W: Coxeter group of affine type;

Y_W orbit configuration space $\left(Y_W = I_{\mathbb{C}} \setminus \bigcup (H_1)_{\mathbb{C}} \right)$

CONJECTURE [Arnol'd, Brieskorn, Pham, Thom ('60-'70)

The orbit space Y_W is a $K(\pi,1)$ space.

<u>Thm.</u> (Paolini - S., Inv. Math., '21) The $K(\pi,1)$-conjecture holds for all Artin groups of affine type.

Tentative proof: try to use the <u>dual structure</u> on W.

So, take R = all reflections, c = Coxeter element,

$P = [1,c]$ interval below c, and try to prove:

1. P is a lattice (so $K = K(P)$ is a $K(\pi,1)$)

2. $G = G(P) \cong \pi_1(K) \cong G_W$

   $G(P)$: dual Artin group; G_W: standard Artin group

3. Prove that $K \simeq Y_W$

(of course, 3. $\Rightarrow$ 2)

Unfortunately: 1. is true $\Leftrightarrow$ $W = \tilde{A}_n$ (for some c), $\tilde{C}_n$, $\tilde{G}_2$

Outline of the proof

1. We proof that the associated complex K is a $K(\pi, 1)$ even if $[1, c]$ is not a lattice.

2. We give homotopy equivalence between K and the orbit configuration space Y_W.

tools: (i) identification of a finite subcomplex $X'_W \subset K$ which has the homotopy type of Y_W $(X'_W \cong X_W)$

(ii) we find an EL-shelling on $[1,c]$ and combinatorial Morse theory to contract $K \searrow X'_W$

Some geometrical properties.

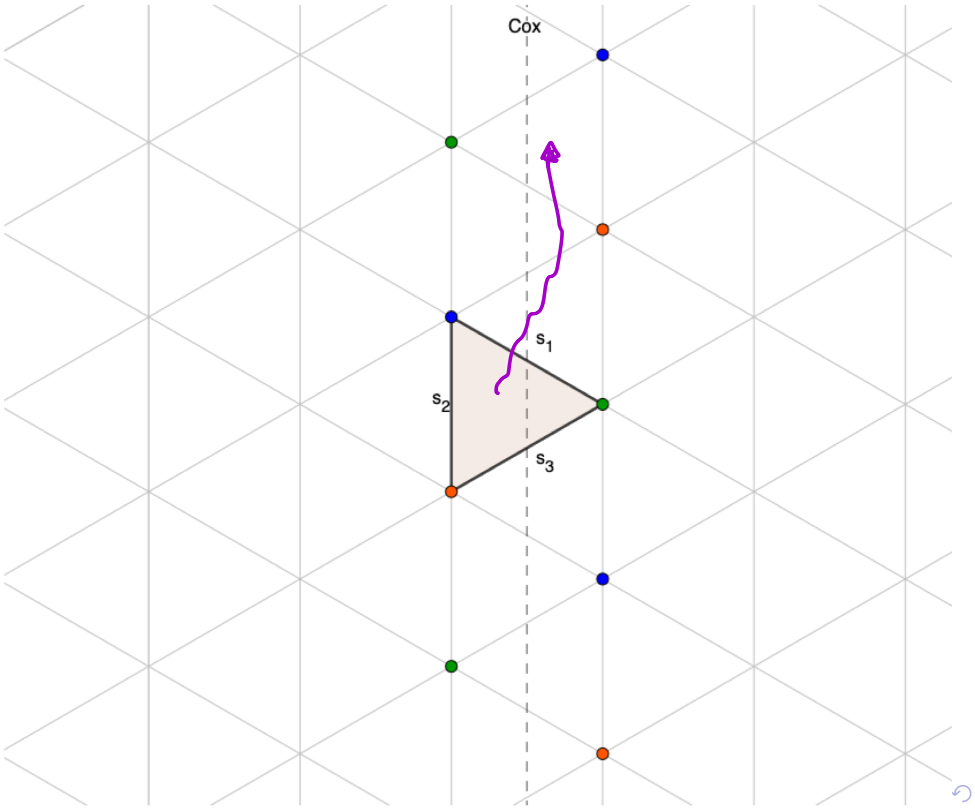
- $w \in W$ is <u>elliptic</u> if it fixes some point;
- w is <u>hyperbolic</u> otherwise.

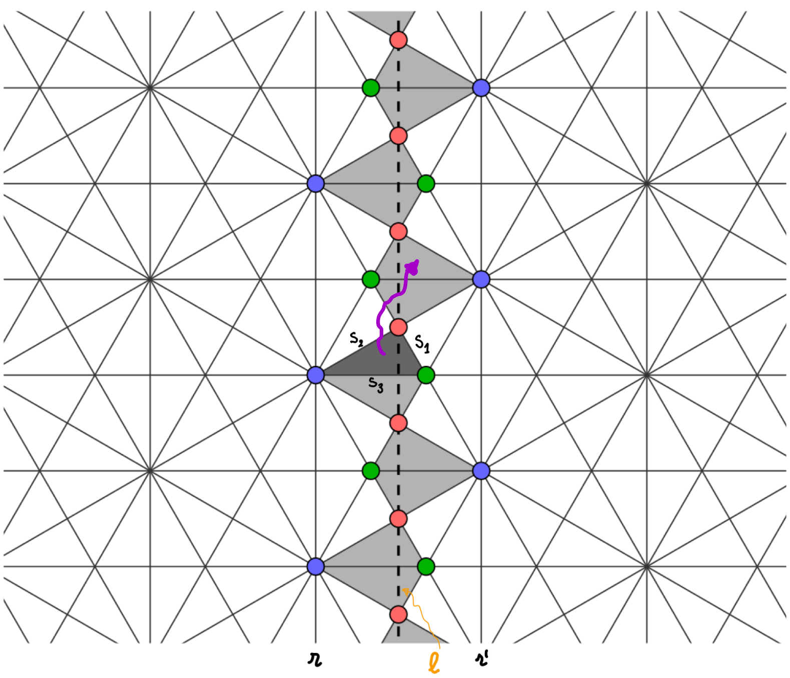
The Coxeter element $c \in W$ is hyperbolic, so every point p in the affine space is moved. Actually, there is a line ℓ of points which are <u>minimally moved</u>, which is called the Coxeter axis;

the restriction $c|_\ell$ is a translation.

$\tilde{A}_2$:

$\mathbf{c} = s_1 s_2 s_3$





$\mathbf{c} = s_1 s_2 s_3$

Coxeter axis ℓ = dotted line

vertical reflections:
$r: \ Fix(r) \cap \ell \neq \emptyset$

horizontal reflections:
$r: \ Fix(r) \cap \ell = \emptyset$

axial chambers (grey):
$C: \ int(C) \cap \ell \neq \emptyset$

$\tilde{G}_2$

McCammond-Sulway, Inv. Math '17

Characterization of the reflections $r \leq c$:

$r \in [1, c] \iff$ r is vertical;

r is horizontal but "close" to ℓ

(it intersects an axial chamber)

They take as much as possible from Garside theory: they add suitable finite set of translations to W, finding a larger group $G \supset W$, and such that

$$\underbrace{[1, c]^G}_{\text{is a lattice}} \longleftrightarrow [1, c]^W$$

Therefore $\tilde{K} = K([1,c]^G)$ is a finite dimensional $K(\pi, 1)$,

There is a diagram of groups:

$$\begin{tikzcd}
 & R_{hor}, R_{ver} & R_{hor}, R_{ver}, T_F \\
 & W \arrow[r, hook] & G \\
 R_{hor} & R_{hor}, T & R_{hor}, T_F \\
H \arrow[r, hook] & D \arrow[r, hook] \arrow[uu, hook] & F \arrow[uu, hook]
\end{tikzcd}$$

where the groups on the right are Garside; this gives a diagram of complexes

These are $K(\pi,1)$

$$\begin{tikzcd}
 & K_c^H \times \mathbb{R} \arrow[dr, "\text{covering}"] & K_c^W \arrow[r, hook] & K_c^G \\
K_c^H \arrow[ur] \arrow[rr, hook] & & K_c^D \arrow[u, hook] \arrow[r, hook] & K_c^F \arrow[u, hook]
\end{tikzcd}$$

$K(\pi,1)$ since $H \cong \prod_i W_{\hat{A}_i}$

Thm: this is a $K(\pi,1)$

and we need that K_c^D is a $K(\pi,1)$.

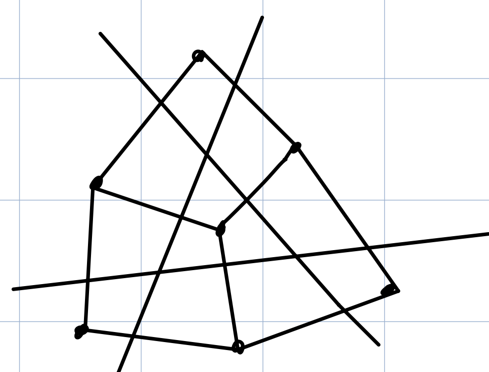
Therefore a Mayer-Vietoris argument gives that K_c^W is a $K(\pi,1)$.

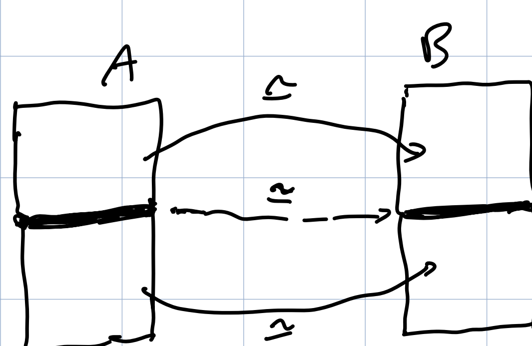
It remains to relate $K = K_c^W$ to Y_W.

Let $X_W' \subset K$ be the subcomplex given by those simplices $[x_1|\cdots|x_q]$ s.t. $x_1 \cdots x_q$ fixes one vertex of the base chamber C_0.

<u>Thm.</u> $X_W' \simeq X_W \; (\simeq Y_W)$

idea: take pieces of both complexes which correspond to parabolic subgroups: use standard theorems in alg. topology and induction to glue homotopy equivalences of the pieces:

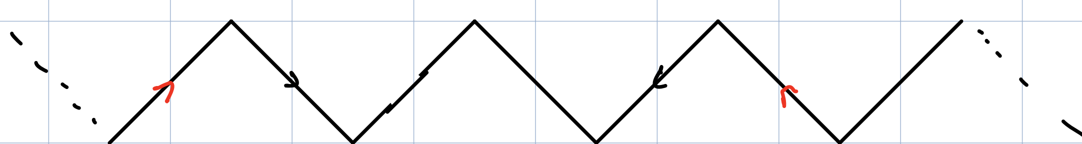




The last step is to contract $K \searrow X_W'$.

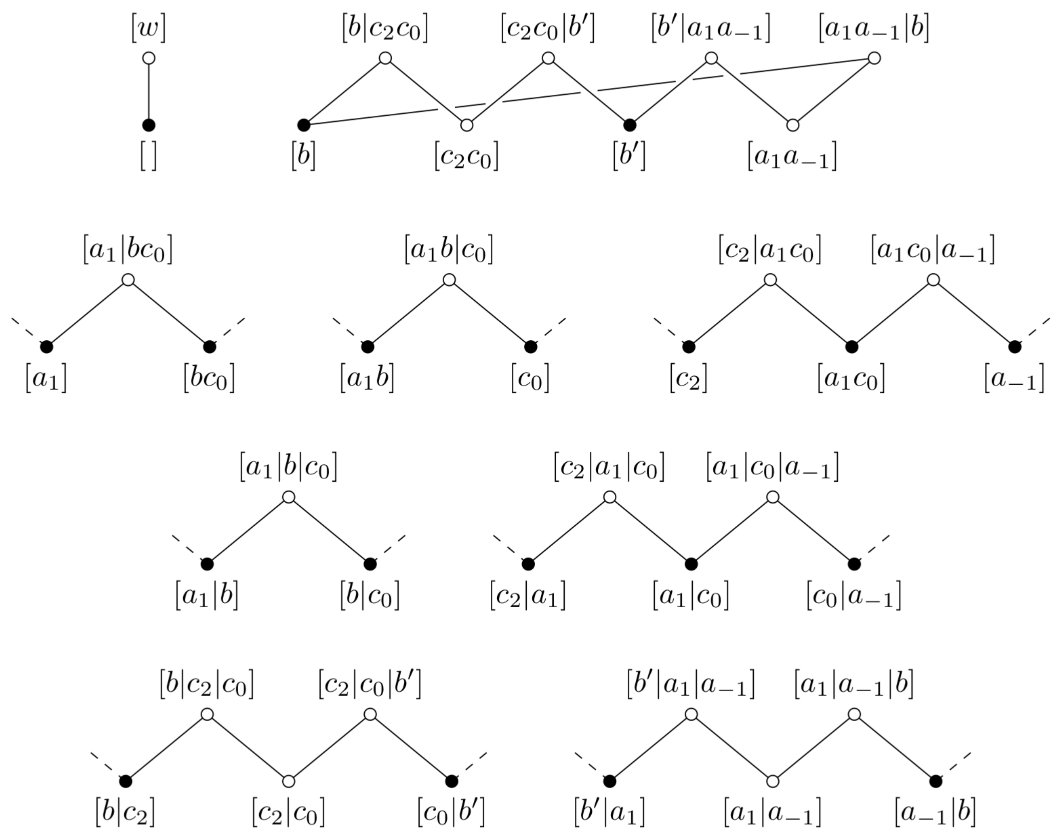
Here we use discrete Morse theory in 2 steps:

1) There is a poset map $\mathcal{F}(K) \longrightarrow \mathbb{N}$
such that each fiber looks like



then we can easily put a matching which reduces K to a finite subcomplex $K' \supset X_W'$

(example in case $\hat{A}_2$:)



2) We find a second matching on K' which is a "perfect matching" on $K' \setminus X'_w$ (so $K' \simeq X'_w$)

To construct this matching, we use:

<u>Thm</u> There exist a total ordering on

$R_0 = R \cap [1,c]$ which makes the poset $[1,c]$ EL-shellable.

EL-shellable: $\forall$ interval $[x<y]$ (i) $\exists$ a unique increasing maximal chain γ; (ii) γ is the minimum in the lexicographical order among the maximal chains.

The ordering of R_0 is an "axial ordering": reflections are ordered according to their intersection with the Coxeter axis.

We use this to "control" cells in H'.

Possible generalizations and problems.

- $\forall$ Coxeter groups W, is the $K(\pi,1)$ conjecture true?

  let c be a Coxeter element;

- is $[1,c]$ a lattice? is it shellable?

- let W_c be the group defined by $[1,c]$ (the dual group). Is it isomorphic to the standard Artin group?

- can one solve the word problem and find the center?

Theorem (Delucchi, Paolini, S. , '22

Let (W, S) be a Coxeter system of rank 3 and G_W the associated Artin group. Let $w \in W$ be any Coxeter element, and consider the associated noncrossing partition poset $[1, w]$.

1. *$[1, w]$ is a lattice.*
2. *$[1, w]$ is EL-shellable.*
3. *Let G_W be an Artin group of rank 3. The dual Artin group associated with $[1, w]$ is isomorphic to G_W.*
4. *G_W is a Garside group.*
5. *The $K(\pi, 1)$ conjecture holds for G_W.*
6. *The word problem for G_W is solvable.*
7. *The center of G_W is trivial unless W is finite.*

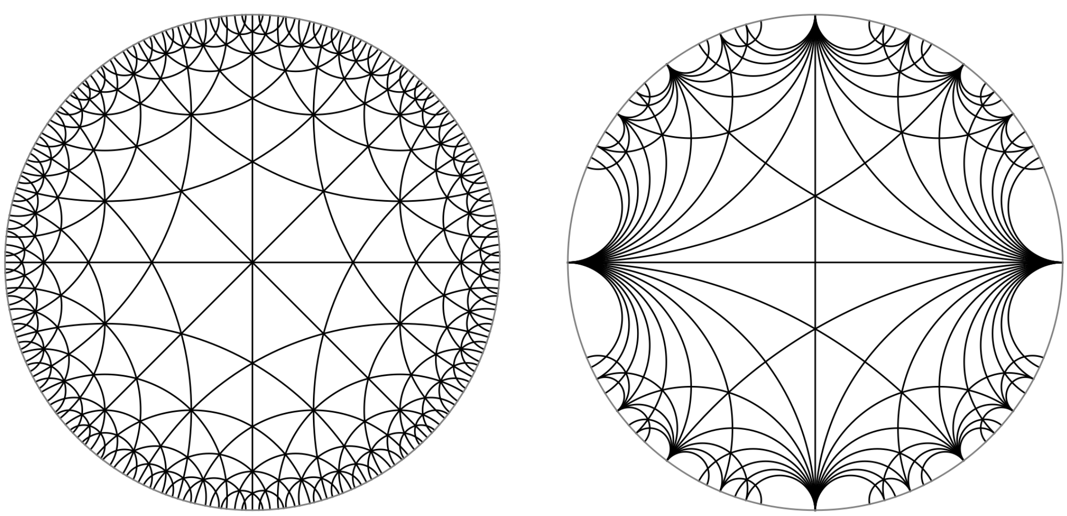


Figure: Examples of arrangements of rank-three hyperbolic Coxeter groups in the Poincaré model. Each picture is captioned with the labels (m_1, m_2, m_3) of the corresponding Coxeter diagram. The hyperbolic plane is tiled by triangles with angles $\frac{\pi}{m_1}, \frac{\pi}{m_2}, \frac{\pi}{m_3}$.

- One can obtain that K is a $K(\pi,1)$ even if $[1,c]$ is not a lattice: how to generalize Garside theory, weakening this condition?

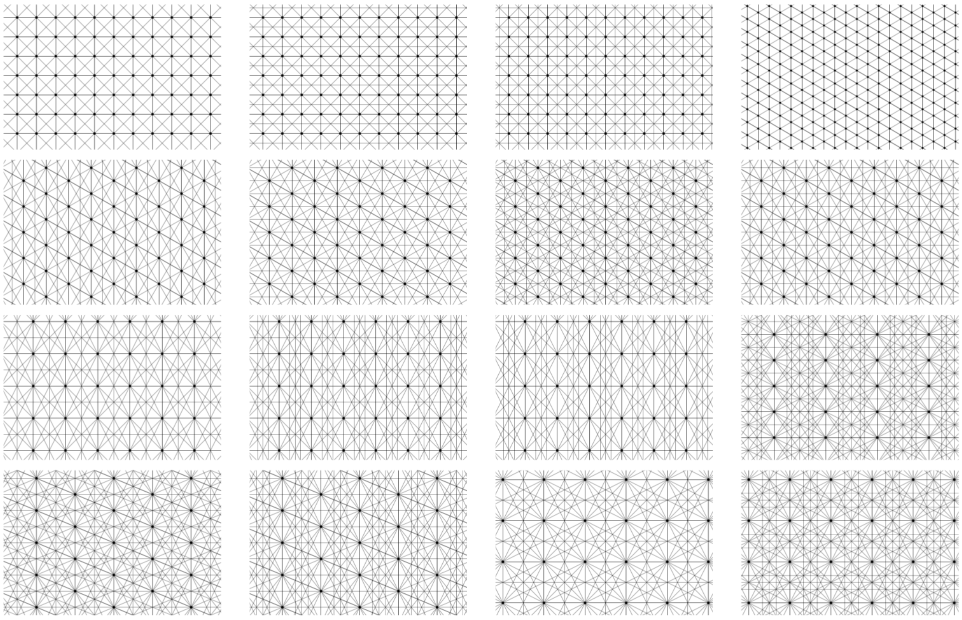
Other generalizations of $K(\pi,1)$ property:

- to complements of discriminant of non-rational (A, D, E) singularities ($\rightsquigarrow$ elliptic)
- to complements of simplicial affine arrangements of hyperplanes. in particular periodic affine arrangements

(see for example papers by Wemyss and collaborators on the space of "Bridgeland stability conditions")

(from Wemyss :)

Theorem 0.5 (Section 4.2). *Suppose that Δ_{aff} is extended ADE Dynkin, and $\mathcal{K} \subseteq \Delta_{\mathsf{aff}}$ satisfies $|\mathcal{K}| = 3$. Then, up to changing the slopes of some of the hyperplanes,* $\mathsf{Level}(\mathcal{K})$ *is one of following sixteen hyperplane arrangements:*



In addition, each of the sixteen arrangements appears as $\mathsf{Level}(\mathcal{J}_{\mathsf{aff}})$ *for some subset of the ADE Dynkin $\mathcal{J} \subseteq \Delta$ satisfying $|\mathcal{J}^{c}| = 2$.*

- toric arrangements

(the end)