Ultrapower of $\mathbb{N}$ and Density Problems

Renling Jin

Abstract. We show how we can apply ultrapower methods to density problems in additive/combinatorial number theory.

1. Introduction and Background Information

In the past decade, the methods from nonstandard analysis have been successfully applied to density problems to obtain many results in additive/combinatorial number theory (cf. [**BJ, Ji1, Ji2, Ji3, Ji4, Ji5, Ji6, Ji7, Ji8, JK**]). Since a nonstandard universe can be constructed by taking an ultrapower of the standard universe, some of the methods used in these papers can also been developed "algebraically" through ultrapower methods without requiring too much knowledge from mathematical logic. In this article, we introduce some of these results through ultrapower methods. In particular, we present results related to Kneser's Theorem and Plünnecke's Theorem.

Let $\mathbb{N}$ be the set of all non-negative integers. To measure the size of a finite set $A \subseteq \mathbb{N}$, one can count the number of elements in A, known as the cardinality of A and denoted by $|A|$. If a is the least element in A and b is the greatest element in A, one can also measure the density of A relative to $[a,b]$ by the ratio $\frac{|A|}{b-a+1}$. When A is infinite, the cardinality of A is no longer useful for distinguishing the size of A from other infinite sets. But the density of a finite set can be extended to the density of A as the asymptotic trend of the densities of $A \cap [a_n, b_n]$ where $[a_n, b_n]$ is a sequence of finite intervals of non-negative integers with $\lim_{n\to\infty}(b_n - a_n) = \infty$. We can compare the "sizes" of two infinite subsets of $\mathbb{N}$ by comparing their densities.

Let $a, b \in \mathbb{N}$. We will denote $[a,b]$ exclusively for the interval of integers between a and b including a and b. The following commonly used densities are under our consideration. Let $A \subseteq \mathbb{N}$. For $a, b \in \mathbb{N}$ let $A(a,b) = |A \cap [a,b]|$ and $A(b) = A(1,b)$. The *Shnirel'man density* $\sigma(A)$, the *lower asymptotic density* $\underline{d}(A)$, the *upper asymptotic density* $\overline{d}(A)$, and the *upper Banach density* $BD(A)$ of A are defined by

$$\sigma(A) = \inf_{n \geq 1} \frac{A(n)}{n},$$

1991 *Mathematics Subject Classification.* Primary 11B05, 11B13, 11U10, 03H15.

Key words and phrases. Shnirel'man density, lower asymptotic density, upper asymptotic density, upper Banach density, Plünnecke's inequality, Kneser's theorem, basis, nonstandard analysis.

The author is supported in part by NSF RUI grant DMS#0500671.

$$\underline{d}(A) = \liminf_{n\to\infty} \frac{A(n)}{n},$$

$$\overline{d}(A) = \limsup_{n\to\infty} \frac{A(n)}{n}, \text{ and}$$

$$BD(A) = \lim_{n\to\infty} \sup_{k\in\mathbb{N}} \frac{A(k, k+n-1)}{n}.$$

Clearly, these densities have the following order by their magnitude.

$$0 \le \sigma(A) \le \underline{d}(A) \le \overline{d}(A) \le BD(A) \le 1$$

for every $A \subseteq \mathbb{N}$. The order of these densities by their popularity among many number theorists seems to be opposite of their magnitudes. Upper Banach density is especially unfamiliar to some number theorists. However, the concept of upper Banach density bears the most resemblance among these densities to the concept of a probability measure space. For example, in [**Fu**, Lemma 3.17] Furstenberg established a correspondence principle between upper Banach density and probability measure and used it to prove many interesting number theoretic results. The author has also developed a general scheme, which establishes a connection between upper Banach density and Shnirel'man density/lower asymptotic density through Loeb probability measure spaces, which enable us to generate many new theorems about upper Banach density (cf. [**Ji2**]).

Shnirel'man density is probably the most popular density among many number theorists. There have been many important classical theorems about Shnirel'man density.

1.1. Kneser's Theorem. Shnirel'man in 1930 proved a theorem that for any $A, B \subseteq \mathbb{N}$, if $0 \in A$ and $1 \in B$, then

$$\sigma(A+B) \ge \sigma(A) + \sigma(B) - \sigma(A)\sigma(B)$$

where $A + B = \{a + b : a \in A \text{ and } b \in B\}$ (cf. [**HR**, Theorem 1 on page 3] or [**Na1**, Theorem 7.5 on page 193]). Let P be the set of all prime numbers and $A = P \cup \{0, 1\}$. By the theorem above Shnirel'man showed that there is a positive integer h such that $\sigma(hA) = 1$ where

$$hA = \underbrace{A + A + \cdots + A}_{h}.$$

This result is the first significant advancement on the famous Goldbach Conjecture. It shows that there is a fixed positive integer h such that every positive integer greater than 1 is the sum of at most h prime numbers.

Mann in 1942 improved Shnirel'man's Theorem by showing that if $0 \in A \cap B$, then $\sigma(A + B) \ge \min\{1, \sigma(A) + \sigma(B)\}$ (cf. [**HR**, Theorem 3 on page 5]). Mann's theorem was included in Khinchin's little book "Three pearls of number theory" as one of the three pearls (cf. [**Kh**]).

It is often the case that after a theorem about Shnirel'man density is proven, people want to know whether it can be generalized to a theorem about lower asymptotic density. However, one cannot replace σ by $\underline{d}$ in either Shnirel'man's Theorem or in Mann's Theorem. Let $2k^2 < g$ and $A = [0, k-1] + \{gn : n \in \mathbb{N}\}$. A is the union of k arithmetic progressions with a common difference g. Clearly, $\underline{d}(A) = \frac{k}{g}$ and

$$\underline{d}(A + A) = \frac{2k-1}{g} = 2\underline{d}(A) - \frac{1}{g} < 2\underline{d}(A) - \left(\frac{k}{g}\right)^2 = 2\underline{d}(A) - \underline{d}(A)^2 < 2\underline{d}(A) \leq 1.$$

However, this counterexample is essentially the only reason why σ cannot be replaced by $\underline{d}$ in Shnirel'man's Theorem or in Mann's Theorem. In 1953 Kneser proved the following theorem.

Theorem 1.1 (M. Kneser, 1953). *Let $A, B \subseteq \mathbb{N}$ such that $\underline{d}(A+B) < \underline{d}(A) + \underline{d}(B)$, then there exist positive integer d and $G \subseteq [0, d-1]$ such that*

(1) $\underline{d}(A + B) \geq \underline{d}(A) + \underline{d}(B) - \frac{1}{d}$,
(2) $A + B \subseteq G + \{dn : n \in \mathbb{N}\}$, *and*
(3) $(G + \{dn : n \in \mathbb{N}\}) \setminus (A + B)$ *is finite.*

The proof of Theorem 1.1 can be found in [**HR**, page 51–75][1]. It is not difficult to prove that Theorem 1.1 is equivalent to the following theorem.

Theorem 1.2. *Let $A, B \subseteq \mathbb{N}$ such that $\underline{d}(A + B) < \underline{d}(A) + \underline{d}(B)$. Then there exist positive integer d and sets $F, F' \subseteq [0, d-1]$ such that*

(1) $A \subseteq F + \{dn : n \in \mathbb{N}\}$, $B \subseteq F' + \{dn : n \in \mathbb{N}\}$, *and*
(2) $\underline{d}(A) + \underline{d}(B) > \frac{|F|+|F'|-1}{d}$.

Theorem 1.2 clearly shows that if $\underline{d}(A+B) < \underline{d}(A)+\underline{d}(B)$, then each of A and B must be large subsets of the union of arithmetic progressions with a common difference d. This formulation of Kneser's Theorem is in the same style as in the so called Freiman's inverse problem for finite sets, which says that if $A + B$ are small, then A and B must have some arithmetic structure (cf. [**Na2**]).

We will present theorems about upper Banach density parallel to Kneser's Theorem in §2 and §3.

1.2. Plünnecke's Theorem. A set $B \subseteq \mathbb{N}$ is called an *essential component* if $\sigma(A + B) > \sigma(A)$ whenever $A \subseteq \mathbb{N}$ and $0 < \sigma(A) < 1$. Since the early time of the last century people have been interested in finding which set $B \subseteq \mathbb{N}$ can be an essential component (cf. [**HR**]). By Shnirel'man's Theorem [**HR**, page 3] it can easily be seen that if $0 \in B$ and $\sigma(B) > 0$, then B is an essential component. However, even if $\sigma(B) = 0$, B can still be an essential component. A set $B \subseteq \mathbb{N}$ is called a *basis* of order h if $hB = \mathbb{N}$. Let $h > 1$. Note that B is a basis of order h iff $\sigma(hB) = 1$. If B is a basis of some finite order, then B is an essential component although such B may have Shnirel'man density 0. For example, $B = \{n^2 : n \in \mathbb{N}\}$ is a basis of order 4 by Lagrange's Theorem and $\sigma(B) = 0$. In 1937 Erdös proved that if B is a basis of order h, then

$$\sigma(A + B) \geq \sigma(A) + \frac{1}{2h} \cdot \sigma(A) (1 - \sigma(A)). \tag{1.1}$$

A short time later, Landau noticed that in Erdös' proof h can be replaced by average order h^* (cf. [**HR**, page 10]). Let $B \subseteq \mathbb{N}$ be a basis of order h. For each $m \in \mathbb{N}$ let

[1] Kneser's Theorem actually deals with the sum of multiple sets. We state only the version for the sum of two sets here for simplicity.

$h_B(m) = \min\{h' \in \mathbb{N} : m \in h'B\}$. The *average order* h^* of B is defined by

$$h^* = \sup_{n \geq 1} \frac{1}{n} \sum_{m=1}^{n} h_B(m).$$

It is easy to see that $h^* \leq h \leq 2h^*$ (cf. [**HR**, page 12]). In 1938 Rohrbach proved a theorem for lower asymptotic density parallel to Erdös–Landau's result. A set $B \subseteq \mathbb{N}$ is called an *asymptotic basis* of order h if hB contains all sufficiently large positive integers. The *average asymptotic order* h^* of an asymptotic basis is defined by

$$h^* = \limsup_{n \to \infty} \frac{1}{n} \sum_{m=1}^{n} h_B(m)$$

where $h_B(m)$ is defined to be 0 if $m \notin hB$. Rohrbach proved (cf. [**HR**, page 45]) that if B is an asymptotic basis of average asymptotic order h^*, then

$$\underline{d}(A+B) \geq \underline{d}(A) + \frac{1}{2h^*} \cdot \underline{d}(A)\left(1 - \underline{d}(A)\right). \tag{1.2}$$

In 1970 Plünnecke obtained the following significant improvement of Erdös–Landau's result (cf. [**Pl**] or [**Na2**, page 225]).

Theorem 1.3 (Plünnecke, 1970). *If B is a basis of order h, then for every $A \subseteq \mathbb{N}$*

$$\sigma(A+B) \geq \sigma(A)^{1-\frac{1}{h}}. \tag{1.3}$$

Since it is easy to show that $x^{1-\frac{1}{h}} \geq x + \frac{1}{h} \cdot x(1-x)$ for any $h > 0$ and $0 \leq x \leq 1$ by elementary calculus, Plünnecke's Theorem implies (1.1) even when $\frac{1}{2h}$ is replaced by $\frac{1}{h}$. Since the average order h^* of a basis B is less than or equal to 2 times the order h of B, Plünnecke's Theorem also implies (1.1) when h is replaced by h^*.

However, we cannot replace σ by $\underline{d}$ and replace the order h by the average asymptotic order h^* in (1.3) as Rohrbach did in (1.2) to Erdös Theorem (1.1). For example, let $A = \{1 + 3n : n \in \mathbb{N}\}$ and $B = \{i + 3n : i = 0, 1 \text{ and } n \in \mathbb{N}\}$. Then $A + B = \{i + 3n : i = 1, 2 \text{ and } n \in \mathbb{N}\}$, $\sigma(A) = \underline{d}(A) = \frac{1}{3}$ and $\sigma(A+B) = \underline{d}(A+B) = \frac{2}{3}$. It is easy to check that B is a basis of order $h = 2$, average order $h^* = \frac{3}{2}$, and average asymptotic order $h^{**} = \frac{4}{3}$. Note that

$$\underline{d}(A)^{1-\frac{1}{h^{**}}} = \left(\frac{1}{3}\right)^{\frac{1}{4}} > \sigma(A)^{1-\frac{1}{h^*}} = \underline{d}(A)^{1-\frac{1}{h^*}}$$

$$= \left(\frac{1}{3}\right)^{\frac{1}{3}} > \frac{2}{3} = \sigma(A+B) = \underline{d}(A+B).$$

We will discuss the generalization of Plünnecke's Theorem to other densities in §2 and §3.

2. Level One Applications

In this section we will introduce ultrapower methods and develop an scheme, which allows us to obtain a theorem about upper Banach density parallel to every existing theorem about Shnirel'man density or lower asymptotic density without making too much effort.

2.1. Ultrapower of $\mathbb{R}$.

DEFINITION 2.1. A collection $\mathcal{F}$ of subsets of $\mathbb{N}$ is called a **filter** if the following are true:

(1) $\emptyset \notin \mathcal{F}$ and $\mathbb{N} \in \mathcal{F}$,
(2) $A \cap B \in \mathcal{F}$ for any $A, B \in \mathcal{F}$,
(3) $A \in \mathcal{F}$ and $A \subseteq B$ imply $B \in \mathcal{F}$ for any $A, B \subseteq \mathbb{N}$.

The filter $\mathcal{F}$ is called a **non-principal ultrafilter** if

(4) $\{n\} \notin \mathcal{F}$ for each $n \in \mathbb{N}$,
(5) for every $A \subseteq \mathbb{N}$, either $A \in \mathcal{F}$ or $\mathbb{N} \setminus A \in \mathcal{F}$.

The existence of a non-principal ultrafilter on $\mathbb{N}$ is guaranteed by the axiom of choice. From now on we fix a non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$. We assume that the reader knows the basic properties of ultrafilters. For example, we assume the reader knows that the intersection of finitely many sets $A_i \in \mathcal{F}$ is again in $\mathcal{F}$ and every cofinite subset of $\mathbb{N}$ is in $\mathcal{F}$.

DEFINITION 2.2. Given any $f, g \in \mathbb{R}^{\mathbb{N}}$, let

(1) $f \sim g$ iff $\{n : f(n) = g(n)\} \in \mathcal{F}$,
(2) $[f] = \{g \in \mathbb{R}^{\mathbb{N}} : g \sim f\}$, and
(3) ${}^*\mathbb{R} = \mathbb{R}^{\mathbb{N}}/\mathcal{F} = \{[f] : f \in \mathbb{R}^{\mathbb{N}}\}$.

It is easy to check that"$\sim$" defined in (1) above is an equivalence relation. Hence $[f]$ is an equivalence class in (2) above and ${}^*\mathbb{R}$ is the set of all equivalence classes. For each $A \subseteq \mathbb{R}$ let ${}^*A = A^{\mathbb{N}}/\mathcal{F} = \{[f] : f \in A^{\mathbb{N}}\}$. A set $B \subseteq {}^*\mathbb{R}$ is called internal if B is an ultraproduct of a sequence of sets $B_n \in \mathbb{R}$, i.e., $B = \Pi_{n\in\mathbb{N}} B_n/\mathcal{F} = \{[f] : f \in \mathbb{R}^{\mathbb{N}}$ and $f(n) \in B_n$ for every $n \in \mathbb{N}\}$. For each $a \in \mathbb{R}$ let f_a be the constant function with value a. If we identify each $a \in \mathbb{R}$ with $[f_a] \in {}^*\mathbb{R}$, we can view $\mathbb{R}$ as a subset of ${}^*\mathbb{R}$. We can extend $\leq$, $+$, $\cdot$, etc. from $\mathbb{R}$ to ${}^*\mathbb{R}$.

DEFINITION 2.3. Given any $[f], [g] \in {}^*\mathbb{R}$, let

(1) $[f] \leq [g]$ iff $\{n : f(n) \leq g(n)\} \in \mathcal{F}$,
(2) $[f] + [g] = [f + g]$, and
(3) $[f]\cdot[g] = [f\cdot g]$

where $f+g$ and $f \cdot g$ are ordinary addition and multiplication of two functions. By the same idea we can extend any relation or function on $\mathbb{R}$ to a relation or function on ${}^*\mathbb{R}$. For example, if $F(x_1, x_2, \ldots, x_k)$ is a k–dimensional function from $\mathbb{R}^k$ to $\mathbb{R}$, then we can extend F to a function from ${}^*\mathbb{R}^k$ to ${}^*\mathbb{R}$ by letting $F([g_1], [g_2], \ldots, [g_k])$ be the equivalence class $[\bar{f}]$ of the function $\bar{f}$ where $\bar{f}(n) = F(g_1(n), g_2(n), \ldots, g_k(n))$. Note that if B is the ultraproduct of a sequence of finite sets $B_n \subseteq \mathbb{R}$, then the cardinality of B is defined by $|B| = [f]$ where $f(n) = |B_n|$.

We can also extend the boolean operations among subsets of $\mathbb{R}$ to internal subsets of ${}^*\mathbb{R}$.

DEFINITION 2.4. Let $A = \Pi_{n\in\mathbb{N}} A_n/\mathcal{F}$ and $B = \Pi_{n\in\mathbb{N}} B_n/\mathcal{F}$ be two internal subsets of ${}^*\mathbb{R}$ and $[f] \in {}^*\mathbb{R}$. Define

(1) $[f] \in A$ if $\{n \in \mathbb{N} : f(n) \in A_n\} \in \mathcal{F}$,
(2) $A \subseteq B$ if $\{n \in \mathbb{N} : A_n \subseteq B_n\} \in \mathcal{F}$,
(3) $A \cap B = \Pi_{n\in\mathbb{N}}(A_n \cap B_n)/\mathcal{F}$,
(4) $A \cup B = \Pi_{n\in\mathbb{N}}(A_n \cup B_n)/\mathcal{F}$, and

(5) $A \setminus B = \Pi_{n\in\mathbb{N}}(A_n \setminus B_n)/\mathcal{F}$.

PROPOSITION 2.5. The relation $\leq$ is a linear order on ${}^*\mathbb{R}$.

PROOF. Let $[f],[g] \in {}^*\mathbb{R}$. Then either $\{n \in \mathbb{N} : f(n) \leq g(n)\} \in \mathcal{F}$ or $\{n \in \mathbb{N} : f(n) > g(n)\} \in \mathcal{F}$ by (5) of Definition 2.1. Hence either $[f] \leq [g]$ or $[f] > [g]$ by (1) of Definition 2.3. □

PROPOSITION 2.6. If $[f],[g] \in {}^*\mathbb{Z}$ such that $[f] \leq [g] \leq [f]+k$ for some $k \in \mathbb{N}$ (k is the equivalence class of a constant function with value k), then there is $m \in \mathbb{N}$ such that $m \leq k$ and $[g] = [f]+m$.

PROOF. Let

$$X = \{n \in \mathbb{N} : f(n) \leq g(n) \leq f(n)+k\}$$
$$= \{n \in \mathbb{N} : f(n) \leq g(n)\} \cap \{n \in \mathbb{N} : g(n) \leq f(n)+k\} \in \mathcal{F}.$$

Since X is the union of $k+1$ sets $X_i = \{n \in \mathbb{N} : g(n) = f(n)+i\}$ for $i = 0,1,\ldots,k$, then one of these X_i's must be in $\mathcal{F}$ because otherwise X would not be in $\mathcal{F}$. Let $m \in \mathbb{N}$ with $m \leq k$ such that $X_m \in \mathcal{F}$. This shows that $[g] = [f]+m$. □

PROPOSITION 2.7. If $A, B \subseteq \mathbb{R}$, then ${}^*(A+B) = {}^*A + {}^*B$.

PROOF. Let $[f] \in {}^*(A+B)$. For each $n \in \mathbb{N}$ choose $a_n \in A$ and $b_n \in B$ such that $a_n + b_n = f(n)$. Define $g_1(n) = a_n$ and $g_2(n) = b_n$. Then $f = g_1 + g_2$. Hence $[f] = [g_1]+[g_2] \in {}^*A + {}^*B$. For each $[g_1]+[g_2] \in {}^*A + {}^*B$ let $f(n) = g_1(n)+g_2(n)$. Then $\{n : f(n) \in A+B\} \in \mathcal{F}$. Hence $[g_1]+[g_2] = [f] \in {}^*(A+B)$. □

From Proposition 2.6 it is not hard to check that $({}^*\mathbb{Z}; \leq, +, \cdot, 0, 1)$ is a discrete ordered ring containing $(\mathbb{Z}; \leq, +, \cdot, 0, 1)$ as a subring. By Proposition 2.6 again we have that if $[f] \in {}^*\mathbb{N}$ and $[f] \neq k$ for any $k \in \mathbb{N}$, then $[f] > k$ for every $k \in \mathbb{N}$. We call $[f] \in {}^*\mathbb{N} \setminus \mathbb{N}$ a hyperfinite integer. For example, if Id is the identity function $Id(n) = n$ for every $n \in \mathbb{N}$, then $[Id]$ is a hyperfinite integer. For each $[f] \in {}^*\mathbb{Z}$, the map $k \mapsto [f]+k$ is an order–isomorphic embedding from $\mathbb{Z}$ onto $[f]+\mathbb{Z} \subsetneq {}^*\mathbb{Z}$. Therefore, for a set $A \subseteq \mathbb{N}$ and $[f] \in {}^*\mathbb{N}$ we can define the densities of ${}^*A$ on $[f]+\mathbb{N}$.

2.2. Densities in a (possibly remote) copy of $\mathbb{N}$.

DEFINITION 2.8. Let $A \subseteq \mathbb{N}$ and $[f] \in {}^*\mathbb{N}$. The Shnirel'man density of ${}^*A$ in $[f]+\mathbb{N}$ is defined by

$$\sigma_{[f]}({}^*A) = \inf_{i\in\mathbb{N}, i\geq 1} \frac{{}^*A([f]+1,[f]+i)}{i}$$

and the lower asymptotic density of ${}^*A$ in $[f]+\mathbb{N}$ is defined by

$$\underline{d}_{[f]}({}^*A) = \liminf_{i\in\mathbb{N}\,\&\, i\to\infty} \frac{{}^*A([f]+1,[f]+i)}{i}.$$

Recall that ${}^*A([f],[g]) = \left|\prod_{n\in\mathbb{N}}(A \cap [f(n),g(n)])/\mathcal{F}\right| = [h]$ where $h(n) = A(f(n),g(n))$ for any $[f],[g] \in {}^*\mathbb{N}$. Note that $\sigma_{[f]}({}^*A) = \sigma(A)$ and $\underline{d}_{[f]}({}^*A) = \underline{d}(A)$ if $[f] = 0$. If $0 \leq [g]-[f] \in \mathbb{N}$, then ${}^*A([f],[g])$ is also in $\mathbb{N}$. The next theorem, although straightforward, is the main tool in this section.

THEOREM 2.9. *Let $A \subseteq \mathbb{N}$ and $\alpha \in \mathbb{R}$. Then the following are equivalent.*

(1) $BD(A) \geq \alpha$.
(2) *There is $[f] \in {}^*\mathbb{N}$ such that $\sigma_{[f]}({}^*A) \geq \alpha$.*
(3) *There is $[f] \in {}^*\mathbb{N}$ such that $\underline{d}_{[f]}({}^*A) \geq \alpha$.*

PROOF. We first prove that (1) implies (2). Let $[a_n, b_n] \subseteq \mathbb{N}$ be such that

$$\lim_{n\to\infty}(b_n - a_n) = \infty \text{ and } \lim_{n\to\infty}\frac{A(a_n, b_n)}{b_n - a_n + 1} = BD(A) \geq \alpha.$$

We intend to find $\{c_k, d_k \in \mathbb{N} : k \in \mathbb{N}\}$ such that

$$\lim_{k\to\infty}(d_k - c_k) = \infty \text{ and } \frac{A(c_k + 1, c_k + i)}{i} > \alpha - \frac{1}{k}$$

for every $i \in [1, d_k - c_k]$, which will imply that $\sigma_{[f]}({}^*A) \geq \alpha$ where f is the function with $f(k) = c_k$.

For positive integer $m, k \in \mathbb{N}$ let

$$l_{m,k} = \max\left\{x \in \mathbb{N} : \exists c \in [a_m, b_m]\, \forall i \in [1, x] \left(\frac{A(c + 1, c + i)}{i} > \alpha - \frac{1}{k}\right)\right\}.$$

Claim For every positive integer k, the sequence $\{l_{m,k} : m \in \mathbb{N}\}$ is upper unbounded in $\mathbb{N}$.

Proof of Claim Suppose the claim is not true. We derive a contradiction. Let k_0 be a positive integer such that l_{m,k_0}'s are bounded by a positive integer $L \in \mathbb{N}$. Let m be large enough so that

$$\frac{A(a_m, b_m)}{b_m - a_m + 1} > \alpha - \frac{1}{2k_0} \text{ and } \frac{L}{b_m - a_m + 1} < \frac{1}{2k_0}.$$

We define a finite sequence $a_m - 1 = d_0 < d_1 < \cdots < d_t \leq b_m$ such that

$$\frac{A(d_i + 1, d_{i+1})}{d_{i+1} - d_i} \leq \alpha - \frac{1}{k_0} \text{ and } b_m - d_t \leq L.$$

Suppose we have found d_i with $b_m - d_i > L$. Let

$$S = \left\{x \in [d_i + 1, b_m] : \frac{A(d_i + 1, x)}{x - d_i} \leq \alpha - \frac{1}{k_0}\right\}.$$

$S \neq \emptyset$ because $l_{m,k_0} < L$. Let $d_{i+1} = \max S$. By induction we can define d_i's until the last term $d_t > b_m - L$. Since

$$\begin{aligned}\frac{A(a_m, b_m)}{b_m - a_m + 1} &\leq \frac{\sum_{i=0}^{t-1} A(d_i + 1, d_{i+1}) + A(d_t + 1, b_m)}{b_m - a_m + 1}\\ &\leq \left(\alpha - \frac{1}{k_0}\right)\frac{\sum_{i=0}^{t-1}(d_{i+1} - d_i)}{b_m - a_m + 1} + \frac{L}{b_m - a_m + 1}\\ &\leq \alpha - \frac{1}{k_0} + \frac{1}{2k_0} = \alpha - \frac{1}{2k_0},\end{aligned}$$

which contradicts the assumption that $\frac{A(a_m,b_m)}{b_m - a_m + 1} > \alpha - \frac{1}{2k_0}$. This completes the proof of the claim.

By the claim we can choose a positive integer m_k for each positive integer k such that $\lim_{k\to\infty} l_{m_k,k} = \infty$ ($d_k = c_k + l_{m_k,k}$ is the number mentioned in the beginning of this proof). Let $f(k) = c_k$ be such that

$$\frac{A(c_k + 1, c_k + i)}{i} > \alpha - \frac{1}{k}$$

for every $i \in [1, l_{m_k,k}]$. We need to show that $\sigma_{[f]}({}^*A) \geq \alpha$. Given $i \in \mathbb{N}$ with $i \geq 1$. Since $A(f(k)+1, f(k)+i)$ for all $k \in \mathbb{N}$ has at most $i+1$ possible values and

$$\frac{A(f(k)+1, f(k)+i)}{i} > \alpha - \frac{1}{k}$$

whenever k is large enough such that $i < l_{m_k,k}$, then there is $K_i \in \mathbb{N}$ such that

$$\frac{A(f(k)+1, f(k)+i)}{i} \geq \alpha$$

for every $k > K_i$. Hence

$$\left\{k \in \mathbb{N} : \frac{A(f(k)+1, f(k)+i)}{i} \geq \alpha\right\} \in \mathcal{F}.$$

This shows that

$$\frac{{}^*A([f]+1, [f]+i)}{i} \geq \alpha$$

is true for any positive integer $i \in \mathbb{N}$. Therefore, $\sigma_{[f]}({}^*A) \geq \alpha$.

It is trivial that (2) implies (3).

We now show that (3) implies (1). To prove $BD(A) \geq \alpha$ it suffices to show that for any positive integer $k \in \mathbb{N}$ there exists an interval $[a, b] \subseteq \mathbb{N}$ such that $b - a > k$ and

$$\frac{A(a, b)}{b - a + 1} > \alpha - \frac{1}{k}.$$

Fix a positive integer k. Since $\underline{d}_{[f]}({}^*A) \geq \alpha$, then there is a positive integer $m > k$ such that

$$\frac{{}^*A([f]+1, [f]+m)}{m} > \alpha - \frac{1}{k}.$$

This implies that

$$S = \left\{n \in \mathbb{N} : \frac{A(f(n)+1, f(n)+m)}{m} > \alpha - \frac{1}{k}\right\} \in \mathcal{F}.$$

In particular, $S \neq \emptyset$. Let $n \in S$, $a = f(n)+1$, and $b = f(n)+m$. The interval $[a, b]$ is what we are looking for. This completes the proof of the theorem □

Theorem 2.9 is a bridge connecting upper Banach density with Shnirel'man density and lower asymptotic density through ultrapower methods. By this connection we have found many theorems about upper Banach density, each of which is parallel to an existing theorem about Shnirel'man density or lower asymptotic density (cf. [**Ji2, Ji3**]). Next we derive two theorems about upper Banach density to demonstrate the idea.

2.3. When Kneser and Plünnecke meet Banach. The first theorem appeared in [**Ji2**] and is parallel to Plünnecke's Theorem.

DEFINITION 2.10. Let $B \subseteq \mathbb{N}$. B is called a piecewise basis of order h if there exists a sequence $\{c_k : k \in \mathbb{N}\}$ of positive integers such that

$$h \cdot (B \cap [c_k, c_k + k]) \supseteq [hc_k, hc_k + k].$$

Note that if B is a basis of order h, then B is a piecewise basis of order at most h because we can choose $c_k = 0$. Note also that $h\cdot(B \cap [c_k, c_k + k]) \supseteq [hc_k, hc_k + k]$ is equivalent to

$$h\cdot((B \cap [c_k, c_k + k]) - c_k) \supseteq [0, k].$$

THEOREM 2.11. *Let $A, B \subseteq \mathbb{N}$. If B is a piecewise basis of order h, then*

$$BD(A + B) \geq BD(A)^{1-\frac{1}{h}}.$$

PROOF. Let $\{c_k : k \in \mathbb{N}\}$ be the sequence associated with B in Definition 2.10. Let $g(k) = c_k$ and Id be the identity function on $\mathbb{N}$. Then we have that

$$h\cdot({}^*B \cap [[g], [g] + [Id]]) \supseteq [h[g], h[g] + [Id]].$$

In particular, we have $h\cdot(({}^*B \cap ([g] + \mathbb{N})) - [g]) \supseteq \mathbb{N}$. This shows that the set $({}^*B \cap ([g] + \mathbb{N})) - [g]$ is a basis of order h.

Let $BD(A) = \alpha$. By Theorem 2.9 there is $[f] \in {}^*\mathbb{N}$ such that $\sigma_{[f]}({}^*A) = \alpha$. This is equivalent to the condition that $\sigma(({}^*A \cap ([f] + \mathbb{N})) - [f]) = \alpha$.

By Plünnecke's Theorem we have that

$$\begin{aligned}\sigma_{[f]+[g]}({}^*(A + B)) &\geq \sigma((({}^*A + {}^*B) \cap ([f] + [g] + \mathbb{N})) - ([f] + [g]))\\ &\geq \sigma((({}^*A \cap ([f] + \mathbb{N})) - [f]) + (({}^*B \cap ([g] + \mathbb{N})) - [g])) \geq \alpha^{1-\frac{1}{h}}.\end{aligned}$$

By Theorem 2.9 again we have $BD(A+B) \geq \alpha^{1-\frac{1}{h}}$. This completes the proof. □

The second theorem appeared in [**Ji3**] and is parallel to Kneser's Theorem.

THEOREM 2.12. *Let $A, B \subseteq \mathbb{N}$. If $BD(A + B) < BD(A) + BD(B)$, then there exists a positive integer d, a set $G \subseteq [0, d-1]$, and a sequence $\{[a_k, b_k] \subseteq \mathbb{N} : k \in \mathbb{N}\}$ of intervals such that*

1. $BD(A + B) \geq \frac{|G|}{d} \geq BD(A) + BD(B) - \frac{1}{d}$,
2. $\lim_{k\to\infty}(b_k - a_k) = \infty$, *and*
3. $(A + B) \cap [a_k, b_k] \supseteq (a_k + G + \{dn : n \in \mathbb{N}\}) \cap [a_k, b_k]$.

PROOF. Let $BD(A) = \alpha$ and $BD(B) = \beta$. By Theorem 2.9 there are $[f], [g] \in {}^*\mathbb{N}$ such that $\underline{d}_{[f]}({}^*A) = \alpha$ and $\underline{d}_{[g]}({}^*B) = \beta$. By Theorem 2.9 again, if $\underline{d}_{[f]+[g]}({}^*A + {}^*B) \geq \alpha + \beta$, then $BD(A + B) \geq \alpha + \beta$, contradicting the assumption, so we have that $\underline{d}_{[f]+[g]}({}^*A + {}^*B) < \alpha + \beta$. By Kneser's Theorem we can find a positive integer d and a set $G \subseteq [0, d - 1]$ such that

$$\begin{aligned}\underline{d}_{[f]+[g]}({}^*(A + B)) &= \underline{d}_{[f]+[g]}({}^*A + {}^*B)\\ &\geq \underline{d}((({}^*A - [f]) \cap \mathbb{N}) + (({}^*B - [g]) \cap \mathbb{N}))\\ &= \frac{|G|}{d} \geq \alpha + \beta - \frac{1}{d},\end{aligned}$$

which implies $BD(A + B) \geq \frac{|G|}{d} \geq \alpha + \beta - \frac{1}{d}$, and

$$\begin{aligned}({}^*A + {}^*B) \cap ([f] + [g] + \mathbb{N}) &\supseteq ({}^*A \cap ([f] + \mathbb{N})) + ({}^*B \cap ([g] + \mathbb{N}))\\ &\supseteq (({}^*A \cap ([f] + \mathbb{N})) + ({}^*B \cap ([g] + \mathbb{N}))) \cap ([f] + [g] + m + \mathbb{N})\\ &= ([f] + [g] + G + \{dn : n \in \mathbb{N}\}) \cap ([f] + [g] + m + \mathbb{N})\end{aligned}$$

for some $m \in \mathbb{N}$. Let $k \in \mathbb{N}$. Since

$$\begin{aligned}&({}^*A + {}^*B) \cap [[f] + [g] + m, [f] + [g] + m + k]\\ &\quad\supseteq ([f] + [g] + G + \{dn : n \in \mathbb{N}\}) \cap [[f] + [g] + m, [f] + [g] + m + k],\end{aligned}$$

there exists $n \in \mathbb{N}$ such that

$$(A+B) \cap [f(n)+g(n)+m, f(n)+g(n)+m+k]$$
$$\supseteq (f(n)+g(n)+G+\{dn : n \in \mathbb{N}\}) \cap [f(n)+g(n)+m, f(n)+g(n)+m+k].$$

Let $a_k = f(n)+g(n)+m$ and $b_k = f(n)+g(n)+m+k$. Clearly, the sequence $\{[a_k, b_k] : k \in \mathbb{N}\}$ is the sequence we desired. This completes the proof. □

3. Level Two Applications

In §2 we developed a general way of deriving a theorem about upper Banach density parallel to each existing theorem about Shnirel'man density or lower asymptotic density via Theorem 2.9. However, a simple application of Theorem 2.9 often results in a theorem, which in some sense is not optimal. For example in Theorem 2.12 the structure of $A+B$ is characterized only on a small portion of $\mathbb{N}$. The characterization of the structure of $A+B$ can be made on a much larger set. In Theorem 2.11 the definition of a piecewise basis seems artificial. In this section we will discuss whether Theorem 1.3 can be generalized to lower asymptotic density, upper asymptotic density, and upper Banach density. The theorem about upper Banach density is in fact a significant improvement of Theorem 2.11. The proofs of the results in this section can be found in [**Ji7**] and [**Ji8**].

Although the ultrapower method introduced in §2 works fine, it is more convenient to work under the full strength of nonstandard analysis. In the ultrapower method, one might view $[f]$ not as a number but as a function (more precisely, as an equivalence class containing f). But from model theoretic point of view $[f]$ is just a single point in the extension $^*\mathbb{R}$ of $\mathbb{R}$. People probably do not consider a real number as a sequence of rational numbers when working on real analysis problems although the Cauchy definition of a real is an equivalence class of a Cauchy sequence of rational numbers. If the reader is interested in mathematical logic, the Łos' Theorem should also be a great help.

For constructing a nonstandard universe we first take $\mathbb{R}$ as a set of atoms. Then let $V_0 = \mathbb{R}$, $V_{n+1} = V_n \cup \mathcal{P}(V_n)$, and $V = \bigcup_{n=0}^{\bar{N}} V_n$, where $\mathcal{P}$ is power set operator, for some sufficiently large positive integer $\bar{N}$. We call $(V, \in)$ the standard universe. The standard universe is large enough to contain every possible mathematical object involved in a standard mathematical argument. For example $\leq$ on $\mathbb{R}$ is a set of ordered pairs. Hence $\leq$ is an element in V_3. The nonstandard universe $^*V$ is the ultrapower of V modulo $\mathcal{F}$. For each $[f], [g] \in {}^*V$ define $[f] \in [g]$ if the set $\{n \in \mathbb{N} : f(n) \in g(n)\}$ is in $\mathcal{F}$. Let $i : V \mapsto {}^*V$ be such that $i(a) = [f_a]$ where f_a is the constant function on $\mathbb{N}$ with value a. Łos' Theorem says that for any first–order formula $\varphi(x_1, x_2, \ldots, x_k)$ in the language of one binary relation $\in$ and any $[f^{(1)}], [f^{(2)}], \ldots, [f^{(k)}] \in {}^*V$, $\varphi([f^{(1)}], [f^{(2)}], \ldots, [f^{(k)}])$ is true in $^*V$ if and only if

$$\{n \in \mathbb{N} : \varphi(f^{(1)}(n), f^{(2)}(n), \ldots, f^{(k)}(n)) \text{ is true in } V\} \in \mathcal{F}.$$

Łos' Theorem implies the famous transfer principle, which says that for any first–order formula $\varphi(x_1, x_2, \ldots, x_k)$ in the language of one binary relation $\in$ and for any $a_1, a_2, \ldots, a_n \in V$ the sentence $\varphi(a_1, a_2, \ldots, a_k)$ is true in $(V, \in)$ if and only if $\varphi([f_{a_1}], [f_{a_2}], \ldots, [f_{a_k}])$ is true in $(^*V, \in)$.

The proofs in [**Ji7**] and [**Ji8**] heavily use nonstandard analysis techniques. We do not intend to include those proofs here. Instead we will explain the general ideas of the proofs.

3.1. Kneser meets Banach again. The following theorem improves Theorem 2.12. The proof of the following theorem can be found in [**Ji7**].

THEOREM 3.1. *Let $A, B \subseteq \mathbb{N}$ be such that $BD(A) = \alpha$, $BD(B) = \beta$, and $BD(A+B) < \alpha+\beta$. Then there are positive $g \in \mathbb{N}$ and $G \subseteq [0, g-1]$ such that*

(1) $BD(A+B) \geq \alpha+\beta-\frac{1}{g}$,
(2) $A+B \subseteq G+g\mathbb{N}$,
(3) *if* $\left\{\left[a_n^{(i)}, b_n^{(i)}\right] : n \in \mathbb{N}\right\}$ *for $i = 1, 2$ are two sequences of intervals such that*

$$\lim_{n\to\infty}\left(b_n^{(i)} - a_n^{(i)}\right) = \infty,$$

$$\lim_{n\to\infty}\frac{A\left(a_n^{(1)}, b_n^{(1)}\right)}{b_n^{(1)} - a_n^{(1)} + 1} = \alpha,\ \lim_{n\to\infty}\frac{B\left(a_n^{(2)}, b_n^{(2)}\right)}{b_n^{(2)} - a_n^{(2)} + 1} = \beta,$$

and

$$0 < \liminf_{n\to\infty}\frac{b_n^{(1)} - a_n^{(1)}}{b_n^{(2)} - a_n^{(2)}} \leq \limsup_{n\to\infty}\frac{b_n^{(1)} - a_n^{(1)}}{b_n^{(2)} - a_n^{(2)}} < \infty,$$

then there exist $\left[c_n^{(i)}, d_n^{(i)}\right] \subseteq \left[a_n^{(i)}, b_n^{(i)}\right]$ *for each $n \in \mathbb{N}$ and $i = 1, 2$ such that*

$$\lim_{n\to\infty}\frac{d_n^{(i)} - c^{(i)}}{b_n^{(i)} - a_n^{(i)}} = 1$$

and

$$(A+B) \cap \left[c_n^{(1)} + c_n^{(2)}, d_n^{(1)} + d_n^{(2)}\right] = (G+g\mathbb{N}) \cap \left[c_n^{(1)} + c_n^{(2)}, d_n^{(1)} + d_n^{(2)}\right].$$

REMARK 3.2. (1) The condition

$$0 < \liminf_{n\to\infty}\frac{b_n^{(1)} - a_n^{(1)}}{b_n^{(2)} - a_n^{(2)}} \leq \limsup_{n\to\infty}\frac{b_n^{(1)} - a_n^{(1)}}{b_n^{(2)} - a_n^{(2)}} < \infty,$$

in (3) of Theorem 3.1 is necessary because otherwise one can choose

$$A = \bigcup_{n=1}^{\infty}\left(\left[2^{(2n)^2}, 1.5\times 2^{(2n)^2} - 2^{(2n-1)^2+1}\right]\right.$$
$$\left.\cup\left[1.5\times 2^{(2n)^2} + 2^{(2n-1)^2+1}, 2\times 2^{(2n)^2}\right]\right),$$

$$B = \bigcup_{n=1}^{\infty}\left(\left[2^{(2n+1)^2}, 1.5\times 2^{(2n+1)^2} - 2^{(2n)^2+1}\right]\right.$$
$$\left.\cup\left[1.5\times 2^{(2n+1)^2} + 2^{(2n)^2+1}, 2\times 2^{(2n+1)^2}\right]\right).$$

Then $BD(A) = \alpha = BD(B) = \beta = 1$, which trivially implies $BD(A+B) < BD(A)+BD(B)$. On the other hand, let $a_n^{(1)} = 2^{(2n)^2}$, $b_n^{(1)} = 2\times 2^{(2n)^2}$, $a_n^{(2)} = 2^{(2n+1)^2}$, and $b_n^{(2)} = 2\times 2^{(2n+1)^2}$. Then all conditions of Theorem 3.1 except the one mentioned above are true. However, the structure described in the last line of (3) in Theorem 3.1 cannot be true because $(A+B) \cap \left[a_n^{(1)} + a_n^{(2)}, b_n^{(1)} + b_n^{(2)}\right]$ has large gaps in the middle of the interval.

(2) Let A achieve its upper Banach density on a sequence of intervals $\left[a_n^{(1)}, b_n^{(1)}\right]$ and B achieve its upper Banach density on a sequence of intervals $\left[a_n^{(2)}, b_n^{(2)}\right]$. We probably shouldn't hope to characterize the structure of $A+B$ outside of the intervals $\left[a_n^{(1)} + a_n^{(2)}, b_n^{(1)} + b_n^{(2)}\right]$ because the upper Banach densities of A and B would not change if we delete all elements of A outside of $\left[a_n^{(1)}, b_n^{(1)}\right]$ and delete all elements of B outside of $\left[a_n^{(2)}, b_n^{(2)}\right]$. On the other hand, we cannot hope to replace $c^{(i)}, d^{(i)}$ by $a^{(i)}, b^{(i)}$ in the last line of Theorem 3.1 by the same reason as in the Kneser's Theorem where the structure of $A+B$ is characterized not in $\mathbb{N}$ but in $\mathbb{N} \setminus [0, m]$ for some $m \in \mathbb{N}$. This is why in Theorem 3.1 the structure of $A+B$ is characterized on $\left[c_n^{(1)} + c_n^{(2)}, d_n^{(1)} + d_n^{(2)}\right]$ instead.

(3) The proof of Theorem 3.1 is much more complicated than the proof of Theorem 2.12. In order to prove Theorem 3.1 one should improve Theorem 2.9 first. In fact $BD(A) \geq \alpha$ implies $\underline{d}_{[f]}({}^*A) \geq \alpha$ for many f's.

Let $[f] < [g]$ be in ${}^*\mathbb{Z}$ such that $[g] - [f]$ is a hyperfinite integer. For each internal subset C of $[[f], [g]]$ the cardinality of C is an element in ${}^*\mathbb{N}$. Define $\mu(C) = \frac{|C|}{[g]-[f]+1}$. Then $0 \leq \mu(C) \leq 1$ and $\mu(C) \in {}^*\mathbb{R}$. Note that for every $r \in {}^*\mathbb{R}$, $0 \leq r \leq 1$ the set S_r of all standard reals $s \in \mathbb{R}$ with $s < r$ has the least upper bound β in $\mathbb{R}$. It is not hard to see that $|r - \beta| < \frac{1}{n}$ for every $n \in \mathbb{N}$. We say that r and β are infinitesimally close. Note also that such β is unique. We call β the standard part of r and denote $st(r) = \beta$. Hence $st{\circ}\mu$ maps every internal subset $C \subseteq [[f], [g]]$ to a standard real number between 0 and 1. In fact $st{\circ}\mu$ is a finitely additive probability measure on the algebra of all internal subsets of $[[f], [g]]$. For any $X \subseteq [[f], [g]]$ we can use $st{\circ}\mu$ to define lower measure and upper measure of X and call X measurable if the lower measure and upper measure of X coincide. By measure–completion process $st{\circ}\mu$ can be extended to a countably additive, complete, atom-less probability measure μ_L on the σ–algebra of all measurable subsets of $[[f], [g]]$. This probability space is called Loeb space. With the idea of Loeb space together with Birkhoff Ergodic Theorem we can improve Theorem 2.9 in the following theorem (cf. [**Ji2**]).

THEOREM 3.3. *Let $A \subseteq \mathbb{N}$ be such that $BD(A) = \alpha > 0$. Suppose $[a_n, b_n] \subseteq \mathbb{N}$ such that*

$$\lim_{n\to\infty}(b_n - a_n) = \infty \text{ and } \lim_{n\to\infty} \frac{A(a_n, b_n)}{b_n - a_n + 1} = \alpha.$$

Let $[f]$ be a hyperfinite integer. Then $[a_{[f]}, b_{[f]}]$ is an interval of hyperfinite length and for μ_L–almost all $x \in [a_{[f]}, b_{[f]}]$ we have $\underline{d}_x({}^*A) = \alpha$ where μ_L is the Loeb probability measure on $[a_{[f]}, b_{[f]}]$. Note that $a_{[f]}$ is the equivalence class $[g]$ where $g(n) = a_{f(n)}$.

By combining Theorem 3.3 and Kneser's Theorem we can pin down the structure of $A + B$ in the interval $\left[a_n^{(1)} + a_n^{(2)}, b_n^{(1)} + b_n^{(2)}\right]$ for each hyperfinite integer n. It takes a small trick to show that the structures of $A + B$ in the intervals $\left[a_n^{(1)} + a_n^{(2)}, b_n^{(1)} + b_n^{(2)}\right]$ for all hyperfinite integers n are actually coherent. Hence we can now characterize the structure of $A+B$ in the union of all these hyperfinite intervals. Finally, by the transfer principle we pull down the nonstandard result to the standard world and obtain Theorem 3.1.

3.2. Plünnecke meets Banach again. It is not hard to show that if B is a basis of order h, then $\underline{d}(A+B) \geq \underline{d}(A)^{1-\frac{1}{h}}$. However, $\underline{d}(A+B) \geq \underline{d}(A)^{1-\frac{1}{h}}$ for B being an asymptotic basis of order h, although true, is not a direct consequence of Theorem 1.3. It is in fact a corollary of Theorem 3.4 which we will present below.

Recall that in Theorem 1.3 we cannot replace σ by $\underline{d}$ and replace a basis B of order h by an asymptotic basis of average asymptotic order h^*. What kind of generalization of Theorem 1.3 to lower asymptotic density can we have?

Let $B \subseteq \mathbb{N}$. The set B is called a *lower asymptotic basis* of order h if

$$\underline{d}(hB) = 1,$$

the set B is called an *upper asymptotic basis* of order h if

$$\overline{d}(hB) = 1,$$

and the set B is called an *upper Banach basis* of order h if

$$BD(hB) = 1.$$

Recall that if $h > 1$, then B is a basis of order h iff $\sigma(hB) = 1$. Hence the style of our definition of the three asymptotic bases above is consistent with that of B being a basis. Note that if B is an asymptotic basis of asymptotic order h_0 and of average asymptotic order h_1, then B is a lower asymptotic basis of order h_2 with $h_1 \leq h_2 \leq h$. Note that if B is a piecewise basis of piecewise order h_0, then B is an upper Banach basis of order $h_1 \leq h_0$.

Let P again be the set of all prime numbers. P is not a basis because P does not contain 0 and 1. If $A = P \cup \{0, 1\}$, then A is a basis of order h for some $h \in \mathbb{N}$. However, the order h may be large ($h = 7$ by a results of Remera). By Vinogradov Three-Prime Theorem P is an asymptotic basis of asymptotic order order 4. By a result in [**Es**] P is a lower asymptotic basis of order 3.[2] Of course, P would be an asymptotic basis of asymptotic order 3 if the famous Goldbach Conjecture for the sum of two prime numbers has a positive answer. This makes Theorem 3.4 below interesting.

The three theorems below are the results of effort for generalizing Plünnecke's Theorem to the three asymptotic densities.

THEOREM 3.4. *Let $A, B \subseteq \mathbb{N}$ and B be a lower asymptotic basis of order h. Then*

$$\underline{d}(A+B) \geq \underline{d}(A)^{1-\frac{1}{h}}.$$

THEOREM 3.5. *There are $A, B \subseteq \mathbb{N}$ with $\overline{d}(A) = \frac{1}{2}$ and B an upper asymptotic basis of order 2 such that*

$$\overline{d}(A+B) = \overline{d}(A).$$

THEOREM 3.6. *Let $A, B \subseteq \mathbb{N}$ and B be a upper Banach basis of order h. Then*

$$BD(A+B) \geq BD(A)^{1-\frac{1}{h}}.$$

[2]This result was discovered independently by Nikolai Chudakov, Johannes van der Corput, and Theodor Estermann at about the same time.

As a corollary of Theorem 3.4, we have that for any set $A \subseteq \mathbb{N}$, $\underline{d}(A+P) \geq \underline{d}(A)^{2/3}$ where P is the set of all prime numbers. It is interesting to see whether the lower bound $\underline{d}(A)^{2/3}$ of $\underline{d}(A+P)$ in this corollary can be improved.

It is not a surprise that the behavior of upper asymptotic density is different from the behavior of lower asymptotic density or the behavior of upper Banach density. We have discovered many instances of that phenomenon.

Theorem 3.6 is a significant improvement of Theorem 2.11 because a piecewise basis of piecewise order h is clearly an upper Banach basis of order at most h and the definition of upper Banach basis seems more natural than the definition of piecewise basis.

The proof of Theorem 3.5 does not involve nonstandard methods. The upper asymptotic basis B for Theorem 3.5 constructed in [**Ji7**] is a modification of the thin basis constructed by Cassels (cf. [**HR**, Theorem 12 on page 39]).

The reader can see that the proof of Theorem 2.11 does not get into Plünnecke's original idea of Plünnecke's graph, which was used to obtain a powerful inequality. It is that Plünnecke's inequality, which leads to Theorem 1.3. The proof of Theorem 3.4 and Theorem 3.6 combines the strength of Plünnecke's inequality and nonstandard methods. In the proof of Theorem 3.4, we apply Plünnecke's inequality to the segments of ${}^*A$ in a hyperfinite interval $[0,n]$. Since the set $A_n = {}^*A \cap [0,n]$ is hyperfinite, it is easy for us to make small adjustments in order to fit the condition required by Plünnecke's inequality. Because of this, a would–be long ϵ–δ argument becomes a very straightforward argument.

The proof of Theorem 3.6 requires the use of Theorem 3.3. Let $BD(A) = \alpha$. Theorem 3.3 shows that there are arbitrarily long intervals $[a,b]$ such that the set ${}^*A \cap [a,b]$ for each such interval $[a,b]$ is homogeneously distributed. Now Theorem 3.6 follows from this homogeneity and Plünnecke's inequality.

References

[BJ] P. Bihani and R. Jin, *Kneser's theorem for upper Banach density*, Journal de theorie des nombres de Bordeaux, vol 18 (2006), no 2, pp. 323–343.

[Es] T. Estermann, *On Goldbach's problem: proof that almost all even positive integers are sums of two primes*, Proceedings of London Mathematical Society, 44 (1938), pp. 307–314.

[Fu] H. Furstenberg, *Recurrence in Ergodic Theory and Combinatorial Number Theory*, Princeton University Press, 1981.

[HR] H. Halberstam and K. F. Roth, *Sequences*, Oxford University Press, 1966

[Ji1] R. Jin, *Sumset phenomenon, The Proceedings of American Mathematical Society*, Vol. 130, No. 3 (2002), pp. 855–861.

[Ji2] R. Jin, *Nonstandard methods for upper Banach density problems*, The Journal of Number Theory, 91 (2001), pp. 20–38.

[Ji3] R. Jin, *Standardizing nonstandard methods for upper Banach density problems*, in the DIMACS series *Unusual Applications of Number Theory*, edited by M. Nathanson, Vol. 64 (2004)

[Ji4] R. Jin, *Inverse problem for upper asymptotic density*, The Transactions of American Mathematical Society, 355 (2003), No. 1, pp. 57–78.

[Ji5] R. Jin, *Solution to the inverse problem for upper asymptotic density*, Journal für die reine und angewandte Mathematik (Crelle's Journal), 595 (2006), pp. 121–166.

[Ji6] R. Jin, *Inverse problem for cuts, Logic and Analysis*, 1, No. 1 (2007), pp. 61–89.

[Ji7] R. Jin, *Pinning down the structure of $A+B$ when $A+B$ has small upper Banach density*, http://jinr.people.cofc.edu/research/publication.html

[Ji8] R. Jin, *Plünnecke's Theorem for asymptotic densities*, http://jinr.people.cofc.edu/research/publication.html

[JK] R. Jin and H. J. Keisler, *Abelian group with layered tiles and the sumset phenomenon*, The Transactions of American Mathematical Society, Vol. 355 (2003), No. 1, pp. 79–97.

[Kh] A. I. Khinchin, *Three pearls of number theory*, Translated from the 2d (1948) rev. Russian ed. by F. Bagemihl, H. Komm, and W. Seidel, Rochester, N.Y., Graylock Press, 1952.

[Na1] M. B. Nathanson, *Additive Number Theory–The Classical Bases*, Springer, 1996.

[Na2] M. B. Nathanson, *Additive Number Theory–Inverse Problems and the Geometry of Sumsets*, Springer, 1996.

[Pl] H. Plünnecke, *Eine zahlentheoretische Anwendung der Graphentheorie*, Journal für die reine und angewandte Mathematik, **234** (1970), 171–183.

Department of Mathematics, College of Charleston, Charleston, South Carolina, 29424

E-mail address: jinr@cofc.edu