, $0 < \varepsilon < 1$ of x is:

$$R_\varepsilon = \{se^{2\pi i\theta} : 2-\varepsilon < s < 2\} \cup \{se^{2\pi i\theta} : 1 < s \leqslant 2, 0 < p - \theta < \varepsilon\}.$$

The sets U_ε, L_ε, L_ε form a base of a topology on X.
For a fixed irrational α define the map $T\colon X \to X$ by the formula:

$$T(re^{2\pi i\theta}) = re^{2\pi i(\theta+\alpha)}.$$

Prove the following statements:
1) X is a connected compact space;
2) the maps T, T^{-1} are continuous;
3) for every point $x \in X$ there exists a neighborhood U of x such that $\{T^n(x), T^{-n}(x)\} \nsubseteq U$ for every natural number n.

§11. Partition of groups and rings.

Let G be an arbitrary group and p an ultrafilter on the group G. For a subset $A \subseteq G$, the set

$$\mathrm{cl}(A,p) = \{x \in G : Px \subseteq A \text{ for some subset } P \in p\}$$

is called the *closure* of A with respect to the ultrafilter p.

11.1. Lemma. *We have* $\mathrm{cl}(A,p) = \bigcap\{P^{-1}A : P \in p\}$.

Proof. If $x \in \mathrm{cl}(A,p)$, then there exists a subset $P_1 \in p$ such that $P_1x \subseteq A$. Consider an arbitrary subset $P \in p$ and put $P_2 = P \cap P_1$. Since $P_2 \subseteq P_1$, we have $P_2x \subseteq A$. Hence, $x \in P_2^{-1}A \subseteq P^{-1}A$.

On the contrary, suppose $x \in P^{-1}A$ for every subset $P \in p$. Suppose that $x \notin \mathrm{cl}(A,p)$. Then $Px \subseteq A$ for every subset $P \in p$. Since p is an ultrafilter, there exists a subset $P_1 \in p$ such that $P_1x \cap A = \varnothing$. Therefore, $x \notin P_1^{-1}A$, a contradiction. □

11.2. Theorem. *For an ultrafilter $p \in \beta G$ the subset $p(\beta G)$ is a minimal right ideal if and only if for every subset $A \in p$ there exists a finite subset $K \subseteq G$ such that $\mathrm{cl}(AK,p) = G$.*

Proof. Let $p(\beta G)$ be a minimal right ideal and e the unit of the group G. Since $e \in \beta G$, we have $p \in p(\beta G)$. Consider an arbitrary ultrafilter $\in \beta G$. Since $pr(\beta G) = p(\beta G)$ and $p \in p(\beta G)$, we have $p \in pr(\beta G)$. Hence, there exists an ultrafilter $t(r)$ such that $p = prt(r)$. Since $A \in prt(r)$ and the subset G is dense in βG, there exists an element $x(r) \in G$ such that $A \in prx(r)$, i.e., $A\big(x(r)\big)^{-1} \in pr$. Using continuity of the multiplication with respect to the second argument, choose a subset $A_r \in r$ such that

$$p\bar{A}_r \subseteq \overline{A(x(r))^{-1}}.$$

The open cover $\bar{A}_r$, $r \in \beta G$, of the space βG contains a finite subcover $\bar{A}_{r_1}, \dots, \bar{A}_{r_n}$. Put

$$K = \left\{\big(x(r_1)\big)^{-1}, \dots, \big(x(r_n)\big)^{-1}\right\}$$

and show that $G = \mathrm{cl}(AK,p)$. Consider an arbitrary element $x \in G$ and choose an index i such that $x \in \bar{A}_{r_i}$. Then $px \in \overline{A(x(r_i))^{-1}}$, i.e.,

$$A(x(r_i))^{-1} \in px.$$

Choose a subset $P \in p$ such that $Px \subseteq A(x(r_i))^{-1}$. Consequently, $Px \subseteq AK$ and $x \in \mathrm{cl}(AK,p)$.

Suppose that the concluding statement of the theorem holds but the ideal $p(\beta G)$ is not minimal. Find an element $r \in \beta G$ such that $p \notin pr(\beta G)$. Since the subset $pr(\beta G)$ is closed, there exists a subset $A \in p$ such that $\bar{A} \cap pr(\beta G) = \varnothing$. Consequently, $A \notin prx$ for every $x \in G$. Choose a finite subset

$$K = \{x_1^{-1}, \dots, x_n^{-1}\}$$

such that $\mathrm{cl}(AK,p) = G$. Since

$$\mathrm{cl}(AK,p) = \mathrm{cl}(Ax_1^{-1},p) \cup \dots \cup \mathrm{cl}(Ax_n^{-1},p),$$

there exists an index i such that $\mathrm{cl}(Ax_i^{-1},p) \in r$. However, the latter relationship is equivalent to $Ax_i^{-1} \in pr$, a contradiction with the choice of the subset A. □

11.3. Corollary. *If p is an ultrafilter from the semigroup βG and $p(\beta G)$ a minimal right ideal of βG, then for every subset $a \in p$ there exists a finite subset $K \subseteq G$ such that $G = A^{-1}AK$.*

Proof. By the above theorem, there exists a finite subset K such that $G = \mathrm{cl}(AK,p)$. Since $A \in p$, by Lemma 11.1, $\mathrm{cl}(AK,p) \subseteq A^{-1}AK$. □

11.4. Corollary. *If p is an ultrafilter from βG, $A \in p$, and $p(\beta G)$ is a minimal right ideal of the semigroup βG, then $A^{-1}AA^{-1}A \in q$ for every idempotent q of the semigroup βG.*

Proof. Using Corollary 11.3, choose a finite subset $K \subseteq G$ such that $G = A^{-1}AK$. Clearly, $G = K^{-1}A^{-1}A$. Choose an element $x \in K^{-1}$ such that $xA^{-1}A \in q$ and put $Q = xA^{-1}A$. Since $qq \in \bar{Q}$, there exists a subset $Q_1 \in q$ such that $q\bar{Q}_1 \subseteq \bar{Q}$. Hence,

$y \in Q_y^{-1}Q \subseteq Q^{-1}Q$. By arbitrarity of the choice of element $y \in Q_1$, we obtain $Q_1 \subseteq Q^{-1}Q$. Since $Q_1 \in q$, we have $Q^{-1}Q \in q$. This proves the theorem. □

11.5. Theorem on partitions of groups. *If $G = A_1 \cup \cdots \cup A_m$ is a finite partition of an arbitrary group G, then there exists a subset $A = A_i$ of the partition such that the following statements hold:*

1) *$G = A^{-1}AK$ for some finite subset $K \subseteq G$;*
2) *$(A^{-1}A)^n$ is a subgroup of finite index of the group G for some natural number n;*
3) *$f(A) \cap A^{-1}AA^{-1}A \neq \varnothing$ for every homomorphism $f: G \to G$.*

Proof. Choose an idempotent p from some minimal right ideal of the semigroup βG. Put $A = A_i$, where A_i is an element of the partition that belongs to the ultrafilter p.

By Corollary 11.3, there exists a finite subset $K \subseteq G$ such that $G = A^{-1}AK$. The first statement of the theorem is proved.

To prove the second statement, put $H = A^{-1}A$ and consider the semigroup F generated by the subset H. Enumerate the elements of the subset $K = \{g_1, \ldots, g_k\}$ so that

$$F \cap Hg_1 \neq \varnothing, \ldots, F \cap Hg_s \neq \varnothing, F \cap Hg_{s+1} \neq \varnothing, \ldots, F \cap Hg_k \neq \varnothing.$$

Since H is symmetric, there exists a minimal natural number m such that

$$H^m \cap Hg_1 \neq \varnothing, \ldots, H^m \cap Hg_s \neq \varnothing.$$

Then $\{g_1, \ldots, g_s\} \subseteq H^{m+1}$ and $Hg_i \subseteq H^{m+2}$ for every $i = 1, \ldots, s$. Put $n = m + 2$. If $g \in F$, then $g \in Hg_i$ for some $i \leqslant s$. Hence, $g \in H^n$ and $H^n = F$. Thus, H^n is the required subgroup.

To prove the third statement, extend the homomorphism $f: G \to G$ to the continuous map $\bar{f}: \beta G \to \beta G$. From the definition of the multiplication operation for ultrafilters it immediately follows that $\bar{f}$ is a homomorphism. Hence $\bar{f}(p)$ is an idempotent of the semigroup βG. By Corollary 11.4, $A^{-1}AA^{-1}A \in \bar{f}(p)$ and it remains to note that $f(A) \in \bar{f}(p)$. □

11.6. Corollary (B. Neumann Theorem). *If a group G is represented as the union of finite number of cosets by subgroups, $G = g_1G_1 \cup \cdots \cup g_kG_k$, then at least one of the subgroups G_i is of finite index.* □

Now we consider partitions of rings. We say that an associative ring with unit R is of the class $\mathcal{U}$, if the following conditions hold:

1) there exists a Banach measure μ (see 2.6) on the additive group of the ring R such that $\mu(R^*) > 0$, where R^* denotes the group of invertible elements of the ring R;
2) there exists an infinite sequence $\langle a_n \rangle$ of elements of the ring R such that $a_i - a_j \in R^*$ for every $i \neq j$.

An associative ring with unit is called a *division ring* (or a *skew field*) provided every its nonzero element is invertible. Note that every infinite associative division ring is of the class $\mathcal{U}$, because, in this case, we can choose μ to be any Banach measure on the additive group of the ring (see Exercise 2 from §2). In [9] it is proved that $R = G - G$ for every subgroup G of finite index from R^*, where R is a ring of the class $\mathcal{U}$. The following lemma is an extension of this statement.

11.7. Lemma. *Let R be a ring of the class $\mathcal{U}$, p an ultrafilter on R^*, and $p(\beta R^*)$ a minimal right ideal of the semigroup βR^*. Then for every subset $A \in p$ the equality*

$$R = A^{-1}AA^{-1}A - A^{-1}AA^{-1}A$$

holds.

Proof. By Theorem 11.5, there exists a finite subset $K \subseteq R^*$ such that $R^* = A^{-1}AK$. Using the Ramsey Theorem and passing to subsequences (see Exercise 2 from §5), we may suppose that $a_i - a_j \in A^{-1}Ag$ for every $i > j$, where g is a fixed element of K. Since $\mu(R^*) > 0$, there exists an element $h \in K$ such that $\mu(A^{-1}Ah) > 0$. Consider an arbitrary nonzero element $x \in R$. Since $\mu(A^{-1}Ah) > 0$, by translation invariantness of μ, there exist indexes i, j, $i > j$, such that

$$(a_ix + A^{-1}Ah) \cap (a_jx + A^{-1}Ah) \neq \varnothing.$$

Then

$$(a_i - a_j)x \in (A^{-1}A - A^{-1}A)h.$$

Since $a_i - a_j \in A^{-1}Ag$, we see that

$$gxh^{-1} \in A^{-1}AA^{-1}A - A^{-1}AA^{-1}A.$$

Since $gRh^{-1} = R$, we have

$$R = A^{-1}AA^{-1}A - A^{-1}AA^{-1}A. \qquad \square$$

11.8. Theorem on partitions of rings. *If $R = A_1 \cup \dots \cup A_n$ is a finite partition of an associative division ring R, that there exists an index m such that for the subset $A = A_m \setminus \{0\}$ we have*

$$R = A^{-1}A - A^{-1}A + A^{-1}A - A^{-1}A = A^{-1}AA^{-1}A - A^{-1}AA^{-1}A.$$

Proof. Consider the subset E of idempotents of the semigroup $\beta(R,+)$ generating minimal right ideals of this semigroup. Since the right multiplication by any element $x \in R^*$ is an isomorphism of the semigroup $\beta(R,+)$, we have $Ex = E$. Therefore, the closure J of the subset E in the Čech-Stone topology is a right ideal of the semigroup βR^*. Consider an arbitrary ultrafilter $q \in J$ generating a minimal right ideal of the semigroup βR^*. Choose an index m so that the subset $A = A_m \setminus \{0\}$ is an element of the ultrafilter q. By Lemma 11.7,

$$R = A^{-1}AA^{-1}A - A^{-1}AA^{-1}A.$$

Since $q \in J$ and $A \in q$, there exists an idempotent $p \in E$ such that $A \in p$. The third statement of Theorem 11.5 implies $Ax \cap (A - A + A - A) \neq \varnothing$ for every $x \in R$. Consequently,

$$R = A^{-1}A - A^{-1}A + A^{-1}A - A^{-1}A. \qquad \square$$

The exposition of this section follows the article [14].

EXERCISES.

1. Prove the following extension of Theorem 11.5 for amenable groups. If $G = A_1 \cup \dots \cup A_k$ is a partition of an amenable group G into k nonempty

subsets, then there exists an index i and a k-element subset $K \subseteq G$ such that $G = A_i^{-1}A_iK$.
2. A *topological group* is a group endowed with a topology such that the group operations are continuous. Prove the following extension of Theorem 11.5 onto topological groups. If $U = A_1 \cup \dots \cup A_k$ is a partition of an arbitrary nieghborhood U of the identity in a topological group G, then there exist an index i and a finite subset K such that $A_i^{-1}A_iK$ is a neighborhood of the identity. The subset K can be chosen from any pregiven neighborhood of the identity.

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INDEX

Additive filter 42,
Additive set 42,
Algebraically isolated point 35,
Amenable group 17,
Antichain 32,
Arnautov Theorem 35,
Banach measure 17,
Base of an ultrafilter 20,
Baumgarter 5,
Bergelson–Hindman Theorem 7,
Blass 6,
Canonical base 21,
Centered family 9,
Chain 9,
Closed set 13,
Closure 13,
Combinatorical line 46,
Compact space 15,
Completely regular space 21,
Continuous map 14,
Diophantine equation 4,
Density 22,
Filter 9,
Finitely additive measure 12,
Folkman 5,
Følner condition 18,
Frechet filter 9,
Free ultrafilter 9,
Function of choice 23,
Furstenberg 7,
Furstenberg–Weiss Theorem 7,57,
Gallai 4,
Gallai Theorem 44,
Galvin 6,
Glaser 6,
Graham 5,
Graham–Leeb–Rothschild
Theorem 5,45,
Hales 5,
Hales–Jewett Theorem 5,43,
Hausdorff space 14,
Hewitt–Marczewski–Pondiczery
Theorem 22,
Hindman 5,
Hindman Theorem 33,
Ideal 26,
Idempotent 26,
Independent family 23,
Invariant subset 53,
Jewett 5,
Joint recurrent point 57,
Kuratowski–Zorn Lemma 9,
Leeb 5,
Left invariant measure 17,
Left topological semigroup 26,
Limit of a filter 14,
Markov Theorem 37,
Minimal set 53,
Neighbourhood 13,
Non-regular partitition 16,
Open set 13,
Orbit 53,
Principal filter 9,
Principal ultrafilter 9,
Product of ultrafilters 25,
Pym 6,
Rado 4,
Rado Theorem 49,
Ramsey 4,
Ramsey number 31,
Ramsey Theorem 29,
Ramsey ultrafilter 32,
Recurrent point 54,
Regular equation 50,
Riesz 11,

Right cancellative semigroup 34,
Right ideal 26,
Rothschild 5,
Sanders 5,
Schur 4,
Schur Theorem 4,
Sierpiński 12,
Solovay 12,
Stone-Čech compactification 6,21,
Strong idempotent 37,
Symbolic dynamics 58,
Syndetic set 54,
Tic-tac-toe 47,
Topological dynamics 53,
Topologizable ring 37,
Topological group 66,
Topological space 13,
Topology 13,
Two-valued measure 12,
Tychonov 16,
Tychonov product 16,
Tychonov space 21,
Tychonov Theorem 16,
Tychonov topology 16,
Ulam 11,
Ultrafilter 9,
Ultrafilter semigroup 25,
Uniform filter 12,
Uniform ultrafilter 12,
Uniformly discrete set 32,
Uniformly recurrent point 54,
van der Waerden 4,
van der Waerden Theorem 40,41,
van Douwen 6,
von Neumann partition 18,
Zermelo-Fraenkel set theory 12,
m-regular partition 16,
FP-set 33,
FS-set 34,