Computing Joins in the Weak Order of Coxeter Groups of type B

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1. Preliminaries and motivation

2. Results and further questions

Coxeter groups background

- (W, S) a Coxeter system: a group W generated by a set of involutions S;

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- (right) weak order: given $u, v \in W$,

$$u \leq_R v \iff \ell(v) = \ell(u) + \ell(u^{-1}v) \iff T_L(u) \subseteq T_L(v);$$

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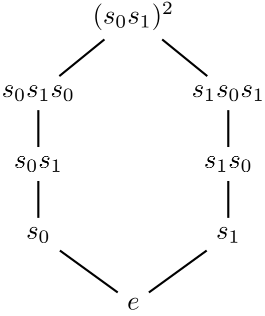
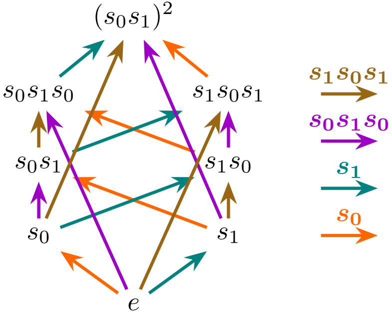
- Bruhat graph: directed graph with W as vertex set and $u \xrightarrow{t} v \iff$ there is $t \in T$ such that $v = tu$ and $\ell(u) < \ell(v)$

Example on B_2

$$B_2 =< s_0, s_1 \mid s_0^2 = s_1^2 = (s_0 s_1)^4 = e >, \quad T = \{s_0, s_1, s_0 s_1 s_0, s_1 s_0 s_1\}$$



Hasse diagram of $(B_2, \leq_R)$.



Bruhat graph of B_2.

Motivating conjecture

Definition (Dermenjian, 2025)

Given $A \subseteq T$, an *A-Bruhat path* is any directed path in the Bruhat graph starting from $e \in W$ whose edges have labels in A.
The *Bruhat preclosure* of A is

$$\overline{A} = \{t \in T \mid \text{there is an } A\text{-Bruhat path from } e \text{ to } t\}.$$

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Note: in general, $\overline{\overline{A}} \neq \overline{A}$ (counterexamples in H_3 and F_4).

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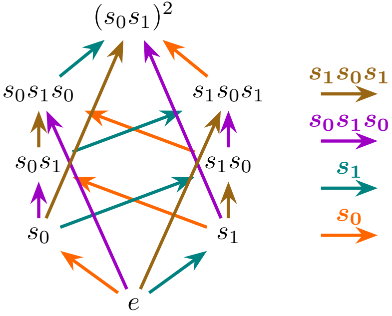

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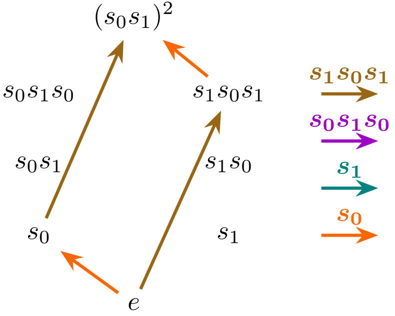
- If $A = \{s_0, s_1 s_0 s_1\}$; then $\overline{A} = A$.



A-Bruhat paths.

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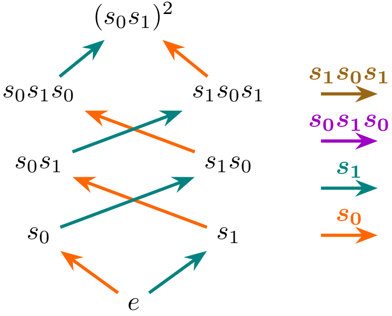

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Conjecture (Dyer, 2019)

Let W be a finite Coxeter group and $u, v \in W$. Then

$$T_L(u \vee_R v) = \overline{T_L(u) \cup T_L(v)}.$$

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Theorem (Dermenjian, 2025)

Let W be a Coxeter group and $w \in W$. Then $\overline{T_L(w)} = T_L(w)$.

This implies $T_L(u \vee_R v) \supseteq \overline{T_L(u) \cup T_L(v)}$.

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  - the swap of a with b and of $\bar{a}$ with $\bar{b}$ if $b \neq \bar{a}$;
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- Consider $S = \{s_0 = (1\ \bar{1})\,, s_1 = (1\ 2)\,, \ldots, s_{n-1} = (n-1\ \ n)\}$;
- (S_n^B, S) is a combinatorial description of the Coxeter system B_n in which reflections coincide with transpositions

$$T = \{(a\ b) \mid a \neq b \in [\pm n]\}.$$

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Some important statistics of a signed permutation $\sigma \in S_n^B$:

- $\mathrm{Inv}(\sigma) = \{(a, b) \in [n] \times [n] \mid a < b, \sigma(a) > \sigma(b)\}$ *inversions*;
- $\mathrm{Neg}(\sigma) = \{a \in [n] \mid \sigma(a) < 0\}$ *negative entries*;
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$$\begin{aligned} T_L(\sigma) = & \{(i\ j) \mid (i,j) \in \mathrm{Inv}(\sigma^{-1})\} \sqcup \{(i\ \bar{i}) \mid i \in \mathrm{Neg}(\sigma^{-1})\} \\ & \sqcup \{(i\ \bar{j}) \mid \{i,j\} \in \mathrm{Nsp}(\sigma^{-1})\}. \end{aligned}$$

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$$\begin{aligned} \sigma \leq_R \tau \iff &\mathrm{Inv}(\sigma^{-1}) \subseteq \mathrm{Inv}(\tau^{-1}), \\ &\mathrm{Neg}(\sigma^{-1}) \subseteq \mathrm{Neg}(\tau^{-1}), \\ &\mathrm{Nsp}(\sigma^{-1}) \subseteq \mathrm{Nsp}(\tau^{-1}). \end{aligned}$$

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Consider $\sigma = \bar{4}\bar{1}23 \in S_4^B$ and its inverse $\sigma^{-1} = \bar{2}34\bar{1}$.

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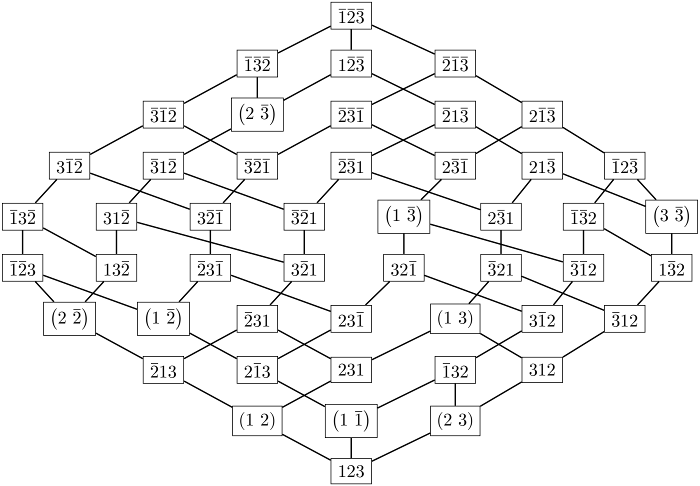
- $T_L(\sigma) = \{(2\ 4), (3\ 4), (1\ \bar{1}), (4\ \bar{4}), (1\ \bar{4})\}$.

Signed permutations



Hasse diagram of $(S_3^B, \leq_R)$

(Markowsky-like) Algorithm: definition by example

Constructing $\sigma \vee_R \tau$ from the complete notations of σ and τ.

(Markowsky-like) Algorithm: definition by example

Input: $\sigma = \bar{5}\bar{3}\bar{1}\bar{2}4\bar{4}1325$ and $\tau = \bar{5}13\bar{4}\bar{2}24\bar{3}\bar{1}5$

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$\sigma^{-1}(1) > 0$ $\tau^{-1}(1) < 0$

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Input: $\sigma = \bar{5}\bar{3}\bar{1}\bar{2}4\bar{4}132 5$ and $\tau = \bar{5}13\bar{4}\bar{2}243\bar{1}5$

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$H_2^{\sigma} = \emptyset$ $H_2^{\tau} = \{\bar{1}\}$

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Input: $\sigma = \bar{5}\bar{3}\bar{1}\bar{2}4\bar{4}132 5$ and $\tau = \bar{5}13\bar{4}\bar{2}243\bar{1}5$

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Input: $\sigma = \bar{5}\bar{3}\bar{1}\bar{2}4\bar{4}1325$ and $\tau = \bar{5}134\bar{2}\bar{2}24\bar{3}\bar{1}5$

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$H_4^\sigma = \{1, 2, 3\}$	$H_4^\tau = \{\bar{3}, \bar{1}\}$	$\pi_4 = 413\bar{2}2\bar{3}\bar{1}\bar{4}$

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Input: $\sigma = \bar{5}\bar{3}\bar{1}\bar{2}4\bar{4}132\textcolor{orange}{5}$ and $\tau = \bar{5}13\bar{4}\bar{2}243\bar{3}\bar{1}\textcolor{orange}{5}$

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Input: $\sigma = \bar{5}\bar{3}\bar{1}\bar{2}4\bar{4}132\textcolor{orange}{5}$ and $\tau = \bar{5}13\bar{4}\bar{2}24\bar{3}\bar{1}\textcolor{orange}{5}$

$\sigma^{-1}(1) > 0$	$\tau^{-1}(1) < 0$	$\pi_1 = 1\bar{1}$
$H_2^\sigma = \emptyset$	$H_2^\tau = \{\bar{1}\}$	$\pi_2 = 1\bar{2}2\bar{1}$
$H_3^\sigma = \{2\}$	$H_3^\tau = \{\bar{2}, \bar{1}, 2\}$	$\pi_3 = 13\bar{2}2\bar{3}\bar{1}$
$H_4^\sigma = \{1, 2, 3\}$	$H_4^\tau = \{\bar{3}, \bar{1}\}$	$\pi_4 = 413\bar{2}2\bar{3}\bar{1}\bar{4}$
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Theorem (Biagioli-P., 2026+)

Let $\sigma, \tau \in S_n^B$*; then* $\pi_n = \sigma \vee_R \tau$.

Characterization through statistics

Let $\sigma, \tau \in S_n^B$;
set $N = \mathrm{Neg}(\sigma^{-1}) \cup \mathrm{Neg}(\tau^{-1})$ and $P = \mathrm{Nsp}(\sigma^{-1}) \cup \mathrm{Nsp}(\tau^{-1})$.

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Conjecture on S_n^B

Theorem (Biagioli-P., 2026+)

Let $\sigma, \tau \in S_n^B$; for any $t \in T_L(\sigma \vee_R \tau)$ there is a palindromic $(T_L(\sigma) \cup T_L(\tau))$-Bruhat path reaching t. In particular,

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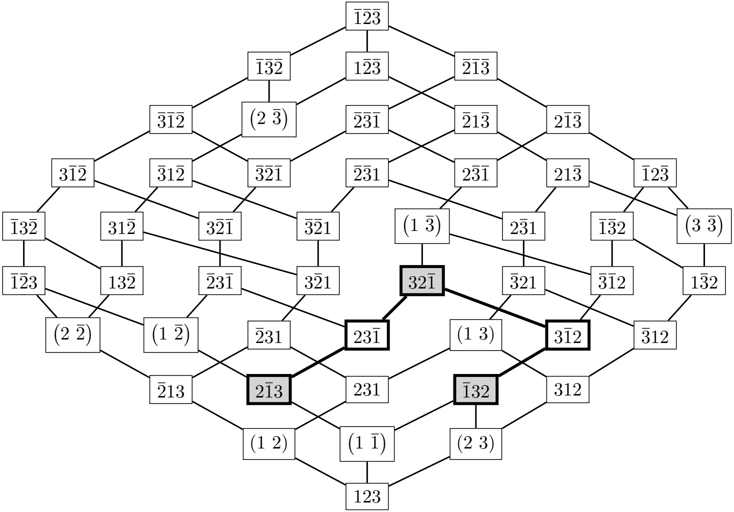
- The proof is divided in 3 cases thanks to the previous characterization;
- The palindromicity comes from the transitive closure of the set of inversions.

Example on the conjecture

Consider $\sigma = 2\bar{1}3$, $\tau = \bar{1}32$ in S_3^B.

Example on the conjecture



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Compute all the $(T_L(\sigma) \cup T_L(\tau))$-Bruhat paths using labels in

$$T_L(\sigma) \cup T_L(\tau) = \{(2\ 3), (1\ \bar{1}), (1\ \bar{2})\}.$$

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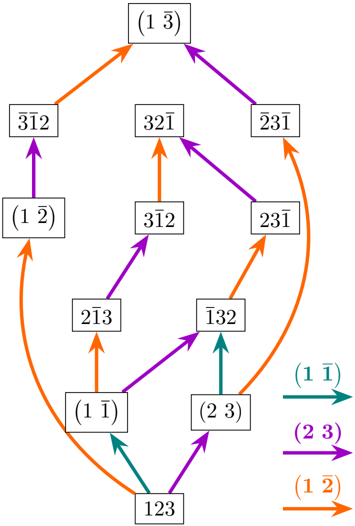

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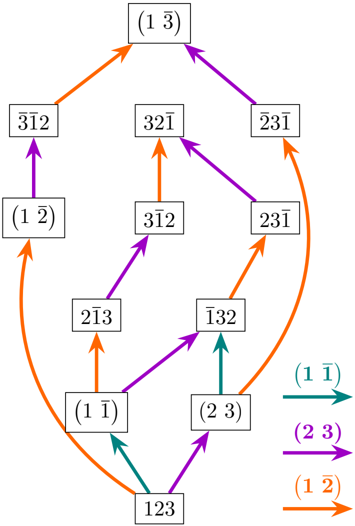
$$\overline{T_L(\sigma) \cup T_L(\tau)} = \{(2\ 3), (1\ \bar{1}), (1\ \bar{2}), (1\ \bar{3})\} = T_L(\sigma \vee_R \tau).$$



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Can we prove it is a closure in type B? (Remember it fails in H_3 and F_4.)

That's all, thank you!

Main references

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