Idempotent ultrafilters and the Rudin-Keisler ordering

Short version: what can we say about the place of idempotent ultrafilters in the Rudin-Keisler ordering?

Longer version:

If U, V are (nonprincipal) ultrafilters on ω, then we write $U \geq_{RK} V$ in case there is some function f :

$$\forall X \subseteq \omega, \quad f^{-1}(X) \in U \iff X \in V.$$

An ultrafilter U is *Ramsey* if given any two-coloring of pairs $c : [\omega]^2 \rightarrow 2$ there is a homogeneous set
whenever $\{C_n\}_{n \in \omega}$ is a partition of ω with each $C_n \notin U$, there is some $H \in U$ such that for all n ∈

$$|H \cap C_n| = 1.$$

An ultrafilter U is *idempotent* if $U \oplus U = U$ where

$$V \oplus W = \{X : \{x : \{y : x + y \in X\} \in W\} \in$$

Idempotent ultrafilters can be proved to exist in ZFC: this amounts to showing that $\oplus$ is left-continuous
$\beta\mathbb{N}$ and then applying Ellis' theorem that every left-continuous semigroup on a compact space has an id
ultrafilters cannot be shown to exist in ZFC, although their existence is equiconsistent with ZFC (in p
or weaker statements - holds, then there are Ramsey ultrafilters).

Now, an easy argument shows that no Ramsey ultrafilter is idempotent. On the other hand, the Ramsey
respect to the RK-ordering: they are precisely the RK-minimal ultrafilters. So combining these facts show
RK-minimal.

My question is, what else can be said about the idempotent ultrafilters in terms of RK-reducibility? For ex
ultrafilter U with exactly one RK-class of (necessarily, Ramsey) ultrafilters strictly RK-below U? This seem
to prove it.

Motivation: a few weeks ago, I taught a one-week course on the proof of Hindman's theorem from additiv
ultrafilters. On the last day, I talked a bit about other types of ultrafilters, and spent a bit of time defining F
proving) their place in the RK-ordering. One of my students asked whether anything similar could be said
the obvious, I couldn't come up with anything, so I'm asking here.

lo.logic | set-theory | gn.general-topology | ultrafilters

edited Aug 4 '13 at 17:28 asked Aug 3 '13 at 5:52

Noah S
4,865 1 11 47

1 The last section of Andreas's "Ultrafilters: Where Dynamics = Algebra = Combinatorics" has some remarks on the relationship between idempotents, Ramsey ultrafilters, and P-points. – Todd Eisworth Aug 3 '13 at 15:34

1 Random Conjecture: Suppose U is idempotent, let f (g) be the map sending n to the position of the rightmost (leftmost) 1 in its binary expansion. Then $f(U)$ and $g(U)$ are not RK-equivalent to each other or to U. – Todd Eisworth Aug 3 '13 at 15:36

1 Todd, see my comment to Andreas's answer -- given any two ultrafilters p, q you'll find an idempotent u with $f(u) = p, g(u) = q$. – Peter Krautzberger Aug 4 '13 at 1:26

1 Answer

The idempotent ultrafilters closest to being Ramsey are the stable ordered-union ultrafilters. These are officially defined as certain ultrafilters on the set F of finite subsets of ω, but they can be transferred to ω via the "binary expansion" map $\mathrm{F} \to \omega : s \mapsto \sum_{n \in s} 2^n$. The image on ω of a stable ordered-union ultrafilter is idempotent and has has exactly three non-isomorphic non-principal ultrafilters RK-below it. Two of these are the ones Todd Eisworth mentioned in his comment; the third is the "pairing" of these two, i.e, the image of the idempotent under the map to ω^2 given by $n \mapsto$ (position of leftmost 1, position of rightmost 1). For details, see my paper "Ultrafilters related to Hindman's finite-unions theorem and its extensions" ["Logic and Combinatorics", Contemporary Math 65 (1987) 89-124] also available at http://www.math.lsa.umich.edu/~ablass/uf-hindman.pdf (this is a scanned picture and therefore not searchable; the stable ordered-union stuff starts on page 113).

answered Aug 3 '13 at 17:42
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I'm curious: is it known to what extent (assuming CH, say) the Ramsey ultrafilters below a given ultrafilter U characterize U? That is, are there non-RK-equivalent U, V which RK-bound the same Ramsey ultrafilters? Alternatively, is it consistent that there are no such pairs of ultrafilters? (It is certainly consistent that such a pair exists, since all pairs of ultrafilters satisfy this property if there are no Ramsey ultrafilters.) – Noah S Aug 3 '13 at 18:25

A more relevant question: do idempotent ultrafilters exist RK-above an arbitrary ultrafilter? I suspect the answer is yes, but I'm having trouble proving it: given a function $f : \omega \to \omega$ and an ultrafilter U, the set of V RK-above U via f is compact as a subset of $\beta\mathrm{N}$, but it's not clear to me that it is closed under $\oplus$ (if it were, I could apply Ellis' theorem and be done). Is this known? – Noah S Aug 3 '13 at 20:39

1 @NoahS for your first comment: Under CH, the Ramsey ultrafilters below a given ultrafilter U don't characterize U (up to isomorphism). For one thing, there are lots of ultrafilters (including some P-points) with no Ramsey ultrafilters below them. Also, if you fix a Ramsey ultrafilter V, then there are several (undoubtedly $2^{\aleph_1}$, but I haven't checked carefully) isomorphism classes of ultrafilters that are RK-above U and no other Ramsey ultrafilters - in fact above no other ultrafilters at all (except of course themselves and principal ultrafilters). – Andreas Blass Aug 3 '13 at 21:32

@NoahS for your second comment: I don't know but I would expect that there are ultrafilters on ω with no idempotent ultrafilters RK-above them. I would be surprised if there exist (say under CH if it helps) any nonprincipal ultrafilter U and any function f such that the set of ultrafilters mapped to U by f is nonempty and closed under addition. – Andreas Blass Aug 3 '13 at 21:37

Thanks - this answers my (admittedly broad) question. – Noah S Aug 3 '13 at 22:57

5 There are idempotents RK-above any ultrafilter, e.g., you can extend the inverse filter under the map that maps each n to the minimum (or maximum) of its binary expension. See my dissertation – Peter Krautzberger Aug 3 '13 at 23:53

(just to be clear: this is old news, see the book by Hindman&Strauss; it's just that my thesis is freely available.) – Peter Krautzberger Aug 4 '13 at 0:04

@PeterKrautzberger Thanks for correcting my erroneous guess, and apologies for not correctly remembering the result. – Andreas Blass Aug 4 '13 at 2:56

No problem at all, glad I could help. – Peter Krautzberger Aug 4 '13 at 16:02