Almost-valuative Invariants of Connected Split Matroids: The cd-index

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Matroid Base Polytopes

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Definition

A *matroid* M of rank k on a ground set E of size n can be defined geometrically via its *base polytope* $\mathcal{P}(M) \subset \mathbb{R}^E$:

- It is a $0/1$-polytope.
- All vertices lie on the *hypersimplex* $\Delta_{k,n}$ (i.e., the sum of coordinates is k).
- Every edge of $\mathcal{P}(M)$ is parallel to $\mathbf{e}_i - \mathbf{e}_j$ for some $i \neq j \in E$.

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The vertices are indicator vectors $\mathbf{e}_B$ corresponding to the *bases* of M.

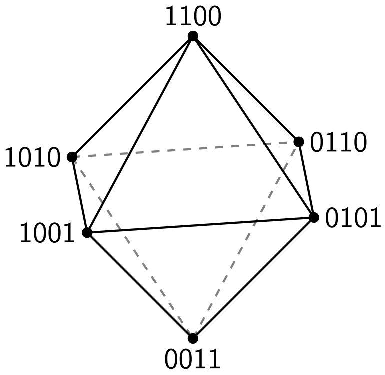
Matroid Base Polytopes and Cuts

- Geometrically, $\mathcal{P}(M)$ is obtained by slicing the hypersimplex $\Delta_{k,n}$ with hyperplanes of the form:

$$H_F = \{x \in \mathbb{R}^E \mid \langle \mathbf{e}_F, x \rangle = \mathrm{rk}(F)\}$$

where $F \subset E$ is a *cyclic flat* of the matroid.

Example: Cuts of Hypersimplex $\Delta_{2,4}$



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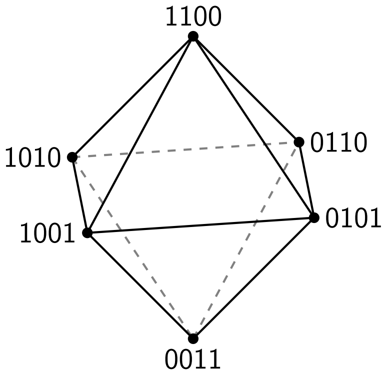

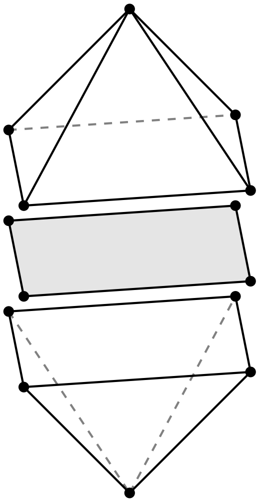

Flag f-vectors and the $\mathtt{ab}$-index

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Definition

For a d-polytope P, the *flag f-vector* counts the number of chains of faces $\mathcal{F} = \{\varnothing = \sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_s \subset \sigma_{s+1} = P\}$ with prescribed dimensions.

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Definition

The *ab-index* of P encodes the flag f-vector into a non-commutative polynomial in variables a, b:

$$\Psi_{ab}(P) = \sum_{\mathcal{F} \text{ face-flag}} w(\mathcal{F}),$$

where $w(\mathcal{F}) = w_0 \ldots w_{d-1}$ with $w_i = b$ if $i \in \{\dim \sigma_1, \ldots, \dim \sigma_s\}$, and $a - b$ otherwise.

The cd-index

Theorem (Bayer-Klapper, 1991)

For any convex polytope, there exists a polynomial $\Psi_{cd}(P)$ in the non commutative variables c and d such that $\Psi_{ab}(P) = \Psi_{cd}(P)|_{c=a+b,d=ab+ba}$.

- The *cd-index* compactly encodes the polytope's flag structure.

Invariants of the Octahedron $\Delta_{2,4}$

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The octahedron is a 3-dimensional polytope with f-vector $f = (6, 12, 8)$.

- **Flag f-vector**: The 8 components f_S for $S \subseteq \{0, 1, 2\}$ are:

$$f_\varnothing = 1,\ f_0 = 6,\ f_1 = 12,\ f_2 = 8,\ f_{0,1} = 24,\ f_{0,2} = 24,\ f_{1,2} = 24,\ f_{0,1,2} = 48$$

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- ab-**index**:

$$\Psi_{ab}(\Delta_{2,4}) = a^3 + 5a^2b + 5aba + 5baa + 7ab^2 + 7bab + 7b^2a + 11b^3$$

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- ab-**index**:

$$\Psi_{ab}(\Delta_{2,4}) = a^3 + 5a^2b + 5aba + 5baa + 7ab^2 + 7bab + 7b^2a + 11b^3$$

- cd-**index**: A much more compact representation!

$$\Psi_{cd}(\Delta_{2,4}) = c^3 + 6cd + 4dc$$

Cuspidal Matroids

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Now we look at matroids obtained by cutting the hypersimplex $\Delta_{k,n}$.

Definition

A *cuspidal matroid* $\Lambda_{k,n}^{r,F}$ is obtained by performing a single cut on $\Delta_{k,n}$ using a hyperplane $H_F = \{x \mid \langle \mathbf{e}_F, x \rangle = r\}$:

$$\mathcal{P}(\Lambda_{k,n}^{r,F}) = \Delta_{k,n} \cap H^{-}(F, r)$$

Connected Split Matroids

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Definition (Connected Split Matroid)

A connected matroid M is *split* if its proper cyclic flats are pairwise incomparable.

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A connected matroid M is *split* if its proper cyclic flats are pairwise incomparable.

- Geometrically, $\mathcal{P}(M)$ is carved out of $\Delta_{k,n}$ by cuts H_F that are mutually non-interfering.
- No point of the hypersimplex $\Delta_{k,n}$ is eliminated by two different cuts!

Examples of Connected Split Matroids

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- **Uniform Matroids** $U_{k,n}$ (for $0 < k < n$).
- **Sparse Paving Matroids**: A matroid of rank k where every k-element subset is either a basis or a circuit-hyperplane, i.e. a cyclic flat of size k of rank $k-1$.

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Conjecture (Crapo and Rota)

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Conjecture (Crapo and Rota)

Almost all matroids are sparse paving.

Conjecture (Ferroni and Schröter)

Almost all non-sparse paving matroids matroids are split matroids.

Valuative Invariants

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Definition

An invariant Ψ is *valuative* if it satisfies an inclusion-exclusion principle for hyperplane cuts. For any hyperplane H splitting a matroid polytope $\mathcal{P}(M)$:

$$\Psi(M) = \Psi(\mathcal{P}(M) \cap H^+) + \Psi(\mathcal{P}(M) \cap H^-) - \Psi(\mathcal{P}(M) \cap H)$$

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- Valuativeness is a very strong geometric property, rarely evident from algebraic definitions.

Valuative Invariants for Split Matroids

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Theorem (Ferroni and Schröter, 2024)

Let Ψ be a valuative invariant. If M is a connected split matroid, then $\Psi(M)$ is completely determined by hypersimplices and cuspidal matroids:

$$\Psi(M) = \Psi(\Delta_{k,n}) + \sum_{F \in \mathcal{Z}(M)} \left(\Psi(\Lambda_M^F) - \Psi(\Delta_{k,n}) \right)$$

- This provides a powerful framework for computing invariants on large classes of matroids!

Almost-Valuative Invariants

However, many important geometric invariants fail to be strictly valuative. We generalize this concept:

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Definition (F., Vargas 2025)

Let $\Psi : \mathrm{MatPoly} \to \mathcal{G}$ be a function, and $(\mathcal{G}, +, \cdot)$ an arbitrary ring. It is *almost-valuative* if for some function $\mathcal{E} : \mathrm{MatPoly} \to \mathcal{G}$ and constant $K \in \mathcal{G}$ we have: for all matroids M and hyperplanes H splitting $\mathcal{P}(M)$ into two matroid polytopes

$$\begin{aligned}\Psi(M) = &\Psi(\mathcal{P}(M) \cap H^+) + \Psi(\mathcal{P}(M) \cap H^-) - \Psi(\mathcal{P}(M) \cap H) \cdot K \\ &- \sum_{\sigma \in \mathcal{T}_H(M)} \mathcal{E}(\sigma \cap H).\end{aligned}$$

Examples of Almost-Valuative Invariants: f-vector

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f-vector

It satisfies the inclusion-exclusion formula up to a well-behaved error term:

$$\begin{aligned}\Psi_f(P) =& \Psi_f(P^+) + \Psi_f(P^-) - \Psi_f(P \cap H) \\ & - \sum_{\sigma \in \mathcal{T}_H(P)} t^{\dim(\sigma \cap H)} \cdot (1+t).\end{aligned}$$

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Ferroni and Schröter (2025) studied the f-vector for this family of matroid.

Examples of Almost-Valuative Invariants: cd-index

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cd-index

By a result of Kim (2010), the cd-index satisfies:

$$\Psi_{cd}(P) = \Psi_{cd}(P^+) + \Psi_{cd}(P^-) - \Psi_{cd}(P \cap H) \cdot c - \sum_{\sigma \in \mathcal{T}_H(P)} \Psi_{cd}(\sigma \cap H) \cdot d \cdot \Psi_{cd}(P \cap H/(\sigma \cap H)).$$

Thus, the cd-index perfectly fits our framework of an **almost-valuative invariant**.

Main Theorem

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Theorem (F., Vargas 2025)

Let M be a connected split matroid with proper cyclic flats $\mathcal{Z}(M)$. Let Ψ be an **almost-valuative** *invariant. Then:*

$$\Psi(M) = \Psi(\Delta_{k,n}) + \sum_{F\in\mathcal{Z}(M)} \left(\Psi(\Lambda_M^F) - \Psi(\Delta_{k,n})\right) - \sum_{(F,G)} W_{F,G}$$

where the last sum ranges over all modular pairs of cyclic flats.

Main Theorem: Remarks

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$$\Psi(M) = \Psi(\Delta_{k,n}) + \sum_{F \in \mathcal{Z}(M)} \Big(\Psi(\Lambda_M^F) - \Psi(\Delta_{k,n})\Big) - \sum_{(F,G)} W_{F,G}$$

- **Valuative part**: Exactly matches the Ferroni-Schröter formula. It requires only evaluations on hypersimplices and cuspidal matroids.

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- **Valuative part**: Exactly matches the Ferroni-Schröter formula. It requires only evaluations on hypersimplices and cuspidal matroids.
- **Error part**: Cuts H_F and H_G intersect inside $\Delta_{k,n}$ *if and only if* F and G form a modular pair.

Structure of the Error Term

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The error term $W_{F,G}$ depends exclusively on the intersection of *modular pairs*:

$$W_{F,G} = \sum_{p=1}^{\alpha} \sum_{q=1}^{\beta} \sum_{i=p+1}^{a-\alpha+p} \sum_{j=q+1}^{b-\beta+q} \binom{a}{i} \binom{b}{j} \binom{a-i}{\alpha-p} \binom{b-j}{\beta-q} \cdot \Psi_{cd}(\Delta_{p,i} \times \Delta_{q,j}) \cdot d \cdot \Psi_{cd}(\Delta_{1,n-i-j}).$$

- Parameters a, b, α, β depend only on sizes and ranks of F, G, and $F \cap G$:
  - $a = |F \setminus G|, \quad b = |G \setminus F|$
  - $\alpha = \mathrm{rk}(F) - |F \cap G|, \quad \beta = \mathrm{rk}(G) - |F \cap G|$

Vanishing Triple Intersections

- A key observation driving the Main Theorem is that the error term does not require triple or higher-order intersections.

Vanishing Triple Intersections

- A key observation driving the Main Theorem is that the error term does not require triple or higher-order intersections.
- **Geometric Insight**: Remarkably, for connected split matroids, no three cuts intersect inside the hypersimplex in a way that contributes to the error.

Application: Sparse Paving Matroids

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Corollary (F., Vargas 2025)

Let M be a connected sparse paving matroid of rank k on n elements. Sparse paving matroids are split matroids where cyclic flats are circuit-hyperplanes. Then:

$$\Psi_{cd}(M) = \lambda \Psi_{cd}(\Lambda_{1,k,n-k,n}) - (\lambda - 1)\Psi_{cd}(\Delta_{k,n}) - \mu(c^2 d + 2d^2)\Psi_{cd}(\Delta_{1,n-4}).$$

- λ: number of circuit-hyperplanes.
- μ: number of modular pairs (intersecting in $k-2$ elements).

Example: The Fano and Vámos Matroids

Let F be the Fano matroid (rank 3, $n = 7$) and V be the Vámos matroid (rank 4, $n = 8$). Both are famous sparse paving matroids.
Using our formula, we easily compute their cd-indices by counting circuit-hyperplanes (λ) and modular pairs (μ):

- **Fano Matroid** ($\lambda = 7, \mu = 21$):

$$\begin{aligned}\Psi_{cd}(F) &= 7\Psi_{cd}(\Lambda_{1,3,4,7}) - 6\Psi_{cd}(\Delta_{3,7}) - 21(c^2d + 2d^2)\Psi_{cd}(\Delta_{1,3}) \\ &= 364cdcd + 98cdc^3 + \cdots + 298d^2c^2 + 482d^3\end{aligned}$$

- **Vámos Matroid** ($\lambda = 5, \mu = 8$):

$$\begin{aligned}\Psi_{cd}(V) &= 5\Psi_{cd}(\Lambda_{1,4,4,8}) - 4\Psi_{cd}(\Delta_{4,8}) - 8(c^2d + 2d^2)\Psi_{cd}(\Delta_{1,4}) \\ &= 3580cdcdc + 1690cdc^2d + \cdots + 1098d^2c^3 + 3772d^3c\end{aligned}$$

Thank You