... $K(\pi,1)$-conjecture for Artin groups ...

Definitions, motivations, connections and examples.

Classical Braids (E. Artin "Theorie der Zöpfe" (1925) Hamb. Abh., " "Theory of Braids", Annals of Math (1947))

(word and conjugacy problem, center ...)

Singularity Theory (Thom, Pham, Arnol'd 60's ... Brieskorn, Deligne ... 70's) $\longrightarrow$ $K(\pi,1)$-conjecture

Proof for finite Coxeter groups (Deligne, 1972)

(A-D-E classification - rational singularities, topology of the universal deformation)

Hyperplane Arrangements and Combinatorics

(Orlik-Solomon algebra, combinatorial problems on lattices, root systems, topological models, cohomological computations

Stanley, Orlik, Solomon ('80), De Concini, S. (90's)

linear, toric, ...)

Development of **Garside theory** (after Deligne)

Dual Artin groups

(Birman, Ko, Lee, Dehornoy, Paris, Bessis ... 90's - 00's)

More geometric methods (McCammond, Sulway, Brady, .. 10's)

& combinatorial methods (discrete Morse Theory, shellings of posets, ...)

Proof of the $K(\pi,1)$-conjecture for affine Artin groups

(G. Paolini-S., Inv.Math '21)

generalizations, open questions

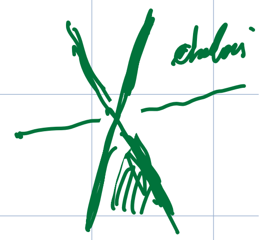
(hyperbolic groups (rank 3, Delucchi-Paolini-S.

other affine non-reflection arrangements

etc...)

EGBERT BRIESKORN
Sur les groupes de tresses
Séminaire N. Bourbaki, 1973, exp. nº 401, p. 21-44
<http://www.numdam.org/item?id=SB_1971-1972__14__21_0>

Bourbaki - Humphreys



W : finite reflection group in $\mathbb{R}^N$.

So, W is generated by (orthogonal) reflections; denote by

R : set of all reflections in W; each $r \in R$ fixes a hyperplane H_r

$\mathcal{A} = \{H_r \,,\ r \in R\}$ the <u>reflection arrangement</u> of W.

- The union $\bigcup_{r \in R} H_r$ stratifies $\mathbb{R}^N$ into "faces"; the connected components of $\mathbb{R}^N \setminus \bigcup_r H_r$ are called <u>chambers</u>
- W acts <u>simply transitively</u> on the set of chambers ;
- $\forall\, p \in \mathbb{R}^N$, $\mathrm{Stab}_W(p) = \langle r \in R \mid p \in H_r \rangle$;

  in particular W <u>acts freely</u> on $\mathbb{R}^N \setminus \bigcup_r H_r$
- $\forall$ chamber C_0, the closure $\overline{C_0}$ is a fundamental domain for the action of W.

- $\overline{C}_0$ is a polyhedral cone, with <u>walls</u> each H_η s.t. $H_\eta \cap \overline{C}_0$ is open in H_η. If the only point which is fixed by all W is 0, (i.e. $T = \bigcap_{\eta \in R} H_\eta = \{0\}$) then $\overline{C}_0$ is <u>simplicial</u>, i.e. $\overline{C}_0$ is linearly isomorphic to the positive octant of $\mathbb{R}^N$.

We can assume that, up to taking $\mathbb{R}^N/T$.

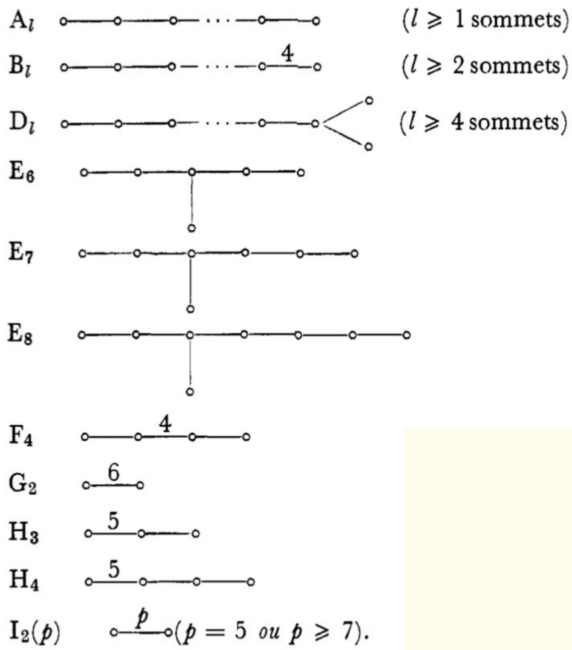
- Let $S \subset R$ be the reflections which correspond to the walls of C_0; then (W, S) is a Coxeter system, meaning that W is presented as:

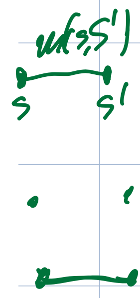
$$W = \langle s \in S \mid (ss')^{m(s,s')} \rangle$$

where $m(s,s) = 1$ and $m(s,s') \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$ if $s \neq s'$.

Classification:

THÉORÈME 1. — *Si* (W, S) *est un système de Coxeter fini irréductible, son graphe de Coxeter est isomorphe à l'un des suivants :*





$= \langle s_i = (i,i+1) \mid (s_i s_j)^{m(s_i,s_j)}$

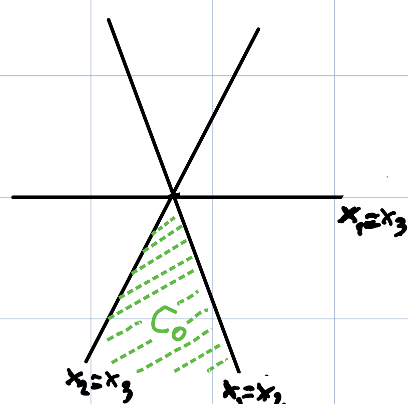
ex: $A_n \leftrightarrow S_{n+1} = \langle s_i = (i,i+1) \mid s_i^2 = 1,\ s_i s_j = s_j s_i$ if $|i-j| \geq 2$, $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \rangle$

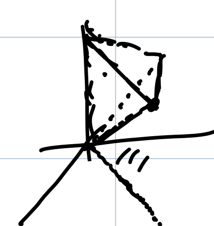
$S_{n+1} \ni \sigma$ acts on $\mathbb{R}^{n+1}$: $\sigma.(x_1,\dots,x_{n+1}) = (x_{\sigma(1)},\dots,x_{\sigma(n+1)})$;

$\mathcal{R} \longleftrightarrow \{ t_{ij} = (i\ j) \}$ t_{ij} fixes $H_{ij} = \{x_i - x_j = 0\}$

$\bigcap_{i,j} H_{ij} = \mathrm{Span}(1,\dots,1)$; base chamber $C_0 = x_1 \leq x_2 \leq \dots \leq x_{n+1}$

A_2





1

- Take a face of $\overline{C}_0$: $F = \bigcap_{s\in J} H_s \cap \overline{C}_0$, for some $J \subset S$;

$s\in J$ generate a subgroup $W_J \subset W$, called <u>parabolic</u> (standard) <u>subgroup</u>. $W_J = \mathrm{Stab}_F$; actually, every face G of the stratification is conjugate to exactly one face $F\subset \overline{C}_0$; the set

$\{ w\in W \mid w.F = G\} = g.W_J$ is a <u>coset</u> of Stab_F; and

Stab_G is a conjugate of W_J by any $w\in g.W_J$.

- Length: the <u>length</u> w.r.t. S is $\ell_S(w) = \min\{k \mid w = s_{i_1}\cdots s_{i_k},\ s_{i_j}\in S\}$

<u>Prop.</u> Every coset $w.W_J$ of a parabolic subgroup contains a <u>unique</u> element of minimal ℓ_S-length. W^J

---

ex.: $A_2 = S_3$, $S = \{s_1 = (1\,2), s_2 = (2,3)\}$, parabolic subgroups:

$W_\emptyset = \{1\}$, $W_{\{s_i\}} = \{1, s_i\}$, $W_S = W$;

cosets of $W_{\{s_1\}}$: $W_{s_1} = \{1, s_1\}$, $s_2 W_{s_1} = \{s_2, s_2 s_1\}$, $s_1 s_2 W_{s_1} = \{s_1 s_2, s_1 s_2 s_1\}$

minimal length elements: 1, s_2, $s_1 s_2$

Complexification. W acts diagonally over $\mathbb{C}^N = \mathbb{R}^N + i\,\mathbb{R}^N$

- let $\mathcal{A}_{\mathbb{C}} = \{ H_n^{(\mathbb{C})},\ H_n \in \mathcal{A} \}$ be the complexified arrangement.

Then:

- $\mathbb{C}^N/W$ is an affine variety isomorphic to the affine space $\mathbb{A}^n_{\mathbb{C}}$.

let $\Delta = \bigcup_n H_{n,\mathbb{C}}$ and

$D = \pi_W(\Delta)$, $\quad \pi_W : \mathbb{C}^N \to \mathbb{C}^N/W$

Then $\mathrm{Stab}_W(p) = \{1\} \iff p \in Y$, so

$\pi_W : \mathbb{C}^N \to \mathbb{C}^N/W$ is a ramified covering

where Δ: ramification locus, D: discriminant.

Set:

$Y = \mathbb{C}^N \setminus \Delta$: <u>Configuration Space</u>

$Y_W = Y/W = (\mathbb{C}^N \setminus \Delta)/W = \mathbb{C}^N/W \setminus D$ <u>Orbit Configuration Space</u>

Then: $\begin{array}{c} Y \\ \downarrow \pi \\ Y_W \end{array}$ is a regular covering (fiber W)

$s^2=1$ $\quad (ss')^{m(s,s')}=1$ $\quad ss'ss'\cdots=1$ $\quad \overbrace{ss'\cdots}^{m(s,s')} = \overbrace{s's\cdots}^{m(s,s')}$

<u>DEF.</u> (ALGEBRAIC) The ARTIN GROUP of type W is

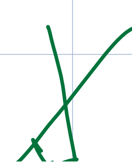
$$G_W = \langle g_s,\ s\in S \mid g_s g_{s'}\cdots = g_{s'} g_s \cdots,\ m(s,s') \text{ factors} \rangle$$

<u>ex.</u> $W = A_n$: $G_W = \left\langle g_1,\dots,g_n \;\middle|\; \begin{array}{l} g_i g_j = g_j g_i, \quad |i-j|\geq 2 \\ g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, \end{array} \right\rangle$

classical braid group.

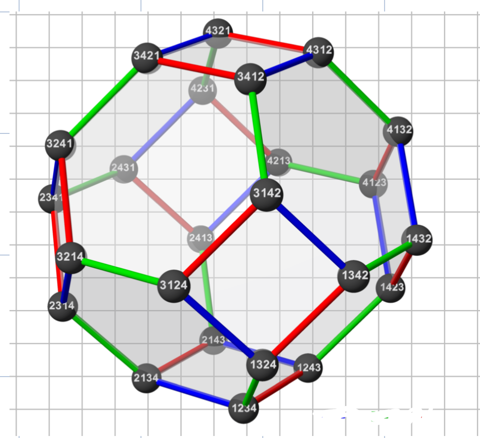
<u>Theorem</u> $\pi_1(Y_W) = G_W$ (Brieskorn)

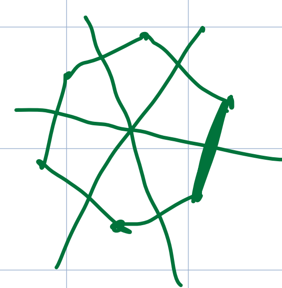
It can be also derived from the following construction.



. Take $x_0 \in C_0$ and let $P = \text{conv. hull}\,[W.x_0]$

It is a convex polyhedron sometimes called PERMUTOHEDRON.





(dim K)-cell $e \in P \longleftrightarrow$ (codim K)-face F of the stratification $\longleftrightarrow$ coset $w.W_J$, $|J| = K$

$\updownarrow$

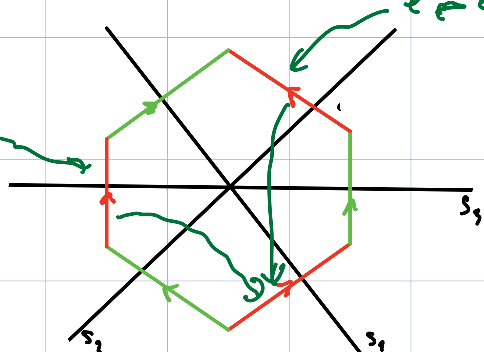
γ_e minimal length

Thm (-, '94) $X_W = P/e \underset{W}{\sim} e'$ $\simeq$ Y_W orbit config. space

where the identification is through: $\gamma_{e'} \circ \gamma_e^{-1}$

$e \sim s_2 W_{\{s_1\}}$

$e' = s_1 s_2 W_{\{s_1\}}$

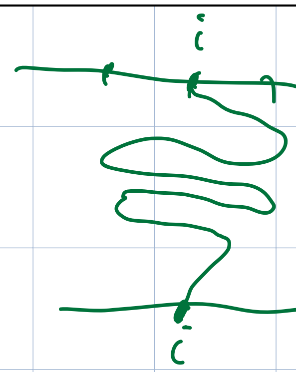


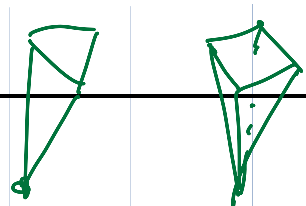
A_2

$$1 \longrightarrow \pi_1(Y) \longrightarrow \pi_1(Y_W) \longrightarrow W \longrightarrow 1$$

pure (or colored) Artin group

Artin group





universal covering contractible

Theorem (Deligne, '72) Y_W (so Y) is a $K(\pi,1)$-space.

He proved more: "If $\mathcal{A}$ is simplicial then Y is $K(\pi,1)$.

"Ad hoc" proof in some cases:

$$\begin{array}{c} Y \leftarrow K(\pi,1) \\ \downarrow \\ Y_W \end{array}$$

<u>Case A_n</u>: $\qquad Y_n = \mathbb{C}^{n+1} \setminus \Delta = \{(z_1, \dots, z_{n+1}) \mid z_i \neq z_j \}$

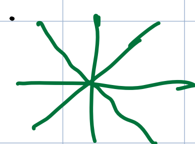
$p : Y_n \longrightarrow Y_{n-1} : (z_1, \dots, z_{n+1}) \longrightarrow (z_1, \dots, z_n)$ is easily seen to be a locally trivial

bundle with fiber $F_n = \mathbb{C} \setminus \{n\text{-pts}\}$. Since the fiber is a $K(\pi,1)$, by

$$\dots \longrightarrow \pi_i(F_n) \longrightarrow \pi_i(Y_n) \longrightarrow \pi_i(Y_{n-1}) \longrightarrow \pi_{i-1}(F_n) \longrightarrow \dots$$

<u>Case $B_n = C_n$</u>: very similar because $Y_n = \{(z_1, \dots, z_n) \mid z_i \neq \pm z_j,\ z_i \neq 0\}$

The fiber of the bundle $p : Y_n \longrightarrow Y_{n-1}$, is $\mathbb{C} \setminus \{n\text{-}n\text{-pts}\}$.



<u>Case D_n</u>: here $Y_n = \{(z_1, \dots, z_n) \mid z_i \neq \pm z_j\}$. $\quad (n \geq 4)$

Here $p : Y_n \longrightarrow Y_{n-1}$ is not a trivial bundle ($p^{-1}(z_1, \dots, z_{n-1}) \cong \mathbb{C} \setminus \{2n\text{-pts}\}$ if $z_i \neq 0, \forall i$,

$p^{-1}(z_1, \dots, z_{n-1}) \cong \mathbb{C} \setminus (2n-1)\text{-pts}$ if one $z_i = 0$).

Here $Z_n = \mathbb{C}^n \setminus \bigcup \{z_i = z_j\} \cup \bigcup \{z_i = 0\}$ is $K(\pi,1)$ (same trick as before). The map

$$Y_n \longrightarrow Z_{n-1} : (z_1, \dots, z_n) \longrightarrow (z_n^2 - z_1^2, \dots, z_n^2 - z_{n-1}^2)$$

is a locally trivial fibering with fiber a $K(\pi,1)$.

In general there is apparently no fibration

Deligne proof is general: based on <u>Garside theory</u>

(Garside, Dehornoy-Paris, McCammond, ...)

EXAMPLES.

· <u>Free abelian group</u>.

$G = \mathbb{Z}^n \supset \underbrace{\mathbb{N}^n}_{\text{MONOID}} \ni \underbrace{\Delta}_{\text{GARSIDE ELEMENT}} = (1, \dots, 1)$, $\underbrace{\mathcal{P}}_{\text{all divisors of } \Delta \text{ in } \mathbb{N}^n} = \{ (\varepsilon_1, \dots, \varepsilon_n) \mid \varepsilon_i \in \{0,1\} \}$

$\mathbb{N}^n$ is a poset with $a = (a_1, \dots, a_n) \leq b = (b_1, \dots, b_n) \iff a_i \leq b_i \ \forall i \iff$

$\iff \exists c \in \mathbb{N}^n \mid ac = b$

(multiplicative notation)

Then: $\mathcal{P} = \{ a \in \mathbb{N}^n \mid a \leq \Delta \}$

$a \vee b$

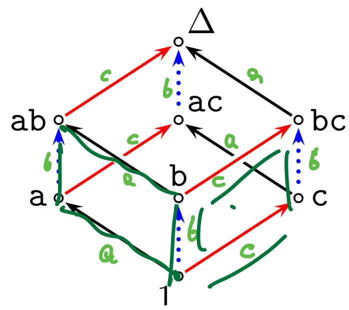
Notice: - $\mathbb{N}^n$ is a cancellative monoid which is a lattice

$a \vee b = (\dots, \max(a_i, b_i), \dots)$, $a \wedge b = (\dots, \min(a_i, b_i), \dots)$

- it is generated by $\mathcal{B}$ (boolean lattice) ← (in this case)
- let generators $a=(1,0,\dots)$, $b=(0,1,0,\dots)$, $c=(0,0,1,0,\dots)$ ..

here is the 3-dim. case :

$\mathcal{B}$:



we label the edges of the Hasse diagram of $\mathcal{B}$ by the generators.

Notice : $G = \langle a,b,c,\dots \mid ab=ba,\ ac=ca,\ bc=cb, \dots \rangle$

i.e. presentation is obtained by reading labels on the maximal chains of the intervals $[x,y]$, $x \leq y$.

So, we have all ingredients of a <u>Garside structure</u> :

- a left and right cancellative monoid M which is a (left and right) lattice ; $\exists\ \nu : M \to \mathbb{N}$ s.t. $\nu(ab) \geq \nu(a)+\nu(b)$ (M 'atomic')
- A special element $\Delta \in M$ (the <u>Garside element</u>) whose left and right divisors coincide and generate M

and the group G is the group of (left) fractions $f^{-1}g$, $f,g\in M$.

First consequence: NORMAL FORM

G Garside group (group of fractions of M with (M,Δ) Garside). Then every $g\in G$ has a unique form: $g=\Delta^n s_1\cdots s_k$,

$n\in\mathbb{Z}$, $s_i\leq\Delta$, $s_1\neq\Delta$, $s_k\neq 1$, $\forall i$: $s_i=\gcd(s_i\cdots s_k,\Delta)$

example:

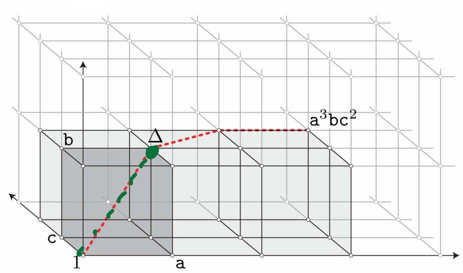


Figure . The free abelian monoid $\mathbb{N}^3$; the cube $\{g\in\mathbb{N}^3 \mid g\leqslant\Delta\}$ is in dark grey; among the many possible ways to go from 1 to a^3bc^2, the distinguished decomposition of Proposition consists in choosing at each step the largest possible element below Δ that lies below the considered element, that is, to remain in the light grey domain.

$$a^3bc^2=\underset{\overset{\parallel}{\Delta}}{(abc)}\;\underset{\overset{\parallel}{s_1}}{(ac)}\;\underset{\overset{\parallel}{s_2}}{a}$$

lattice

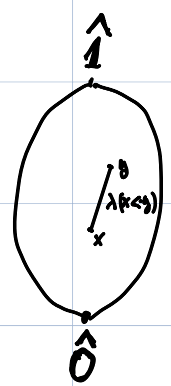
"atomic" subposet

So, $G\supset M\supset[1,\Delta]=\mathcal{P}$ subposet

Try to get all informations from the poset $\mathcal{P}$.

COMBINATORIAL GARSIDE THEORY

Start from a finite, bounded, edge-labeled poset $\mathcal{P}$:

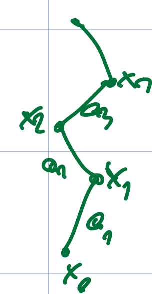


A : set of labels

$x \lessdot y \Rightarrow \lambda(x \lessdot y) = a \in A$

cover relation

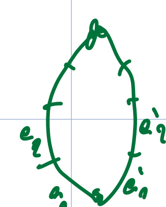
edge of the Hasse diagrams



$\forall$ maximal chain $\gamma = (x_0 \overset{a_1}{\lessdot} x_1 \overset{a_2}{\lessdot} x_2 \lessdot \cdots \overset{a_k}{\lessdot} x_k)$ has a labeling

$\lambda(\gamma) = a_1 \cdots a_k \in A^*$

we associate a monoid



$M = M(\mathcal{P}) = \langle a \in A \mid \lambda(\gamma) = \lambda(\gamma')$, γ, γ' maximal chains in some interval $[x,y] \rangle$

and a group

$G = G(\mathcal{P})$ defined by the same presentation.

Main example: G group, $M \subset G$ monoid, $\Delta \in M$ some element,

A = set of generators of M (and of G) s.t. M is atomic. So

M is a finite-height poset w.r.t. divisibility; and

$\mathcal{P} = [1, \Delta]$ is edge-labeled by A.

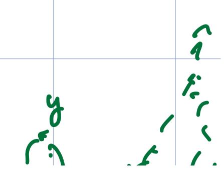
<u>def</u> Say that $\mathcal{P}$ is a <u>combinatorial Garside structure</u> if

$\mathcal{P}$ is (1) group-like; (2) balanced; (3) graded; (4) lattice

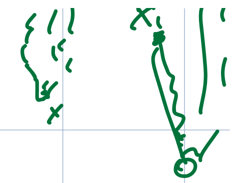
[combinatorial counterparts of cancellative, left and right divisors of Δ coincide, ...]

Then:

<u>Theorem</u> $\mathcal{P}$ combinatorial Garside structure. Then:



1) There are inclusions: $\mathcal{P} \overset{i}{\hookrightarrow} M(\mathcal{P}) \overset{j}{\hookrightarrow} G(\mathcal{P})$.



here: $\forall\, x < y$ one has the "language":

$$\mathcal{L}(x,y) = \{ \lambda(\gamma) \in A^* \mid \gamma \text{ maximal chain between } x \text{ and } y \}$$

then $\forall x \in \mathcal{P}$, every $\lambda(\gamma) \in \mathcal{L}(\hat{0}, x)$ represents the same element of $M(\mathcal{P})$ (by definition of $M(\mathcal{P})$). So $i(x) = \lambda(\gamma)$

2) Setting $\Delta = i(\hat{1})$ $(= \lambda(\gamma),\ \gamma$ maximal chain in $\mathcal{P})$, (M, Δ) is a Garside structure (M cancellative, atomic, lattice; G Garside group)

3) There is a CW-complex (Δ-complex)

$$K(\mathcal{P}) = \Delta(\mathcal{P})/\sim \;\simeq K(G,1)$$

$\Delta(\mathcal{P})$: order complex of $\mathcal{P}$

MOST USEFUL Let G be a group, $R = R^{-1}$ generating set and take $g \in G$ "balanced" (left divisors = right divisors). G is a poset with:

$x \leq g \Leftrightarrow \ell_R(x) + \ell_R(x^{-1}g) = \ell_R(g)$ ($\Leftrightarrow \exists$ minimal length factorization of g beginning by a minimal length factorization of x). Let $\mathcal{P}=[1,g]$, which is edge-labeled by $\mathcal{P}_0 = \mathcal{P}\cap[1,g]$. Then if $\mathcal{P}$ is a combinatorial Garside structure ($\Leftrightarrow$ $\mathcal{P}$ is a lattice) then $G(\mathcal{P})$ is a Garside group with Garside structure $(M(\mathcal{P}), g)$.

interval group

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rmk. If G is a Garside group with Garside structure (M,Δ), then $\mathcal{P}=[1,\Delta]$ is a poset (w.r.t. divisibility) edge-labeled by the atoms $a\leqslant\Delta$, and $M = M(\mathcal{P})$, $G=G(\mathcal{P})$ (therefore M (and G) has a presentation read from the labels of chains in $\mathcal{P}$).

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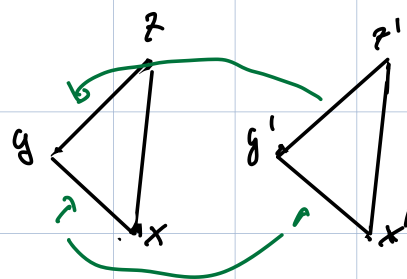
group-like: $x\le y\le z$, $x'\le y'\le z'$

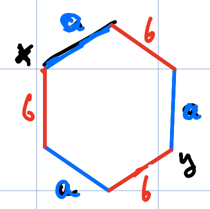


$\mathcal{L}(x,y)=\mathcal{L}(x',y')$, $\mathcal{L}(y,z)=\mathcal{L}(y',z')$ $\Rightarrow$ $\mathcal{L}(x,z)=\mathcal{L}(x',z')$.

sufficient: $\mathcal{P}=[1,g]$, (right) Cayley graph of a group.

balanced: $\{\mathcal{L}(\hat{0},x) \mid x\in\mathcal{P}\} = \{\mathcal{L}(u,\hat{1}) \mid u\in\mathcal{P}\}$

For example: $\left(\mathfrak{S}_3, \; \begin{matrix} a=(12) \\ b=(23) \end{matrix}\right)$



$\mathcal{L}(1,x) = \{a\,b\} = \mathcal{L}(y,1)$

<u>Examples</u>

1. Standard Garside structure.

(W,S) finite Coxeter system. $\ell_S : W \longrightarrow \mathbb{N}$ the length function.

$\mathcal{R} = \bigcup_{s\in S} w^{-1} s w$ all reflections, $\mathcal{A} = \{H_r\}$ reflection arrangement.

C_0 : base chamber. Then

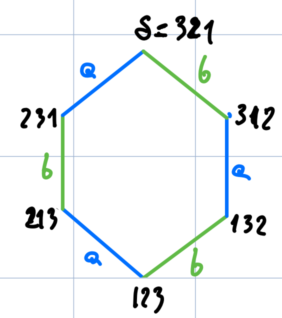
- $\ell_S(w) = \#\{H_r \mid H_r$ separates C_0 from $w.C_0\}$
- $\exists$ unique element $\delta \in W$ of maximal length;
- $\ell_S(w) = \#\mathcal{A} = \#\mathcal{R}$
- for $w \in W$, set $\mathcal{R}(w) = \{ r \in \mathcal{R} \mid H_r$ separates C_0 from $w.C_0\}$.

and $w \preccurlyeq w' \Leftrightarrow \ell_S(w) + \ell_S(w^{-1}w') = \ell_S(w')$ $[\Leftrightarrow \mathcal{R}(w) \subset \mathcal{R}(w')]$

(weak Bruhat order). Then

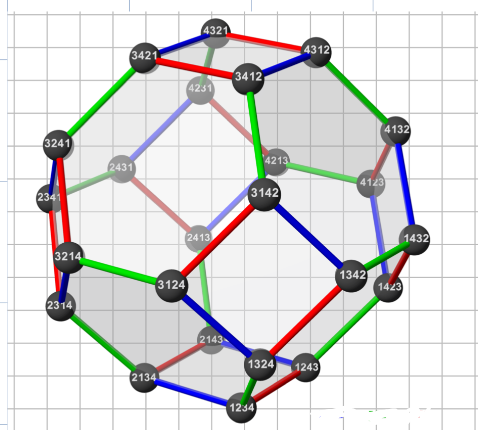
$W=[1,\delta]$ is an S-labeled poset (an interval)

$W=\sigma_3$, $S=\{a=(12),\ b=(23)\}$



$\delta = aba = bab$

$W=\sigma_4$, $S=\{a=(12),\ b=(23),\ c=(34)\}$



$\delta = abcaba = (4321)$

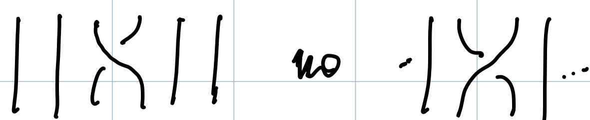
NOTICE: $G([1,\delta]) = \pi_1(X_W) =$ ARTIN GROUP G_W !

same presentation $G_W = \langle g_s,\ s\in S \mid \underbrace{g_s g_{s'} \cdots}_{} = \underbrace{g_{s'} g_s \cdots}_{}$ $m(s,s')$ factors $\rangle$

monoid $M([1,\delta]) = G_W^+ = \langle \text{same presentation} \rangle_+$

ex: for $W = \sigma_n$, $G_W = Br_n$,

$G_W^+ = Br_n^+$ is given by "positive braids"

composition of 

This implies that there is a section of the natural map

$$\begin{tikzcd} G_W^+ \arrow[r, "\pi"] & W \arrow[l, bend right, "s"'] \end{tikzcd}$$

Now to conclude the Deligne proof in finite case, one can

1) show that $P = [1,\delta] = W$ is a lattice (it derives from the Deletion condition [exercise])

2) use lattice property and a CW-model (as X_W above) to prove that the universal covering of Y_W is contractible; alternatively, one can proof that the associated complex K (which is a $K(\pi,1)$ by part 3 of the thm above) is homotopy equivalent to the orbit configuration space (see for ex. [Delucchi, '09])