

Algebraic and enumerative invariants of finite graded bounded posets

Lecture 1: Flag enumeration, labelings, and extended ab-positivity

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In parts based on joint work with
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I will not give precise credits but
rather try to tell a coherent story

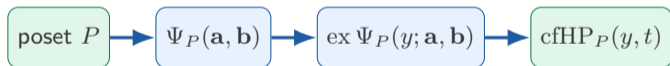
The slides were generated by playing pingpong with ClaudeCode and ChatGPT

The three lectures

Algebraic and enumerative invariants of finite graded bounded posets

- L1. Flag enumeration, labelings, and extended ab-positivity
- L2. Chow polynomials, γ -positivity, and real-rootedness
- L3. The ω -transform, QSym/Sym, and Schur functions

Today (Lecture 1) we meet the main objects and how they refine one another:



R-labelings turn each arrow into a *positivity* statement.

Finite graded bounded posets

Let P be a finite poset (partially ordered set).

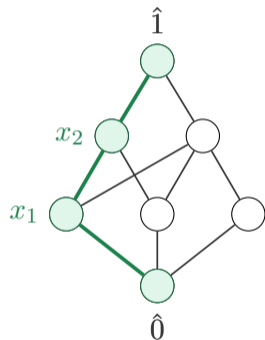
- ▶ P is **bounded** if it has $\hat{0}$ and $\hat{1}$.
- ▶ P is **graded** if it has a rank function

$$\rho : P \longrightarrow \mathbb{N}, \quad \rho(\hat{0}) = 0,$$

increasing by 1 along cover relations.

- ▶ We write

$$\text{rk}(P) = \rho(\hat{1}).$$



A **chain** is a totally ordered subset

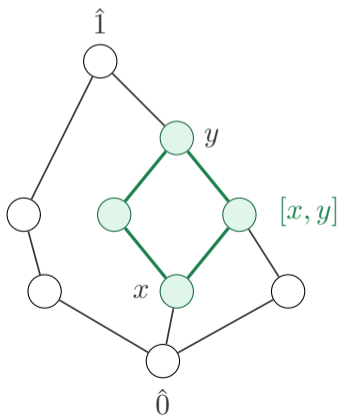
$$\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1}.$$

For instance, $\hat{0} < x_1 < x_2 < \hat{1}$ is a maximal chain in the poset above.

Intervals in posets

For $x \leq y$ in P , the interval is

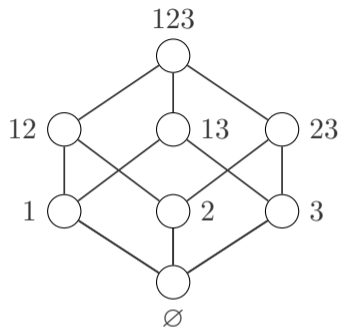
$$[x, y] = \{z \in P : x \leq z \leq y\}.$$



The interval point of view gives rise to an incidence algebra structure on posets.

Important examples

- ▶ Boolean lattices B_n
- ▶ Partition lattices Π_n
- ▶ Intersection posets $L(\mathcal{A})$ of arrangement \mathcal{A}
- ▶ Lattices of flats $\mathcal{L}(M)$ of matroids
- ▶ Simplicial posets

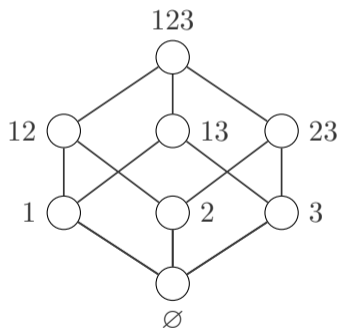


The language of posets—in this context—captures structures from:

algebraic geometry, topology, matroid theory, hyperplane arrangements, and enumerative combinatorics.

Example: the Boolean lattice

The Boolean lattice B_n consists of all subsets of $[n]$ ordered by inclusion.



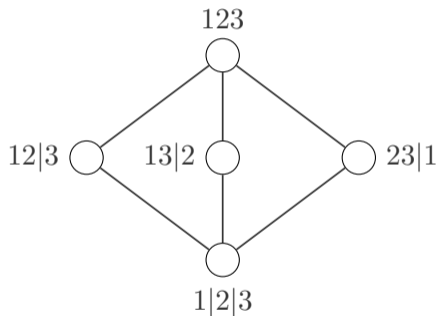
$$\hat{0} = \emptyset, \quad \hat{1} = [n], \quad \rho(S) = |S|.$$

A maximal chain is the same as a permutation $\pi = (\pi_1, \pi_2, \dots, \pi_n)$ of $[n]$:

$$\emptyset \subset \{\pi_1\} \subset \{\pi_1, \pi_2\} \subset \dots \subset [n].$$

Example: the partition lattice

The partition lattice Π_n consists of all set partitions of $[n]$, ordered by refinement.



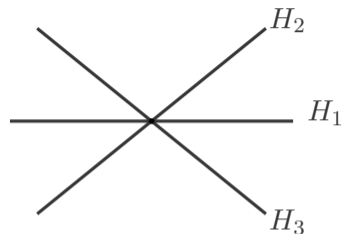
$$\hat{0} = 1|2|3, \quad \hat{1} = 123,$$

$$\rho(\sigma) = n - (\#\text{blocks of } \sigma).$$

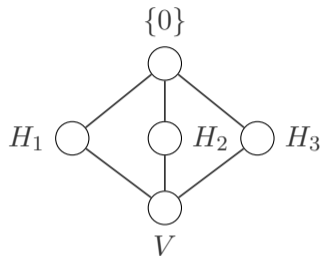
So $\text{rk}(\Pi_3) = 2$, with three coatoms.

Π_n is a geometric lattice: the lattice of flats of the braid arrangement.

Example: arrangements and matroids



$$\mathcal{A} \rightsquigarrow L(\mathcal{A})$$



The **intersection poset** $L(\mathcal{A})$ consists of all intersections of subcollections of hyperplanes (the *flats*), ordered by **reverse inclusion**: the ambient space V is $\hat{0}$, and intersecting more hyperplanes moves *upward*.

Intersection poset of hyperplane arrangements \rightsquigarrow lattice of flats of realizable matroids

From flats to faces

For the braid arrangement

$$\mathcal{A}_{n-1} = \{x_i = x_j : 1 \leq i < j \leq n\},$$

flats and faces give two natural partition posets.

Flats

$$L(\mathcal{A}_{n-1}) \cong \Pi_n$$

Set partitions record
equal coordinate blocks.

Faces

$$\Sigma(\mathcal{A}_{n-1}) \leftrightarrow \text{ordered set partitions}$$

Ordered partitions record the
strict order of the blocks.

A face encoded by $B_1 | \cdots | B_k$ is given by

$$x_i = x_j \text{ within each } B_m, \quad x_{B_1} < x_{B_2} < \cdots < x_{B_k}.$$

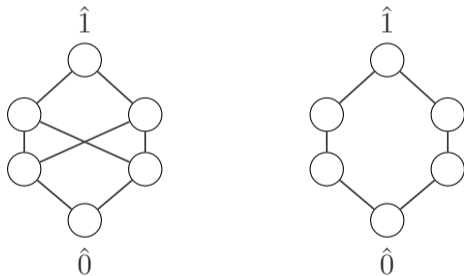
Full-dimensional regions correspond to ordered singleton partitions, hence to permutations.

Rank enumeration is not fine enough

The rank numbers

$$f_i(P) = \#\{x \in P : \rho(x) = i\}$$

are useful, but they forget incidences:



Instead of counting elements of each rank, count chains using their rank sets.

Flag f-vector

Let P be graded of rank n . For $S \subseteq [n - 1]$, the **flag f-vector** is

$$\alpha_P(S) = \#\{\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1} : \{\rho(x_1), \dots, \rho(x_{k-1})\} = S\}.$$

- ▶ $\alpha_P(\emptyset) = 1$.
- ▶ $\alpha_P(\{i\})$ counts elements of rank i .
- ▶ $\alpha_P(S)$ counts chains passing through exactly the ranks in S .

Flag enumeration records which ranks occur together in chains.

Can you find two non-isomorphic posets with the same flag f -vector?

Flag h -vector

The **flag h -vector** is the inclusion-exclusion transform of the flag f -vector:

$$\beta_P(S) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} \alpha_P(T),$$

analogously to the usual f - vs. h -vector of a simplicial complex. Equivalently,

$$\alpha_P(S) = \sum_{T \subseteq S} \beta_P(T).$$

The $\beta_P(S)$ are the structurally more interesting invariants:

They carry the interesting symmetries and provide the right coordinates for further investigations.

A first computation: B_3

For B_n , maximal chains are permutations of $[n]$, and (under the standard labeling) the **descent set** of a chain is $\text{Des}(\pi)$. Hence the flag h -vector counts permutations by descent set:

$$\beta_{B_n}(S) = \#\{\pi \in \mathfrak{S}_n : \text{Des}(\pi) = S\}.$$

Eulerian numbers:

$$A(n, k) = \#\{\pi \in \mathfrak{S}_n : \text{des}(\pi) = k\} = \sum_{|S|=k} \beta_{B_n}(S).$$

$$A(3, 0), A(3, 1), A(3, 2) = 1, 4, 1.$$

π	$\text{Des}(\pi)$	S	$\beta_{B_3}(S)$
123	\emptyset	\emptyset	1
132	$\{2\}$	$\{1\}$	2
213	$\{1\}$	$\{2\}$	2
231	$\{2\}$	$\{1, 2\}$	1
312	$\{1\}$		
321	$\{1, 2\}$		

For B_n the flag h -vector is the descent statistic.

From subsets to words

Let $S \subseteq [n - 1]$. Associate to S a word in the noncommutative alphabet

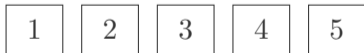
$$u_S = u_1 u_2 \cdots u_{n-1} \quad \text{with} \quad u_S \in \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle.$$

by

$$u_i = \begin{cases} \mathbf{b}, & i \in S, \\ \mathbf{a}, & i \notin S, \end{cases} \quad \text{and} \quad w_i = \begin{cases} \mathbf{b}, & i \in S, \\ \mathbf{a} - \mathbf{b}, & i \notin S, \end{cases}$$

The **chain monomial** is $u_S = u_1 \cdots u_{n-1}$ and **chain weight** is $wt_S = w_1 \cdots w_{n-1}$.

$$S = \{2, 5\}$$



$$\mathbf{a} \quad \mathbf{b} \quad \mathbf{a} \quad \mathbf{a} \quad \mathbf{b} \quad u_S = \mathbf{a} \mathbf{b} \mathbf{a} \mathbf{a} \mathbf{b}$$

$$\mathbf{a} - \mathbf{b} \quad \mathbf{b} \quad \mathbf{a} - \mathbf{b} \quad \mathbf{a} - \mathbf{b} \quad \mathbf{b} \quad wt_S = (\mathbf{a} - \mathbf{b}) \mathbf{b} (\mathbf{a} - \mathbf{b}) (\mathbf{a} - \mathbf{b}) \mathbf{b}$$

The ab-index

The **ab-index** of P is the sum of chain weights over all chains from $\hat{0}$ to $\hat{1}$:

$$\Psi_P(\mathbf{a}, \mathbf{b}) = \sum_{\mathcal{C}} \text{wt}_{\mathcal{C}}(\mathbf{a}, \mathbf{b}) \in \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle.$$

Expanding the weights and collecting terms gives, as a consequence,

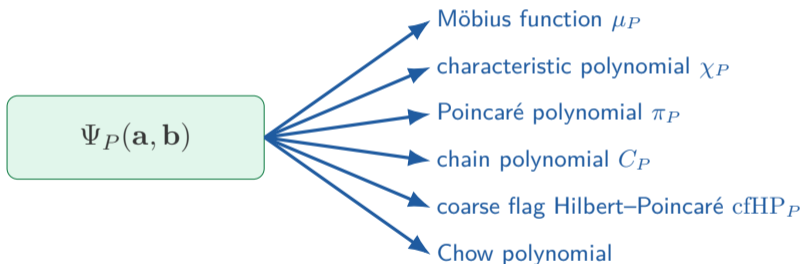
$$\Psi_P(\mathbf{a}, \mathbf{b}) = \sum_{S \subseteq [n-1]} \alpha_P(S) \text{wt}_S = \sum_{S \subseteq [n-1]} \beta_P(S) u_S.$$

The ab-index is the flag h -vector written in noncommutative coordinates.

What the ab-index remembers

Two non-isomorphic posets can share the same flag f -vector, and hence the same ab-index $\Psi_P(\mathbf{a}, \mathbf{b})$.

Nonetheless, many important enumerative invariants are obtained from the ab-index:



Many classical one-variable invariants are shadows of richer flag data.

Example: the ab-index of B_3 and Π_3

Using $\Psi_P = \sum_{S \subseteq [n-1]} \beta_P(S) u_S$ with the flag h -vectors we already computed.

For B_3 :

$$(\beta_{B_3}(\emptyset), \beta_{B_3}(\{1\}), \beta_{B_3}(\{2\}), \beta_{B_3}(\{1, 2\})) = (1, 2, 2, 1):$$

$$\Psi_{B_3}(\mathbf{a}, \mathbf{b}) = \mathbf{aa} + 2\mathbf{ba} + 2\mathbf{ab} + \mathbf{bb}.$$

For Π_3 :

$$(\beta_{\Pi_3}(\emptyset), \beta_{\Pi_3}(\{1\})) = (1, 2):$$

$$\Psi_{\Pi_3}(\mathbf{a}, \mathbf{b}) = \mathbf{a} + 2\mathbf{b}.$$

We read the ab-index directly from the descent counts $\beta_P(S)$.

Examples of classical poset invariants

Möbius function : $\mu_P(x, y),$

characteristic polynomial : $\chi_P(y) = \sum_{x \in P} \mu_P(\hat{0}, x) y^{\text{rk}(P) - \rho(x)},$

Poincaré polynomial : $\pi_P(y) = \sum_{x \in P} \mu_P(\hat{0}, x) (-y)^{\rho(x)},$

chain polynomial : $C_P(y) = \sum_{\mathcal{C}} y^{\#\mathcal{C} - 2},$

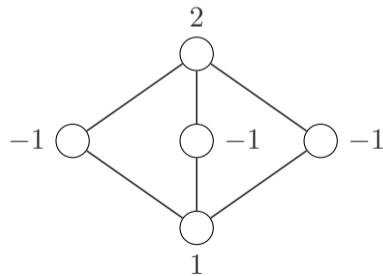
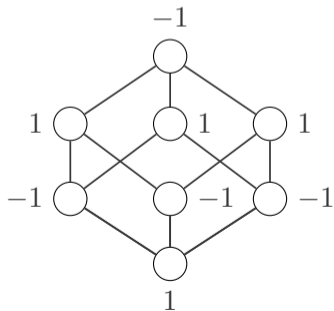
	B_3	Π_3
$\mu_P(\hat{0}, \hat{1})$	-1	2
$\chi_P(y)$	$(y - 1)^3$	$(y - 1)(y - 2)$
$\pi_P(y)$	$(1 + y)^3$	$(1 + y)(1 + 2y)$

Möbius labels for B_3 and Π_3

The label at a vertex x is $\mu_P(\hat{0}, x)$.

The Möbius function is defined recursively by

$$\mu_P(x, x) = 1, \quad \mu_P(x, y) = - \sum_{x \leq z < y} \mu_P(x, z) \quad (x < y).$$



Edge labelings

Let \mathbb{N} be the linearly ordered label set. An edge labeling is a map

$$\lambda : \{x \triangleleft y\} \longrightarrow \mathbb{N}.$$

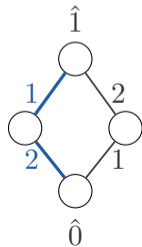
A maximal chain

$$\mathcal{M} : \hat{0} = x_0 \triangleleft x_1 \triangleleft \cdots \triangleleft x_n = \hat{1}$$

gets a label word

$$\lambda(\mathcal{M}) = \lambda_1 \lambda_2 \cdots \lambda_n, \quad \lambda_i = \lambda(x_{i-1}, x_i).$$

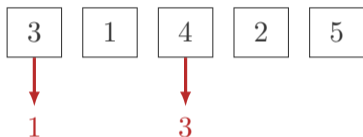
$$\lambda(\mathcal{M}) = 21$$



Descents

For a label word $\lambda_1\lambda_2\cdots\lambda_n$, define

$$\text{Des}(\lambda) = \{i \in [n-1] : \lambda_i > \lambda_{i+1}\}.$$



$$\text{Des}(31425) = \{1, 3\}$$

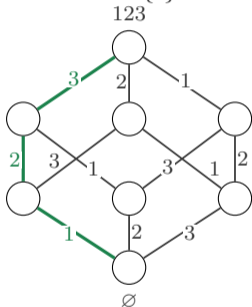
We will again see:

Descents are the link between labelings and flag h-vectors.

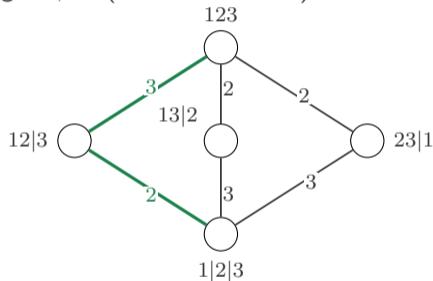
R-labelings

An edge labeling is an **R-labeling** if every interval $[x, y]$ has a unique increasing maximal chain.

cover $S \prec S \cup \{i\}$ labeled i



merge B, B' ($\min B < \min B'$) labeled $\min B'$



EL-labelings: the unique increasing chain is also lex-smallest.

The descent theorem

If P admits an R-labeling, then

$$\beta_P(S) = \#\{\text{maximal chains } \mathcal{M} \text{ of } P : \text{Des}(\mathcal{M}) = S\}.$$

Equivalently,

$$\Psi_P(\mathbf{a}, \mathbf{b}) = \sum_{\mathcal{M}} u_{\text{Des}(\mathcal{M})}.$$

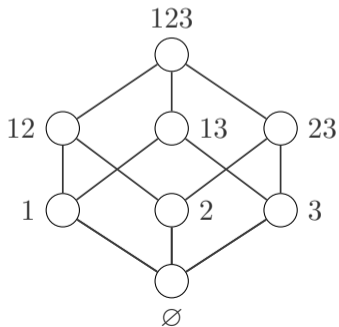
recall $u_S = u_1 \cdots u_{n-1}$ with $u_i = \mathbf{b}$ for $i \in S$ and $u_i = \mathbf{a}$ otherwise.

P admits an R-labeling \Rightarrow The ab-index has nonnegative coefficients.

Boolean lattices: the prototype

For B_n , label the cover relation $S \lessdot S \cup \{i\}$ by i .

This is clearly an R-labeling: in any interval $[S, T]$ the unique increasing chain adds the elements of $T \setminus S$ in increasing order.



A maximal chain corresponds to

$$\pi = \pi_1 \pi_2 \cdots \pi_n \in \mathfrak{S}_n.$$

Then

$$\text{Des}(\mathcal{M}) = \text{Des}(\pi).$$

Thus

$$\beta_{B_n}(S) = \#\{\pi \in \mathfrak{S}_n : \text{Des}(\pi) = S\}.$$

More examples of R-labeled posets

- ▶ Boolean lattices B_n
- ▶ Partition lattices Π_n
- ▶ Lattices of flats $\mathcal{L}(M)$ of matroids

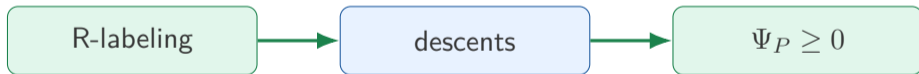
Lattices of flats of matroids (= geometric lattices) are R-labeled by the **standard atom labeling**: fix a linear order on the atoms and label each cover $x \lessdot y$ by the smallest atom a with $x \vee a = y$.

Positivity of the ab-index

The R-labeling story gives

$$P \text{ R-labeled} \implies \Psi_P(\mathbf{a}, \mathbf{b}) \in \mathbb{N}\langle \mathbf{a}, \mathbf{b} \rangle.$$

More precisely, the coefficient of u_S is a descent number.



Commercial Break

Challenge for everyone

- ▶ Send me a mathematics question or a problem with a **unique, well-defined answer which is known to you**.

It may involve computation, but its core must be **theoretical**.

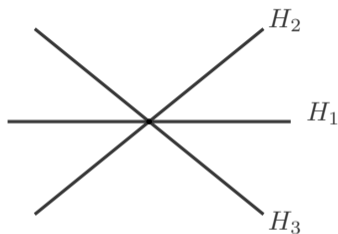
- ▶ Make it as hard as you can — I will try to solve it using AI.
- ▶ I send you back my answer & you tell me if my answer is correct.
- ▶ If I get it wrong, you receive 20 credit points for the AI chat system on my project **math.ScienceBench.ai**.

[example submissions](#)

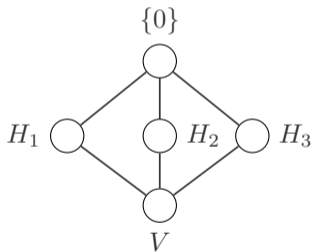
[example chat](#)

Two posets from one arrangement

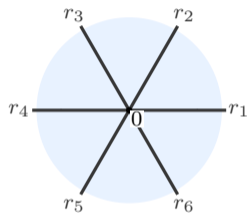
A real hyperplane arrangement has two closely related posets.



$L(\mathcal{A}) =$ intersection poset



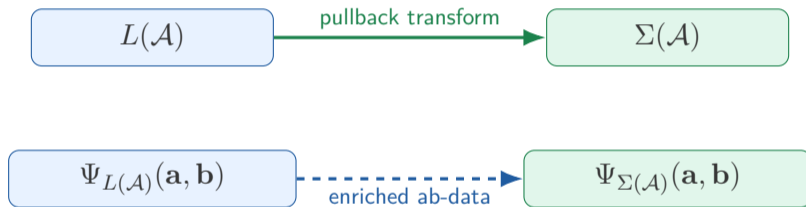
$\Sigma(\mathcal{A}) =$ face poset



How much of the face side is already encoded by the intersection side?

The pullback philosophy

For arrangements, the ab-index of the face poset is controlled by data on the intersection poset.



Based on this ideal, we now discuss how to enrich the ab-index.

Why an extension is needed

The ordinary ab-index records rank sets of chains.

The geometry asks for more:

- ▶ Poincaré polynomials of intervals,
- ▶ pullback transformations from intersections to faces,
- ▶ Chow and augmented Chow polynomials,
- ▶ coarse flag Hilbert–Poincaré series,
- ▶ positivity after inserting extra polynomial parameters.

We extend the ab-index by

inserting Poincaré polynomials of intervals into the gaps of each chain.

$$\Psi_P(\mathbf{a}, \mathbf{b}) \rightsquigarrow \text{ex } \Psi_P(y; \mathbf{a}, \mathbf{b}).$$

Poincaré polynomial of an interval

Remember:

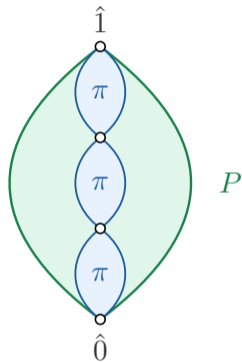
$$\pi_P(y) = \sum_{x \in P} \mu_P(\hat{0}, x) (-y)^{\rho(x)}.$$

For a chain

$$\mathcal{C} : \hat{0} = c_0 < c_1 < \dots < c_k < c_{k+1} = \hat{1},$$

put

$$\pi_{\mathcal{C}}(y) = \prod_{i=0}^k \pi_{[c_i, c_{i+1}]}(y).$$



The extra parameter y measures Poincaré data in the gaps of a chain.

Example: Poincaré polynomials in B_3 and Π_3

Every interval of B_n is again Boolean, so $\pi_{[S,T]}(y) = (1+y)^{|T|-|S|}$.

For the chain $\mathcal{C} : \emptyset < \{1\} < \{1, 2, 3\}$ in B_3 , the two gaps give

$$\pi_{\mathcal{C}}(y) = \underbrace{(1+y)}_{[\emptyset, \{1\}]} \cdot \underbrace{(1+y)^2}_{[\{1\}, \{1, 2, 3\}]} = (1+y)^3.$$

For the full interval of Π_3 ,

$$\pi_{[\hat{0}, \hat{1}]}(y) = 1 + 3y + 2y^2 = (1+y)(1+2y).$$

The extended ab-index

The **extended ab-index** is the Poincaré-weighted chain sum

$$\text{ex } \Psi_P(y; \mathbf{a}, \mathbf{b}) = \sum_{\mathcal{C}} \pi_{\mathcal{C}}(y) \text{ wt}_{\mathcal{C}}(\mathbf{a}, \mathbf{b}) \in \mathbb{Z}[y]\langle \mathbf{a}, \mathbf{b} \rangle.$$

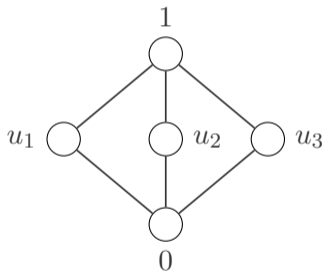
It is the same chain enumeration as the ab-index, but with Poincaré polynomials inserted into the intervals of a chain.

Since $\pi_{\mathcal{C}}(0) = 1$, we have that the extended ab-index is indeed an extension:

$$\text{ex } \Psi_P(0; \mathbf{a}, \mathbf{b}) = \Psi_P(\mathbf{a}, \mathbf{b}).$$

Example: a rank two poset

Let P have three elements between $\hat{0}$ and $\hat{1}$.



With the internal-rank convention used above,

$$\Psi_P(\mathbf{a}, \mathbf{b}) = \mathbf{a} + 2\mathbf{b}.$$

Moreover,

$$\pi_P(y) = 1 + 3y + 2y^2,$$

$$\pi_{[0, u_i]}(y)\pi_{[u_i, 1]}(y) = (1 + y)^2,$$

so

$$\text{ex } \Psi_P(y; \mathbf{a}, \mathbf{b}) = (1 + 3y + 2y^2)\mathbf{a} + (2 + 3y + y^2)\mathbf{b}.$$

Specializations

The extended ab-index specializes:

$$\text{ex } \Psi_P(0; \mathbf{a}, \mathbf{b}) = \Psi_P(\mathbf{a}, \mathbf{b}).$$

The chain polynomial (= f-polynomial of the order complex) is recovered by

$$C_P(y) = \text{ex } \Psi_P(0; 1 + y, y) = \Psi_P(1 + y, y)$$

and the h-polynomial of the order complex by

$$h_P(y) = \text{ex } \Psi_P(0; 1, y) = \Psi_P(1, y).$$

The coarse flag Hilbert–Poincaré series and the Chow polynomial. The latter as

$$H_P(y) = \text{ex } \Psi_P(-y; 1, y) (1 - y)^{-\text{rk}(P)}$$

Lecture 2: Chow and augmented Chow polynomials are evaluations of extended ab-indices.

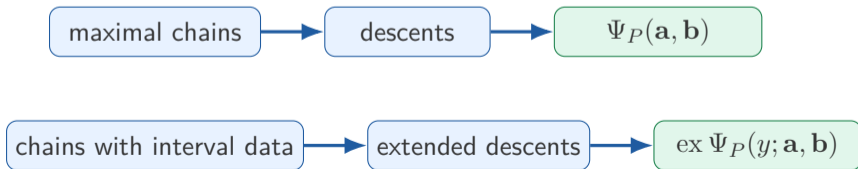
Extended positivity for R-labeled posets

For R-labeled posets the ordinary descent story lifts to the extended ab-index.

If P admits an R-labeling, then

$$\text{ex } \Psi_P(y; \mathbf{a}, \mathbf{b}) \in \mathbb{N}[y]\langle \mathbf{a}, \mathbf{b} \rangle$$

with a positive combinatorial expansion refining descents.



Omega preview

There is a linear operator $\omega : \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle \longrightarrow \mathbb{Z}[y]\langle \mathbf{a}, \mathbf{b} \rangle$ such that

$$\text{ex } \Psi_P(y; \mathbf{a}, \mathbf{b}) = (1 + y) \omega(\Psi_P(\mathbf{a}, \mathbf{b})).$$

The point for now is conceptual:

ordinary flag enumeration $\xrightarrow{\omega}$ Poincaré-enriched flag enumeration.

The ab-index knows the extended ab-index.

The explicit formulas for ω are slightly technical, so we postpone it.

Coarse flag Hilbert–Poincaré series

Define

$$\text{cfHP}_P(y, t) = \sum_{\mathcal{C}} \pi_{\mathcal{C}}(y) \frac{1}{1-t} \left(\frac{t}{1-t} \right)^{\#\mathcal{C}-2}.$$

Equivalently, write it as a rational function

$$\text{cfHP}_P(y, t) = \frac{\mathcal{N}_P(y, t)}{(1-t)^{\text{rk}(P)}}.$$

This is a coarsening of the flag Hilbert–Poincaré series motivated by work on **local Igusa zeta functions** \rightsquigarrow algebraic number theory.

Relation with the extended ab-index

The numerator is obtained from the extended ab-index by the specialization

$$\mathcal{N}_P(y, t) = \text{ex } \Psi_P(y; 1, t).$$

We thus obtain the positivity

$$(1 - t)^{\text{rk}(P)} \text{cfHP}_P(y, t) = \text{ex } \Psi_P(y; 1, t) \in \mathbb{N}[y, t].$$

The same noncommutative invariant controls both flag enumeration and Hilbert–Poincaré numerators.

Two basic specializations

The numerator $\mathcal{N}_P(y, t)$ specializes to familiar objects.

$$\mathcal{N}_P(y, 0) = \pi_P(y).$$

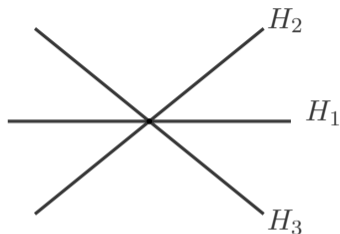
For simplicial hyperplane arrangements \mathcal{A} of rank n , we have

$$\mathcal{N}_{L(\mathcal{A})}(1, t) = \#\{\text{regions}\} \cdot E_n(t),$$

where $E_n(t)$ is the Eulerian polynomial.

If \mathcal{A} is the reflection arrangement of a finite reflection group W , its regions are in bijection with W . Is there a companion statistic on $W \times \mathfrak{S}_n$ whose two-variable distribution realizes $\mathcal{N}_{L(\mathcal{A})}(y, t)$?

Arrangement picture: three lines



For the braid arrangement A_2 (three lines through the origin):

regions \leftrightarrow permutation labels

123, 132, 213, 231, 312, 321.

In this example the numerator is

$$\mathcal{N}(y, t) = (1 + 3y + 2y^2) + (2 + 3y + y^2)t.$$

For braid arrangements, the companion-statistic problem from the previous slide has been solved in joint work with Elena Hoster.

References and credits

Thank you for listening!

- ▶ R. Stanley; M. Bayer–L. Billera; M. Bayer–A. Klapper: flag enumeration, flag h -vectors, and the ab-index.
- ▶ A. Björner–M. Wachs: edge labelings, EL-labelings, and shellability.
- ▶ M. Bayer–B. Sturmfels; R. Ehrenborg–M. Readdy: arrangements, face posets, and ab-index transformations.
- ▶ G. Dorpalen-Barry–J. Maglione–C. Stump: Poincaré-extended ab-index and positivity for R -labeled posets.
- ▶ E. Hoster–C. Stump–L. Vecchi: the ω -viewpoint and extensions of the ab-index.
- ▶ E. Hoster–C. Stump: companion statistics for braid arrangements.

These are the credits relevant for the narrative of Lecture 1.