How $K(P)$ is defined?

P poset

start from the ORDER COMPLEX

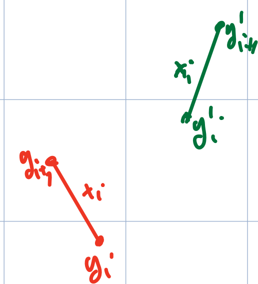
$$\Delta(P) = \{\sigma^q = (y_0 < y_1 < \cdots < y_q) \mid y_i \in P\}$$

and identify two chains $y_0 < \cdots < y_q$, $y'_0 < \cdots < y'_q$ $\Longleftrightarrow$

$$\mathcal{L}(y_i, y_{i+1}) = \mathcal{L}(y'_i, y'_{i+1}) \quad , \forall i$$

In the case of a group G and poset $\mathcal{P} = [1, \delta]$

this means $x_i = y_i^{-1} y_{i+1} = (y'_i)^{-1} y'_{i+1}$;



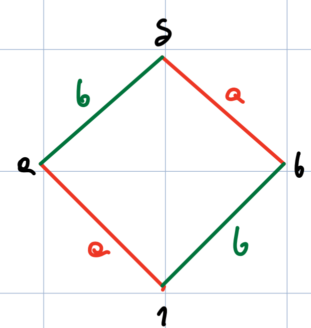
so the q-simplices correspond to sequences:

$$[x_1 | \cdots | x_q] \quad , \quad x_i \le \delta \quad , \quad x_1 \cdots x_i \le \delta , \ \forall i$$

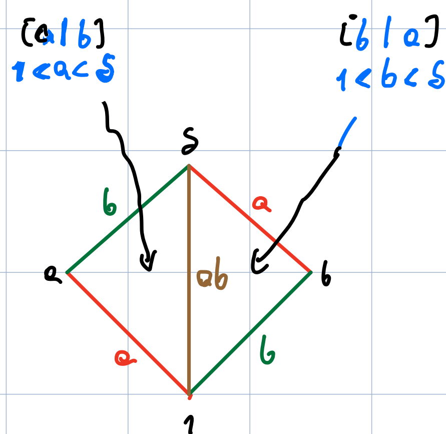
Examples.

$G = \mathbb{Z} \times \mathbb{Z}$, a, b generators, $\delta = ab = ba$

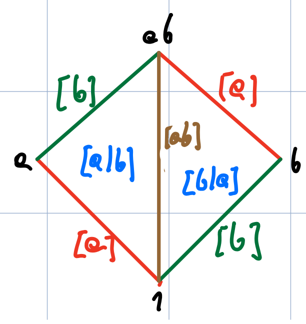
$\mathcal{P}$:



$\Delta(\mathcal{P}) =$



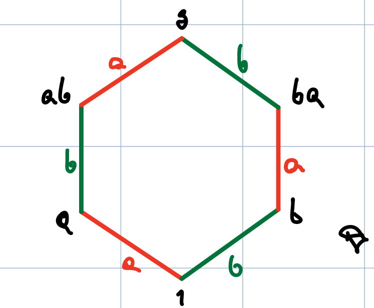
$K = \Delta(\mathcal{P})/\sim$:

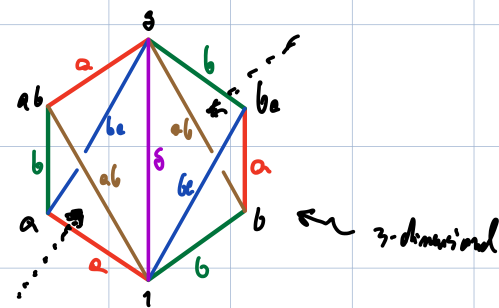


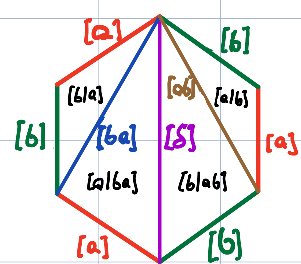
$\parallel$

$S^1 \times S^1$

$G = G_{\sigma_3} = Br_3$, a, b generators, $\delta = aba = bab$

$\mathcal{P}$: 

$\Delta(\mathcal{P})$: 

$K(\mathcal{P})$: 

$\simeq \; X_W \simeq Y_W$

(exercise)

<u>Thm</u> $\pi_1(K) = G$ (by examining the 2-skeleton of K)

<u>Dual Garside structures on Coxeter groups</u>

Recall: a <u>Coxeter element</u> $c \in W$ is the product (in some order) of all simple reflections. For example if $(W, S) = (\sigma_n, \; s_j = (j, j+1), \; j = 1, \dots, n-1)$ then one Coxeter element is $c = (12) \cdots (n-1, n) = (1\,2 \cdots n)$.

Consider $T = \bigcup_{w \in W} w^{-1} S w$, all reflections, as set of generators of W. and the associated length $\ell_T : W \to \mathbb{N}$

Then W is a T-labeled poset with ordering $w \le w' \Leftrightarrow \ell_T(w) + \ell_T(w^{-1}w') = \ell_T(w')$

which means that a reduced expression for w' begins by a reduced expression for w.

We consider the poset $\mathcal{P} = [1, c]$, labeled by T.

<u>Thm</u> $\mathcal{P} = [1, c]$ is a combinatorial Garside structure.

Michel, Bessis

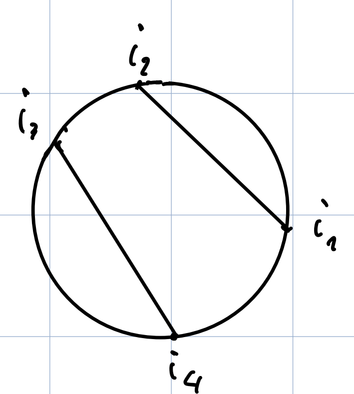
The group $G(\mathcal{P}) \cong G_W$ (Birman, Ko, Lee /

Case $W = \mathfrak{S}_n$. Coxeter element $c = (1\ 2 \cdots n)$

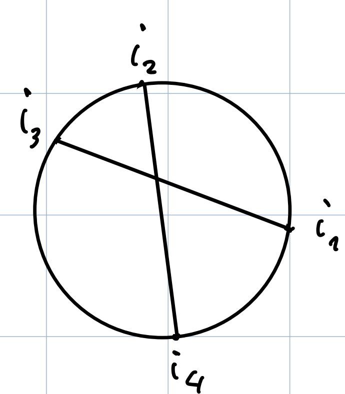
<u>def</u>. A partition $(\lambda_1, \cdots, \lambda_m)$ of $\{1, \cdots, n\}$ is called <u>noncrossing</u> if

$\forall\ 1 \le i_1 < i_2 < i_3 < i_4 \le n$, $\{i_1, i_3\} \subset \lambda_j \Rightarrow \{i_2, i_4\} \not\subset \lambda_k$, $j \neq k$

non-crossing:



crossing:

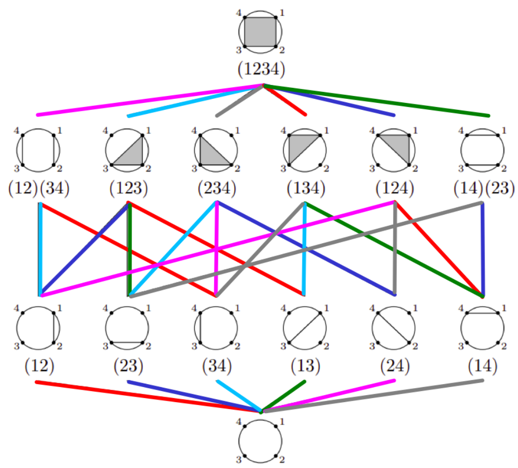


the convex hulls of the blocks in S^1 are disjoint.

Thm The map $\mathfrak{S}_n \ni \sigma \longrightarrow$ partition of $\{1,\dots,n\}$ into σ-orbits.

induces poset-bijection: $[1,c] \longrightarrow$ non-crossing partitions (with natural ordering)

ex:



$$\#[1,c] = \frac{1}{n+1}\binom{2n}{n}$$

$$G = \left\langle a_{ij},\ 1 \le i < j \le n \ \middle|\ \begin{array}{ll} a_{ij}\, a_{kl} = a_{kl}\, a_{ij} & \text{if } [ij] \cap [kl] = \emptyset \\ a_{ij}\, a_{jk} = a_{jk}\, a_{ik} = a_{ik}\, a_{ij} & \forall \text{ triangle } i<j<k \end{array} \right\rangle$$

Corollary The associated complex $K = K(\mathcal{P})$ is a $K(\pi,1)$, so by Deligne $\Rightarrow K \simeq Y_W$

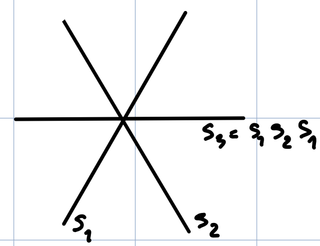
So, $\mathcal{P} = [1,c]$ gives a different Garside structure on G_W, W finite.

Usually called "dual Garside structure". The poset is called (generalized) non-crossing partition lattice. In the case $W = \mathfrak{S}_n$

	standard $\mathfrak{S}_n$	dual NC_n
# atoms	$n-1$	$\binom{n}{2}$

length	$\binom{n}{2}$	$n-1$

$W = A_2$

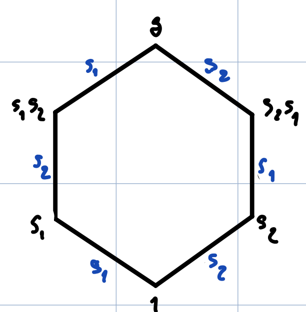


STANDARD

$R = \{s_1, s_2\}$

$\delta = s_1 s_2 s_1$

$[1, \delta]$:

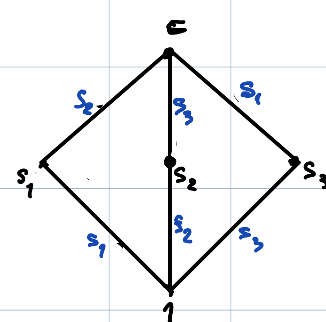


DUAL

$R = \{s_1, s_2, s_3\}$

$c = s_1 s_2$

$[1, c]$:



...

$W_\delta = \langle s_1, s_2 \mid s_1 s_2 s_1 = s_2 s_1 s_2 \rangle$

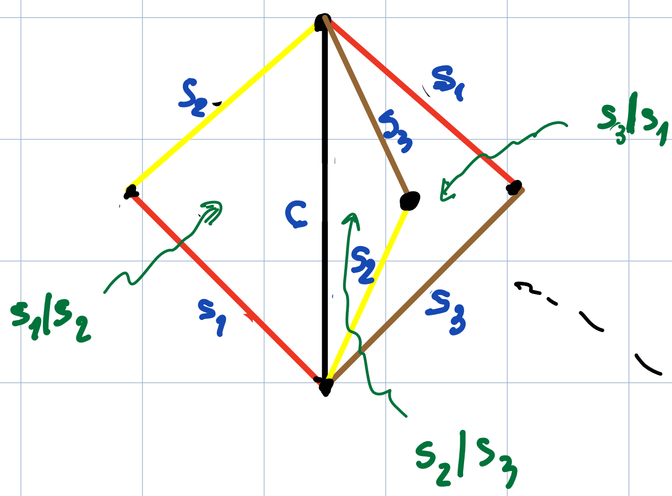
'standard' Artin group

$W_c = \langle s_1, s_2, s_3 \mid s_1 s_2 = s_2 s_3 = s_3 s_1 \rangle$

'dual' Artin group

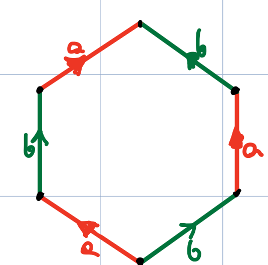
simplices of $K_c = \overset{0\text{-dim}}{[\ ]}, \overbrace{[s_1],[s_2],[s_3],[c]}^{1\text{-dim}}, \overbrace{[s_1|s_2],[s_2|s_3],[s_3|s_1]}^{2\text{-dim}}$

K_c:



exercise:

find an homotopy equivalence between K_c and



General case.

Given $W = \langle s \in S \mid (ss')^{m(s,s')} = 1 \rangle$ (S finite)

Tits representation: in $V = \mathbb{R}^{|S|} = \mathrm{Span}\langle e_s, s \in S \rangle$ take the bilinear form

$$B(e_s, e_{s'}) = -\cos \frac{\pi}{m(s,s')}$$

the map $s \longrightarrow [\rho_s(x) = x - 2B(x, e_s)\, e_s]$ extends to a linear representation of W into V preserving B.

W is spherical $\iff$ $B > 0$

affine $\iff$ $B \geq 0$, $\mathrm{rk}\, B = |S| - 1$

hyperbolic $\iff$ B non degenerate with signature $(|S|-1, 1)$

In the contragradient representation $W \longrightarrow GL(V^*)$ one has:

- A fundamental domain is $D = \{ x_i \geq 0,\ i = 1, \dots, |S| \}$.
- $I = W . D$ is a cone (the Tits cone) in V^*
- I is stratified by the reflection arrangement

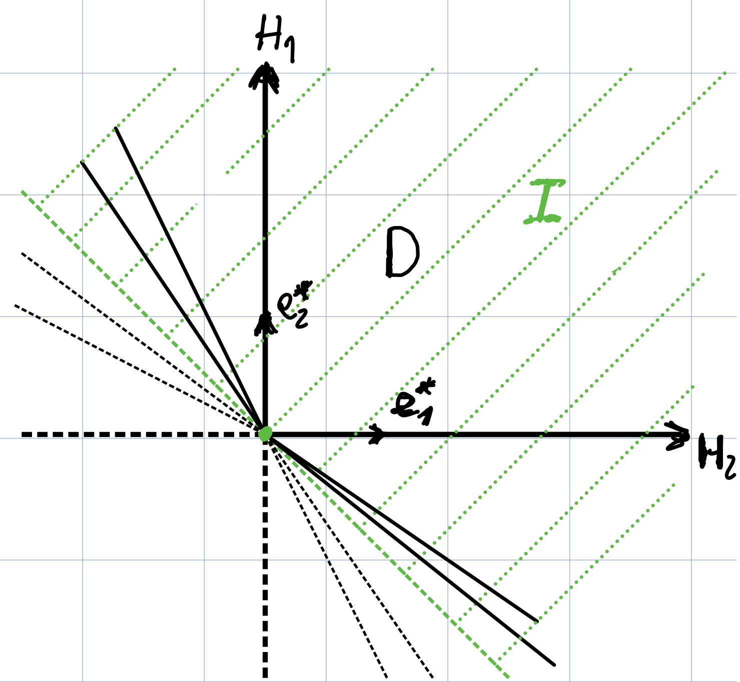
$$\mathcal{A}_W = \{ H_r \mid r \in R = \bigcup_{w \in W} w^{-1} H_s w \}$$

where $\{\text{chambers}\} \xleftrightarrow{1:1} \{ w . \mathring{D} \mid w \in W \}$ and

a face $\overset{\in D}{F} \subset \mathrm{int}\, I \iff \mathrm{Stab}_F$ is finite.

example.

$\tilde{A}_1$



$$\operatorname{int}(D) = \{ x e_1^* + y e_2^* : x, y > 0 \}$$

$$\sigma_1^* = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$\sigma_2^* = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$$

$$I = \{ x e_1^* + y e_2^* \mid x + y > 0 \} \cup \{ (0,0) \}$$

Now take the Configuration Space

$$Y = I_{\mathbb{C}} \setminus \bigcup_{\eta \in R} (H_\eta)_{\mathbb{C}} \qquad \left[= I \times \mathbb{R}^N \setminus \bigcup_{\eta \in R} (H_\eta \times H_\eta) \right]$$

which has a free W-action, so one has an

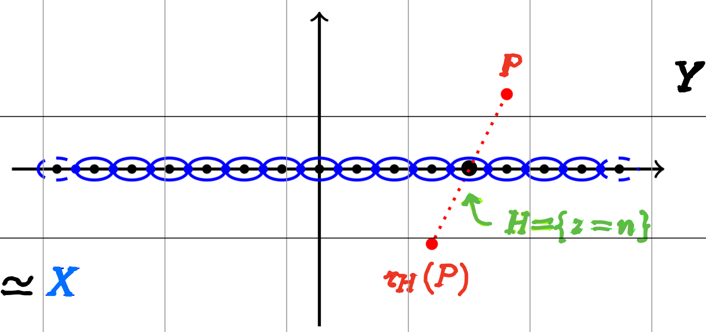
Orbit Configuration Space

$$Y_W = Y/W$$

Thm $\pi_1(Y_W) = G_W = \langle g_s,\ s \in S \mid g_s g_{s'} \cdots = g_{s'} g_s \cdots \ (m(s,s') \text{ factors}) \rangle$

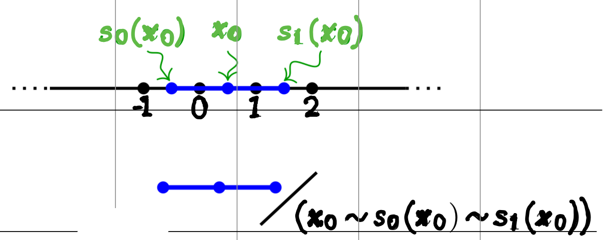
(Van der Lek, '83; easily deduced also by looking at the 2-skeleton of the complex X_W, which exists in general)

•—∞—• $\tilde{A}_1$ $\quad Y = \mathbb{C} \setminus \bigcup_{n\in\mathbb{Z}} \{z = n\}$ $\quad Y_W \simeq \mathbb{C} \setminus \{2 \text{ pts}\}$



$Y \simeq X$

$Y_W = Y/W \simeq X_W = X/W = S^1 \vee S^1 =$



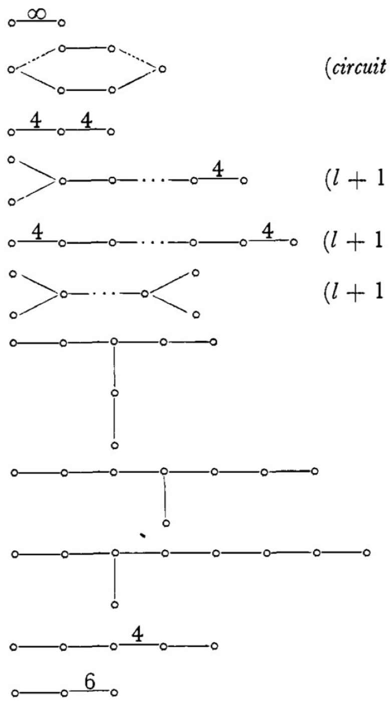
$G_W = \langle s_0, s_1 : \text{no relations} \rangle = F[s_0, s_1]$

CONJECTURE [Arnol'd, Brieskorn, Pham, Thom ('60-'70)

The orbit space Y_W is a $K(\pi, 1)$ space

Next important case after the spherical case: AFFINE TYPE

THÉORÈME 4. — *Soit* (W, S) *un système de Coxeter irréductible, avec* S *fini. Pour que la forme quadratique associée* (chap. V, § 4, n° 1) *soit positive et dégénérée, il faut et il suffit que le graphe de Coxeter de* (W, S) *soit isomorphe à l'un des suivants :*



$\tilde{A}_1$

$\tilde{A}_l$ $(l \geqslant 2)$ (*circuit à* $l+1$ *sommets*)

$\tilde{B}_2$

$\tilde{B}_l$ $(l \geqslant 3)$ ($l+1$ *sommets*)

$\tilde{C}_l$ $(l \geqslant 3)$ ($l+1$ *sommets*)

$\tilde{D}_l$ $(l \geqslant 4)$ ($l+1$ *sommets*)

$\tilde{E}_6$

$\tilde{E}_7$

$\tilde{E}_8$

$\tilde{F}_4$

$\tilde{G}_2$

Case $\hat{A}_n, \hat{C}_n$ (Okonek '79)

Case $\tilde{G}_2$ (Charney-Davis, '95 - rk 2)

Case $\tilde{B}_n$ (Callegaro-Moroni-S. '10)

Thm. (G. Paolini - S., Inv. Math., '21)

The $K(\pi,1)$-conjecture holds for all Artin groups of affine type.

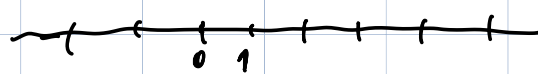
Try to generalize Deligne proof.

First obstacle: no standard Garside structure:

W has no element δ of maximal S-length.

Then, try dual structure.

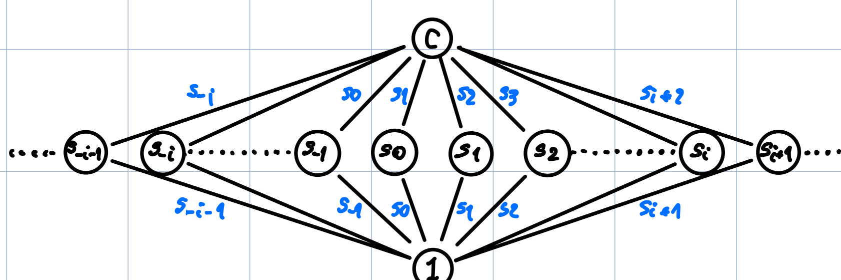
$c = s_0 s_1$



Dual structure

Coxeter element $c = s_0 s_1 = s_1 s_2 = \cdots = s_i s_{i+1} = \ldots \quad (i \in \mathbb{Z})$

$[1, c]$:



$W_c = \langle s_i \mid s_i s_{i+1} = s_j s_{j+1}, \forall i, j \rangle = G_W$

simplices of K_c :

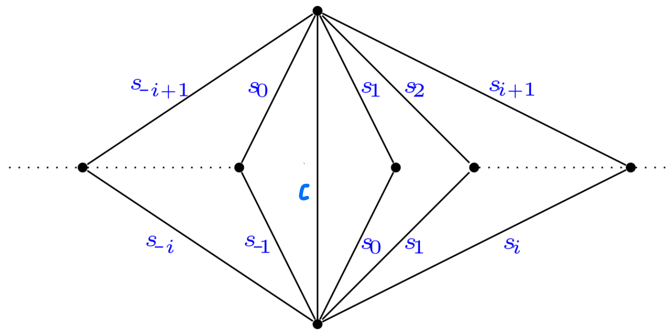
0 – cell $[\ ]$

1 – cells $[s_i]$, $i \in \mathbb{Z}$

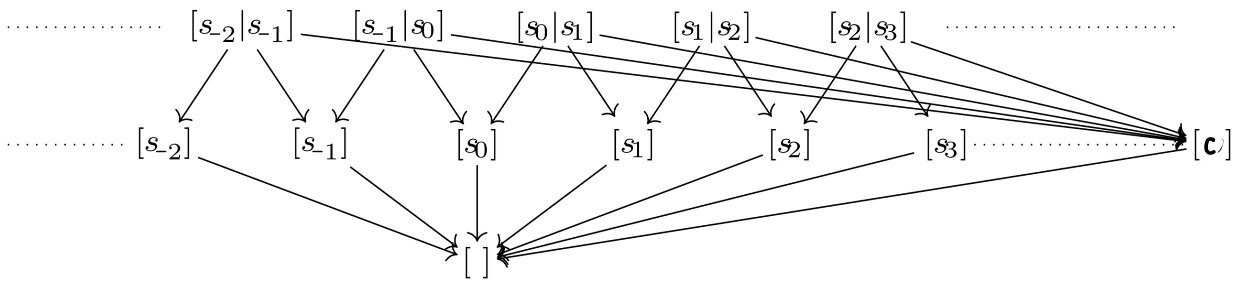
$[c]$

2 – cells $[s_i|s_{i+1}]$, $i \in \mathbb{Z}$

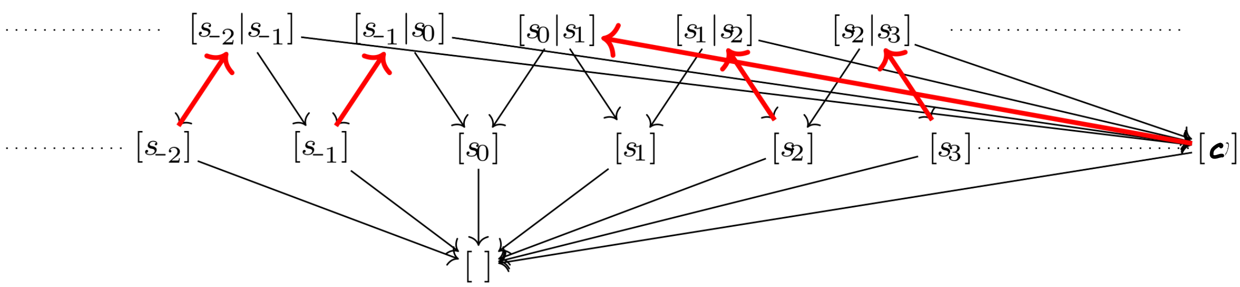
$\partial[s_i|s_{i+1}] = [s_i] \cup [s_{i+1}] \cup [c]$



Face poset of K_c :



Take a matching on the face poset of K_c:



critical cells: $[s_0]$, $[s_1]$, so $K_c \cong S^1 \vee S^1$

Unfortunately, even the "dual approach" doesn't work in general.

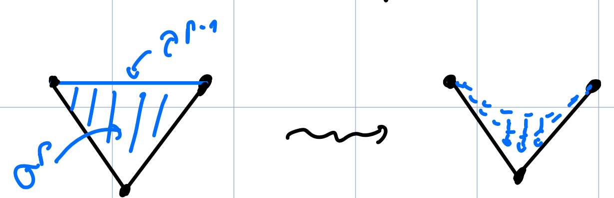
Prop (Digne, '04; McCammond '15) In case $\hat{A}_n$ (good choice of Coxeter element) $\hat{C}_n$, $\hat{G}_2$ the interval poset $[1,c]$ is a lattice ($\Rightarrow$ Garside). In no other cases $[1,c]$ is a lattice.

Therefore, except for $\hat{A}_n, \hat{C}_n$, the associated complex K was not known to be a $K(\pi,1)$.

Let's see why $K(\mathcal{P})$ is $K(\pi,1)$ when $\mathcal{P}$ lattice.

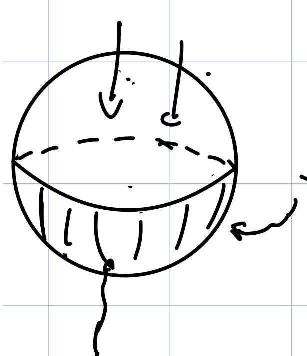
It's a good point to introduce COMBINATORIAL MORSE THEORY, which plays a basic role in the proof of the conjecture.

Basic: X CW-complex; perform a sequence of "elementary collapses" related to some pairs $\sigma^p > \tau^{p-1}$ (σ^p p-cell, $\tau^{p-1} \subset \partial\sigma^p$ p-1-cell) obtaining a smaller *Morse complex* X^m.



push down $\tau^{p-1} \searrow \overline{\partial\sigma^p \setminus \tau^{p-1}}$

Def X CW-complex, $\tau^{p-1} < \sigma^p$ is a regular face of σ if

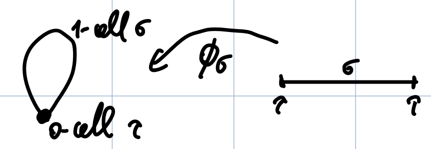


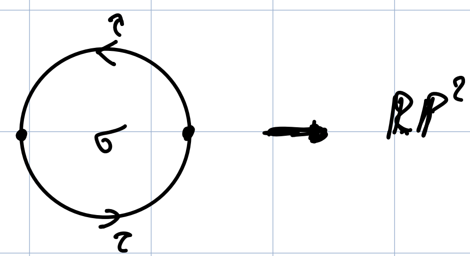
$$
\begin{tikzcd}
S^{d-1}_{\leq 0} \arrow[r, "\cong"] \arrow[d, hook] & B^{d-1} \arrow[dr, "\Phi_\tau"] & \\
 & & X \\
B^d \arrow[r, "\cong", "\Psi"'] & B^d \arrow[ur, "\Phi_\sigma"'] &
\end{tikzcd}
$$

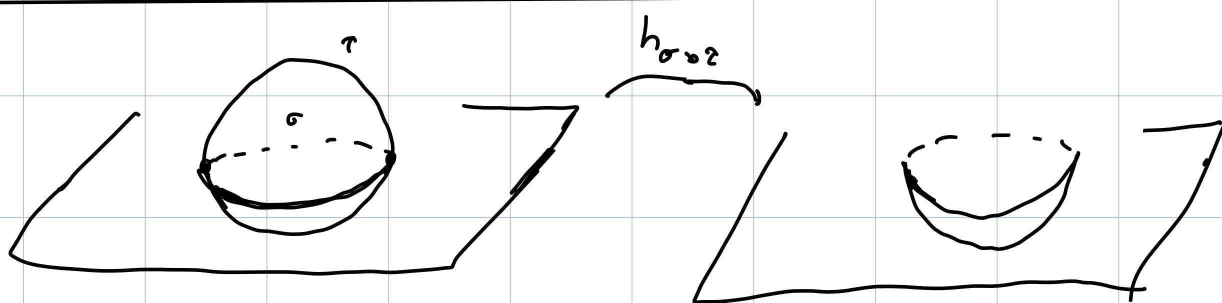
commutative

and $$(\Psi \circ \Phi_\sigma)^{-1}(\tau) = S^{d-1}_{\geq 0}$$

e.g.: not regular





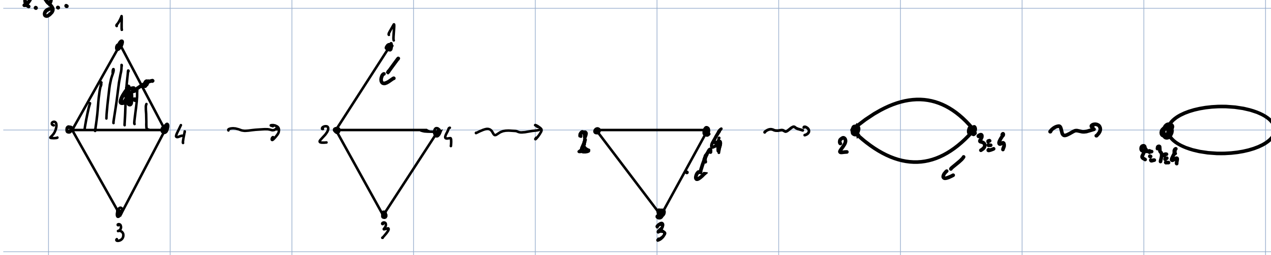


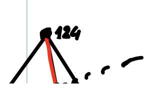
Let $\mathcal{F}(X)$ be the poset given by the face-poset of X, where

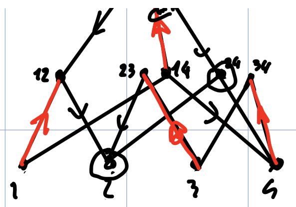
$\tau \leq \sigma$ means $\tau \subset \partial\sigma$; write $\tau^{p-1} \lessdot \sigma^p$ for a p-1-face of σ.

An elementary collapse is encoded by an edge of the Hasse diagram of $\mathcal{F}(X)$

e.g.:







red arrows:

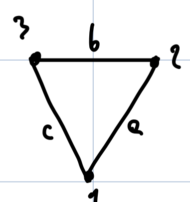
M **matching**: set of disjoined edges $(\tau < \sigma)$ with τ regular face of σ

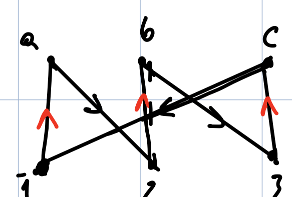
Let Γ_M be the graph where

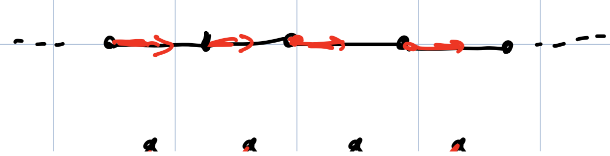
$\sigma \downarrow \tau$ if $(\tau < \sigma) \notin M$, $\sigma \uparrow \tau$ if $(\tau < \sigma) \in M$

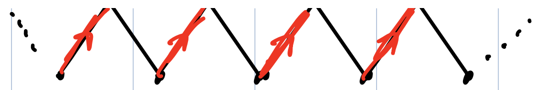
M is **acyclic** if Γ_M is acyclic (no directed closed circuits)

A cell σ is **critical** if it doesn't come in any pair in M.

e.g.:  $M = \{(1<a), (2<b), (3<c)\}$ is not acyclic

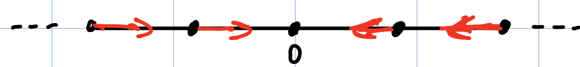


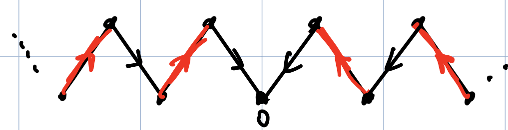
Infinite CW-complexes: 



everything collapses.

Proper matching: from any σ a finite number of cells is reacheable through a path in $\Gamma_{\mathcal{M}}$.





<u>Theorem</u> Let $\mathcal{M}$ be a proper acyclic matching on the CW-complex X.

1) $X \simeq X^{\mathcal{M}}$ (the Morse complex) whose cells are in dimension-preserving bijection with the critical cells of X

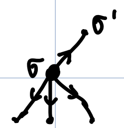
2) If the critical cells of X form a subcomplex $X_{\mathcal{M}}$, then

   $X \searrow X_{\mathcal{M}}$ (deformation retraction)

"Proof" (finite case). By induction on the number of cells.

In $\mathcal{F}(X)$ introduce an ordering $\sigma \leq \sigma' \iff$ there is a path in $\Gamma_{\mathcal{M}}$ from σ to σ'.

Let σ minimal w.r.t. $\leq_m$. Then

either  or 

$(\sigma<\sigma')\in M$ $\qquad\qquad$ σ critical

σ is a free-face of σ'  $\qquad$ $\mathcal{F}(X) \setminus \mathrm{int}\,\sigma \underset{h}{\overset{\text{induction}}{\simeq}} X'_m$

can cancel σ, σ' $\qquad\qquad$ $\Downarrow$

$$\mathcal{F}(X) = (\mathcal{F}(X) - \mathrm{int}\,\sigma) \cup_{\varphi} \sigma \simeq X'_m \cup_{\varphi\circ h} \sigma$$

Application: $K(\mathcal{P}) = \Delta(\mathcal{P})/\sim$ is $K(\pi,1)$ if $\mathcal{P}=[1,\delta]$ in a comb. Gar. structure.

Recall: K is a simplicial complex with n-simplices $[x_1|\cdots|x_n]$, $x_1\cdots x_k$ initial factorization of δ.

Then the universal cover $\tilde{K}$ is also a simplicial complex with simplices:

$$[g\,\|\,x_1|\cdots|x_k] \quad , g\in G, \ [x_1|\cdots|x_k]\in K.$$

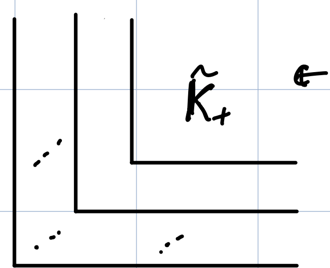
Notice: G acts on $\hat{K}$ by $\quad g\cdot[g'\,\|\,x_1|\cdots|x_k]=[gg'\,\|\,x_1|\cdots|x_k]$

and $\hat{K} \to \hat{K}/G = K$ is a covering map

Let $\hat{K}_+$ be the subcomplex of $\tilde{K}$ consisting of all simplices $[x \| x_1|\cdots|x_k]$, $x \in M$.

Since every $g \in G$ has the shape $g = \delta^{-n} x$, $x \in M$, $n \geq 0$, one can invade $\hat{K}$ by

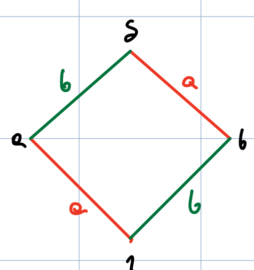
$$\hat{K} = \bigcup_{n \geq 0} \delta^{-n} \tilde{K}_+$$



Therefore it is enough to prove that $\hat{K}_+$ is contractible.

Let's take a matching:

$$\mathcal{M} = \{ [x x_1 \| x_2|\cdots|x_k] \prec [x \| x_1|\cdots|x_k] \mid \gcd(\delta, x x_1 \cdots x_k) = x_1 \cdots x_k \}$$

example: $\Gamma =$ 

$K \simeq S^1 \times S^1$, $\tilde{K} \simeq \mathbb{R} \times \mathbb{R}$

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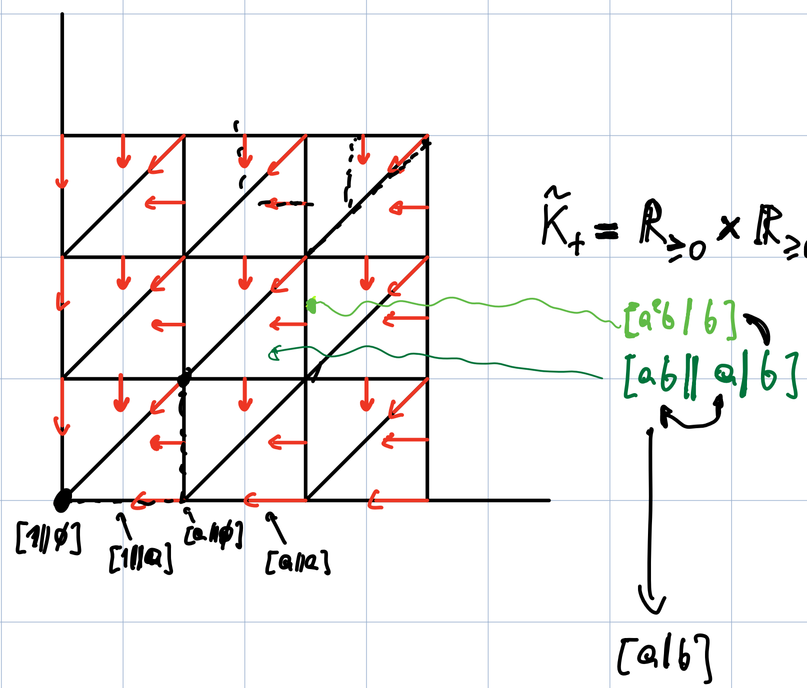

$\tilde{K}_+ = \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$

check: $\mathcal{M}$ is proper and acyclic.