Logarithmic geometry and tropicalization

"piecewise linear" AG

"decorated" version of algebraic geometry, where varieties are equipped with a sheaf of monoids ⇝ combinatorics, polyhedral geometry

commutative

Plan:

1. Toric varieties
2. Log geometry
3. Tropical geometry and tropicalization.

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1. Toric varieties [Cox-Little-Schenck, Toric varieties]

Def: a monoid is a set P with $+ : P \times P \to P$ associative and commutative, with a neutral elt. $0 \in P$.

Ex:
- every abelian gp is a monoid
- $\mathbb{N}$ with $+$ and 0 is a monoid, $\mathbb{N}^k$ with component-wise $+$ and $(0, \dots, 0)$.

  (products $P \times Q$ are as you'd expect)
- $P = \{0, 1\}$, with $1 + 1 = 1$

Def: a homomorphism of monoids $f: P \to Q$ is a function compatible with $+$ and 0.

Associated group: $P \rightsquigarrow P^{gp} = {P \times P}/{\sim}$

idea: $(p,q) \Leftrightarrow p-q$

$$(p,q) \sim (p',q') \iff p+q'+r = p'+q+r$$
$$(\exists r \in P \text{ s.t.})$$

This becomes a group with the induced operation.

There is a homom. $P \to P^{gp}$, $p \mapsto [(p,0)]$ and this is universal,

in the sense that $\forall$ other ab. group G, every hom.

$P \to G$ will factor.

$$\begin{tikzcd} P \arrow[r] \arrow[d] & G \\ P^{gp} \arrow[ur, dotted, "\exists!"'] & \end{tikzcd}$$

($(-)^{gp}: (\mathrm{Mon}) \to (\mathrm{Grp})$ is the left adjoint of the inclusion

$(\mathrm{Grp}) \hookrightarrow (\mathrm{Mon})$ $\qquad \mathrm{Hom}_{Ab}(P^{gp}, G) = \mathrm{Hom}_{Mon}(P, G)$)

Def: a monoid is <u>integral</u> if $p+r = q+r$ in $P \Rightarrow p = q$

(equivalent to saying that $P \to P^{gp}$ is injective)

and <u>saturated</u> if it is integral, and

$$x \in P^{gp},\ nx \in P \Rightarrow x \in P$$

$n \in \mathbb{N}$ $\quad n \neq 0$

<u>Ex</u>: • $\mathbb{N}^k$ is saturated, $(\mathbb{N}^k)^{gp} = \mathbb{Z}^k$

• $P = \langle 2, 3 \rangle \subseteq \mathbb{N}$, $P^{gp} = \mathbb{Z}$

$= \{a \cdot 2 + b \cdot 3 \mid a, b \in \mathbb{N}\}$ $\quad 1 = 3 - 2$

but $1 \notin P$, so P is not saturated.

• $P = \{0, 1\}$ with $1 + 1 = 1$ is not even integral.

<u>Def</u>: a monoid is <u>finitely generated</u> if there is a surjection $\mathbb{N}^k \twoheadrightarrow P$ for some $k \in \mathbb{N}$

$$\left(\exists\, p_1, \dots, p_k \in P \text{ s.t. } p \in P \rightsquigarrow p = \sum_{i=1}^{k} a_i p_i \right), \quad a_i \in \mathbb{N}$$

<u>fine</u> = integral and finitely generated.

<u>fs</u> = fine and saturated

A <u>presentation</u> of a monoid is a coequalizer diagram

$$\begin{tikzcd} \mathbb{N}^k \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & \mathbb{N}^n \arrow[r, "f"] \arrow[dr] & P \arrow[d, dotted, "\exists!"] \\ & & P' \end{tikzcd}$$

$P = \langle \text{generators} \mid \text{relations} \rangle$

$f(e_i)$ $\quad g(e_j) = h(e_j)$

<u>Thm</u> (Redei): every finitely generated monoid is finitely presented.

Ex: $P = \langle p,q,r \mid p+q = 2r \rangle$ $\left(\cong \langle (1,0),(1,2),(1,1) \rangle \subseteq \mathbb{Z}^2\right)$

$P = \langle p,q,r,s \mid p+q = r+s \rangle$ $\left(\cong \langle (1,0,0), (0,1,0), (0,0,1), (1,1,-1) \rangle\right)$

Def: a monoid is toric if it is fs + torsion free

i.e. $nx = ny \Rightarrow x = y$

$n \in \mathbb{N}$, $x,y \in P$

$n \neq 0$

$k = \bar{k}$, think

$k = \mathbb{C}$ if you want

Def: the monoid algebra of P is $k[P] = \bigoplus_{p \in P} k \cdot t^p$

is the free k-vect. space on the set P

with the multiplication defined by $t^p \cdot t^q = t^{p+q}$, extended by linearity.

Every k-algebra gives a monoid using its multiplication, and the obvious
$$\begin{aligned} P &\to k[P] \\ p &\mapsto t^p \end{aligned}$$
(hom. of monoids)

is initial w.r.t. hom $P \to A$ any k-algebra

$$\begin{tikzcd} P \arrow[r] \arrow[d] & A \\ k[P] \arrow[ur, dashed, "\exists !"'] & \end{tikzcd}$$

(i.e. the functor $P \mapsto k[P]$ is the left adjoint to the forgetful)

In particular a presentation $\mathbb{N}^k \rightrightarrows \mathbb{N}^n \to P$

will give a coequalizer in k-algebras

$$k[\mathbb{N}^k] \underset{h}{\overset{g}{\rightrightarrows}} k[\mathbb{N}^n] \to k[P]$$

$$\begin{array}{ccc} \shortparallel & & \cong \\ k[y_1,\dots,y_k] & & k[x_1,\dots,x_m] \end{array}$$

$$\Rightarrow k[P] \cong \frac{k[x_1,\dots,x_m]}{(g(y_i)-h(y_i))}$$

<u>Ex</u>: $P = \langle p,q,r \mid p+q = 2r \rangle \rightsquigarrow k[P] = \frac{k[x,y,z]}{xy-z^2}$

$P = \langle p,q,r,s \mid p+q = r+s \rangle \rightsquigarrow k[P] = \frac{k[x,y,z,w]}{(xy-zw)}$

From now on P will be toric, in part.

$$k[P] \cong k[x_1,\dots,x_m]/I$$

$k[P]$ has an associated affine alg. variety,

$$\mathrm{Spec}(k[P]) = V(I) \subseteq \mathbb{A}^n_k \leftarrow \text{n dim'l affine space}$$

$\uparrow$ vanishing locus

($X = \mathrm{Spec}\, A \iff X$ is affine, and $A = k[X]$ is its coordinate ring)

Def: an (normal) <u>affine toric variety</u> (TV) is an affine var. $/k$

with a dense open immersion $T \subseteq X$, where

T is an alg. torus, and the translation action

$T \times T \to T$ extends to an action $T \times X \to X$.
$(x, y) \mapsto x \cdot y$

__Thm__: $\mathbb{A}_P := \operatorname{Spec} k[P]$ is an affine TV, and all affine TVs arise like this.

Where is the torus of $\mathbb{A}_P$? $P \subseteq P^{gp} \rightsquigarrow k[P] \subseteq k[P^{gp}]$

$P^{gp} \cong \mathbb{Z}^r$ for some $r \in \mathbb{N}$, and

$$k[P^{gp}] \cong k[\mathbb{Z}^r] = k[x_1^{\pm 1}, \ldots, x_r^{\pm 1}]$$

$$\operatorname{Spec} k[x_1^{\pm 1}, \ldots, x_r^{\pm 1}] = (\operatorname{Spec} k[t, t^{-1}])^r \cong (k^*)^r$$

$$k[x_1^{\pm 1}, \ldots, x_r^{\pm 1}] = k[x_1^{\pm 1}] \otimes_k k[x_2^{\pm 1}] \otimes_k \cdots \otimes_k k[x_r^{\pm 1}]$$

$\operatorname{Spec} k[t, t^{-1}] \cong \mathbb{A}^1 \setminus \{0\}$ (k^*)

A (split) algebraic torus is a group variety isomorphic to $(k^*)^r$.

Since you can realize $k[P] \subseteq k[P^{gp}]$ by localizing at $\prod t^{p_i}$ where $\{p_i\}$ is a generating set of P,

$T_P = \operatorname{Spec} k[P^{gp}] \subseteq \operatorname{Spec} k[P]$ is a principal open.

The action $T_P \times \mathbb{A}_P \to \mathbb{A}_P$ is given by

$$k[P] \longrightarrow k[P^{gp}] \otimes_k k[P]$$
$$t^p \longmapsto t^p \otimes t^p$$

<u>Ex</u>: $P = \langle (1,0), (1,2), (1,1) \rangle \subseteq \mathbb{Z}^2$, $P^{gp} = \mathbb{Z}^2$

$$k[P] = k[x,y,z]/_{xy - z^2} \subseteq k[P^{gp}] = k[s^{\pm 1}, t^{\pm 1}]$$

$s \leftrightarrow (1,0)$
$t \leftrightarrow (0,1)$

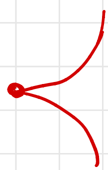
$$(1,0) \leftrightsquigarrow x \longmapsto s$$
$$(1,2) \leftrightsquigarrow y \longmapsto st^2$$
$$(1,1) \leftrightsquigarrow z \longmapsto st$$

So that $T_P \times \mathbb{A}_P \to \mathbb{A}_P$ looks like

$$(s,t) \cdot (a,b,c) = (sa, st^2 b, stc)$$
$$ab = c^2$$

<u>Remark</u>: normality of $\mathbb{A}_P \leftrightarrow$ saturatedness of P

<u>Ex</u>: $\operatorname{Spec} k[\langle 2,3 \rangle] \cong \operatorname{Spec} k[x,y]/_{x^3 - y^2}$, a non-normal affine TV



(normal)

<u>Def</u>: a <u>toric variety</u> is a normal variety (and integral red, sep. scheme of finite type over k), w/ a dense open torus $T \subseteq X$ s.t. the translation action of

T on itself extends to X.

<u>Example</u>: $\mathbb{P}^n_k$ is a toric variety: $(k^*)^n \subseteq \mathbb{A}^n \subseteq \mathbb{P}^n_k$,

the action is $(k^*)^n \times \mathbb{P}^n \to \mathbb{P}^n$

$$(a_1,\dots,a_n)\bullet[x_0,\dots,x_n] = [x_0, a_1x_1,\dots,a_nx_n]$$

Any TV has an canonical affine open cover of the form $\mathbb{A}_{p_i}$.

This is completely described by a fan, a collection of cones in the vector space $N\otimes_{\mathbb{Z}}\mathbb{R}$, where N is the cocharacter lattice.

(character of T : $T \to k^*$ morphism of alg. groups

cocharacter of T : $k^* \to T$ " " "

characters of T form a lattice, denoted by $M = p^{op}$

cocharacters " " " " " " " N

and $N \cong M^\vee$ canonically via $\langle\,,\,\rangle : M\times N \to \mathbb{Z}$)

$T\to k^*,\ k^*\to T \mapsto$

$k^* \to T \to k^*$

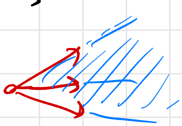
$t \longmapsto t^k$

RPC

<u>Def</u>: a <u>convex rational polyh. cone</u> in $\mathbb{R}^n$ is : $N\otimes_{\mathbb{Z}}\mathbb{R} =: N_{\mathbb{R}}$

$$\sigma = \Big\{ \sum_{u\in S} \lambda_u u \;\Big|\; \lambda_u \in \mathbb{R}_{\geq 0} \Big\} = \mathrm{Cone}(S)$$

where $S \subseteq \mathbb{Z}^n$ is finite.



The <u>dual</u> of $\sigma \subseteq N_{\mathbb{R}}$ is $\sigma^\vee = \{ m \in M_{\mathbb{R}} \mid \langle m,n \rangle \geq 0 \ \forall n \in \sigma \} \subseteq M_{\mathbb{R}}$

This is again a RPC, and $(\sigma^\vee)^\vee = \sigma$.

The point is that every toric monoid P can be written as $\sigma^\vee \cap M$ for a strongly convex RPC in $N_{\mathbb{R}}$

doesn't contain any >0-dim'l lin. subspace.

<u>Ex</u>: $P = \langle (1,0), (1,2), (1,1) \rangle = \sigma^\vee \cap M$

where $\sigma = \mathrm{Cone}((0,1), (2,-1)) \subseteq \mathbb{R}^2$

<u>Def</u>: a <u>fan</u> is a finite collection of strongly convex RPC $\Sigma = \{\sigma\}$ s.t.

- $\sigma \cap \sigma'$ is a face of both σ and σ' $\forall \sigma, \sigma' \in \Sigma$.

  (a face of σ, $\tau \leq \sigma$, is the intersection of σ with a "supporting hyperplane")

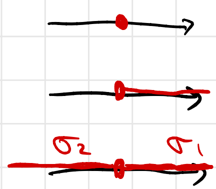
- if $\tau \leq \sigma$, $\sigma \in \Sigma \Rightarrow \tau \in \Sigma$

From a fan Σ in $N_{\mathbb{R}}$, get a TV X_Σ:

- $\sigma \in \Sigma \rightsquigarrow X_\sigma := \mathbb{A}_P = \operatorname{Spec} k[P]$, $P = \sigma^\vee \cap M$
- $\tau \le \sigma \rightsquigarrow X_\tau \hookrightarrow X_\sigma$ an open embedding.
- $\sigma \cap \sigma' = \tau \rightsquigarrow X_\sigma, X_{\sigma'}$ can be glued along X_τ, a common open subvariety.
- do all the gluings prescribed by the fan, to get X_Σ.

<u>Thm</u>: X_Σ is a TV and every TV arises like this.

<u>Ex</u>: • toric curves:



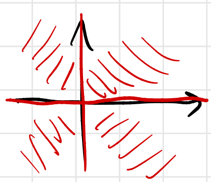
k^*

$\mathbb{A}^1$

$\mathbb{A}^1 \cup \mathbb{A}^1 = \mathbb{P}^1$
$\sigma_1 \; (k^*) \; \sigma_2$

•  $\mathbb{R}^2_{\ge 0} = \sigma \rightsquigarrow \mathbb{A}^2$

•  $\rightsquigarrow \mathbb{P}^2$ ($\mathbb{P}^n$ has a similar higher dim'l fan)

•  $\rightsquigarrow \mathbb{P}^1 \times \mathbb{P}^1$ ($X_\Sigma \times X_{\Sigma'} \cong X_{\Sigma \times \Sigma'}$ in general)

<u>Slogan</u>: the (equivariant) geometry of X_Σ is controlled

by the "combinatorics" of Σ.

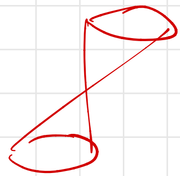
Ex:

- X_Σ is proper (↔ compact) $\Leftrightarrow$ $|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma = N_{\mathbb{R}}$

- X_Σ is smooth $\Leftrightarrow$ every $\sigma \in \Sigma$ is smooth, i.e. the minimal ray generators are part of a $\mathbb{Z}$-basis of N.

for example, the cone $\sigma = \mathrm{Cone}((0,1),(2,-1))$ of yesterday is not smooth, and the corresponding $X_\sigma = \mathrm{Spec}\, k[x,y,z]/_{xy-z^2}$ has singularity at $(0,0,0)$

"quadric cone"



(real trace)

You can show that σ is smooth $\Leftrightarrow$ $X_\sigma \cong \mathbb{A}^k \times (k^*)^{n-k}$.

This implies that smooth toric varieties are covered by affine spaces, and that $X_\Sigma \setminus T$, the "toric boundary" is a simple normal crossings divisor

codim. $=1$

irred. components are smooth

the components intersect transversely.

- π_1, divisors/line bundles, (co)homology, Chow ring, equivariant vector bundles, ... of X_Σ.

can all be described/computed from the fan Σ

Rmk: projective TVs are encoded by lattice polytopes, via the "normal fan" construction.

Orbit-cone correspondence

there is a $\underset{\text{inclusion-reversing}}{\wedge}$ 1:1 correspondence $\Sigma \leftrightarrow$ orbit closures for the action of T on X_Σ

($\leftrightarrow$ orbits of the action)

$n = \mathrm{rk}\, N$

$\sigma \leftrightarrow V(\sigma) = \overline{O(\sigma)} \subseteq X_\Sigma$

$\dim V(\sigma) = n - \dim \sigma$

$(O(\sigma))$

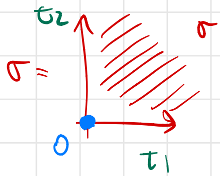
$\nwarrow$ torus

$V(\sigma)$ is actually a $O(\sigma)$-toric variety.

Ex: $X_\Sigma = \mathbb{A}^2 \supseteq (k^*)^2$

$(a,b)\cdot(x,y) = (ax, by)$
$\uparrow$
$(k^*)^2$

orbits: $\overset{\sigma}{\{(0,0)\}}$, $\overset{\tau_2}{x\text{-axis} \setminus \text{origin}}$, $\underset{\tau_1}{y\text{-axis} \setminus \text{origin}}$, $\underset{0}{(k^*)^2}$



Rmk: the boundary $X_\Sigma \setminus T$ is a union of toric varieties of lower dimension, the $V(\sigma)$ for $\sigma \neq 0$

Logarithmic geometry [Abramovich et al - Logarithmic geometry and moduli]

The geometry of a TV is quite constrained, for example TVs are all rational (birational to $\mathbb{P}^n$).

Log structures aim to generalize the theory of TVs by retaining some of the combinatorial aspects, but on more complicated varieties.

Recall that on an affine TV $\mathbb{A}_P$ ($= \operatorname{Spec} k[P]$) we have $P \to k[P]$

$p \mapsto t^p$

$p + p' \mapsto t^p \cdot t^{p'}$

"exponential"

Idea: globalize this homomorphism by "sheafifying".

Def: a log structure on a scheme X

(think algebraic variety, or even complex analytic space)

is (M, α) where:

pre-log structure

- M is a sheaf of monoids (for the étale topology, think Euclidean top. / $\mathbb{C}$)
- $\alpha : M \to (\mathcal{O}_X, \cdot)$ is a homom. of monoids

sheaf of regular functions, i.e. morphisms to $\mathbb{A}^1$

- $\alpha|_{\alpha^{-1}(\mathcal{O}_X^*)} : \alpha^{-1}(\mathcal{O}_X^*) \xrightarrow{\cong} \mathcal{O}_X^*$ is an isomorphism.

A log scheme is a (X, M, α) of a scheme w/ a log str.

Rmk: every pre-log str. can be canonically turned into a log str, as follows: given $\alpha : M \to \mathcal{O}_X$

pushout

$$\begin{tikzcd}
\alpha^{-1}(\mathcal{O}_X^*) \arrow[r, hook] \arrow[d, "\alpha"'] & M \arrow[d] \arrow[dr, bend left, "\alpha"] \arrow[dl, phantom, "\lrcorner"] & \\
\mathcal{O}_X^* \arrow[r] \arrow[rr, hook, bend right, "i"'] & M^a \arrow[r, dashed, "\alpha^a"] & \mathcal{O}_X
\end{tikzcd}$$

(M^a, α^a) is the "associated log structure".

A morphism $(M, \alpha) \to (M', \alpha')$ is a homom. $M \to M'$ s.t.

$$\begin{tikzcd}
M \arrow[r] \arrow[dr, "\alpha"'] & M' \arrow[d, "\alpha'"] \\
 & \mathcal{O}_X
\end{tikzcd}$$

commutes.

Ex: • $M = \mathcal{O}_X^*$ and $\alpha = i : \mathcal{O}_X^* \hookrightarrow \mathcal{O}_X$ is the trivial log structure on X.

simple normal crossing

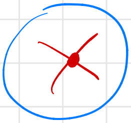
• if (X, D) is a SNC pair, then one can construct a log str. as follows

the divisorial log str. ind. by D

$$M_D(U) = \{ f \in \mathcal{O}_X(U) \mid f|_{(X \setminus D) \cap U} \in \mathcal{O}^* \} \overset{\alpha}{\hookrightarrow} \mathcal{O}_X(U)$$

$U \subseteq X$

~ local equations for the components of D.



If at $x \in X$, $f_1 = 0, \dots, f_k = 0$ are local eqns around x of the components of D, then

$$M_{D,x} \cong \mathcal{O}^*_{X,x} \oplus \mathbb{N}^k \quad \text{``} \langle (f_1)_x, \dots, (f_k)_x \rangle$$

This also applies to toric varieties, by taking the pair $(X_\Sigma, X_\Sigma \setminus T)$.

(and also to "toroidal embeddings" i.e. $U \subseteq X$ that locally look like the embedding of $T_P \subseteq \mathbb{A}_P$).

- log points : $X = \operatorname{Spec} k$ (a point),

$$M = k^* \oplus P \xrightarrow{\alpha} k$$

$$(a, p) \longmapsto a \cdot 0^p = \begin{cases} a & \text{if } p = 0 \\ 0 & \text{if } p \neq 0 \end{cases}$$

$P = \mathbb{N} \rightsquigarrow$ "standard log point" ("punctured point")

Log structures can be pulled back (and pushed forward):

If $f : X \to Y$ is a morphism, and (M, α) is a log structure on Y, we can obtain a pullback log str. on X: $\quad M \xrightarrow{\alpha} \mathcal{O}_Y$

$f^{-1}M \xrightarrow{f^{-1}\alpha} f^{-1}\mathcal{O}_Y \longrightarrow \mathcal{O}_X$ is a pre-log structure, and we can take the associated log str. $(f^{-1}M)^a \to \mathcal{O}_X$.

Ex: on $(\mathbb{A}^1, 0)$ we have the divisorial log structure.

The only non-trivial stalk of M is at 0, and

$$M_0 \cong \mathcal{O}^*_{\mathbb{A}^1,0} \oplus \mathbb{N} \leftarrow \langle x \rangle \qquad 0 = \{x = 0\}$$

We can pull this log str. back to the origin itself along $\{0\} \hookrightarrow \mathbb{A}^1$. If you do this, you get exactly the standard log point.

Def: a <u>morphism</u> of log schemes $(X, M, \alpha) \to (Y, N, \beta)$ is a $f: X \to Y$ + $f^\flat: f^*N \to M$ of log str. on X.

Ex: • if (X, D), (Y, E) are SNC pairs, a morphism

$$f: (X, M_D) \to (Y, M_E) \text{ is exactly a}$$

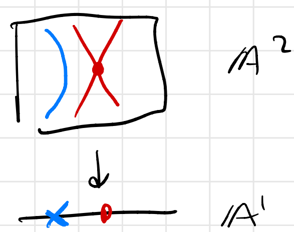
$$f: X \to Y \quad \text{s.t.} \quad f^{-1}(E) \subseteq D \quad (\Leftrightarrow f(X \setminus D) \subseteq Y \setminus E)$$

• $\mathbb{A}^2 \to \mathbb{A}^1$ is of this form

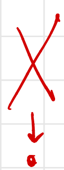
$V(xy-t) \subset \mathbb{A}^3$

$(x, y) \mapsto xy = t$

"versal deformation of a node"



Can restrict to the origin with the pullback log structures:



the log str. remember "something" about the whole family.

Def: the <u>characteristic sheaf</u> of (M,α) is $\overline{M} = M/\mathcal{O}_X^*$.

This has a "discrete/combinatorial" nature, in the presence of good local models.

Def: a <u>chart</u> for (M,α) is a hom. $P \to M(X)$ s.t.

$$\underbrace{\underline{P} \to M \xrightarrow{\alpha} \mathcal{O}_X}_{\text{pre-log str}} \rightsquigarrow (\underline{P})^a \xrightarrow{\cong} M \xrightarrow{\alpha} \mathcal{O}_X$$

constant sheaf

Rmk: $P \to M(X) \to \mathcal{O}_X(X) \rightsquigarrow k[P] \to \mathcal{O}_X(X) \rightsquigarrow X \to \operatorname{Spec} k[P] = \mathbb{A}_P$

$P \to M(X)$ is a chart $\Leftrightarrow$ $X \to \mathbb{A}_P$ is <u>strict</u>, i.e.

the log str on X is the pullback of the divisorial one on $\mathbb{A}_P$.

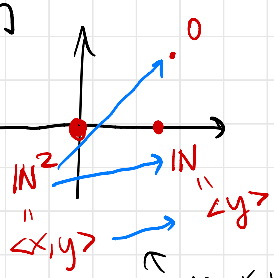
Def: a log scheme is <u>fs</u> if it locally admits charts from fs monoids.

Rmk: if $P \to M(X)$ is a chart, then the stalks of $\overline{M}$ are quotients of P by <u>faces</u>.

If (X, M, α) is fs, then $\overline{M}$ is locally constant on a stratification of X. (this generalizes the orbit stratification of a toric variety).

Ex: $\mathbb{A}^2 = \operatorname{Spec} k[\mathbb{N}^2]$

$\overline{M} =$



kills x which becomes a unit.

(analogous picture for $\mathbb{A}_P$)

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log smoothness and log curves

There is a notion of smoothness for maps of log schemes, which is more flexible than the classical one.

The definition is via an "infinitesimal lifting criterion", but

<u>Thm</u>: $f: (X,M) \to (Y,N)$ of fs log schemes is log smooth iff locally on X and Y there are charts $P \to M(X)$, $Q \to N(Y)$, $Q \to P$ s.t.

$$\begin{tikzcd}
X \arrow[r, dashed] \arrow[rr, bend left, "\text{strict}"] \arrow[dr, bend right] & X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathbb{A}_P \arrow[d] \\
& Y \arrow[r, "\text{strict}"'] & \mathbb{A}_Q
\end{tikzcd}$$

P
$\uparrow$
Q

$X \to X'$ is (classically) smooth

(+ technical condition on $Q \to P$, true e.g.

if $\operatorname{char} k = 0$ and $Q \hookrightarrow P$)

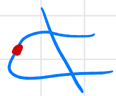
"log smooth = smooth + combinatorial".

<u>Ex</u>: • if $\operatorname{char} k = 0$ and $Q \hookrightarrow P$ is injective $\Rightarrow \mathbb{A}_P \to \mathbb{A}_Q$ is log smooth

• $Q = 0 \rightsquigarrow \mathbb{A}_P \to \operatorname{Spec} k$ is $\overset{\text{log}}{\checkmark}$ smooth

• in particular $\mathbb{A}^2 \to \mathbb{A}^1$ above is log smooth (it is induced by $\mathbb{N} \to \mathbb{N}^2$, $1 \mapsto (1,1)$), and this stays true for the central fiber

In particular, you can study log smooth curves and their moduli spaces.



<u>Thm</u>: log smooth curves are precisely nodal (marked) curves. Around a node p in a fiber of a family $(C, M_C) \to (S, M_S)$ of log smooth curves, the family is locally pulled back from $\mathbb{A}^2 \to \mathbb{A}^1$ along $(S, M_S) \to (\mathbb{A}^1, 0)$ ($\leftrightsquigarrow m \in M_S(S)$)

"logarithmic deformation parameter of the node"

The moduli space (stack) of log curves is isomorphic to the log stack given by the

Deligne-Mumford → DM compactification $\overline{M}_{g,n}$, equipped w/ the divisorial log str. from the boundary $\overline{M}_{g,n} \setminus M_{g,n}$ (a NC divisor).

Working with log smooth objects sometimes gives a compact mod. space. This principle has been applied e.g. to log GW invariants and log compactification of the universal Jacobian.

Tropical geometry

[Chan - lectures on tropical curves and their moduli spaces]

"piecewise-linear / polyhedral AG"

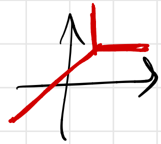
History: if K in a non-archimedean field w/ valuation

$$v : K^* \to \mathbb{R} \qquad \left(\overset{e.g.}{K} = \overline{\mathbb{C}((t))} = \bigcup_{n \in \mathbb{N}} \mathbb{C}((t^{\frac{1}{n}})) \right)$$

and a closed $X \subseteq (K^*)^n \rightsquigarrow \mathrm{Trop}(X) = \overline{v(X)} \subseteq \mathbb{R}^n$

$v : (K^*)^n \to \mathbb{R}^n$ component-wise valuation.

Thm: $\mathrm{Trop}(X)$ is a (balanced) polyhedral complex.

e.g. a line $\ell \subseteq (K^*)^2$ becomes something like 

Curves get tropicalized to metric graphs.

Abstracting:

<u>Def</u>: a <u>tropical curve</u> is a finite metric graph Γ, possibly with legs (i.e. unbounded edges w/ only 1 endpoint) and a weighting $g: V(\Gamma) \to \mathbb{N}$.

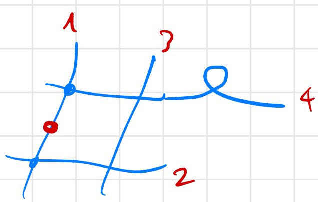
These things show up as the "dual graph" of a nodal curve:

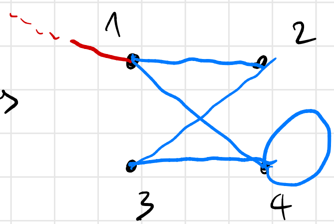
C nodal curve $\rightsquigarrow$ $\Gamma(C)$ vertices $\leftrightarrow$ irr. comp. of C

$g(v) =$ genus

edges $\leftrightarrow$ nodes

legs $\leftrightarrow$ markings

E.g.: 

$\rightsquigarrow$ 

The length of an edge come from val of a deformation parameter of the corresponding node, because the stable curve that you tropicalize will come as the central fiber of a stable model of a smooth curve over $\mathbb{K}$ (whose residue field is the k over which the stable curve lives).

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The genus of Γ is $\sum_{v \in V} g(v) + b_1(\Gamma)$.

There is a moduli space M_g^{trop} of tropical curves of genus g, and this is a generalized cone complex. There is a tropicalization map

$$M_g^{an} \longrightarrow M_g^{trop}$$

Berkovich analytification.

There is a version of tropicalization in log geometry, modeled on $X_\Sigma \leftrightarrow \Sigma$ for toric varieties.

Given an fs (X, M, α), let x_S be the generic point of the log stratum S.

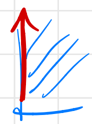
$$\overline{M}_{x_S} \rightsquigarrow \sigma_S = (\overline{M}_{x_S})^\vee_{\mathbb{R}} = \mathrm{Hom}(\overline{M}_{x_S}, \mathbb{R}_{\geq 0}) \text{ a RPC}$$

if $x_S \rightsquigarrow x_{S'}$ (i.e. $\overline{S'} \subseteq \overline{S}$), there is a quotient map $\overline{M}_{x_{S'}} \twoheadrightarrow \overline{M}_{x_S}$, that dualizes to a face inclusion $\sigma_s \hookrightarrow \sigma_{s'}$

e.g.: the quotient $\underset{\langle x,y\rangle}{\overset{\shortparallel}{\mathbb{N}^2}} \xrightarrow{\pi_2} \underset{\langle y\rangle}{\overset{\shortparallel}{\mathbb{N}}}$ on $\mathbb{A}^2$ that we had before

dualizes to $\mathbb{R}_{\geq 0} \hookrightarrow \mathbb{R}^2_{\geq 0}$

$r \longmapsto (0, r)$



Define the tropicalization Σ_X as the gluing of all these cones along these face inclusions.

This is a (generalized) cone complex (with integral structure).

<u>Ex</u>: • $\Sigma_{X_\Sigma} = |\Sigma|$, but without the embedding in $N_{\mathbb{R}}$.

• $\Sigma_{\mathbb{A}_P} = P^\vee_{\mathbb{R}} = \mathrm{Hom}(P, \mathbb{R}_{\geq 0}) = \Sigma_{(\mathrm{Spec}\, k,\, k^* \oplus P)}$

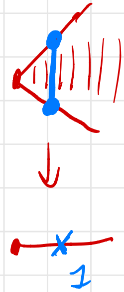
Tropicalizing log curves gives (families) of tropical curves. When you take the dual graph, the "length" of an edge is the log. def. parameter of the corresponding node.

Ex: $\mathbb{A}^2 \longrightarrow \mathbb{A}^1$ $\rightsquigarrow$

$(x,y) \mapsto xy$

$xy = t$

$$\begin{array}{ccc} \Sigma_{\mathbb{A}^2} & = & (\mathbb{R}_{\geq 0})^2 \\ \downarrow & & \downarrow \\ \Sigma_{\mathbb{A}^1} & = & \mathbb{R}_{\geq 0} \end{array}$$



This stuff was fundamental for various recent developments e.g. in log GW theory, the Gross-Siebert program, and compactifications of the universal Jacobian.