Automorphisms with some topological triviality, as those acting trivially on ($\mathbb{Q}$ or $\mathbb{Z}$)-cohomology: the benchmark case of surfaces, according to Kodaira dimension.

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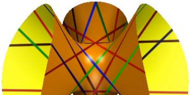
(Last Lecture of the course 'Finite automorphism groups of complex and algebraic manifolds', Dobbiaco-February 24-28, 2025.



Outline

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Automorphisms

If X = compact complex manifold, Bochner and Montgomery proved that
$Aut(X)$ (the group of biholomorphisms of X) is a finite dimensional complex Lie Group,
with Lie Algebra the space $H^0(\Theta_X)$ of holomorphic vector fields.
Let $Aut_0(X)$ be the connected component of the identity: then the quotient group $Aut(X)/Aut_0(X)$ is called the group of components: it is at most countable, and can be infinite.

Example

Let E be an elliptic curve, and let $X = E^n$.
Then $Aut_0(X) = E^n$, while the group $Aut(X)/Aut_0(X)$ contains $GL(n, \mathbb{Z})$, acting in the obvious way:

$$g \in GL(n, \mathbb{Z}), x = (x_1, \dots, x_n) \mapsto gx = (\sum_j g_{1j} x_j, \dots, \sum_j g_{nj} x_j).$$

Automorphisms with some topological triviality.

I will concentrate mostly on the group of cohomologically trivial automorphisms,

$$\mathrm{Aut}_{\mathbb{Z}}(X) := \{\sigma \in \mathrm{Aut}(X) \mid \sigma \text{ induces a trivial action on } H^*(X;\mathbb{Z})\},$$

and the larger group $\mathrm{Aut}_{\mathbb{Q}}(X)$, of numerically trivial automorphisms (important for the theory of period maps).

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For Teichmüller theory is also important the smaller subgroup

$$\mathrm{Aut}_{iso}(X) = \{\sigma \in \mathrm{Aut}(X) \mid \sigma \in \mathrm{Diff}_0(X)\},$$

of differentiably-isotopically trivial automorphisms, contained in the group of homotopically trivial automorphisms

$$\mathrm{Aut}_{hom}(X) = \{\sigma \in \mathrm{Aut}(X) \mid \sigma \text{ is homotopic to } \mathrm{id}_X\}$$

Open Question: is there an example with $\mathrm{Aut}_{iso}(X) \neq \mathrm{Aut}_{hom}(X)$?

A chain of subgroups.

We have a chain (of normal subgroups)

$$\mathrm{Aut}_0(X) \lhd \mathrm{Aut}_{iso}(X) \lhd \mathrm{Aut}_{hom}(X) \lhd \mathrm{Aut}_{\mathbb{Z}}(X) \lhd \mathrm{Aut}_{\mathbb{Q}}(X) \lhd \mathrm{Aut}(X).$$

For complex dimension $n = 1$ everything simplifies: in fact here $\mathrm{Aut}_0(X) = \mathrm{Aut}_{\mathbb{Q}}(X)$.
But already for $n = 2$ the situation is extremely delicate (in fact there are many wrong theorems and assertions in the literature).
An important REMARK is: $\mathrm{Aut}_{\mathbb{Z}}(X) = \mathrm{Aut}_{\mathbb{Q}}(X)$ if the cohomology $H^*(X, \mathbb{Z})$ is torsion-free, since then $H^*(X, \mathbb{Z}) \subset H^*(X, \mathbb{Q})$.

The case of curves.

If X has dimension $n = 1$, and genus g, then, as we saw:

1. $g = 0 \Leftrightarrow X = \mathbb{P}^1 \Rightarrow Aut(X) = Aut_0(X) = \mathbb{P}GL(2, \mathbb{C})$,
2. $g = 1 \Leftrightarrow X = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$,
   $X \cong Aut_0(X)$ is the subgroup of translations,
   $\Gamma(X) := Aut(X)/Aut_0(X)$ equals $\mathbb{Z}/2$ unless $\tau = i$
   $(\Gamma(X) = \mathbb{Z}/4)$, or $\tau^3 = 1$ $(\Gamma(X) = \mathbb{Z}/6)$.
3. $g \geq 2 \Rightarrow \ |Aut(X)| \leq 84(g-1)$, by Hurwitz' theorem.
4. $\mathrm{Aut}_{\mathbb{Q}}(X) = \mathrm{Aut}_0(X)$, as shown by Lefschetz (hence, for $g \geq 2$, $\mathrm{Aut}_{\mathbb{Q}}(X) = \{Id_X\}$).

The case of Kähler manifolds.

The case where X is a compact Kähler Manifold (this case includes the case of projective manifolds) was considered around 1978 by Lieberman and Fujiki; in particular, follows from their results:

Theorem

The quotient group

$$\mathrm{Aut}_{\mathbb{Q}}(X)/\mathrm{Aut}_0(X)$$

is a finite group.

Hence a natural question is to see when the group $\mathrm{Aut}_{\mathbb{Q}}(X)/\mathrm{Aut}_0(X)$ can be nonzero, and to give an upper bound for its cardinality in terms of the numerical invariants of X; ditto for $\mathrm{Aut}_{\mathbb{Z}}(X)/\mathrm{Aut}_0(X)$.

$\mathrm{Aut}_{\mathbb{Q}}(X)/\mathrm{Aut}_0(X) =?$

Around 1975 Piatetski-Shapiro and later Burns and Rapoport proved that, for a K3 surface X, $\mathrm{Aut}_{\mathbb{Q}}(X)$ is a trivial group. Recall that a K3 surface is a surface with trivial canonical divisor, and simply connected (this distinguishes the case of K3 surfaces from the case of complex tori, which have $\pi_1(X) = \mathbb{Z}^4$).

Peters began the study of $\mathrm{Aut}_{\mathbb{Q}}(X)$ for more general compact Kähler surfaces. Automorphisms of surfaces were also investigated by Ueno and Maruyama in the 70's, Mukai and Namikawa in the 80's, and other authors later.

Outline

Kodaira dimension

Definition

The Kodaira dimension of a compact complex manifold X*, denoted* $Kod(X)$*, is the maximal dimension of the image* $\Phi_m(X)$*of the m-th pluricanonical map* Φ_m*,* $(m \geq 1)$*.*
Φ_m *is associated to a basis* $s_1, \dots s_{P_m}$ *of the space of holomorphic sections in* $H^0(\mathcal{O}_X(mK_X))$ *as follows:*
$\Phi_m : X \to \mathbb{P}^{P_m - 1}$ *is the rational map given by*

$$\Phi_m(x) := (s_1(x), \dots s_{P_m}(x)).$$

If $P_m = 0 \forall m \geq 1$*, then we say that* $Kod(X) := -\infty$

Obviously $Kod(X) \leq dim(X)$, and, if equality holds, one says that X is **of general type**.

Classification of algebraic surfaces according to Kodaira dimension

Two algebraic manifolds X_1, X_2 are said to be **birational** if their fields of meromorphic functions are isomorphic, $\mathbb{C}(X_1) \cong \mathbb{C}(X_2)$. For surfaces S_1, S_2 this is equivalent to be obtained one from the other via a sequence of one point blow ups and their inverses.
S is said to be **minimal** if it is not the blow up of another surface.

1. $Kod(S) = -\infty \Leftrightarrow S$ is **ruled**, that is, birational to a product $C \times \mathbb{P}^1$. S is **rational** if it is birational to $\mathbb{P}^1 \times \mathbb{P}^1 \sim_{bir} \mathbb{P}^2$.
2. $Kod(S) = 0$ and S minimal $\Leftrightarrow 12K_S \equiv 0$ (tori, K3, Enriques, hyperelliptic surfaces).
3. $Kod(S) = 1 \Leftrightarrow \Phi_{12}$ is a fibration with fibres elliptic curves.
4. $Kod(S) = 2$: by definition, S is of general type.

Surfaces of general type

Theorem

(Cai) For surfaces S of general type, $Aut(S)$ is a finite group, and there is an absolute constant C such that

$$|\mathrm{Aut}_{\mathbb{Q}}(S)| \leq C.$$

For the Beauville surface $|\mathrm{Aut}_{\mathbb{Q}}(S)| = 25$.
Recall that this is a surface isogenous to a product, $S = (C \times C)/G$, $G \cong (\mathbb{Z}/5)^2$, where C is the Fermat quintic curve, and the action is free. $\mathrm{Aut}_{\mathbb{Q}}(S)$ is induced by $G \times Id$, and Frapporti calculated that in this case $|\mathrm{Aut}_{\mathbb{Z}}(S)| = 1$.
Is 25 the maximum for surfaces of general type?
What about $max|\mathrm{Aut}_{\mathbb{Z}}(S)|$ (again for S of general type) ?
There seem to be only examples with $|\mathrm{Aut}_{\mathbb{Z}}(S)| = 2$. (?)

First unboundedness Result for Surfaces not of general type

In my pre-covid joint work with Wenfei Liu the following theorem (contradicting earlier assertions of other authors) answered two questions raised by Meersseman in 2017:

Theorem

For each positive integer m there exists a rational surface X_m such that $Aut_{\mathbb{Q}}(X_m) \cong \mathbb{Z}/m$.

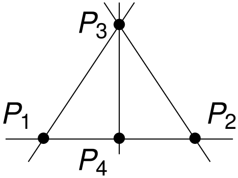
The surface X_m is an iterated blow-up of the projective plane. A key point is that, if $X' \to X$ is the blow up of a point P, then $\mathrm{Aut}_{\mathbb{Q}}(X') \subset \mathrm{Aut}_{\mathbb{Q}}(X)$ is the subgroup fixing the point P. In fact..

Some elementary observations.

(1) Let $g \in \mathrm{Aut}_{\mathbb{Q}}(X)$, where X is a surface, and let C be an irreducible curve with $C^2 < 0$.
Then $g(C) = C$.
Proof: If $g(C) \neq C$, then $C \cdot g(C) \geq 0$ (and > 0 if $C \cap g(C) \neq \emptyset$).
But since $g(C)$ has the same class of C, we have
$C \cdot g(C) = C^2 < 0$, a contradiction.
(2) Let $f : X \to B$ be a fibration of the surface X onto a curve, and $\sigma \in \mathrm{Aut}_{\mathbb{Q}}(X)$: then σ preserves the fibration, that is, there is an action of σ on B such that $\sigma \circ f = f \circ \sigma$.
Moreover, if F''_{red} is a reducible fibre, then $f(F'') = F''$.
Proof: Let C be an irreducible fibre of f: since
$C \cdot g(C) = C^2 = g(C)^2 = 0$, follows that $g(C)$, which is irreducible, is another fibre of f.
Zariski proved that the components C of F'' satisfy $C^2 < 0$. $\square$

$Aut_{\mathbb{Q}}(X_m) = Aut_*(X_m) \cong \mathbb{Z}/m$

Let $P_1 = (1:0:0)$, $P_2 = (0:1:0)$, $P_3 = (0:0:1)$ be the coordinate points of $\mathbb{P}^2$, and let $P_4 = (1:1:0)$.



Let $\pi\colon X_4 \to \mathbb{P}^2$ be the blow-up of the four points $P_i, 1 \leq i \leq 4$. Let $G_4 = \{\sigma \in \mathbb{P}GL(3) \mid \sigma(P_i) = P_i \text{ for } 1 \leq i \leq 4\}$. Then

$$\mathrm{Aut}_{\mathbb{Q}}(X_4) \cong G_4 = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{bmatrix} \middle| \; a \in \mathbb{C}^* = \mathbb{C} \setminus \{0\} \right\}$$

$Aut_{\mathbb{Q}}(X_m) = Aut_*(X_m) \cong \mathbb{Z}/m$

Blowing up further $m+1$ points, the first m ones infinitely near to P_4 on the proper transform of the line joining P_3, P_4, and the last one a different point on the last exceptional line, one reaches the conclusion that $Aut_{\mathbb{Q}}(X_m) = \{a \in \mathbb{C}^* | a^m = 1\}$. That $Aut_{\mathbb{Q}}(X_m) = Aut_*(X_m)$ follows argueing like this: differentiably X_m is the same as the blow up X'_m where the last point continues to lie on the proper transform of the line joining P_3, P_4. In the latter case $Aut_{\mathbb{Q}}(X'_m) = \mathbb{C}^*$, and $Aut(X_m) \cong \mathbb{Z}/m$ corresponds under the diffeomorphism to a subgroup of $Aut(X'_m) = \mathbb{C}^*$: hence these diffeomorphisms are isotopic to the identity.

Rational and ruled surfaces

Theorem

(C.-Liu) Let X be a smooth projective rational surface. Then

$$\mathrm{Aut}_{iso}(X) = \mathrm{Aut}_{\mathbb{Z}}(X) = \mathrm{Aut}_{\mathbb{Q}}(X).$$

The same unboundedness phenomenon for $\mathrm{Aut}_{iso}(X)/\mathrm{Aut}_0(X)$ can happen also for minimal ruled surfaces:

Theorem

For E an elliptic curve, let $X := \mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E(D))$, where D is a divisor of positive degree d: then

$$\mathrm{Aut}_{\mathbb{Q}}(X) = \mathrm{Aut}_{iso}(X), \ |\mathrm{Aut}_{iso}(X)/\mathrm{Aut}_0(X)| \geq d^2.$$

More unboundedness Results for Surfaces of special type (that is, not of general type)

Wenfei Liu and I showed unboundedness also for other quotients of the standard chain of subgroups:

Theorem

i) For each positive integer n there exists a minimal surface S_n of Kodaira dimension 1 *such that* $[\mathrm{Aut}_{\mathbb{Q}}(S_n) : \mathrm{Aut}_{\mathbb{Z}}(S_n)] \geq n$.
ii) For each positive integer n there exists a (non minimal) surface S'_n of Kodaira dimension 1 *such that* $\mathrm{Aut}_{iso}(S'_n) = \{\mathrm{id}_{S'_n}\}$, *and*

$$\mathrm{Aut}_{\mathbb{Z}}((S'_n)) = \mathbb{Z}/n.$$

For (ii) is the same blow-up game, starting with minimal surfaces S with $Aut_0(S)$ infinite.
For (i) these are surfaces S with $\chi(S) = 0$, $\chi := 1 - q + p_g$.

Boundedness fo Kodaira dimension 0

The case $Kod(X) = 0$ is pretty well understood (and we have boundedness, that is, an absolute upper bound for all such surfaces).

- For complex tori and their blow ups X, $Aut_0(X) = \mathrm{Aut}_{\mathbb{Q}}(X)$.
- For K3 surfaces $\mathrm{Aut}_{\mathbb{Q}}(X) = 0$.
- For minimal Enriques surfaces (quotients of a K3 by a fixpoint free involution) Mukai and Namikawa proved that $|\mathrm{Aut}_{\mathbb{Q}}(X)| \leq 4$, and exist examples with $\mathrm{Aut}_{\mathbb{Z}}(X) = \mathbb{Z}/2$.
- For hyperelliptic surfaces X, Wenfei Liu and I showed that $\mathrm{Aut}_{\mathbb{Z}}(X) = Aut_0(X)$ is a quotient of $Alb(X)$, while $\mathrm{Aut}_{\mathbb{Q}}(X)/\mathrm{Aut}_{\mathbb{Z}}(X)$ can be described in each case. It is a group of order ≤ 12, and $\mathfrak{A}_4$ occurs.

Hyperelliptic surfaces and surfaces isogenous to a product

Definition

A surface S is said to be isogenous to a product of unmixed type if

1. *there is a finite group G,*
2. *there are curves C_1, C_2 of genera $g_1, g_2 \geq 1$ on which G acts faithfully, such that*
3. *the diagonal action $g(x,y) = (gx, gy)$ is free on $C_1 \times C_2$,*
4. *$S = (C_1 \times C_2)/G$.*
5. *S is said to be* **hyperelliptic** *(bielliptic) if $g_1 = g_2 = 1$,*
6. **isogenous to a higher product** *if $g_1, g_2 \geq 2$,*
7. **isogenous to a higher elliptic product** *if $g_1 \geq 2, g_2 = 1$.*

Outline

The situation for Kodaira dimension 1

Properly elliptic surfaces are minimal surfaces with Kodaira dimension $\mathcal{K}(X) = 1$. Recall that $\chi(S) := 1 - q(S) + p_g(S)$, and that for these surfaces $12\chi = (Noether) = e(S) + K_S^2 = e(S)$. There is a canonical fibration $f : X \to B$ over a curve B and with general fibre a smooth elliptic curve; $Aut(X)$ acts equivariantly on X, B.

We have several cases (and $Aut_0(X)$ is nontrivial only for a subcase of (2), the peudo-elliptic surfaces):

1. $\chi(X) > 0$ and $p_g(X) > 0$,
2. $\chi(X) = 0$
3. $\chi(X) = 1, p_g(X) = q(X) = 0$.

Peters claimed in 1980 that $\mathrm{Aut}_{\mathbb{Q}}(X) = 0$ in case (1), Cai gave a new proof in 2009, but with Liu and Schütt we gave counterexamples, showing unboundedness, as we shall see.

Properly elliptic surfaces with $\chi(X) = 0$

In this initial, yet quite complicated case, we got very near to a complete classification.
Here, by surface classification, the pluricanonical fibration is an elliptic quasi-bundle (all reduced fibres are smooth elliptic) and the surfaces are isogenous to a higher elliptic product.
This means that S is the quotient of the product of a curve C of genus $g' \geq 2$ with an elliptic curve E by the free action of a finite group G

$$S = (C \times E)/G.$$

G acts on C and on E, and on the product via
$g(x, y) = (g(x), g(y))$.
S is said to be pseudo-elliptic if G acts on E via translations, i.e., if E/G has genus 1.

Pseudo-elliptic surfaces

$S = (C \times E)/G$ is said to be pseudo-elliptic if G acts on E via translations, i.e., if E/G has genus 1.
In this case E acts on S via translations

$$t \in E \Rightarrow t(x, y) := (x, y + t),$$

hence $Aut^0(S)$ has dimension 1.

Theorem

If S is pseudo elliptic, $Aut^0(S)$ is infinite, and either
(1) $Aut_{\mathbb{Z}}(S) = Aut^0(S)$ or
(2) $|Aut_{\mathbb{Z}}(S)/Aut^0(S)| = 2$.
Case (2) occurs precisely when $G = \mathbb{Z}/2m$, with m an odd integer, $C/G = \mathbb{P}^1$ and $C \to \mathbb{P}^1$ is branched in four points with local monodromies $\{m, m, 2, -2\}$:

Other elliptic surfaces with $\chi = 0$

Theorem

Assume that S is a properly elliptic surface with $\chi(S) = 0$ *and with* $Aut^0(S)$ *trivial (i.e., S is not pseudo-elliptic).*
If $Aut_{\mathbb{Z}}(S)$ *is nontrivial, then*

(II-a) $B := C/G$ *has genus h at least* 1

(II-b) $\mathrm{Aut}_{\mathbb{Z}}(S)$ *is isomorphic to one of the following groups:*

$$\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}, (\mathbb{Z}/2\mathbb{Z})^2.$$

(II-c) *The cases where* $\mathrm{Aut}_{\mathbb{Z}}(S) = \mathbb{Z}/2$, *respectively* $\mathrm{Aut}_{\mathbb{Z}}(S) = \mathbb{Z}/3$, *do actually occur.*

Key observations

By Poincaré duality $H^i(S,\mathbb{Z}) \cong H_{4-i}(S,\mathbb{Z})$.
$H^1(S,\mathbb{Z})$ is free abelian, while we have a non canonical splitting $H_1(S,\mathbb{Z}) = Tors(S) \oplus \mathcal{A}$, where $\mathcal{A}$ is free abelian. Moreover, the torsion subgroup of $H^2(S,\mathbb{Z}) = H_2(S,\mathbb{Z})$ is canonically equal to $Tors(S)$.

Corollary

If $Tors(S) = 0$, then $\mathrm{Aut}_{\mathbb{Z}}(S) = \mathrm{Aut}_{\mathbb{Q}}(S)$.

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Corollary

If $Tors(S) = 0$, then $\mathrm{Aut}_{\mathbb{Z}}(S) = \mathrm{Aut}_{\mathbb{Q}}(S)$.

General strategy: describe $\pi_1(S)$ and its abelianization $H_1(S, \mathbb{Z})$ and determine the group $\mathcal{H}$ of automorphisms in $\mathrm{Aut}_{\mathbb{Q}}(S)$ acting trivially on it; then, in the case $Tors(S) \neq 0$, see whether $\mathcal{H}$ acts non trivially also on $H^2(S, \mathbb{Z})$.

Tips for $\chi(S) = 0$

If $S = (C \times E)/G$ is isogenous to a higher elliptic product, then G acts on E, hence there is a 2-generated abelian group T such that

$$G = T \rtimes \mu_r, \ r \in \{2, 3, 4, 6\}.$$

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Finally, $\mathrm{Aut}(S) = \mathcal{N}_G/G$, where $\mathcal{N}_G$ is the Normalizer of G inside $Aut(C \times E)$; and this formula implies that $Aut_{\mathbb{Z}}(S)$ is contained in the centre of G.
The fundamental group $\pi_1(S)$ can be determined using the method of Reidemeister-Schreier, once G and the Monodromy are given. This gives rise to an infinite number of cases, and makes a complete classification cumbersome.

Examples for $\chi(S) = 0$ of $Aut_{\mathbb{Z}}(S) = \mathbb{Z}/3, \mathbb{Z}/2$.

Take $G = \mathbb{Z}/3 \times \mu_3$ or $G = \mathbb{Z}/2 \times \mu_4$, take B a genus two curve and $C \to B$ unramified with values of the Monodromy homomorphism $\pi_1(B) \hookrightarrow G$ on the four generators: $(0, \epsilon), (1, Id), (0, Id)(0, Id)$,(here ϵ is a generator of μ_r).
Then $S \to B$ is a fibre bundle with fibre an elliptic curve E, hence we have an exact sequence
$1 \to \pi_1(E) \to \pi_1(S) \to \pi_1(B) \to 1$.
Taking the Abelianizations, we get the exact sequence

$$H_1(E, \mathbb{Z}) \to H_1(S, \mathbb{Z}) \to H_1(B, \mathbb{Z}) \to 0.$$

We show then that $H_1(E, \mathbb{Z})$ maps to 0, hence $Tors(S) = 0$.
We get $Aut_{\mathbb{Z}}(S) = \mathbb{Z}/3, \mathbb{Z}/2$, taking automorphisms which are the identity on C, and translations by T on E.

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We get $Aut_{\mathbb{Z}}(S) = \mathbb{Z}/3, \mathbb{Z}/2$, taking automorphisms which are the identity on C, and translations by T on E.
We have also examples with $Tors(S) \neq 0$. To prove the upper bound we must use computer algebra for implementing the algorithm of Reidemeister-Schreier.

Properly elliptic isotrivial surfaces with $\chi(S) > 0$

Here $f : S \to B$ isotrivial means that S is birational to $(C \times E)/G$, with non free action by $G = T \rtimes \mu_r$, $r \in \{2, 3, 4, 6\}$. Since $Bir(S) = Aut(S)$, with similar methods to the case $\chi(S) = 0$, with Liu and Schütt we proved:

Theorem

If S properly elliptic, isotrivial with $\chi(S) > 0$, then
$|Aut_{\mathbb{Z}}(S)| \leq 3$,
and exist infinitely many cases with order $2, 3$. Moreover
$|Aut_{\mathbb{Q}}(S)| \leq r, \text{ for } r \leq 4, \leq 3 \text{ for } r = 6$,
except for $q(S) = p_g(S) = 0$, where, s being the number of multiple fibres
$|Aut_{\mathbb{Q}}(S)| \leq 4s \leq 4(P_2(S) + 1)$,
and for each value of $s \in \mathbb{N}$, exist cases with
$|Aut_{\mathbb{Q}}(S)| = 4s = 4(P_2(S) + 1)$.

Elliptic algebraic surfaces with $\chi(S) > 0$: the Mordell-Weil method.

Since $f : S \to B$ yields a curve $\mathcal{C}$ of genus 1 over the field $\mathcal{K} := \mathbb{C}(B)$, and since $Bir(S) = Aut(S)$, we have to study $Aut(\mathcal{C})$, except that we allow automorphisms of the base field. In other words, we have an exact sequence

$$1 \to Aut_{\mathcal{K}}(\mathcal{C}) \to Aut(\mathcal{C}) \to Aut(\mathcal{K}).$$

Now, after a (finite) Galois field extension $\mathcal{K}' \supset \mathcal{K}$, the curve $\mathcal{E} := \mathcal{C} \otimes \mathcal{K}'$ acquires a $\mathcal{K}'$-rational point, hence it is an elliptic curve, hence $\mathcal{E} \cong Pic^d(\mathcal{E})$ for each integer d.
In particular

$$Aut_{\mathcal{K}}(\mathcal{C}) \subset Aut_{\mathcal{K}'}(\mathcal{E}) \cong \mathcal{E} \rtimes \mu_r, r \in \{2, 4, 6\}.$$

Instead, $\mathcal{C}$ is not isomorphic to $Pic^0(\mathcal{C})$ if $\mathcal{C}$ has no rational point (corresponding to a section of the fibration f).

The Mordell-Weil method.

At any rate, $\mathcal{C}$ is a principal homogeneous space over $Jac(\mathcal{C}) := Pic^0(\mathcal{C})$, and on $\mathcal{C}$ operates the Mordell-Weil group $MW(\mathcal{C})$, which is defined as the group of $\mathcal{K}$-rational points of $Jac(\mathcal{C}) = Pic^0(\mathcal{C})$.
Geometrically, to $Jac(\mathcal{C})$ corresponds the Jacobian surface $Jac(f)$, $MW(f)$ is the group of sections of the Jacobian fibration, which acts on S inducing the identity on the base curve B.
An important feature is that S and $J := Jac(f)$ have the same singular fibres, except for the multiple fibres: the multiple fibres of F which are of the form mF' with F' singular, lose their multiplicity, and become just F'.
To bound $|Aut(S)|$ we use the analogous exact sequence

$$1 \to Aut_B(S) \to Aut(S) \to Aut(S)_{|B} \to 1, Aut(S)_{|B} < Aut(B).$$

Elliptic surfaces with $\chi(S) > 0$: the Mordell-Weil method.

A construction introduced by Kondo for Enriques surfaces can be used to construct numerically trivial automorphisms:

- start with a finite subgroup $H < MW(f)$, and take a suitable H-Galois cover $B' \to B$.
- define $S = (B' \times_B S)/H$, where H acts diagonally, and freely for a suitable choice of H and the covering B'.
- take the action of H on S induced by the product of the identity times the action of H on S.

There remains to see whether H has a numerically trivial action: the main point is to show that it does not permute the irreducible components of the singular fibres, and that these and the multisection associated to H generate $H^2(S, \mathbb{Q})$.

Non isotrivial elliptic surfaces with $\chi(S) > 0$: the main results.

Theorem

1) If $p_g(S) = 0$ and S is not isotrivial (not all smooth fibres are isomorphic) $|Aut_{\mathbb{Q}}(S)| \leq 9$ and equality can hold.
2) If $p_g(S) \neq 0$, $Aut_{\mathbb{Q}}(S)$ is isomorphic to a subgroup of $MW(f)_{tors}$, and conversely for any finite 2-generated abelian group H there is such a surface S with $H < Aut_{\mathbb{Q}}(S)$.
3) If $p_g(S) \neq 0$ then $|Aut_{\mathbb{Q}}(S)| \leq 12\pi^2(q(S)+2)$.
4) $Aut_{\mathbb{Q}}(S)$ is trivial if there is an additive fibre, or all multiple fibres have smooth support, or the fibration is isotrivial.

Non isotrivial elliptic surfaces with $\chi(S) > 0$: the main results.

Theorem

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1) For $p_g(S) = 0$ then $Jac(f)$ is a rational elliptic surface, and $MW(f)$ has been described by Miranda, Oguiso-Shioda ..
2-3) For $p_g(S) > 0$ in the non isotrivial case we view $f : S \to B$ as pull-back of an elliptic modular surface to get upper bounds.

Non isotrivial elliptic surfaces with $\chi(S) > 0$ and $p_g(S) > 0$: triviality.

We have shown in particular:

4) $Aut_{\mathbb{Q}}(S)$ is trivial if there is an additive fibre, or all multiple fibres have smooth support, or the fibration is isotrivial.

This result rescues somehow the claims by Peters and Cai (that the group is always trivial) making clear the problems especially with the more refined arguments by Cai, and confirming to us that surfaces were created by the devil..

Examples of non isotrivial elliptic surfaces with $Aut_{ZZ}(S) = /ZZ/2, \mathbb{Z}/3$.

Theorem

(I) For any $s \in \mathbb{N}$ there is a $2s$-dimensional family of non-isotrivial elliptic surfaces $Y \to \mathbb{P}^1$ with $p_g(Y) = q(Y) = 0$ and $h^0(2K_Y) = 2s - 1$ admitting a cohomologically trivial involution.
(II) There is a 1-dimensional family of non-isotrivial properly elliptic surfaces $Y \to \mathbb{P}^1$ with $q = p_g = 0$ admitting a cohomologically trivial automorphism of order 3.

In (I), for $s = 1$, the previous theorem recovers the family of Enriques surfaces first studied by Barth–Peters.
Difficulty of the proof : to find an explicit basis of $H^2(S, \mathbb{Z})$.

$Aut_{\mathbb{Z}}(S)$ versus $Aut_{\mathbb{Q}}(S)$.

One main difference is first of all that, while numerical automorphisms preserve the reducible fibres, cohomologically trivial automorphisms preserve also the multiple fibres in the case where $q(S) = genus(B)$.

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One main difference is first of all that, while numerical automorphisms preserve the reducible fibres, cohomologically trivial automorphisms preserve also the multiple fibres in the case where $q(S) = genus(B)$.
There are no known examples, in the case where K_S is nef, where the group $Aut_{\mathbb{Z}}(S)$ has at least 4 elements.
A trivial but important remark is that, even if one establishes that an automorphism $\sigma \in Aut_{\mathbb{Q}}(S)$ acts trivially on $H_1(S, \mathbb{Z})$ (hence trivially also on $Tors(S) \subset H^2(S, \mathbb{Z})$), the action of σ is trivial on $H^2(S, \mathbb{Z})/Tors(S)$ but not necessarily on $H^2(S, \mathbb{Z})$.
For $p_g(S) = 0$, it is difficult but sometimes possible to find an explicit basis of $H^2(S, \mathbb{Z})$ to see how σ acts; for $p_g(S) > 0$ we have no hint on how to complete the sublattice $Num(S) \oplus Transc(S)$ to the full lattice $H^2(S, \mathbb{Z})$.

The end

THANKS FOR YOUR ATTENTION!