Multiplicities for finite reductive groups, Macdonald polynomials and character varieties for Riemann surfaces

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- One of hardest problem in algebraic combinatorics.

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- Representation theory of H " $=$ " rep. theory of Weyl group of G + alg. geom. G.

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- Bijection $\widehat{\mathrm{GL}_n(\mathbb{F}_q)}$ and $Cl(\mathrm{GL}_n(\mathbb{F}_q))$.

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- Irreducible if $\alpha \neq \beta \rightsquigarrow$ cong. class of diag. matrix with 2 distinct eigen. (regular ss.)

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- Unipotent character $R_\mu \leftrightarrow$ unipotent conj. class C_μ.
- $\mu =$ size of the Jordan blocks.

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- Geometric picture $\rightsquigarrow$ some results.
- Generic k-tuple $(\chi_1, \dots, \chi_k)$
- Results of Hausel, Letellier, Rodriguez-Villegas:
  i) Combinatorial expression of $\langle \chi_1 \otimes \cdots \otimes \chi_k, 1 \rangle$
  ii) related to character varieties and quiver varieties

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- SL_n= Langlands dual of PGL_n.

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- Local monodromy around D by C.

Cohomology of character varieties

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- $H_c(M_C; q, t) = \sum_{m,k} q^{\frac{m}{2}} t^k \dim W_m^k / W_{m-1}^k H_C^k(M_C, \mathbb{Q}) =?$.

Cohomology of character varieties II

- Hausel, Letellier, R.Villegas result:
  i) Conjecture $H_c(M_C; q, t) = \mathbb{H}_\mu\left(\frac{t}{\sqrt{q}}, t\sqrt{q}\right)$
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- $\Omega(z, w)$ encodes all cohomology of generic character varieties.

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- Combinatorial definition:

  i)$\tilde{H}_\lambda(\mathrm{x}(1-z); z, w) \in \mathbb{Q}(z,w)\{s_\mu\}_{\mu \geq \lambda}$
  ii) $\tilde{H}_{\lambda'}(\mathrm{x}; w, z) = \tilde{H}_\lambda(\mathrm{x}; z, w)$
  iii) $\langle \tilde{H}_\lambda(\mathrm{x}, z, w), s_{(n)} \rangle = 1.$

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- Values of characters of $\mathrm{GL}_n(\mathbb{F}_q)$ expressed through $\widetilde{K}_{\lambda,\mu}(z, w)$.

$$\widetilde{K}_{\lambda,\mu}(0, q) = R_\lambda(C_\mu).$$

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- $\mathcal{C}$ generic $\rightsquigarrow$ $M_{\mathcal{C}}$ smooth (Deligne-Mumford).

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- $C_x, \mathcal{C}_x \subseteq \mathrm{GL}_2, \mathrm{PGL}_2$ cong. classes
- If $x \neq -1$, $p_2 : C_x \to \mathcal{C}_x$ isomorphism (*non-degenerate*).

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- Joint work with Emmanuel Letellier:

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  ii) Computation of $E(M_{\mathcal{C}}, \mathcal{L}; q)$.

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- Character table $\mathrm{SL}_n(\mathbb{F}_q)$ (2003). (Shoji, Bonnafe).

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- Character tables and point counting way harder.