THE HOMOLOGY OF THE BRAID GROUP MODULO ITS CENTER

(based on *Homology operations for gravity algebras,* arXiv:2404.10639)

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Geometry, Algebra and Combinatorics of Moduli Spaces and Configurations VIII

Dobbiaco

PLAN OF THE TALK

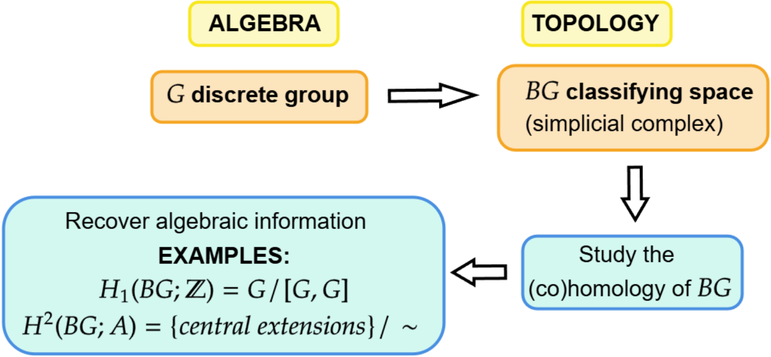
1) **Preliminaries:** (co)homology of groups

2) **Braid groups and configuration spaces**

3) **Computation:** $H_*(B_n / Z(B_n); \mathbb{F}_p)$

PRELIMINARIES: (CO)HOMOLOGY OF GROUPS

INTRODUCTION TO GROUP (CO)HOMOLOGY



HOMOTOPICAL CHARACTERIZATION OF BG

Theorem: $\pi_*(BG) = \begin{cases} G & * = 1 \\ 0 & else \end{cases}$

Moreover, this condition **characterize** BG **up to homotopy**

EXAMPLES:

1) $B(\mathbb{Z}) \simeq S^1$

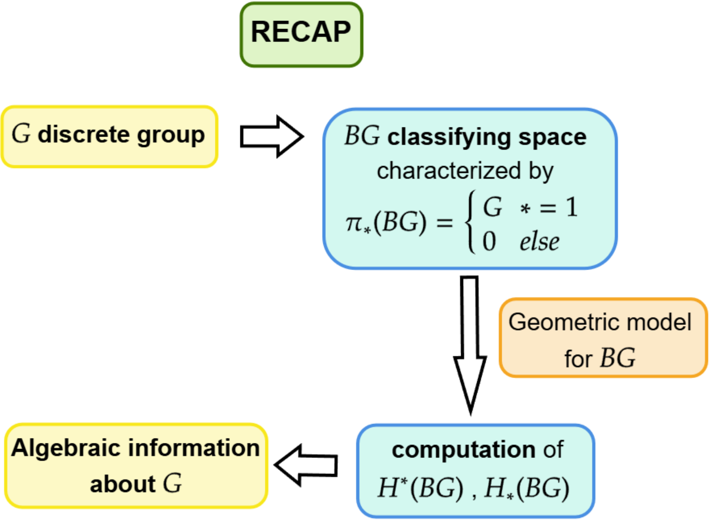
2) $B(C_2) \simeq \mathbb{RP}^\infty$

⇨

COMPUTATIONS:

1) $H^*(B\mathbb{Z};\mathbb{Z}) = \mathbb{Z}[x]/(x^2)$

2) $H^*(BC_2;\mathbb{Z}) \simeq \mathbb{Z}[x]/(2x)$


RECAP
G discrete group
BG classifying space
characterized by
$\pi_*(BG) = \begin{cases} G & * = 1 \\ 0 & else \end{cases}$
Geometric model
for BG
computation of
$H^*(BG)$, $H_*(BG)$
Algebraic information
about G

CONFIGURATION SPACES AND BRAID GROUPS

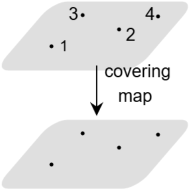
CONFIGURATION SPACES

Ordered configuration space:

$$F_n(\mathbb{C}) := \left\{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j\right\}$$

Unordered configuration space:

$$C_n(\mathbb{C}) := F_n(\mathbb{C}) / \Sigma_n$$



Threorem:

1) $\pi_*(F_n(\mathbb{C})) = \begin{cases} PB_n & * = 1 \\ 0 & else \end{cases}$

2) $\pi_*(C_n(\mathbb{C})) = \begin{cases} B_n & * = 1 \\ 0 & else \end{cases}$

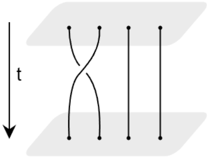
⇒

The (ordered) configuration space is a **classifying space** (for its fundamental group)

BRAID GROUPS

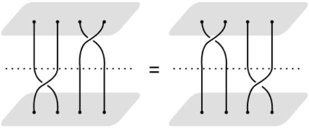
Pure braid group:
$PB_n := \pi_1(F_n(\mathbb{C}))$

Braid group:
$B_n := \pi_1(C_n(\mathbb{C}))$



Presentation (Artin):

$$B_n = < \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i \; if \; |i-j| \geq 2, \; \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} >$$



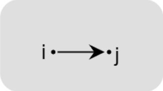
THE COHOMOLOGY OF PURE BRAID GROUPS (Arnol'd)

Pure braid group/ordered configuration space:

$$H^*(F_n(\mathbb{C});\mathbb{Z}) = \frac{\Lambda(w_{ij} \mid i < j)}{(w_{ij}w_{jk} + w_{jk}w_{ki} + w_{ki}w_{ij})} \quad \text{with } |w_{ij}| = 1$$

Geometric interpretation: $PD : H^*(F_n(\mathbb{C})) \xrightarrow{\cong} H^{BM}_{2n-*}(F_n(\mathbb{C}))$

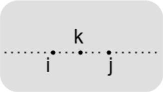
$$PD(w_{ij}) =$$



$$\partial W_{ijk} =$$

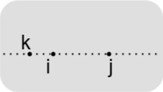
$PD(w_{ij}w_{jk})$

$\pm$



$PD(w_{ik}w_{kj})$

$\pm$



$PD(w_{ki}w_{ij})$

THE HOMOLOGY OF BRAID GROUPS (Cohen, Arnol'd)

Rationally:
$H_*(C_n(\mathbb{C}); \mathbb{Q}) = \mathbb{Q}x_0 \oplus \mathbb{Q}x_1$

Over the integers:
complicated, but explicit!

IDEA: it is convenient to study $C_n(\mathbb{C})$ all together!

$$C(\mathbb{C}) := \bigsqcup_{n=1}^{\infty} C_n(\mathbb{C})$$

$H_*(C(\mathbb{C}))$ has a **rich algebraic strucure** (E_2-algebra):

1) $\mu_* : H_*(C(\mathbb{C})) \otimes H_*(C(\mathbb{C})) \to H_*(C(\mathbb{C}))$ **commutative product**

2) $[-,-] : H_*(C(\mathbb{C})) \otimes H_*(C(\mathbb{C})) \to H_*(C(\mathbb{C}))$ **Lie bracket**

Geometric interpretation:

$\mu : C(\mathbb{C}) \times C(\mathbb{C}) \to C(\mathbb{C})$ associative, commutative (up to homotopy)

Theorem (Cohen):

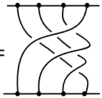
$$H_*(C(\mathbb{C}); \mathbb{F}_2) = \mathbb{F}_2\left[\iota, Q\iota, Q^2\iota, \ \dots\ \right]$$

$$H_*(C(\mathbb{C}); \mathbb{F}_p) = \mathbb{F}_p\left[\iota, \beta Q[\iota,\iota], Q^2[\iota,\iota], \ \dots\ \right] \otimes \Lambda_{\mathbb{F}_p}\left([\iota,\iota], Q[\iota,\iota], Q^2[\iota,\iota], \ \dots\right)$$

THE HOMOLOGY OF $B_n / Z(B_n)$

THE CENTER OF THE BRAID GROUP

Fact: $Z(B_n) \cong \mathbb{Z} \cdot \Delta^2$

$\Delta =$ 

GOAL: compute $H_*(B_n / Z(B_n))$

We need a good geometric model for the classifying space $B(B_n / Z(B_n))$

ALGEBRA

Short exact sequence

$0 \to Z(B_n) \to B_n \to B_n / Z(B_n) \to 0$

$\Longleftrightarrow$

TOPOLOGY

Fiber sequence

$B\mathbb{Z} \to B(B_n) \to B(B_n / Z(B_n))$

HOMOTOPY QUOTIENTS

PROBLEM:

ordinary quotients are not homotopy invariant!

e.g. $\mathbb{R} \simeq *$ but $\mathbb{R}/\mathbb{Z} \not\simeq */\mathbb{Z}$

PROPERTIES:

1) $X \simeq_G Y \implies X//G \simeq Y//G$
2) If G acts **freely** on X, then $X/G \simeq X//G$
3) **Fiber sequence**: $X \hookrightarrow X//G \to BG$

RESULTS

Theorem 1 (R. 2023): let x be a variable of degree 2. For any $n \in \mathbb{N}$ and p prime we have:

$$H_*(B_n / Z(B_n); \mathbb{F}_p) = \begin{cases} H_*(C_n(\mathbb{C})) \otimes \mathbb{F}_p[x] & if\ n = 0, 1 \mod p \\ H_*(C_n(\mathbb{C})) / Im(d_2) & else \end{cases}$$

Theorem 2 (R. 2023): for any $n \in \mathbb{N}$ and p prime $H_*(B_n / Z(B_n); \mathbb{F}_p(\pm 1))$ is computed explicitly (additive basis described)

SKETCH OF PROOF FOR $H_*(B_n / Z(B_n); \mathbb{F}_p)$

MEMO: $B(B_n / Z(B_n) \simeq C_n(\mathbb{C}) // S^1 \Rightarrow$

We can try to compute
$H_*\left(C_n(\mathbb{C}) // S^1; \mathbb{F}_p\right)$
via the Serre sp. sequence associated to
$C_n(\mathbb{C}) \hookrightarrow C_n(\mathbb{C}) // S^1 \to BS^1$

Degeneration Theorem: X space with an S^1-action, consider the Serre sp. seq. associated to $X \hookrightarrow X // S^1 \to BS^1$ ($\mathbb{F}_p$ coefficients). Then:

$$\sum_{i=0}^{\infty} dim_{\mathbb{F}_p} H_i(X) = \sum_{i=0}^{\infty} dim_{\mathbb{F}_p} H_i\left(X^{\mathbb{Z}/p}\right) \implies E_2 = E_\infty$$

$\Downarrow$

Theorem 1 (R. 2023): for any $n \in \mathbb{N}$, p prime

$$H_*\left(C_n(\mathbb{C}) // S^1; \mathbb{F}_p\right) \cong \begin{cases} H_*(C_n(\mathbb{C})) \otimes H_*\left(BS^1\right) & if\ n = 0, 1 \bmod p \\ H_*(C_n(\mathbb{C})) / Im(d_2) & else \end{cases}$$

SKETCH FOR $H_*(B_n/Z(B_n);\mathbb{F}_p(\pm 1))$

Key fact 1:

$$H_*\left(E(\lambda;S^1);\mathbb{F}_p\right) \cong \bigoplus H_*(B_n/Z(B_n);\mathbb{F}_p(\pm 1))$$

Key fact 2:

$$H_*\left(E(\lambda;S^1);\mathbb{F}_p\right) \cong H_*\left(C(\mathbb{C};S^1)\right)\otimes H_*(BS^1)$$