

# Algebraic and enumerative invariants of finite graded bounded posets

Lecture 2: Chow polynomials,  $\gamma$ -positivity, and real-rootedness

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Based on joint work with

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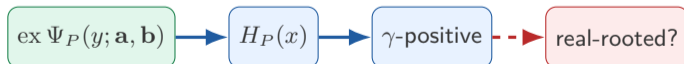
I will tell the story and collect  
precise credits at the end.

# The three lectures

## Algebraic and enumerative invariants of finite graded bounded posets

- L1. Flag enumeration, labelings, and extended ab-positivity
- L2. Chow polynomials,  $\gamma$ -positivity, and real-rootedness
- L3. The  $\omega$ -transform, QSym/Sym, and Schur functions

Today (Lecture 2) we study one-variable evaluations of the extended ab-index:



## Recap: flag vectors

We start with a recap of Lecture 1

Let  $P$  be finite graded bounded of rank  $n$ . For  $S \subseteq [n - 1]$ , the **flag  $f$ -vector** is

$$\alpha_P(S) = \# \left\{ \hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1} : \{\rho(x_1), \dots, \rho(x_{k-1})\} = S \right\}.$$

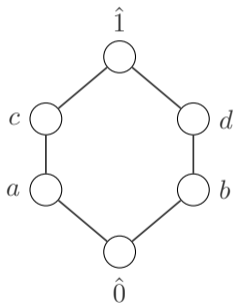
The **flag  $h$ -vector** is the inclusion-exclusion transform

$$\beta_P(S) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} \alpha_P(T), \quad \alpha_P(S) = \sum_{T \subseteq S} \beta_P(T).$$

## Recap: answering a question

The smallest example is the **six-element hexagon poset**  $X_6$ :

$$\hat{0} < a < c < \hat{1}, \quad \hat{0} < b < d < \hat{1}.$$



It has rank 3, so  $S \subseteq \{1, 2\}$ .

$S$	$\emptyset$	$\{1\}$	$\{2\}$	$\{1, 2\}$
$\alpha_{X_6}(S)$	1	2	2	2

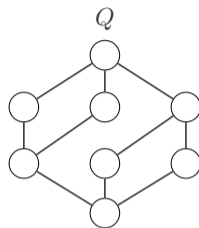
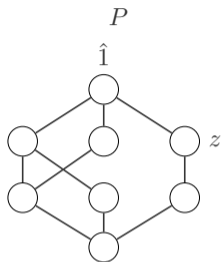
$S$	$\emptyset$	$\{1\}$	$\{2\}$	$\{1, 2\}$
$\beta_{X_6}(S)$	1	1	1	<b>-1</b>

Indeed,

$$\beta_{X_6}(\{1, 2\}) = 2 - 2 - 2 + 1 = -1.$$

The flag  $h$ -vector of an arbitrary graded bounded poset can be negative.

## Recap: answering a second question



When restricting to ranks 1 & 2, they have 3 resp. 4 connected components  $\rightsquigarrow$  non-isomorphic.

<i>S</i>	$\emptyset$	$\{1\}$	$\{2\}$	$\{1,2\}$
$\alpha_P(S) = \alpha_Q(S)$	1	3	3	4

## Recap: chain weights

Let  $\mathcal{C}$  be a chain from  $\hat{0}$  to  $\hat{1}$  with rank set

$$S = S(\mathcal{C}) \subseteq [n - 1].$$

Define two words, one for flag  $h$ -coordinates and one for chain weights:

$$u_i = \begin{cases} \mathbf{b}, & i \in S, \\ \mathbf{a}, & i \notin S, \end{cases} \quad w_i = \begin{cases} \mathbf{b}, & i \in S, \\ \mathbf{a} - \mathbf{b}, & i \notin S. \end{cases}$$

$$u_S = u_1 \cdots u_{n-1}, \quad \text{wt}_{\mathcal{C}}(\mathbf{a}, \mathbf{b}) = w_1 \cdots w_{n-1}.$$

Interchanging  $\mathbf{a} - \mathbf{b} \leftrightarrow \mathbf{a}$  corresponds to the inclusion-exclusion between the flag  $f$ -vector and the flag  $h$ -vector.

## Recap: the three incarnations of the ab-index

The **ab-index** is the noncommutative polynomial

$$\Psi_P(\mathbf{a}, \mathbf{b}) = \sum_{\mathcal{C}} \text{wt}_{\mathcal{C}}(\mathbf{a}, \mathbf{b}) = \sum_S \alpha_P(S) w_S = \sum_S \beta_P(S) u_S \in \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle.$$

The ab-index is the generating function of the flag  $h$ -vector.

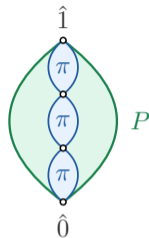
# Recap: the extended ab-index

For a chain

$$\mathcal{C} : \hat{0} = c_0 < c_1 < \cdots < c_k < c_{k+1} = \hat{1},$$

put

$$\pi_{\mathcal{C}}(y) = \prod_{i=0}^k \pi_{[c_i, c_{i+1}]}(y).$$



The **extended ab-index** is the noncommutative polynomial

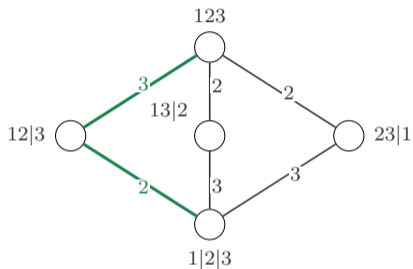
$$\text{ex } \Psi_P(y; \mathbf{a}, \mathbf{b}) = \sum_{\mathcal{C}} \pi_{\mathcal{C}}(y) \text{ wt}_{\mathcal{C}}(\mathbf{a}, \mathbf{b}) \in \mathbb{Z}[y]\langle \mathbf{a}, \mathbf{b} \rangle.$$

There is **no flag  $h$  incarnation**: Poincaré weights depend on the gap intervals of a chain, not only on its rank set.

Since  $\pi_{\mathcal{C}}(0) = 1$  for every chain  $\mathcal{C}$ , we have  $\text{ex } \Psi_P(0; \mathbf{a}, \mathbf{b}) = \Psi_P(\mathbf{a}, \mathbf{b})$ .

## Recap example: the rank-two lattice $\Pi_3$

Let  $P = \Pi_3$ , the rank-two lattice with three atoms.



The ordinary ab-index is

$$\Psi_P(\mathbf{a}, \mathbf{b}) = \mathbf{a} + 2\mathbf{b}.$$

The Poincaré polynomials of the two possible gap types give

$$\begin{aligned} \text{ex } \Psi_P(y; \mathbf{a}, \mathbf{b}) &= (1 + 3y + 2y^2)\mathbf{a} \\ &\quad + (2 + 3y + y^2)\mathbf{b}. \end{aligned}$$

## Recap: the companion-statistic problem

The **coarse flag Hilbert-Poincaré polynomial** is

$$\text{cfHP}_P(y, t) := \sum_{\mathcal{C}} \pi_{\mathcal{C}}(y) \frac{1}{1-t} \left( \frac{t}{1-t} \right)^{\#\mathcal{C}-2} = \frac{\text{ex } \Psi_P(y; 1, t)}{(1-t)^{\text{rk}(P)}} = \frac{\mathcal{N}_P(y, t)}{(1-t)^{\text{rk}(P)}}.$$

If  $\mathcal{A}$  is the reflection arrangement of a finite reflection group  $W$  of rank  $n$  and  $P = L(\mathcal{A})$ , then

$$\{\text{regions of } \mathcal{A}\} \longleftrightarrow W, \quad \mathcal{N}_{L(\mathcal{A})}(1, t) = \#W \cdot E_n(t),$$

where  $E_n(t) = \sum_{u \in \mathfrak{S}_n} t^{\text{des}(u)}$  is the **Eulerian polynomial**.

Find a companion statistic  $\text{st}_{\mathcal{A}} : W \times \mathfrak{S}_n \rightarrow \mathbb{N}$  such that

$$\mathcal{N}_{L(\mathcal{A})}(y, t) = \sum_{w \in W, \sigma \in \mathfrak{S}_n} y^{\text{st}_{\mathcal{A}}(w, \sigma)} \cdot t^{\text{des}(\sigma)}.$$

## Recap: the braid-arrangement case

For the braid arrangement  $\mathcal{A}_{n-1} = \{x_i = x_j : 1 \leq i < j \leq n\}$ , we have

$$L(\mathcal{A}_{n-1}) = \Pi_n, \quad \text{rk}(\mathcal{A}_{n-1}) = n - 1, \quad \{\text{regions}\} = \mathfrak{S}_n.$$

Then

$$\mathcal{N}_{\Pi_n}(y, t) = \sum_{(w, \sigma) \in \mathfrak{S}_n \times \mathfrak{S}_{n-1}} y^{\text{st}(w, \sigma)} \cdot t^{\text{des}(\sigma)}.$$

We solved this in joint work with Elena Hoster.

Solving the problem would in particular solve the longstanding open problem to generalize left-to-right maxima on permutations to reflection groups.

We now turn to the main structure of this lecture:

## **Chow polynomials**

# The Chow polynomial

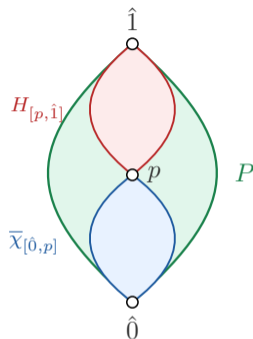
For an interval  $[u, v]$  in a finite graded bounded poset, set

$$\chi_{[u,v]}(x) = \sum_{u \leq z \leq v} \mu(u, z) x^{\rho(v) - \rho(z)},$$

$$\bar{\chi}_{[u,v]}(x) = \chi_{[u,v]}(x) / (x - 1) \quad (u < v).$$

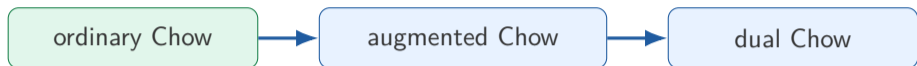
The **Chow polynomial** is defined recursively  
(**Ferroni–Matherne–Vecchi**, 2024) by

$$H_P(x) = \sum_{\hat{0} < p \leq \hat{1}} \bar{\chi}_{[\hat{0}, p]}(x) H_{[p, \hat{1}]}(x), \quad H_{\{\hat{1}\}}(x) = 1.$$



This is the definition of the ordinary Chow polynomial in this lecture.

# Chow-type invariants around matroids



We consider here the ordinary Chow polynomial and show what we can learn about it by identifying it as a specialization of the extended ab-index.

**augmented Chow polynomials** and the recently introduced **dual Chow polynomials** (last week by Caiolo–Ferroni–Hoster) also fit into the extended ab-index picture and are evaluations thereof.

# The Hilbert series of the Chow ring

Let  $\text{CH}(M)$  denote the **Chow ring** of the matroid  $M$  without definition. Set

$$H_M(x) = \sum_{i \geq 0} \dim_{\mathbb{Q}} \text{CH}^i(M) x^i.$$

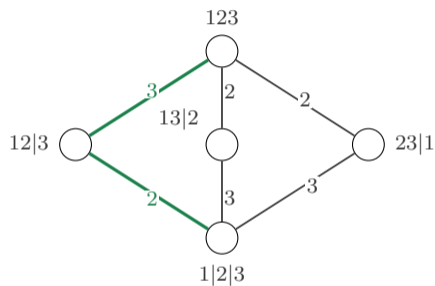


The Feichtner–Yuzvinsky basis shows that this Hilbert series satisfies the recursion; hence

$$H_M(x) = H_{\mathcal{L}(M)}(x).$$

- ▶ Introduced by **Feichtner–Yuzvinsky** (2004); generalizes the Chow ring of the wonderful compactification of an arrangement (**De Concini–Procesi**, 1995).
- ▶  $\text{CH}(M)$  has a **Kähler package** (**Adiprasito–Huh–Katz**, 2018): Poincaré duality and hard Lefschetz, hence  $H_M(x)$  is **palindromic and unimodal** — the beginning of the story.

## Example: the partition lattice $\Pi_3$



The interval to an atom has

$$\bar{\chi}_{[\hat{0}, p]}(x) = 1.$$

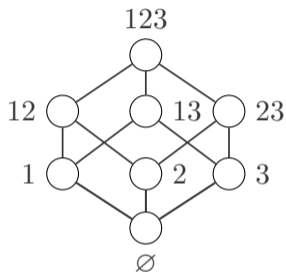
There are three atoms.

For the top element:

$$\chi_{\Pi_3}(x) = (x - 1)(x - 2), \quad \bar{\chi}_{\Pi_3}(x) = x - 2.$$

$$H_{\Pi_3}(x) = 3 + (x - 2) = 1 + x$$

## Example: the Boolean lattice $B_3$



Boolean intervals satisfy

$$\chi_{B_r}(x) = (x-1)^r, \quad \bar{\chi}_{B_r}(x) = (x-1)^{r-1}.$$

The recursion gives

$$\begin{aligned} H_{B_3}(x) &= 3H_{B_2}(x) + 3(x-1)H_{B_1}(x) + (x-1)^2 \\ &= 1 + 4x + x^2. \end{aligned}$$

$H_{B_n}(x) = A_n(x)$  is the Eulerian polynomial.

# A hierarchy of positivity

For palindromic polynomials with nonnegative coefficients:

$$\text{real-rooted} \implies \gamma\text{-positive} \implies \text{unimodal}.$$

$\gamma$ -positivity of the Chow polynomial of a matroid follows from the Kähler package.

Or, more briefly:  $\gamma$ -positivity is a deeper strengthening beyond hard Lefschetz.

The theory's master conjecture: Chow polynomials of matroids are real-rooted.

# Evaluation theorem

## Evaluation theorem (St 2024)

For every finite graded bounded poset  $P$ ,

$$\text{ex } \Psi_P(-x; 1, x) = (1 - x)^{\text{rk}(P)} H_P(x).$$

Iterating the recursion gives the explicit chain formula  $H_P(x) = \sum_{\mathcal{C}} \bar{\chi}_{\mathcal{C}}(x)$ .

Compare this with

$$\text{ex } \Psi_P(y; \mathbf{a}, \mathbf{b}) = \sum_{\mathcal{C}} \pi_{\mathcal{C}}(y) \text{wt}_{\mathcal{C}}(\mathbf{a}, \mathbf{b}).$$

Both invariants are built by summing over chains and their gap intervals.

Once we see this, the evaluation theorem can be worked out.

## Example: $\Pi_3$ from the extended ab-index

From Lecture 1:

$$\text{ex } \Psi_{\Pi_3}(y; \mathbf{a}, \mathbf{b}) = (1 + 3y + 2y^2)\mathbf{a} + (2 + 3y + y^2)\mathbf{b}.$$

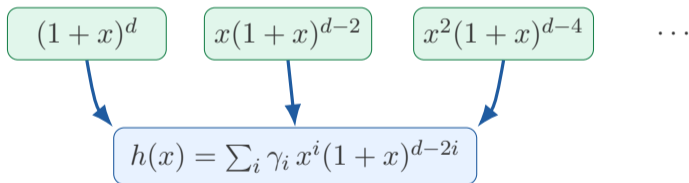
Evaluate at  $y = -x$ ,  $\mathbf{a} = 1$ ,  $\mathbf{b} = x$ :

$$\begin{aligned} \text{ex } \Psi_{\Pi_3}(-x; 1, x) &= (1 - 3x + 2x^2) + (2 - 3x + x^2)x \\ &= 1 - x - x^2 + x^3 \\ &= (1 - x)^2(1 + x) \\ &= (1 - x)^{\text{rk}(\Pi_3)} \cdot H_{\Pi_3}(x). \end{aligned}$$

## $\gamma$ -positivity

Every palindromic polynomial  $h(x) = h_0 + h_1x + \cdots + h_dx^d$  has a unique  $\gamma$ -**expansion**

$$h(x) = \sum_{i=0}^{\lfloor d/2 \rfloor} \gamma_i x^i (1+x)^{d-2i}. \quad [\text{with } \gamma_0 = h(0)].$$



$h(x)$  is  $\gamma$ -**positive** if all  $\gamma_i \geq 0$ .

## Small examples

$$H_{\Pi_3}(x) = 1 + x.$$

$$\begin{aligned} H_{B_3}(x) &= 1 + 4x + x^2 \\ &= (1 + x)^2 + 2x. \end{aligned}$$

Our goal is to now reprove  $\gamma$ -positivity without using the Kähler package.

**Mathematics Break**

## Leiden Declaration on Artificial Intelligence and Mathematics

Released **2 June 2026** and endorsed by the **International Mathematical Union**. It addresses the challenges AI poses to mathematical research and calls on mathematicians to preserve the discipline's core values — **proof-based certainty, proper attribution, transparency, and self-directed research** — with recommendations for individual mathematicians, academic organizations, policymakers, and commercial AI developers.

*“Mathematics is, and should always remain, a profoundly human endeavour.”*

Signed by **844 mathematicians in the past day**, including Fields Medalists **Terence Tao, Peter Scholze, Martin Hairer, James Maynard, Hugo Duminil-Copin**.

[leidendeclaration.ai](https://www.leidendeclaration.ai)

[New York Times article](#)

## Back to yesterday's challenge

So far, **two** problem have been submitted.

Every **unsolvable** problem that reaches me before **Friday, 10:30** will be featured in the Friday break.

[example submissions](#)

[example chat](#)

## Isolated descent sets

Let  $P$  admit an R-labeling. Maximal chains have descent sets

$$\text{Des}(\mathcal{M}) \subseteq \{1, \dots, n-1\}.$$

$S$  is **isolated** if  $i \in S \implies i+1 \notin S$ .

$\{2, 5, 8\}$

isolated

$\{2, 3, 7\}$

not isolated

# The combinatorial $\gamma$ -expansion

## Theorem (St. 2024)

Let  $P$  be a rank  $n$  poset admitting an R-labeling. Then

$$H_P(x) = \sum_{\substack{\mathcal{M} \text{ maximal chain} \\ \{0\} \cup \text{Des}(\mathcal{M}) \text{ isolated}}} x^{|\text{Des}(\mathcal{M})|} (1+x)^{n-1-2|\text{Des}(\mathcal{M})|}.$$

The summands are precisely the  $\gamma$ -basis terms.

In particular,  $H_P(x)$  is  $\gamma$ -positive for R-labeled posets.

We next prove this theorem.

# The $\omega$ -transform

$$\omega : \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle \rightarrow \mathbb{Z}[y]\langle \mathbf{a}, \mathbf{b} \rangle.$$

Word-level definition: first replace every occurrence of

$$\mathbf{ab} \mapsto (1 + y)\mathbf{ab} + y(1 + y)\mathbf{ba},$$

then replace all remaining letters by

$$\mathbf{a} \mapsto \mathbf{a} + y\mathbf{b}, \quad \mathbf{b} \mapsto \mathbf{b} + y\mathbf{a}.$$

## Theorem (Hoster–St–Vecchi 2025)

For every finite graded bounded poset  $P$ ,

$$\text{ex } \Psi_P(y; \mathbf{a}, \mathbf{b}) = (1 + y)\omega(\Psi_P(\mathbf{a}, \mathbf{b})).$$

Surprise: the ordinary ab-index already knows the extended ab-index.

# The $\omega$ -transform

$$\omega : \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle \rightarrow \mathbb{Z}[y]\langle \mathbf{a}, \mathbf{b} \rangle, \quad \text{ex } \Psi_P(y; \mathbf{a}, \mathbf{b}) = (1 + y)\omega(\Psi_P(\mathbf{a}, \mathbf{b})).$$

Word-level definition: first replace every occurrence of

$$\mathbf{ab} \mapsto (1 + y)\mathbf{ab} + y(1 + y)\mathbf{ba},$$

then replace all remaining letters by

$$\mathbf{a} \mapsto \mathbf{a} + y\mathbf{b}, \quad \mathbf{b} \mapsto \mathbf{b} + y\mathbf{a}.$$

The **Chow evaluation** yields

$$y = -x, \quad \mathbf{a} = 1, \quad \mathbf{b} = x$$

$$\omega(\mathbf{ab}) \mapsto x(1 - x)^2, \quad \omega(\mathbf{a}) \mapsto (1 - x)(1 + x), \quad \omega(\mathbf{b}) \mapsto 0$$

# Chow from maximal chains

Lecture 1:  $\Psi_P(\mathbf{a}, \mathbf{b}) = \sum_{\mathcal{M} \text{ maximal}} u_{\text{Des}(\mathcal{M})}(\mathbf{a}, \mathbf{b})$ . Using  $\text{ex } \Psi_P = (1 + y) \omega(\Psi_P)$ , we get

$$(1 - x)^n H_P(x) = \sum_{\mathcal{M} \text{ maximal}} (1 + y) \omega(u_{\text{Des}(\mathcal{M})})_{y=-x, \mathbf{a}=1, \mathbf{b}=x}$$

Recall the three evaluations:

$$\omega(\mathbf{a}) \mapsto (1 - x)(1 + x), \quad \omega(\mathbf{b}) \mapsto 0, \quad \omega(\mathbf{ab}) \mapsto x(1 - x)^2.$$

This is the mechanism behind the isolated-descent  $\gamma$ -expansion.

# Real-rootedness conjecture for posets

The theory's master conjecture: Chow polynomials of matroids are real-rooted.

This now generalizes to

Which evaluations of the extended ab-index are real-rooted?

Other real-rootedness conjectures come from evaluations of  $\text{ex } \Psi_P$ :

$$\mathcal{N}_P(1, t) = \text{ex } \Psi_P(1; 1, t) \quad \text{Maglione–Voll 2023}$$

$$C_P(q) = \text{ex } \Psi_P(0; 1 + q, q) \quad \text{Athanasiadis–Kalampoglia–Evangelinou 2024}$$

For which posets is the real-rootedness known?

$$H_{B_n}(x) = A_n(x),$$

where

$$A_n(x) = \sum_{\sigma \in \mathfrak{S}_n} x^{\text{des}(\sigma)}$$

is the Eulerian polynomial. Equivalently, in the isolated-subset form that exhibits the  $\gamma$ -expansion,

$$A_n(x) = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \{0\} \cup \text{Des}(\sigma) \text{ isolated}}} x^{|\text{Des}(\sigma)|} (1+x)^{n-1-2|\text{Des}(\sigma)|}.$$

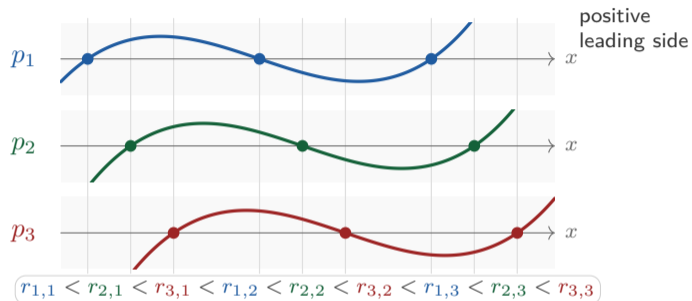
$$H_{B_3}(x) = 1 + 4x + x^2.$$

Eulerian polynomials are real-rooted.

# For which posets is the real-rootedness known?

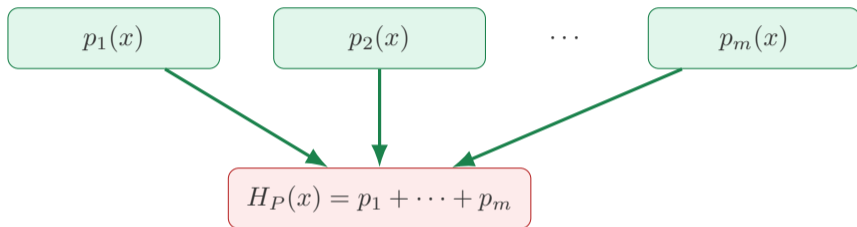
- ▶ Uniform matroids: Brändén–Vecchi (2025)
- ▶ Simplicial posets with positive  $h$ -vector: Hoster–Stump (2025).
- ▶ Maximally ranked posets / totally-nonnegative families: Brändén–Vecchi (2026)
- ▶ Real-Rootedness of the Poincaré polynomials of the Deligne-Mumford moduli space of stable  $n$ -pointed rational curves: an AI-assisted proof (last week)

# Interlacing roots



Real-rooted polynomials with positive leading coefficients **interlace** if their roots alternate.

## A decomposition philosophy



If the summands form an interlacing sequence, then  $H_P(x)$  is real-rooted.

# A conjecture for you to work on: first-step polynomials

**$\gamma$ -polynomial of the Chow polynomial**

$$\gamma(H_M)(x) = \sum_{\substack{\mathcal{M} \\ \text{ddeg}(\lambda_{\mathcal{M}})=0, \lambda_1 < \lambda_2}} x^{\text{des}(\lambda_{\mathcal{M}})}.$$

$\gamma(H_M)$  is real-rooted if and only if  $H_M$  is real-rooted.

We conjecture each of the sequences  $\{a_{M,k}\}_k$ ,  $\{b_{M,k}\}_k$ ,  $\{c_{M,k}\}_k$  to be interlacing for some atom ordering.

(Hoster–St 2025)

For a possible first label  $\lambda_1 = k$ :

$$a_{M,k}(x) = \sum_{\lambda_1 < \lambda_2}^{\mathcal{M}} x^{\text{des}(\lambda_{\mathcal{M}})},$$

$$b_{M,k}(x) = \sum_{\lambda_1 > \lambda_2}^{\mathcal{M}} x^{\text{des}(\lambda_{\mathcal{M}})},$$

$$c_{M,k}(x) = a_{M,k}(x) + b_{M,k}(x)$$

$$\gamma(H_M) = \sum_k a_{M,k}.$$

## Thank you for listening!

- ▶ Feichtner-Yuzvinsky: Chow rings of atomic lattices and building sets.
- ▶ Adiprasito-Huh-Katz: Hodge theory for combinatorial geometries.
- ▶ Braden-Huh-Matherne-Proudfoot-Wang: augmented Chow rings and semi-small decompositions.
- ▶ Ferroni-Matherne-Stevens-Vecchi: matroid Chow Hilbert-Poincaré series and  $\gamma$ -positivity.
- ▶ Ferroni-Matherne-Vecchi: Chow functions for partially ordered sets.
- ▶ Dorpalen-Barry-Maglione-Stump: the Poincaré-extended ab-index.
- ▶ Stump: Chow and augmented Chow polynomials as evaluations of Poincaré-extended ab-indices.
- ▶ Hoster-Stump: real-rootedness for simplicial posets with positive  $h$ -vector.
- ▶ Hoster-Stump-Vecchi: extending the ab-index via the  $\omega$ -transform.
- ▶ Kühne-Maglione: geometry of flag Hilbert-Poincaré series for matroids.
- ▶ Athanasiadis-Kalampoglia-Evangelinou: chain polynomials and real-rootedness conjecture.
- ▶ Brändén-Vecchi: real-rootedness results for uniform matroids and totally nonnegative posets.
- ▶ Caiolo-Ferroni-Hoster: dual Chow polynomials of matroids and posets.