Algebraic and enumerative invariants of finite graded bounded posets

Lecture 3: The ω-transform, QSym/Sym, and Schur functions

Christian Stump

Dobbiaco Spring School, June 2026

Based on joint work with

Galen Dorpalen-Barry, Elena Hoster, Josh Maglione, and Lorenzo Vecchi.

I will tell the story and collect precise credits at the end.

The three lectures

L1. Flag enumeration, labelings, and extended ab-positivity

↓

L2. Chow polynomials, γ-positivity, and real-rootedness

↓

L3. The ω-transform on quasi-symmetric functions, and Schur functions

Recap: the extended ab-index

Let P be finite graded bounded of rank n.

Lecture 1:

$$\Psi_P(\mathbf{a},\mathbf{b}) = \sum_{\mathcal{C}} \mathrm{wt}_{\mathcal{C}}(\mathbf{a},\mathbf{b})$$

$$= \sum_{S} \alpha_P(S)\,\mathrm{wt}_S = \sum_{S} \beta_P(S) u_S$$

$$\mathrm{ex}\,\Psi_P(y;\mathbf{a},\mathbf{b}) = \sum_{\mathcal{C}} \pi_{\mathcal{C}}(y)\,\mathrm{wt}_{\mathcal{C}}(\mathbf{a},\mathbf{b}).$$

For Π_3:

$$\Psi_{\Pi_3} = \mathbf{a} + 2\mathbf{b} \qquad \mathrm{ex}\,\Psi_{\Pi_3}(y;\mathbf{a},\mathbf{b}) = (1+3y+2y^2)\mathbf{a} + (2+3y+y^2)\mathbf{b}.$$

Recap: the Chow polynomial

Lecture 2:

$$H_P(x) = \sum_{\hat{0}<p\le\hat{1}} \overline{\chi}_{[\hat{0},p]}(x)\, H_{[p,\hat{1}]}(x), \qquad H_{\{\hat{1}\}}(x) = 1.$$

The extended ab-index evaluation gives

$$H_P(x) = \mathrm{ex}\,\Psi_P(-x;1,x)\cdot(1-x)^{-\operatorname{rk}(P)}$$

and if P admits an R-labelling

$$H_P(x) = \sum_{\substack{\mathcal{M}\text{ maximal}\\ \{0\}\cup\mathrm{Des}(\mathcal{M})\text{ isolated}}} x^{|\mathrm{Des}(\mathcal{M})|}(1+x)^{n-1-2|\mathrm{Des}(\mathcal{M})|}.$$

For example,

$$H_{B_3}(x) = 1+4x+x^2, \qquad H_{\Pi_3}(x) = 1+x\,.$$

Recap: the ω-transform

The word-level rule was:

first replace every occurrence

$$\mathbf{ab} \mapsto (1+y)\mathbf{ab} + y(1+y)\mathbf{ba},$$

then replace all remaining letters by

$$\mathbf{a} \mapsto \mathbf{a} + y\mathbf{b}, \qquad \mathbf{b} \mapsto \mathbf{b} + y\mathbf{a}.$$

Theorem (Hoster–St–Vecchi 2025)

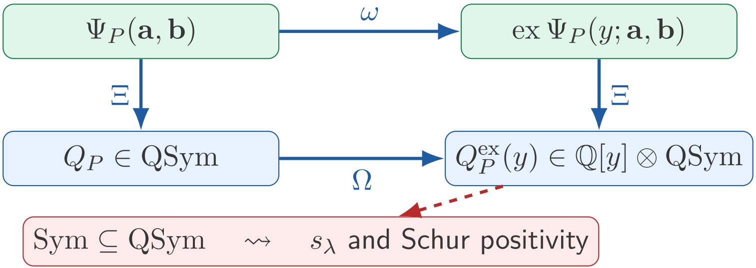
For every finite graded bounded poset P,

$$\mathrm{ex}\,\Psi_P(y;\mathbf{a},\mathbf{b}) = (1+y)\,\omega\big(\Psi_P(\mathbf{a},\mathbf{b})\big).$$

Story of Lecture 3

We now translate:

$$\mathbb{Q}\langle \mathbf{a}, \mathbf{b} \rangle \rightsquigarrow \mathrm{QSym}, \qquad \mathbb{Q}[y]\langle \mathbf{a}, \mathbf{b} \rangle \rightsquigarrow \mathbb{Q}[y] \otimes \mathrm{QSym}.$$

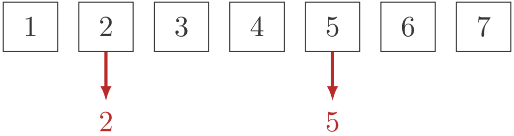


The ω-transform interacts nicely with (quasi-)symmetric functions.

From subsets to compositions

Fix degree n and $S = \{s_1 < \cdots < s_k\} \subseteq [n-1]$. The associated integer composition is

$$\mathbf{c}(S) = (s_1,\ s_2 - s_1,\ \ldots,\ s_k - s_{k-1},\ n - s_k), \qquad \mathbf{c}(\varnothing) = (n).$$



$S = \{2, 5\} \subseteq [6]$

$\mathbf{c}(S) = (2, 3, 2)$

This is the same data as a rank set, an ab-word, or a fundamental QSym index.

Monomial and fundamental quasisymmetric functions

$$M_{\mathbf{c}} = \sum_{i_1 < \cdots < i_k} x_{i_1}^{c_1} \cdots x_{i_k}^{c_k}, \qquad \mathbf{c} = (c_1, \ldots, c_k) \qquad \mathrm{wt}_S \longleftrightarrow M_S$$

For $S \subseteq [n-1]$, write $M_S := M_{\mathbf{c}(S)}$. The fundamental basis is

$$F_S = \sum_{T \supseteq S} M_T \qquad u_S = \sum_{T \supseteq S} \mathrm{wt}_T \longleftrightarrow F_S.$$

M_S corresponds to flag f-coordinates.

F_S corresponds to flag h-coordinates.

Moving from 2 non-commuting variables to quasi-symmetric functions

Define a vector-space isomorphism

$$\Xi : \mathbb{Q}\langle \mathbf{a}, \mathbf{b} \rangle_{n-1} \longrightarrow \mathrm{QSym}_n$$

by

$$\mathrm{wt}_S \longmapsto M_S \qquad \text{or, equivalently,}$$
$$u_S \longmapsto F_S.$$

The identities

$$\Psi_P = \sum_S \alpha_P(S)\,\mathrm{wt}_S = \sum_S \beta_P(S) u_S$$

become the monomial and fundamental expansions of the same QSym function.

The QSym invariant of a poset

Set $Q_P := \Xi\big(\Psi_P(\mathbf{a}, \mathbf{b})\big)$. Then:

The identities

$$\Psi_P = \sum_S \alpha_P(S)\,\mathrm{wt}_S = \sum_S \beta_P(S) u_S$$

become the identities

$$Q_P = \sum_S \alpha_P(S) M_S = \sum_S \beta_P(S) F_S$$

Example: Π_3 in QSym

For Π_3 we have

$$\Psi_{\Pi_3}(\mathbf{a}, \mathbf{b}) = \mathbf{a} + 2\mathbf{b}.$$

We consider $n = 2$ being the rank of Π_3 and thus subsets of $\{1\}$:

$$Q_{\Pi_3} = F_{\varnothing} + 2F_{\{1\}}.$$

$$F_{\varnothing} = M_{\varnothing} + M_{\{1\}} \qquad F_{\{1\}} = M_{\{1\}}$$

This example now lives in QSym.

The extended QSym invariant

$$\Xi : \mathbb{Q}[y]\langle \mathbf{a}, \mathbf{b} \rangle_{n-1} \longrightarrow \mathbb{Q}[y] \otimes \mathrm{QSym}_n,$$

$$\mathrm{ex}\, \Psi_P(y; \mathbf{a}, \mathbf{b}) \longmapsto Q_P^{\mathrm{ex}}(y).$$

For Π_3,

$$Q_{\Pi_3}^{\mathrm{ex}}(y) = (1 + 3y + 2y^2) F_{\varnothing} + (2 + 3y + y^2) F_{\{1\}}.$$

Transporting ω to QSym

The word-level ω-transform gives a linear map

$$\Omega : \mathrm{QSym}_n \longrightarrow \mathbb{Q}[y] \otimes \mathrm{QSym}_n$$

by transporting ω through Ξ:

$$\Omega(F_S) := (1 + y)\, \Xi\big(\omega(u_S)\big).$$

Thus

$$Q_P^{\mathrm{ex}}(y) = \Omega(Q_P).$$

The internal coproduct on QSym

For two ordered alphabets X, Y, let XY be the lexicographically ordered alphabet of products $x_i y_j$. The internal coproduct is defined by

$$f[XY] = \sum_i g_i[X] h_i[Y], \qquad \Delta(f) = \sum_i g_i \otimes h_i.$$

Gessel's formula (composition $(\sigma\tau)(i) = \sigma(\tau(i))$):

$$\Delta(F_{\mathrm{Des}(\pi)}) = \sum_{\tau\sigma=\pi} F_{\mathrm{Des}(\sigma)} \otimes F_{\mathrm{Des}(\tau)}.$$

This is the Kronecker coproduct on QSym.

A QSym formula for Ω

Define the linear functional $\varphi_y : \mathrm{QSym}_n \to \mathbb{Q}[y]$ by

$$\varphi_y(F_S) = \begin{cases} y^k, & S = \{1, 2, \ldots, k\} \text{ for some } 0 \leq k \leq n-1, \\ 0, & \text{otherwise.} \end{cases}$$

Liu's QSym formula (2024)

$$\Omega = (1 + y)\,(\mathrm{id} \otimes \varphi_y) \circ \Delta.$$

Symmetric functions inside QSym

The ring of symmetric functions is a subring

$$\mathrm{Sym} \subseteq \mathrm{QSym}\,.$$

For $\lambda \vdash n$, the internal coproduct restricts to symmetric functions:

$$\Delta(s_\lambda) = \sum_{\mu,\nu \vdash n} g_{\lambda,\mu,\nu}\, s_\mu \otimes s_\nu\,.$$

The Ω-transform connects to the representation theory of the symmetric group via the Kronecker coefficients $g_{\lambda,\mu,\nu}$.

Schur-positivity theorem

Theorem (Liu 2024)

$\Omega : \mathrm{Sym} \longrightarrow \mathbb{Q}[y] \otimes \mathrm{Sym}$ is Schur-positive, and for $\lambda \vdash n$ we have

$$\Omega(s_\lambda) = (1+y)\sum_{k=0}^{n-1} (s_\lambda * s_{(n-k,1^k)})y^k.$$

Here $*$ is the internal, or Kronecker, product of symmetric functions.
Michele D'Adderio and Giovanni Interdonato showed me this week:

Theorem (D'Adderio–Interdonato 2026)

$$\Omega(s_\lambda)[X] = s_\lambda[X(1-\epsilon y)] = s_\lambda[X - \epsilon y X] \in \mathbb{Q}[y] \otimes \mathrm{Sym}$$

Kronecker coefficients

Expanding Liu's theorem in the Schur basis gives, for $\lambda \vdash n$,

$$\Omega(s_\lambda) = (1+y) \sum_{\mu \vdash n} \left(\sum_{k=0}^{n-1} g_{\lambda,\mu,(n-k,1^k)} y^k \right) s_\mu \in \mathbb{N}[y] \otimes \mathrm{Sym}\,.$$

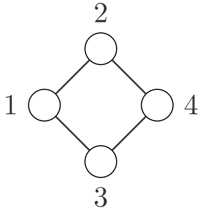
Since the Kronecker coefficients $g_{\lambda,\mu,\nu} \in \mathbb{N}$ are nonnegative, the coefficients are in $\mathbb{N}[y]$.

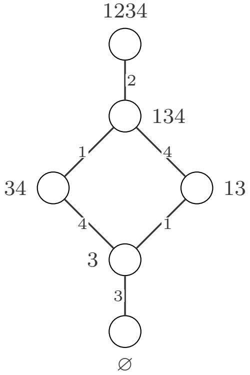
One may compute:

$$\Omega(s_{(2,2)}) = (y + 2y^2 + y^3)\big(s_{(2,1,1)} + s_{(3,1)}\big) + (1 + y + y^3 + y^4)s_{(2,2)}.$$

From ab-indices to P-partitions

Let $J(P)$ be the distributive lattice of order ideals:





Maximal chains in $J(P)$ are linear extensions of P;
the edge labels record the element added at each step.

From P-partitions to Schur functions

Stembridge's P-partition construction realizes Schur functions from Ferrers posets.

1	2
3	4

Ferrers diagram $\lambda = (2, 2)$

For the labeled Ferrers poset of shape λ,

$$s_\lambda = \sum_{T \in \mathrm{SYT}(\lambda)} F_{\mathrm{Des}(T)}.$$

Standard Young tableaux are linear extensions !

Schur functions via ab-indices

Using that standard Young tableaux are linear extensions, we obtain that

$$\Xi : \mathbb{Q}\langle \mathbf{a}, \mathbf{b} \rangle \to \mathrm{QSym}$$

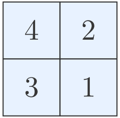
$$\sum_{T \in \mathrm{SYT}(\lambda)} u_{\mathrm{Des}(T)} \mapsto s_\lambda$$

Schur functions are ab-indices inside QSym !

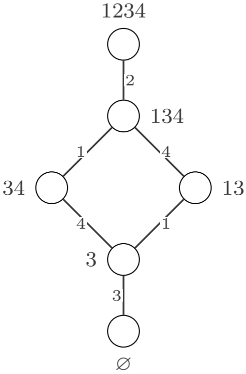
Example: $\lambda = (2,2)$

For $\lambda = (2,2)$, the two standard tableaux give descent sets $\{1,3\}$ and $\{2\}$.

$$s_{(2,2)} = F_{\{1,3\}} + F_{\{2\}} = \Xi(\mathbf{bab} + \mathbf{aba}).$$



Ferrers poset $\lambda = (2,2)$



The two maximal chains have label words

$$3412,$$
$$3142.$$

Thus

$$\mathrm{Des}(3412) = \{2\},$$
$$\mathrm{Des}(3142) = \{1,3\}.$$

A direct computation gives

$$\Omega(s_{(2,2)}) = (y + 2y^2 + y^3)\big(s_{(2,1,1)} + s_{(3,1)}\big) + (1 + y + y^3 + y^4)s_{(2,2)}.$$

The third open problem

Is the extended ab-index related to known
or new extensions of Schur polynomials?

More concretely:

$$\Xi(\Psi_P) \in \mathrm{Sym} \quad \Longrightarrow \quad \Xi(\mathrm{ex}\,\Psi_P) = \Omega\big(\Xi(\Psi_P)\big) \in \mathbb{Q}[y] \otimes \mathrm{Sym}\,.$$

If $\Xi(\Psi_P)$ is Schur-positive, so is $\Xi(\mathrm{ex}\,\Psi_P)$ in $\mathbb{Q}[y] \otimes \mathrm{Sym}$.

How does it relate to other deformations of Schur functions, such as

- Hall–Littlewood polynomials,
- Grothendieck polynomials,
- Schur P- and Q-functions, or
- Macdonald polynomials?

This piece of music was actually the

Musical Epilogue

Thank you for listening!

Gustav Mahler in Dobbiaco

Gustav Mahler (1860–1911) spent his last three summers, **1908–1910**, in Toblach / Dobbiaco — then Austria-Hungary, today South Tyrol, northern Italy. In a small *Komponierhäuschen* just outside the village he wrote **Das Lied von der Erde**, the **Ninth Symphony**, and sketches for the unfinished **Tenth**. Dobbiaco still honours him every summer with the Gustav Mahler Music Weeks in the Kulturzentrum.

Das Lied von der Erde — Kaufmann · Abbado · Berliner Philh.

Kulturzentrum Grand Hotel, Dobbiaco

References and credits

Thank you for listening!

- Dorpalen-Barry, Maglione, Stump: the Poincaré-extended ab-index and the QSym translation.
- Hoster, Stump, Vecchi: the identity $\mathrm{ex}\,\Psi_P = (1+y)\,\omega(\Psi_P)$ for finite graded bounded posets.
- Gessel: quasisymmetric functions, P-partitions, and the internal coproduct formula.
- Stembridge: enriched P-partitions, peak algebra, and Schur functions from P-partitions.
- Ricky Ini Liu: proof of the Schur-positivity conjecture via hook Kronecker products.
- Grinberg and Vassilieva: related extended peak and enriched QSym structures.