Weighted projective space bundles

Plan 1) Introduce w.p.s., graded rings & hypersurfaces in w.p.s.

2) Develop machinery over a smooth curve, multigraded rings
Cox coordinates

3) Some applications

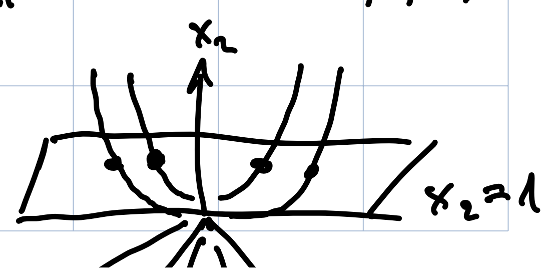
Have some fun computations

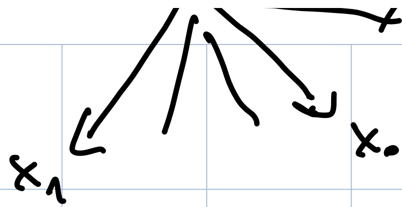
$/\mathbb{C}$ $\quad a_0, \dots, a_n$ positive integers - weights

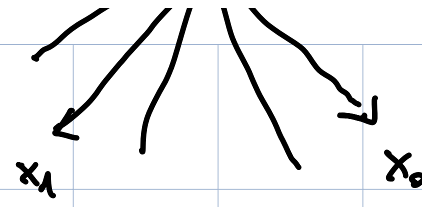
w.p.s. $\quad \mathbb{P}^n(a_0, \dots a_n) := (\mathbb{A}^{n+1} - 0)/\mathbb{G}_m$

$$\lambda \cdot (x_0, \dots x_n) = (\lambda^{a_0} x_0, \dots, \lambda^{a_n} x_n)$$

Examples $\quad \mathbb{P}^n(1, \dots 1) = \mathbb{P}^n \qquad \mathbb{P}^2(1,1,2)$







Set theoretically one point of $\mathbb{P}^n(\underline{a})$

$\Updownarrow$

one orbit of $\mathbb{G}_m$-action

$\mathbb{P}^2(1,1,2)$ image of morphism

$$\varphi : \mathbb{A}^3 \setminus 0 \longrightarrow \mathbb{P}^3$$
$$(x_0, x_1, x_2) \longmapsto (\underline{x_0^2}, \underline{x_0 x_1}, \underline{x_1^2}, \underline{x_2})$$

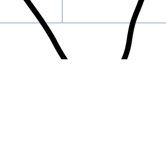
image of φ $\cong \mathbb{P}(1,1,2)$

$\mathbb{G}_m$-orbits in $\mathbb{A}^3 \setminus 0$ $\qquad$ φ well-defined on

$(\alpha_0, \alpha_1, \alpha_2) \in \mathbb{A}^3$ $\qquad$ $\varphi(\underline{\alpha}) = \varphi(\lambda \cdot \underline{\alpha}) \ \forall \lambda \in \mathbb{G}_m$

choose coords $z_0 = x_0^2$ $z_1 = x_0 x_1$, $z_2 = x_1^2$
$z_2 = x_2$

image$(\varphi) \subset \mathbb{P}^3_{z_i}$ $\qquad$ $(z_0 z_2 = z_1^2)$



$\mathbb{G}_m$-orbit $\quad$ x_2-axis in $\mathbb{A}^3$

$X \longleftarrow$ maps to vertex

3 affine charts covering $\mathbb{P}(1,1,2)$

$D(x_0) \cong \mathbb{A}^2 \left(\frac{x_1}{x_0}, \frac{x_2}{x_0^2} \right) = \left(\frac{z_1}{z_0}, \frac{z_3}{z_0} \right)$

$D(x_1) \cong \mathbb{A}^2$ similar

$D(x_2) \subset \mathbb{A}^3 \left(\frac{x_0^2}{x_2}, \frac{x_0 x_1}{x_2}, \frac{x_1^2}{x_2} \right)$ $\frac{z_i}{z_3}$ in

$\frac{z_0}{z_3}, \frac{z_1}{z_3}, \frac{z_2}{z_3}$

Graded rings $\quad R = \bigoplus_{k \geq 0} R_k, \quad R_0 = \mathbb{C}$

R f.g. $/\mathbb{C}$ generators have degrees $a_i > 0$

mult. $R_p \times R_q \longrightarrow R_{p+q}$

$$R \cong \mathbb{C}[x_0, \dots, x_n] / I$$

I homogeneous ideal $\quad I = \bigoplus_{k \geq 0} I_k$

$\forall f = \sum f_k \quad f_k \in I_k$

$M := \bigoplus R_k$ irrelevant ideal

$k > 0$ $\qquad I \supset M^k$

$\mathbb{G}_m$-action on R $\qquad \lambda \cdot R_k$ is given by $\lambda^k \cdot f$

$f \in R$ is weighted homogeneous of degree k

iff $\quad f \in R_k$

iff $\quad f(\lambda^{a_0} x_0, \ldots, \lambda^{a_n} x_n) = \lambda^k \cdot f(x_0, \ldots, x_n)$

<u>Example</u> $\quad R = \mathbb{C}[\underset{1}{x_0}, \underset{1}{x_1}, \underset{2}{x_2}]$

$$R_1 = \langle x_0, x_1 \rangle_{\mathbb{C}}$$

$$R_2 = \langle x_0^2, x_0 x_1, x_1^2, x_2 \rangle_{\mathbb{C}}$$

$$R \not\cong \operatorname{Sym} R_1$$

Proj of a polynomial ring

$$S = \mathbb{C}[x_0, \ldots, x_n] \qquad \deg x_i = a_i$$

$$\operatorname{Proj} S = \mathbb{P}(a_0, \ldots, a_n)$$

principal affine charts on Proj S

$f \in R_k$, $D(f) = \left(\mathbb{A}^{n+1} \setminus (f=0)\right) / \mathbb{G}_m$

$$\mathcal{O}_{D(f)} = S\left[\frac{1}{f}\right]^{(0)} = \left\{ \frac{g}{f^{l}} \,\middle|\, g \in S_{kl} \right\}$$

in particular

$$\mathcal{O}_{D(x_i)} = \left\{ \frac{g}{x_i^{\,l}} \,\middle|\, g \in S_{a_i l} \right\}$$

Eg $\mathbb{P}(1,1,2)$

$$\mathcal{O}_{D(x_2)} \cong \mathbb{C}\left[\frac{x_0^2}{x_2}, \frac{x_0 x_1}{x_2}, \frac{x_1^2}{x_2} \right]$$

Proposition The chart $D(x_j)$ in $\mathbb{P}(\underline{a})$ is isomorphic to $\mathbb{A}^n / (\mathbb{Z}/a_j)$

action is by
$$\begin{pmatrix} \varepsilon^{a_0} & & & & 0 \\ & \ddots & & & \\ & & \widehat{\varepsilon^{a_j}} & & \\ & & & \ddots & \\ 0 & & & & \varepsilon^{a_n} \end{pmatrix}$$

ε a primitive a_j th root of unity.

Notation $\quad D(x_j) \cong \frac{1}{a_j}(a_0, \dots \hat{a}_j, \dots, a_n)$

eg $\quad \mathbb{P}(1,1,2) \quad D(x_2) \cong \frac{1}{2}(1,1)$ A_1 surf sing

$$\frac{1}{a_j}(a_0, \dots a_n) = \frac{1}{a_j}(\bar{a}_0, —, \bar{a}_n) \quad \bar{a}_0 \equiv a_0 \ (a_j)$$

$$\frac{1}{3}(1,1,2) \cong \frac{1}{3}(2,2,1)$$

$$P \qquad \frac{1}{4}(2,2,2)$$

Proof $\quad \mathcal{O}_{D(x_j)}$ gen'd Laurent monomials

$$\frac{\prod_i x_i^{m_i}}{x_j^{\sum a_i m_i / a_j}} \qquad \underline{\underline{\sum a_i m_i \equiv 0 \ (a_j)}}$$

Alternative define $\quad z_i := \dfrac{x_i}{x_j^{a_i/a_j}} \quad i \neq j$

$$\prod z_i^{m_i} \qquad \sum a_i m_i \equiv 0 \ (a_j)$$

$$\mathcal{O}_{D(x_j)} \cong \mathbb{C}[z_0, \dots, z_n]^{\mathbb{Z}/a_j}$$ $\square$

<u>Ex</u> $\mathbb{P}(1,1,2,3)$

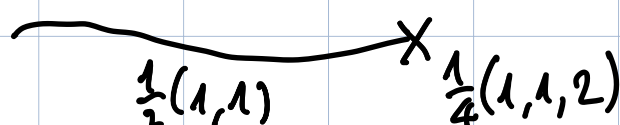
$D(x_2) \cong \frac{1}{2}(1,1,1) \leftarrow$ sing

$D(x_3) \cong \frac{1}{3}(1,1,2) \leftarrow$ sing

$\mathbb{P}(1,1,2,4)$

$D(x_2) \cong \frac{1}{2}(1,1,0)$ curve of $\frac{1}{2}(1,1)$

$D(x_3) \cong \frac{1}{4}(1,1,2)$ $\frac{1}{4}(1,1,2)$ at origin



dissident points

Ex $\mathbb{P}(2,3,6)$

$D(x_1) \cong \frac{1}{2}(1,0)$ $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$\mathbb{C}[z_0, z_1]^{\mathbb{Z}/2} \cong \mathbb{C}[z_0^2, z_1]$

$D(x_1) = \mathbb{A}^2$.

$\frac{1}{2}(1,0)$ has quasireflections — action fixes a hyperplane

Truncations & veronese embeddings.

$$S^{[d]} := \bigoplus_{k \geq 0} S_{kd}$$

$$S^{[d]}_k := S_{kd}$$

$S^{[d]}_k \ni f$ has degree k

<u>Proposition</u> $\operatorname{Proj} S^{[d]} \cong \operatorname{Proj} S$

eg $\mathbb{C}[\underset{1}{x_0}, \underset{1}{x_1}, \underset{2}{x_2}]^{[2]} \cong \mathbb{C}[x_0^2, x_0x_1, x_1^2, x_2,$

<u>Pf</u> $\mathcal{O}_{\operatorname{Proj} S, D(f)} = \left\{ \frac{g}{f^m} \,\middle|\, g \in S_{mk} \right\}$

$$\mathcal{O}_{\operatorname{Proj} S^{[d]}, D(f^d)} = \left\{ \frac{h}{(f^d)^\ell} \,\middle|\, h \in S^{[d]}_{k\ell} \right\}$$

$$\mathcal{O}_{\operatorname{Proj} S, D(f)} = \mathcal{O}_{\operatorname{Proj} S^{[d]}, D(f^d)}$$

$\supseteq$ easy

$$\cong \quad \frac{g}{f^m} = \frac{g f^{dl-m}}{f^{dl}} = \frac{h}{f^{dl}} \qquad \square$$

<u>Corollary</u> 1) $\mathbb{P}(da_0, \ldots, da_n) \cong \mathbb{P}(a_0, \ldots, a_n)$

2) $\mathbb{P}(a_0, da_1, \ldots, da_n) \cong \mathbb{P}(a_0, \ldots, a_n)$

1) $$S = \underset{da_0 \quad\quad da_n}{\mathbb{C}[x_0, \ldots, x_n]} \cong \underset{a_0 \quad\quad a_n}{\mathbb{C}[x_0, \ldots, x_n]} \cong S^{[d]}$$

2) $$\mathbb{C}[x_0^d, x_1, \ldots, x_n]$$

Given $\mathbb{P}(b_0, - b_n)$ $\exists$ $\mathbb{P}(a_0, - a_n)$

isomorphic s.t. $(a_0, - a_n) = 1$

$(a_0, - \hat{a_j}, - a_n) = 1 \quad \forall j$

well-formed w.p.s.

$$\mathbb{P}(2, \underline{3, 6}) \cong \mathbb{P}(2, \underbrace{1, 2}) \cong \mathbb{P}^2$$

$\Pi(a, b) = 1$

guarantees that $\mathbb{P}(\underline{a})$ has singularities in codim ≥ 2

§ Hypersurfaces in w.p.s.

$$R = \mathbb{C}[\underset{\substack{\uparrow \\ a_0}}{x_0} - \underset{\substack{\uparrow \\ a_n}}{x_n}] / F_d \leftarrow \text{wtd homog.}$$

a_i well-formed.

Ex

$$X(10) \subset \mathbb{P}(1,1,2,5)$$
$$X(6) \subset \mathbb{P}(1,2,3)$$
$$X(7) \subset \mathbb{P}(1,1,1,3)$$

punctured affine cone over X

$$CX^* = \pi^{-1}(X)$$

$$\pi : \mathbb{A}^{n+1} \setminus 0 \longrightarrow \mathbb{P}(\underline{a})$$

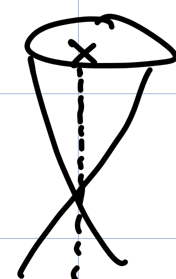
X is <u>quasismooth</u> if CX^* is nonsing.

X is <u>well-formed</u> if $X \cap \operatorname{Sing} \mathbb{P}(\underline{a})$ has codim 2 in X.

quasismooth means "X inherits its sings. from $\mathbb{P}(\underline{a})$"

Ex $\mathbb{P}(1,1,2) \supset X : (x_0 = 0)$

curve <u>not</u> well-formed b/c meets $(0:0:1)$



nonsingular despite meeting the vertex.

$X(10) \subset \mathbb{P}(1,1,2,5)$

$$x_0^{10} + x_1^{10} + x_2^5 + x_3^2 = 0$$

quasismooth — partial derivatives

well-formed — $\operatorname{Sing} \mathbb{P}(\underline{a})$ is 0-diml.

X is smooth b/c $(0:0:1:0)$ & $(0:0:0:1) \notin X$

Eg $\quad X(7) \subset \mathbb{P}(1,1,1,3)$

$$0 = \underline{\underline{x_0 x_3^2}} + f_4(x_1, x_2)x_3 + g_7(x_0, x_1, x_2)$$

X meets $\frac{1}{3}(1,1,1)$ $\quad (0:0:0:1)$

locally $D(x_3)$ $\quad z_0, z_1, z_2$

$$z_0 + f_4(z_1, z_2) + g_7(z_0, z_1, z_2)$$

$/\mathbb{Z}/3$

local analytically eliminates z_0

X has singularity $\cong \frac{1}{3}(1,1)$

§ Differential forms & adjunction.

Euler sequence

$$\boxed{0 \to \Omega_{\mathbb{P}(\underline{a})} \to \bigoplus_i \mathcal{O}(-a_i) \xrightarrow{\varphi} \mathcal{O} \to 0}$$

$$\bigoplus S\, dx_i \longrightarrow S$$
$$(f_i dx_i)_{i=1}^n \longrightarrow \sum f_i x_i$$

$$S\, dx_i \longrightarrow S$$
$$(f_i\, dx_i)_{i=1}^n \longrightarrow \sum a_i f_i x_i$$

Euler relation F homogeneous

$$\deg F \cdot F = \left(\sum_i x_i \frac{\partial}{\partial x_i}\right) F$$

weighted version

$$\deg F \cdot F = \left(\sum_i a_i x_i \frac{\partial}{\partial x_i}\right) F$$

$$x_0^6 + x_1^3 + x_2^2$$

Dualising sheaf

$$\omega_{\mathbb{P}(\underline{a})} = \mathcal{O}_{\mathbb{P}(\underline{a})}\left(-\sum a_i\right)$$

Adjunction for $X(d) \subset \mathbb{P}(\underline{a})$

$$\omega_X = \mathcal{O}_X(d - \sum a_i)$$

What is $\mathcal{O}_{\mathbb{P}(\underline{a})}(1)$?

$$S = \mathbb{C}[x_0, \dots x_n]$$

$$S(k) = \mathbb{C}[x_0 - x_n](k)$$

$$= \bigoplus_{l \geq 0} S_{k+l}$$

$$\mathcal{O}(k) = \widetilde{S(k)}$$

$\mathbb{P}(1,1,2)$ $\qquad$ $\mathcal{O}(1)$ assoc to ruling

$\mathcal{O}(2)$ is a line bundle.

$$H^0(\mathcal{O}(k)) \cong S_k$$

$$H^0(\mathbb{P}(1,1,2), \mathcal{O}(1)) = \langle x_0, x_1 \rangle$$

$$, \mathcal{O}(2)) = \langle x_0^2, x_0x_1, x_1^2, x_2 \rangle$$

$\mathbb{P}(1,3,4,5)$, $\mathcal{O}(2)$ has just one section!

$\mathcal{O}(1) \otimes \mathcal{O}(1) \cong \mathcal{O}(2)$

<u>Example</u> $C(6) \subset \mathbb{P}(1,2,3)$

curve, quasismooth, smooth

$$\omega_C = \mathcal{O}_C(6-1-2-3) = \mathcal{O}_C$$

elliptic curve.

$$x_2^2 = x_1^3 + a_4(x_0)x_1 + b_6(x_0)$$

Weierstrass normal form for an elliptic curve

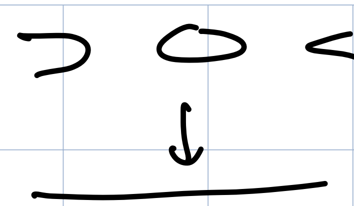
2nd veronese embedding:

$$\mathbb{C}[x_0^2, x_1, x_0x_2]$$

$$\begin{array}{ccc} 1 & 1 & 2 \\ y_0 & y_1 & y_2 \end{array}$$

$$y_2^2 = F_4(y_0, y_1) \subset \mathbb{P}(1,1,2)$$

$$\downarrow$$
$$\mathbb{P}^1_{y_0, y_1}$$



3rd veronese embedding

plane cubic.

$$\mathcal{O}(k)^\vee \overset{?}{=} \mathcal{O}(-k)$$

$$\mathcal{H}om(\mathcal{O}(k), \mathcal{O}) = \mathcal{O}(k)^\vee$$

Weil divisorial sheaf

Last time $\quad \mathbb{P}(\underline{a}) = (\mathbb{A}^{n+1} \setminus 0)/\mathbb{G}_m$

$\lambda \in \mathbb{G}_m \quad \lambda \cdot \underline{x} = (\lambda^{a_0} x_0, \dots, \lambda^{a_n} x_n)$

Can assume
$$\left.\begin{array}{l} (a_0, \dots, a_n) = 1 \\ (a_0, \dots \hat{a_j}, \dots a_n) = 1 \end{array}\right\} \quad \mathbb{P}(\underline{a}) \text{ well-formed}$$

$\mathbb{P}(\underline{a})$ has quotient sings in codim 2

$$\frac{1}{a_j}(a_0, \dots \hat{a_j}, \dots a_n) \qquad \mathbb{A}^n/(\mathbb{Z}/a_j)$$

$\mathbb{P}(\underline{a}) = \operatorname{Proj} S \qquad S = \bigoplus S_k \quad \mathbb{C}[x_0 - x_n]$

$\deg x_0 = a_0$

$X(d) \subset \mathbb{P}(\underline{a})$ def. by wtd hom. poly.

quasismooth $\qquad CX^*$ smooth

well-formed $\qquad \operatorname{codim}_X (X \cap \operatorname{Sing} \mathbb{P}(a)) \geq 2$

X inherits sings from $\mathbb{P}(\underline{a})$

$X(7) \subset \mathbb{P}(1,1,1,3) \qquad \frac{1}{3}(1,1)$

sheaves $\mathcal{O}(k)$ not invertible, rk 1

$\omega_{\mathbb{P}} = \mathcal{O}_{\mathbb{P}(\underline{a})}(-\sum a_i)$, $\omega_X = \mathcal{O}_X(d - \sum a_i)$

$X(d) \subset \mathbb{P}(\underline{a})$ study

Philosophy: abstract variety X + ample Weil divisor D

$\longrightarrow$ graded ring $R = \bigoplus_{n \geq 0} H^0(X, nD)$

$H^0(X, nD) =$ gl. sections $\mathcal{O}_X(nD)$

$\mathcal{O}_X(nD) =$ rational functions on X with poles at worst along nD

R fin gen. $\longrightarrow$ choose gen's $R \cong \mathbb{C}[x_0, \dots, x_n]/I$

$\leadsto X \subset \mathbb{P}(a_0, \dots, a_n)$

Example $C(6) \subset \mathbb{P}(1, 2, 3)$

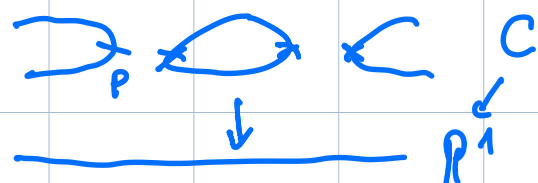
$$\omega_C = \mathcal{O}_C(6-1-2-3) = \mathcal{O}_C$$

$$x_2^2 = x_1^3 + a x_0^4 x_1 + b x_0^6$$

nonsingular elliptic curve.

C elliptic curve

P ramification pt



$\mathcal{O}_C(2P) = g^1_2$ $\quad H^0(\mathcal{O}(2P))$ give map to $\mathbb{P}^1$

$$R = \bigoplus_{n \geq 0} H^0(C, nP)$$

Riemann-Roch + vanishing $\quad H^0(C, nP) = n$ $\quad n \geq 1$

Castelnuovo bpf pencil trick

$$H^0(C, 2P) \times H^0(C, mP) \twoheadrightarrow H^0((m+2)P)$$

for all $m \geq 3$

n	$H^0(nP)$	
1	x_0	pole of order 2 @ P
2	x_0^2, x_1	$\leftarrow g^1_2$ $\quad x_0^2, x_1$ coords on $\mathbb{P}^1$

3	$\underbrace{x_0^3,\ x_0x_1}$, x_2 pole of order 3
4	$x_0^4,\ \underset{2}{\underline{x_0^2x_1}},\ \underset{4}{\underline{x_1^2}},\ x_0x_2$ pole order 3
6	$x_0^6,\ x_0^4x_1,\ x_0^2x_1^2,\ x_1^3,\ x_0^3x_2,\ x_2^2$
	$x_0x_1x_2$

$x_2 + * x_0^3 + * x_0x_1$

$x_1 + x_0^2 \qquad X_2^2 = x_1^3 + \qquad ! \qquad \mathbb{P}(1,2,3)$

In general hyperelliptic curves

$C \xrightarrow{2:1} \mathbb{P}^1 \qquad C(4g+2) \subset \mathbb{P}(1,2,2g+1)$

$g = \text{genus}(C)$

www.grdb.co.uk

Brown-Kasprzyk et al

§ Proj. space bundles over $\mathbb{P}^1$ ← (B smooth variety)

$t_0 \ t_1 \quad x_0 \quad x_1 \quad \cdots \quad x_n$

$$\left(\begin{array}{cc|cccc} 1 & 1 & -b_0 & -b_1 & \cdots & -b_n \\ 0 & 0 & a_0 & a_1 & \cdots & a_n \end{array}\right)$$

a_i coprime & $(n-1)$ of a_i coprime

$b_i \in \mathbb{Z}_{\geq 0}$

$$(\mathbb{A}^2 \setminus 0) \times (\mathbb{A}^{n+1} \setminus 0) \big/ (\mathbb{G}_m)^2 =: \mathbb{F}\left(\frac{\underline{b}}{\underline{a}}\right)$$

$$(\lambda, \mu) \cdot (\underline{t} ; \underline{x}) = (\lambda t_0, \lambda t_1 ; \lambda^{-b_0} \mu^{a_0} x_0 - \lambda^{-b_n} \mu^{a_n} x_n)$$

- $(t_0 : t_1)$ gives $\mathbb{F} \longrightarrow \mathbb{P}^1$ fibration
- Fix $(t_0 : t_1) = (\alpha_0 : \alpha_1)$ use $\mathbb{G}_m$-action of λ to scale $\alpha_1 = 1$

fibre over $(\alpha : 1)$ is $\cong \mathbb{P}(\underline{a})$

Example

0) $b_0 = b_1 = \cdots = b_0$

$$\mathbb{F} = \mathbb{P}^1 \times \mathbb{P}(\underline{a})$$

1)

1) $a_0 = \dots = a_n = 1$

$\mathbb{F} = \mathbb{P}^n\text{-bundle}/\mathbb{P}^1$

$\mathbb{P}_{\mathbb{P}^1}\left(\mathcal{O}_{\mathbb{P}^1}(b_0) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^1}(b_n)\right)$

Segre-Hirzebruch scroll.

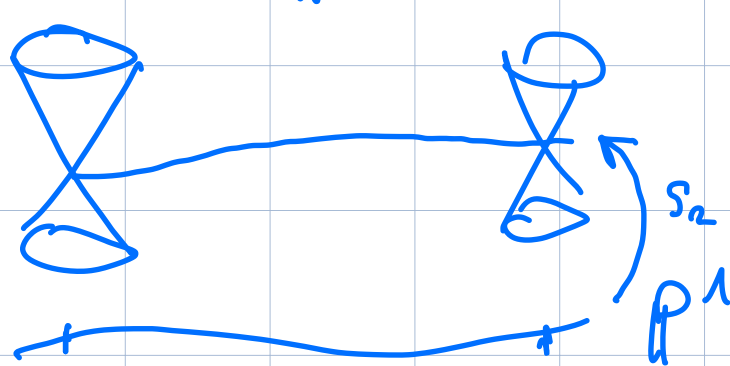
2) $\begin{pmatrix} 1 & 1 & | & 0 & -1 & -1 \\ 0 & 0 & | & 1 & 1 & 2 \end{pmatrix}$ $\quad \mathbb{P}(1,1,2)$ bundle

$\begin{array}{cccccc} & & & x_0 & x_1 & x_2 \end{array}$

"veronese" $\begin{pmatrix} 1 & 1 & | & 0 & -1 & -2 & -1 \\ 0 & 0 & | & 1 & 1 & 1 & 1 \end{pmatrix} \supset (z_0 z_2 = z_1^2)$

$\begin{array}{ccccccc} & & & x_0^2 & x_0x_1 & x_1^2 & x_2 \end{array}$

$\mathbb{P}^3\text{-bundle}/\mathbb{P}^1 \supset (z_0 z_2 = z_1^2)$



charts

$D(x_0, t_0) \cong \mathbb{A}^1_{t_1} \times \mathbb{A}^2(t_0 x_1, t_0 x_2)$

$$D(x_2, t_0) \subseteq \mathbb{A}^1_{\frac{t_1}{t_0}} \times \mathbb{A}^3\left(\frac{x_0^2}{t_0 x_2}, \frac{x_0 x_1}{x_2}, \frac{t_0 x_1^2}{x_2}\right)$$

§ Bigraded rings

$$S = \bigoplus_{(k,l)\in\mathbb{Z}^2} S_{k,l} \qquad S_{k,l} \times S_{m,n} \to S_{k+m,\, l+n}$$

$$\| $$

$$\mathbb{C}[t_0, t_1, x_0, \dots, x_n] = \mathbb{C}[\underline{t}][\underline{x}]$$

$$t_0 \in S_{1,0} \qquad x_i \in S_{-b_i, a_i}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} -b_i \\ a_i \end{pmatrix}$$

bihomogeneity defined in usual way

$$(\lambda, \mu) . f = \lambda^k \mu^l f \iff f \in S_{k,l}$$

$$S_{\bullet, 0} = \mathbb{C}[t_0, t_1].$$

$$\uparrow$$

$$\mathbb{C}[t_0, t_1, t_2]/$$

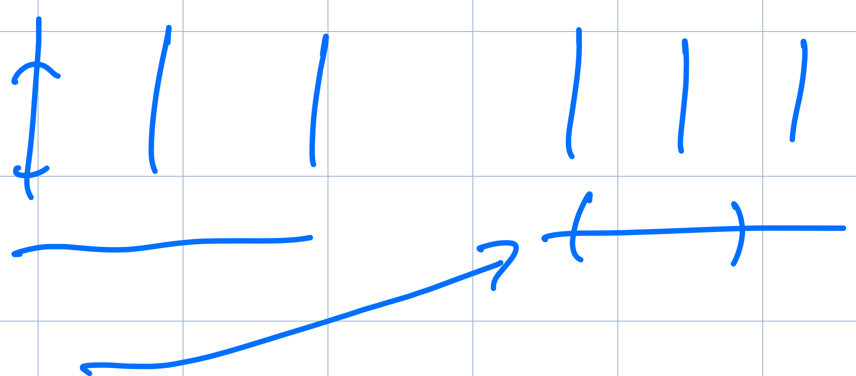
$/(g)$

$$\begin{array}{ccccc} 1 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 2 \\ \hline t_0 & t_1 & x_0 & x_1 & x_2 \end{array}$$

$$S_{0,1} = x_0,\ t_0x_1,\ t_1x_1$$

$$S_{0,2} = x_0^2,\ t_0x_0x_1,\ t_1x_0x_1,\ (t_0^2, t_0t_1, t_1^2)x_1^2,\ t_0x_2,\ t_1x_2$$

can look at charts in different ways



$$\left(\begin{array}{cc|ccc} 1 & 1 & -b_0 & \dots & -b_n \\ 0 & 0 & a_0 & — & a_n \end{array}\right)$$

$D(t_0)$

$$\mathcal{O}_{D(t_0)} = S\left[\frac{1}{t_0}\right]^{(0,\cdot)} = \left\{ \frac{f}{t_0^k} \,\middle|\, f \in S_{k,\cdot} \right\}$$

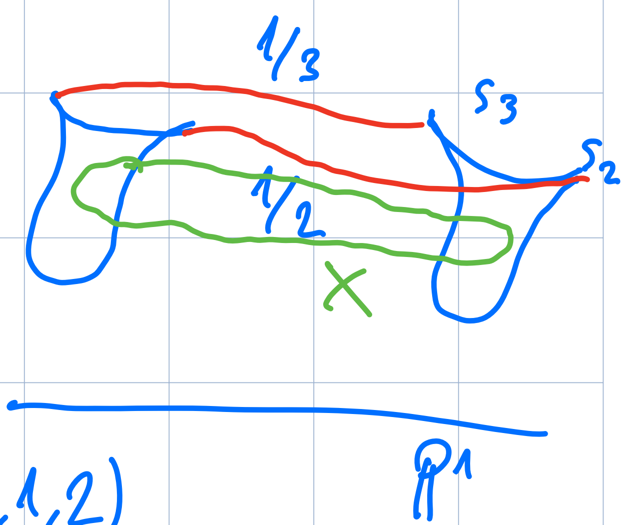
generators $\frac{t_1}{t_0}, \; t_0 x_0, \ldots, t_0 x_n$

$$\left(\mathbb{A}^1 \times \mathbb{P}^n(\underline{a})\right)$$

$$\mathrm{Proj}_{\mathbb{P}^1} S$$

$$\left(\begin{array}{cc|cccc} 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & 1 & 2 & 3 \end{array}\right)$$

$\frac{1}{2}(1,1,1)$ $\quad$ $\frac{1}{3}(1,1,2)$



We have sheaves

$$\mathcal{O}_{\mathbb{F}}(m) = \widetilde{S(m)}$$

$$S(m) = \bigoplus_{k,l} S_{k,l+m}$$

$$H^0(\mathcal{O}_{\mathbb{F}}(m)) \cong S_{0,m}$$

$$\left(\begin{array}{cc|ccc} 1 & 1 & 0 & -1 & -1 \\ \ldots \end{array}\right)$$

$$H^0(\mathcal{O}(1)) = x_0,\ t_0x_1,\ t_1x_1$$

$$H^0(\mathcal{O}(2)) = x_0^2,\ (1)x_0x_1,\ (2)x_1^2,\ (1)x_2$$

$\mathcal{O}_{\mathbb{F}}(n)$ usually not a line bundle.

$$\eta := \pi^*\mathcal{O}_{\mathbb{P}^1}(1) = \widetilde{S(1,0)}$$

$H^0(\eta) = t_0, t_1$ is a line bundle

§ Hypersurface in a $\mathbb{P}(1,1,2,3)$-bundle

$$\mathbb{F} : \begin{array}{c} \left(\begin{array}{cc|cccc} 1 & 1 & -2 & -3 & \boxed{0} & \boxed{0} \\ 0 & 0 & 1 & 1 & 2 & 3 \end{array}\right) \\ \begin{array}{cccccc} t_0 & t_1 & x_0 & x_1 & x_2 & x_3 \end{array} \end{array} \supset X\begin{pmatrix} 0 \\ 6 \end{pmatrix}$$

equation $\quad x_3^2 = x_2^3 + \underbrace{(18)}_{\alpha_{18}}x_1^6 + \underbrace{(12)}_{\beta_{12}}x_0^6$

$$\begin{tikzcd} \mathbb{F} \arrow[dr] & \\ X \arrow[u, phantom, "\cup" rotate=90] \arrow[r] & \mathbb{P}^1 \end{tikzcd}$$

Sings of X $\longleftarrow$ partial derivatives do not vanish simult. on X

CX^* is nonsingular

$\cap$

$(\mathbb{A}^2 \setminus 0) \times (\mathbb{A}^n \setminus 0)$

$$S_2 : \quad \mathbb{P}^1 \times \begin{array}{c} x_1 \\ (0\ 0\ 1\ 0) \end{array}$$

$$S_3 : \quad \mathbb{P}^1 \times \begin{array}{c} x_2 \\ (0\ 0\ 0\ 1) \end{array}$$

S_0 $\quad X \cap S_2 = X \cap S_3 = 0$

§ Canonical sheaf & adjunction

$$\omega_{\mathbb{F}} = \mathcal{O}_{\mathbb{F}}\left(\left(\sum b_i - 2\right)M\right)\left(-\sum a_i\right)$$

x-dir

$$\begin{pmatrix} 1 & 1 & -b_0 & \cdots & -b_n \\ 0 & 0 & a_0 & \cdots & a_n \end{pmatrix}$$

1st

– sum of columns

2nd

$X\begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$

$$\omega_X = \mathcal{O}_{\mathbb{F}}\big((\textstyle\sum b_i - 2 + d_1)M\big)(d_2 - \textstyle\sum a_i)$$

Ex — elliptic K3 surface

$$\begin{array}{ll} X & \text{K3} \qquad \omega_X = \mathcal{O}_X \\ \downarrow & \text{fibres ell. curve.} \\ \mathbb{P}^1 & \end{array}$$

choose P

$C(6) \subset \mathbb{P}(1,2,3)$

2

$$\left(\begin{array}{cc|ccc} 1 & 1 & -* & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \end{array}\right) \supset \begin{pmatrix} 0 \\ 6 \end{pmatrix}$$

$$\omega_X = \mathcal{O}_X\big((* - 2)M\big)(6-1-2-3)$$

nonsingular surface

$x_0 = 0$

$$x_2^2 = x_1^3 + \boxed{\alpha_8 x_0^4 x_1 + \beta_{12} x_0^6}$$

$\mathbb{P}^1_x(0:1:1)$

$\Delta \; (\alpha^3 \, , \, \beta^2)$ deg 24

$$\begin{tikzcd} X \arrow[r] & \mathbb{P}^1 \arrow[l, bend left, "\sigma"] \end{tikzcd}$$ has 24 singular fibres

How can we constr. $X \subset \mathbb{F}$ without guessing?

$X \to \mathbb{P}^1$ ell. K3 w/ section σ

build bigraded ring

$$\bigoplus H^0(kM + \ell\sigma)$$

$M = \pi^* \mathcal{O}_{\mathbb{P}}(1)$, σ = section

$$\sigma \longrightarrow \mathcal{O}_{\mathbb{F}}(1)\big|_X$$

$$\begin{tikzcd} \mathbb{F}\left(\begin{array}{cc|ccc} 1 & 1 & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & 2 & 3 \\ & & x_0 & x_1 & x_2 \end{array}\right) \arrow[d, "\pi"'] & \supset \; (6) \; X \arrow[d] \\ \mathbb{P}^1 & \mathbb{P}^1 \end{tikzcd}$$

" $\pi_* \mathcal{O}_{\mathbb{F}}(1), \ \mathcal{O}_{\mathbb{F}}(2), \ \mathcal{O}_{\mathbb{F}}(3)$ "

$$H^0(\mathcal{O}_{\mathbb{F}}(1)) = (\cdot)\, x_0 \ \longleftarrow$$

$$\big\downarrow {\scriptstyle \times 2}$$

$$H^0(\mathcal{O}_{\mathbb{F}}(2)) = (\cdot)\, x_0^2 \,, \ (\cdot)\, x_1$$

$$\pi_* \mathcal{O}_{\mathbb{F}}(1) = \mathcal{O}_{\mathbb{P}^1}(b_0)$$

$$\pi_* \mathcal{O}_{\mathbb{F}}(2) = \mathcal{O}_{\mathbb{P}^1}(2b_0) \oplus \mathcal{O}_{\mathbb{P}^1}(b_1)$$

$$\pi_* \mathcal{O}_{\mathbb{F}}(3) = \underset{x_0^3}{\mathcal{O}_{\mathbb{P}^1}(3b_0)} \oplus \underset{x_0 x_1}{\mathcal{O}_{\mathbb{P}^1}(b_0+b_1)} \oplus \underset{x_2}{\mathcal{O}_{\mathbb{P}}(b_2)}$$

What do we know about $\pi_*(n\sigma)$?

RR fibre & RR for K3

$$\operatorname{rank} \pi_*(n\sigma) = n$$

$$\dim H^0(\pi_*(n\sigma)) = 2 + n^2$$

n	$\dim$	$\pi_* \mathcal{O}(n\sigma)$

1	3	$\mathcal{O}(2)$	$\leftarrow b_0 = 2$
2	$6 \leftarrow$	$\mathcal{O}(4) \oplus \mathcal{O}$	$\leftarrow b_1 = 0$
3	11	$\underset{x_0^3}{\mathcal{O}(6)} \oplus \underset{x_0 x_1}{\mathcal{O}(2)} \oplus \mathcal{O}$	$\leftarrow b_2 = 0$

$$\mathbb{P}(2, \ldots \qquad (\mathbb{A}^n - H)/\mathbb{G}_m$$

$$1 + t + 2t^2 + 3t^3 + 4t^4 + \ldots$$

$$\frac{1 - t^6}{(1-t)(1-t^2)(1-t^3)}$$

$$0 \to \underline{R(-6)} \to R \to R/f \to 0$$

$$\frac{1}{(1-t)(1-t^2)(1-t^3)} \leftarrow \frac{1}{(1-x_0)(1-x_1)(1-x_2)}$$

References for w.p.s.

I. Dolgachev Weighted proj. vars ~'70

A. Iano-Fletcher ~'00

Cox-Little-Schenck

jt R. Pignatelli

3-dim'l alg var. of general type

$\omega_X \sim K_X$ ample

$$\bigoplus_{n\geq 0} H^0(X, nK_X) \sim \text{canonical model of } X$$

$$\underbrace{\dim H^0}_{h^0}(X, nK_X) \sim O(n^3)$$

$$\mathrm{vol}(X) := \limsup_{m\to\infty}\left(h^0(mK_X)\cdot\frac{3!}{m^3}\right)$$

$$p_g(X) = h^0(X, K_X)$$

$\mathrm{vol}(X) \in \mathbb{Q}_{>0} \leftarrow$ (X can be singular)

$\in 2\mathbb{Z}_{>0}$ $\quad X$ nonsingular

Geography question:

what values can (p_g, vol) take?

$$\underbrace{\frac{4}{3} p_g - \frac{10}{3} \le \mathrm{vol}(X)}_{\text{Noether inequality}} \underset{?}{\le} 72\, p_g$$

J. Chen
M. Chen
C. Jiang

Sharp $\sim$ M. Kobayashi $\sim$ 92

3-folds with $\boxed{\mathrm{vol}(X) = \frac{4}{3} p_g - \frac{10}{3}}$

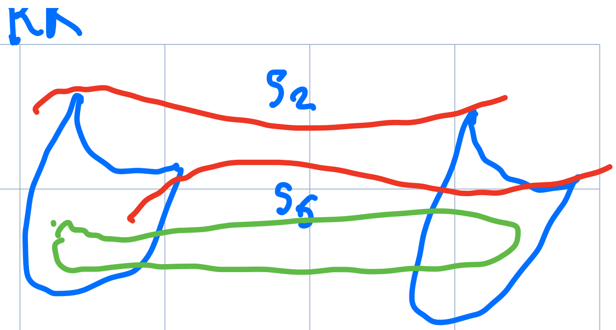
$d \in \mathbb{Z}_{\ge 2}$, vary d_0

$$X\binom{0}{10} \subset \left(\begin{array}{cc|cccc} 1 & 1 & d-d_0 & d_0-2d & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 5 \end{array}\right)$$

$p_g = 3d-2$, $\quad \mathrm{vol}\, X = 4d-6$

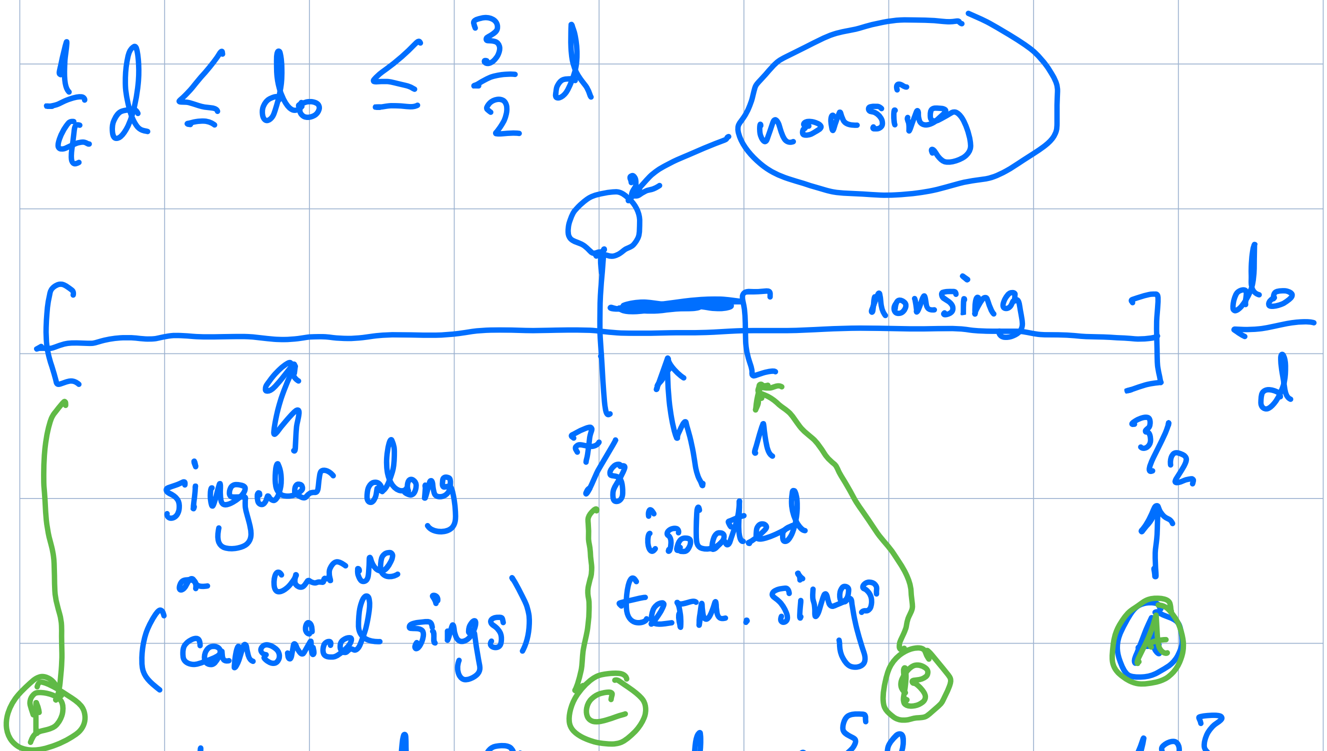
adjunction

$\mathbb{P}(1,1,2,5)$ - bundle



What about d_0?

$$\frac{1}{4}d \le d_0 \le \frac{3}{2}d$$



Examples $\quad d=8 \quad d_0 \in \{2, \dots, 12\}$

$$\begin{array}{ccccccc} & t_0 & t_1 & x_0 & x_1 & y & z \end{array}$$

(A) $\left(\begin{array}{cc|cccc} 1 & 1 & -4 & -4 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 5 \end{array}\right) \supset X\begin{pmatrix} 0 \\ 10 \end{pmatrix}$ nonsing.

$$z^2 = y^5 + (\cdot)x_1^{10} + (\cdot)x_0^{10}$$

(B) $\left(\begin{array}{cc|cccc} 1 & 1 & 0 & -8 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 5 \end{array}\right) \supset X\begin{pmatrix} 0 \\ 10 \end{pmatrix}$ nonsing

$$z^2 = y^5 + (80)\,x_1^{10} + x_0^{10}$$

(C) $\left(\begin{array}{cc|cccc} 1 & 1 & 1 & -9 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 5 \end{array}\right) \supset X\begin{pmatrix} 0 \\ 10 \end{pmatrix}$ nonsing

$$z^2 = y^5 + (\cdot)\,x_1^{10} + \dots + x_0^9 x_1 \qquad \cancel{x_0^{10}}$$

const.

(D) $\left(\begin{array}{cc|cccc} 1 & 1 & 6 & -14 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 5 \end{array}\right) \supset X\begin{pmatrix} 0 \\ 10 \end{pmatrix}$

$$z^2 = y^5 + (\cdot)\,x_1^{10} + \dots + x_0^7 x_1^3 \qquad \cancel{x_0^9 x_1, \; x_0^{10}}$$

Singular along $S_0 : \mathbb{P}^1 \times (1:0:0:0)$

x_0

$$z^2 = y^5 + x_0^3 + \text{h.o.t.}$$

$E_8 \times \mathbb{P}^1$ canonical 3-fold sing.

Stratification of moduli space

$(d=8)$ $\quad M(d_0)$ – locus in moduli space

$\mathcal{M}(8) \to \cdots \mathcal{M}(10) \to \mathcal{M}(11) \to \mathcal{M}(12)$

dense

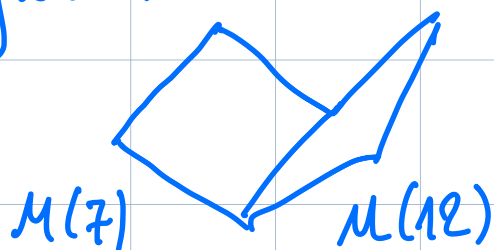
$\mathcal{M}(k) \subset \overline{\mathcal{M}(k+1)}$

What about $\mathcal{M}(7)$?

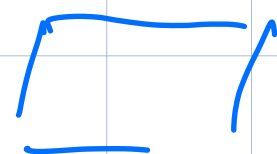
$\mathcal{M}(7) \cap \overline{\mathcal{M}(8)} \neq \emptyset$

but X is singular.

if $d \equiv 0 \ (8)$



$d \not\equiv 0 \ (8)$



[Surfaces on the Noether line ← Horikawa]

How to think about this stratification?

(Hirzebruch-Segre

$\mathbb{F}_0 \sim \mathbb{F}_2 \sim \mathbb{F}_4 \sim \mathbb{F}_6 \sim \dots$)

$$\left(\begin{array}{cc|cccc} 1 & 1 & -3 & -5 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 5 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|cccc} 1 & 1 & -2 & -6 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 5 \end{array}\right)$$

$$d_0 = 10 \qquad\qquad d_0 = 11$$

$$\widetilde{\mathbb{F}} : \begin{array}{c} \left(\begin{array}{cc|ccccc} 1 & 1 & -2 & -3 & \boxed{-5} & 0 & 0 \\ 0 & 0 & 1 & 1 & \boxed{1} & 2 & 5 \end{array}\right) \\ \begin{array}{ccccc} \tilde{x}_0 & x_0 & x_1 & y & z \end{array} \end{array} \supset \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$\widetilde{\mathbb{F}} \times \Delta \supset \mathcal{Y} = \{ \delta \tilde{x}_0 = t_0 x_0 - t_1^3 x_1 \}$$

$$\mathcal{Y} \downarrow \Delta \ni \delta$$

$$\delta \neq 0 \qquad \mathcal{Y}_\delta \cong \left(\begin{array}{cc|cccc} 1 & 1 & -3 & -5 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 5 \end{array}\right)$$

$$\delta = 0 \qquad \mathcal{Y}_0 \qquad t_0 x_0 = t_1^3 x_1$$

$$\tilde{x}_1 := \frac{x_0}{t_1^3} = \frac{x_1}{t_0} \in \begin{pmatrix} -6 \\ 1 \end{pmatrix}$$

$$\mathcal{Y}_0 \cong \begin{array}{c} \left(\begin{array}{cc|cccc} 1 & 1 & -2 & -6 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 5 \end{array}\right) \\ \begin{array}{cc} \tilde{x}_0 & \tilde{x}_1 \end{array} \end{array}$$

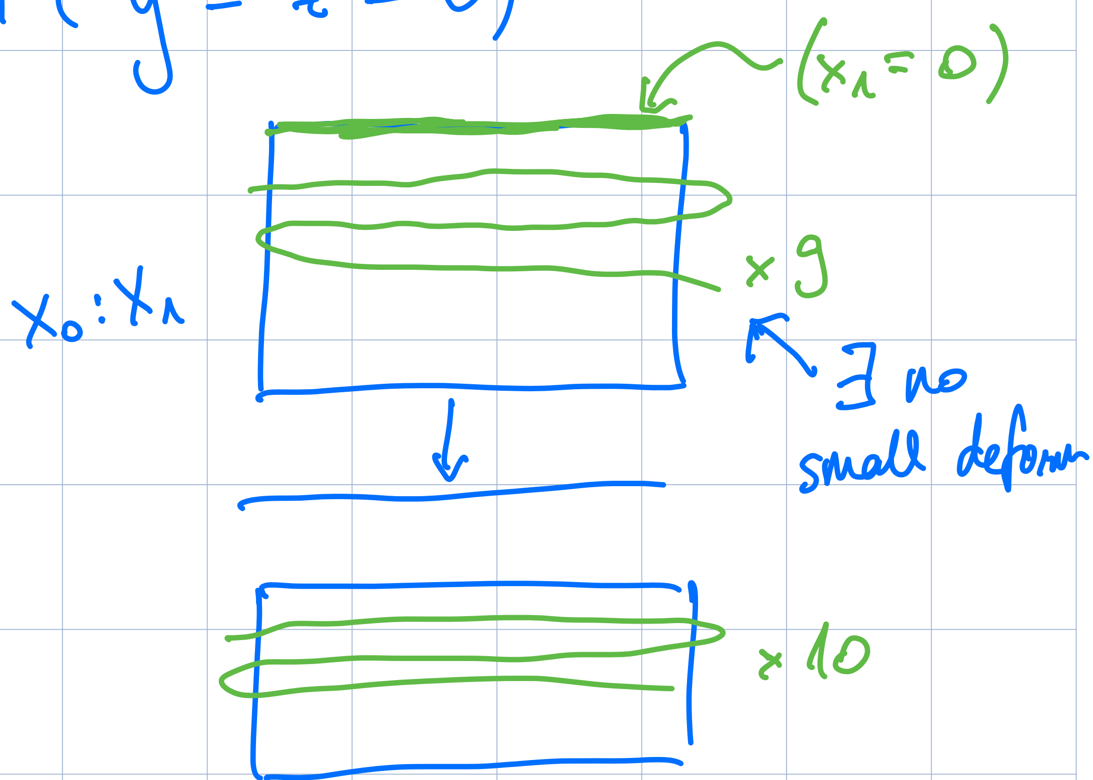
Advantage – we can analyse how $X_\delta \in \mathcal{F}_\delta$ degenerates to $X_0 \in \mathcal{F}_0$

What is different about $\mathcal{M}(7)$?

$$X_7 : \cancel{z}^2 = \cancel{y}^5 + (\cdot)x_1^{10} + \dots \boxed{+ \underbrace{\cdot}_{\text{constant.}} x_0^9 x_1}$$

$$X_7 \cap (y = z = 0)$$

Picture



$\nexists$ family $\mathcal{X} \longrightarrow \Delta$ s.t.

- $\mathcal{X}_\delta$ smooth $\forall \delta$
- $\mathcal{X}_\delta \in \mathcal{M}(8)$ $\delta \neq 0$
- $\mathcal{X}_\delta \in \mathcal{M}(7)$ $\delta = 0$

[Horikawa]

What about 3-folds of gen. type

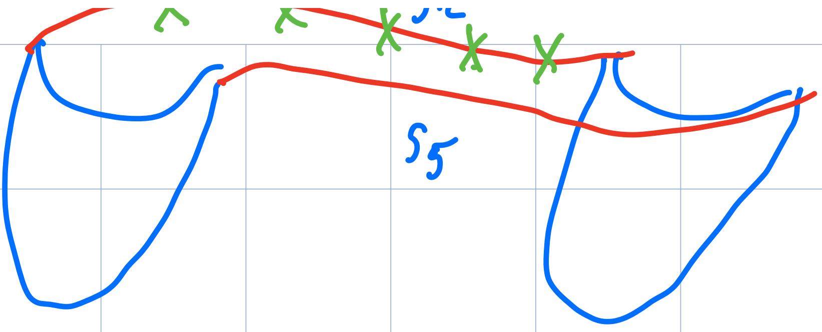
$$\mathrm{vol} = \frac{4}{3} p_g - \frac{10}{3} + \varepsilon, \ \varepsilon \text{ small } ?$$

1 sing $\to$ minimal $\varepsilon > 0$

One easy approach

$$\begin{array}{cccccc} & & x_0 & x_1 & y & z \\ \left(\begin{matrix} 1 & 1 \\ 0 & 0 \end{matrix}\right| & & \begin{matrix} * \\ 1 \end{matrix} & \begin{matrix} * \\ 1 \end{matrix} & \begin{matrix} -1 \\ 2 \end{matrix} & \left.\begin{matrix} 0 \\ 5 \end{matrix}\right) \end{array} \supset \begin{pmatrix} 0 \\ 10 \end{pmatrix} : X$$

$$z^2 = \boxed{\alpha_5 \, y^5} + (x_0, x_1)$$


s5