An introduction to Nichols algebras

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Abstract

We will give an overview of Nichols algebras and their connections with geometry, Lie theory, combinatorics, moduli spaces and hyperplane arrangements.

1 Braided Vector Spaces

For simplicity we will work over the field of complex numbers $\mathbb{C}$ throughout the text, even if most results hold in a wider generality.

Definition 1.1. *A braided vector space is the datum of a vector space V together with an element $c \in \mathrm{GL}(V \otimes V)$, called braiding, satisfying the so-called braid equation:*

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c). \tag{1.1}$$

Example 1.2.

1. *(V, τ), where $\tau\colon V \otimes V \to V \otimes V$ is the standard flip $v \otimes w \mapsto w \otimes v$ for $v, w \in V$*
2. *$V = V_0 \oplus V_1$ is a $\mathbb{Z}_2$-graded vector space and $c(v \otimes w) = (-1)^{lj} w \otimes v$ for $v \in V_l$ and $w \in V_j$.*
3. *Let $\{v_1, \ldots, v_n\}$ be a basis for V and let $M \in \mathrm{M}_n(\mathbb{C})$ be a matrix with non-zero entries. Then the automorphism of $V \otimes V$ given by $c(v_k \otimes v_l) = m_{kl} v_l \otimes v_k$ satisfies (1.1). A braiding of this form is called* diagonal.

1.1 A class of examples: Yetter-Drinfeld modules

Definition 1.3. *Let G be a group. A Yetter-Drinfeld module for G is the datum of a G-graded G-module $V = \bigoplus_{g \in G} V_g$ satisfying the compatibility condition $gV_h = V_{ghg^{-1}}$. The support of V is the G-stable subset*

$$\mathrm{supp}(V) = \{h \in G \mid V_h \neq 0\}.$$

Yetter-Drinfeld modules form a category ${}^G_G\mathcal{YD}$, where morphisms are the G-module morphisms preserving the grading, and Yetter-Drinfeld submodules and simple Yetter-Drinfeld submodules are defined in the usual way. If G is finite, it follows from [44, Section 1] that ${}^G_G\mathcal{YD}$ is semisimple, that is, every Yetter-Drinfeld module is the direct sum of simple Yetter-Drinfeld modules.

Example 1.4. *Examples of Yetter-Drinfeld modules:*

1. *The trivial G-module with grading concentrated in 1_G.*

2. *Let $G = \mathbb{Z}_2 = \{\overline{0}, \overline{1}\}$ and let V be a finite-dimensional representation of G, with decomposition into isotypical compenents $V = V_{triv} \oplus V_{sign}$. Then taking $V_{triv} = V_{\overline{1}}$ and $V_{sign} = V_{\overline{0}}$ or $V_{triv} = V_{\overline{0}}$ and $V_{sign} = V_{\overline{1}}$ gives a Yetter-Drinfeld module structure on V.*

3. *If G is abelian and V is a G-module, then any decomposition of V as a direct sum of G-stable subspaces labeled by $g \in G$ equips V with a Yetter-Drinfeld module structure.*

4. *Let $g \in G$ and let W be a representation of $C_G(g)$. Then $V = \operatorname{Ind}_{C_G(g)}^G(W)$ is a Yetter-Drinfeld module, with grading determined by setting $V_g = W$.*

Proposition 1.5. *Let G be a group and let $V = \bigoplus_{g \in G} V_g$ be a G-graded* faithful *G-module. Then the linear map $c \in \operatorname{End}(V^{\otimes 2})$ given by*

$$c(v \otimes w) = gw \otimes v, \quad v \in V_g, \quad w \in V \tag{1.2}$$

is a solution of the braid equation if and only if V is a Yetter-Drinfeld module for G.

Proof. Exercise. The "if" direction does not require faithfulness. □

If V is a Yetter-Drinfeld module, c will denote the braiding as in Proposition 1.5.

Example 1.6. *If V is the trivial representation with trivial braiding then c is the usual flip τ. If $G = \mathbb{Z}_2 = \{\overline{0}, \overline{1}\}$, and V is a direct sum of sign representations, concentrated in degree* 1*, then $c = -\tau$.*

Let G be finite and let $\mathcal{O}_g$ be the conjugacy class of $g \in G$ and let $V = \bigoplus_{h \in G} V_h$ be a Yetter-Drinfeld module for G. Then, $V_{\mathcal{O}_g} := \bigoplus_{h \in \mathcal{O}_g} V_h$ is G-stable and it is again a Yetter-Drinfeld module, and V decomposes as $V = \bigoplus_{\mathcal{O}_g \text{ class in } G} V_{\mathcal{O}_g}$. This leads to a decomposition of the category ${}^G_G\mathcal{YD}$ into blocks, parametrized by the conjugacy classes of G. The objects in the block corresponding to $\mathcal{O}_g$ are the Yetter-Drinfeld modules whose support is $\mathcal{O}_g$. Hence, each simple object in ${}^G_G\mathcal{YD}$ lies in one of these blocks (and it is finite-dimensional).

Proposition 1.7. *Let G be a finite group. The irreducible Yetter-Drinfeld modules are parametrized by the G-conjugacy classes of pairs $(\mathcal{O}_g, W)$ where $\mathcal{O}_g$ is the conjugacy class of g in G and W is an irreducible representation of $C_G(g)$. The correspondence is obtained by taking $V = \operatorname{Ind}_{C_G(g)}^G W$ and $V_g = W$.*

Proof. (Sketch) Assume that V is simple. Then, $V = V_{\mathcal{O}_g}$ for some $g \in G$ and V_g is $C_G(g)$-stable. In addition, V is a quotient of $\mathrm{Ind}_{C_G(g)}^G V_g = \mathbb{C}G \otimes_{\mathbb{C}C_G(g)} V_g$. By dimensional reasons, $V = \mathrm{Ind}_{C_G(g)}^G V_g$, and V_g is necessarily irreducible. Conversely, any Yetter-Drinfeld module constructed in this way is simple. Equivalence up to conjugacy is obtained by standard arguments. □

Exercise 1.8. *Complete the details in the proof of Proposition 1.7.*

Remark 1.9. Yetter-Drinfeld modules can be interpreted as G-equivariant sheaves on the (possibly finite) variety G: the homogeneous component V_g is the stalk of the sheaf at g. The simple equivariant sheaves for G finite were identified in [44], and re-discovered in many different setups: irreducible representations of Drinfeld quantum double, conformal field theory, Hopf modules.

1.2 The braid groups

Let $n \geq 2$. The Braid group $\mathbb{B}_n$ on n strands is the group generated by $\sigma_1, \dots, , \sigma_{n-1}$ with relations $\sigma_j\sigma_{j+1}\sigma_j = \sigma_{j+1}\sigma_j\sigma_{j+1}$ for $j = 1, \dots, n-2$ and $\sigma_j\sigma_l = \sigma_l\sigma_j$ for $|l-j| > 1$ and $1 \leq l < j \leq n-1$.

The group $\mathbb{B}_2 \simeq \mathbb{Z}$ is abelian, but $\mathbb{B}_n$ not abelian for $n > 2$ since there is a surjective group homomorphism $\pi\colon \mathbb{B}_n \to \mathbb{S}_n$, determined by $\pi(\sigma_j) = (j\ j+1)$.

The morphism π has a set-theoretic section $M\colon \mathbb{S}_n \to \mathbb{B}_n$ called the Matsumoto section, that is uniquely determined following the recipe: take $\sigma \in \mathbb{S}_n$, decompose it as a product of transpositions of the form $(j\ j+1)$ using a minimal number of terms, so $\sigma = (j_1\ j_1 + 1)(j_2\ j_2+1)\cdots(j_l\ j_l+1)$. Then, $M(\sigma) := \sigma_{j_1}\cdots\sigma_{j_l}$. Matsumoto's theorem states that the map M is well-defined.

Remark 1.10. The group $\mathbb{B}_n$ is the fundamental group of the space $\mathrm{S}ym^n_{\neq}(\mathbb{C}) := \mathbb{C}^n_{\neq}/\mathbb{S}_n$ of unordered configurations of n distinct points in $\mathbb{C}$.

Any braided vector space (V, c) gives naturally a representation ρ_n of $\mathbb{B}_n$ on $V^{\otimes n}$ for any $n \geq 2$ according to the following rule:

$$\rho_n(\sigma_i) := \mathrm{id}_V^{\otimes(j-1)} \otimes c \otimes \mathrm{id}_V^{\otimes(n-j-1)}. \tag{1.3}$$

2 Tensor algebra, shuffle algebra and Nichols algebra

In this section (V, c) is a braided vector space.

We recall that the tensor algebra $T_!(V)$ of a vector space V is the graded algebra whose underlying vector space is $\mathbb{C} \oplus \bigoplus_{n\geq 1} V^{\otimes n}$, with unit $1_{\mathbb{C}}$ and product given by juxtaposition of tensors. It has the universal property that any linear map $V \to A$, where A is an associative algebra, extends to a unique associative algebra morphism $T_!(V) \to A$.

The space $\mathbb{C} \oplus \bigoplus_{n\geq 1} V^{\otimes n}$ can be equipped with another graded associative algebra structure: the shuffle algebra (or cotensor algebra), that we denote by $T_*(V)$.

First, let $n, k, l \in \mathbb{N}$ with $n \geq 2$ and $n = k + l$. We say that $\sigma \in \mathbb{S}_n$ is a (k, l)-shuffle if $\sigma(a) < \sigma(b)$ if $1 \leq a < b \leq k$ or $k + 1 \leq a < b \leq n$. We denote by $\Sigma_{k,l}$ the subset of $\mathbb{S}_n$ consisting of (k, l)-shuffles in $\mathbb{S}_{k+l}$. So $\Sigma_{1,1} = \mathbb{S}_2$ and $\Sigma_{1,2} = \{\mathrm{id}, (12), (132)\}$ and $\Sigma_{2,1} = \{\mathrm{id}, (23), (123)\}$.

The product $\bullet$ of two homogeneous elements $v_{a_1} \otimes \cdots \otimes v_{a_k}$ and $v_{a_{k+1}} \otimes \cdots \otimes v_{a_{k+l}}$ in $T_*(V)$ is given by

$$(v_{a_1} \otimes \cdots \otimes v_{a_k}) \bullet (v_{a_{k+1}} \otimes \cdots \otimes v_{a_{k+l}}) = \sum_{\sigma \in \Sigma_{k,l}} \rho_{k+l}(M(\sigma))(v_{a_1} \otimes \cdots \otimes v_{a_k} \otimes v_{a_{k+1}} \otimes \cdots \otimes v_{a_{k+l}}).$$

where ρ_n is as in (1.3) and M is Matsumoto section. It is a non-trivial result that this algebra is associative.

Example 2.1. *For any $u, v, w \in V$ there holds*

$$\begin{aligned} v \bullet w &= v \otimes w + c(v \otimes w) \\ (u \otimes v) \bullet w &= u \otimes v \otimes w + u \otimes c(v \otimes w) + (c \otimes \mathrm{id})(\mathrm{id} \otimes c)(u \otimes v \otimes w). \end{aligned} \tag{2.1}$$

1. *If $c = \tau$ is the usual flip, then* (2.1) *become*

$$\begin{aligned} v \bullet w &= v \otimes w + w \otimes v \\ (u \otimes v) \bullet w &= u \otimes v \otimes w + u \otimes w \otimes v + w \otimes u \otimes v. \end{aligned}$$

2. *If $G = \mathbb{Z}_2$ and $V = V_0 \oplus V_1$ is a Yetter-Drinfeld module for G with G acting trivially on V_0 and by -1 on V_1, then $c(v \otimes w) = (-1)^{|v||w|} w \otimes v$ for all homogeneous $v, w \in V$ and* (2.1) *become*

$$\begin{aligned} v \bullet w &= v \otimes w + (-1)^{|w||v|} w \otimes v \\ (u \otimes v) \bullet w &= u \otimes v \otimes w + (-1)^{|v||w|} u \otimes w \otimes v + (-1)^{|v||w|+|u||w|} w \otimes u \otimes v. \end{aligned}$$

Remark 2.2. The product in Example 2.1 (1) has a long history: it was introduced in [50] to retrieve in a uniform way several results concerning the Baker–Campbell–Hausdorff formula.

Exercise 2.3. *Compute the number of k, l shuffles in $\mathbb{S}_n$.*

The universal property of $T_!(V)$ applied to the natural inclusion $V \to T_*(V)$ guarantees that there is a unique algebra morphism $Q\colon T_!(V) \to T_*(V)$ extending id_V.

The Nichols algebra of (V, c), denoted by $T_{!*}(V)$ for simplicity, is the image of Q, so $T_{!*}(V) = \mathrm{Im}(Q) \simeq T_!(V)/\mathrm{Ker}(Q)$. Nichols algebras firstly appeared in the work of Nichols [49], and were re-discovered in [56], see also [29].

Example 2.4. *1. If c is the usual flip τ, then for any $u, v, w \in V$ we have*

$$\begin{aligned} Q(v \otimes w) &= Q(v) \bullet Q(w) = v \otimes w + w \otimes v, \\ Q(u \otimes v \otimes w) &= Q(u \otimes v) \bullet Q(w) = (u \otimes v + v \otimes u) \bullet w \\ &= u \otimes v \otimes w + u \otimes w \otimes v + w \otimes u \otimes v + v \otimes u \otimes w + v \otimes w \otimes u + w \otimes v \otimes u. \end{aligned}$$

One can verify that $T_{!*}(V)$ is the symmetric algebra $S(V)$, that is a quadratic, infinite dimensional algebra.

2. If $G = \mathbb{Z}_2$, and V carries the sign representation and is concentrated in odd degree $V = V_1$, then for any $u, v, w \in V$ we have

$$\begin{aligned} Q(v \otimes w) &= Q(v) \bullet Q(w) = v \otimes w - w \otimes v, \\ Q(u \otimes v \otimes w) &= Q(u \otimes v) \bullet Q(w) = (u \otimes v - v \otimes u) \bullet w \\ &= u \otimes v \otimes w - u \otimes w \otimes v + w \otimes u \otimes v - v \otimes u \otimes w + v \otimes w \otimes u - w \otimes v \otimes u. \end{aligned}$$

One can verify that $T_{!*}(V) = \wedge V$, the exterior algebra, that is a quadratic, finite dimensional algebra.

More generally, the morphism Q is explicitly given by

$$\begin{aligned} &Q \colon T_!(V) \to T_*(V), && Q := \mathrm{id}_{\mathbb{C}} \oplus \mathrm{id}_V \oplus \bigoplus_{n \geq 2} Q_n, \\ &Q_n \colon V^{\otimes n} \to V^{\otimes n}, && Q_n := \sum_{\sigma \in \mathbb{S}_n} \rho_n(M(\sigma)). \end{aligned}$$

The component Q_n is called the n-th quantum symmetrizer, [52].

Example 2.5. Let $G = \mathbb{Z}^n$, let $V = \mathbb{C}^n$ be a Yetter-Drinfeld module for G with basis $\{v_1, \ldots, v_n\}$ of homogeneous common eigenvectors of V. Let $v_l \in V_{g_l}$ and $g_l v_j = \chi_j(g_l) v_j$ for $l, j \in \{1, \ldots, n\}$. Then, $c(v_l \otimes v_j) = \chi_j(g_l) v_j \otimes v_l$ and $Q(v_l \otimes v_j) = v_l \otimes v_j + \chi_j(g_l) v_j \otimes v_l$. By direct calculation, $Q(v_j^{\otimes m}) = \prod_{l=1}^m (1 + \chi_j(g_j) + \cdots \chi_j(g_j)^{l-1}) v_j^{\otimes m}$ for any $l \in \mathbb{N}$. If $\chi_j(g_j)$ is a non-trivial N-th root of 1, then $v_j^{\otimes N} \in \mathrm{Ker} Q$. If $\chi_j(g_j)$ is 1 or it is not a root of unity, then $\mathrm{Im}(Q)$ contains all tensor powers of v_j and it is therefore infinite-dimensional.

Exercise 2.6. Let $A = \left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)$, let $q \in \mathbb{C}^*$ and let c be the diagonal braiding on $V = \mathrm{span}(v_1, v_2)$ given by $c(v_l \otimes v_j) = q^{a_{lj}} v_j \otimes v_l$. Show that

$$\begin{aligned} Q\left(v_1 \otimes v_2 \otimes v_2 - (q + q^{-1}) v_2 \otimes v_1 \otimes v_2 + v_2 \otimes v_2 \otimes v_1\right) = 0, \\ Q\left(v_2 \otimes v_1 \otimes v_1 - (q + q^{-1}) v_1 \otimes v_2 \otimes v_1 + v_1 \otimes v_1 \otimes v_2\right) = 0. \end{aligned}$$

Example 2.7. [29, 51] Let $q \in \mathbb{C}^*$, let $D, A \in \mathsf{M}_n(\mathbb{Z})$ where A is the Cartan matrix of a semisimple Lie algebra $\mathfrak{g}$ and D is an invertible diagonal matrix with coefficients in $\{0, 1, 2, 3\}$ such that DA is symmetric. Let then $m_{lj} := q^{d_l a_{lj}}$ for $l, j \in \{1, \ldots, n\}$. Let $V = \mathrm{span}(v_1, \ldots, v_n)$ be a braided vector space with diagonal braiding given by the matrix

$M = (m_{lj})$. Then, $T_{!*}(V)$ is the algebra generated by $v_1, \ldots, v_n$ subject to the so-called quantum Serre relations

$$\sum_{l=0}^{1-a_{lj}} (-1)^l \frac{[1-a_{lj}]_{q^{d_l}}!}{[1-a_{lj}-l]_{q^{d_l}}![l]_{q^{d_l}}!} e_l^n e_j e_l^{1-a_{lj}-l} = 0, \qquad l \neq j$$

where $[t]_a := \frac{q^a - q^{-a}}{q-q^{-1}}$ for $t, a \in \mathbb{Z}_{\geq 0}$ and $[t]_a! := \prod_{s=1}^t [s]_a$. In other words, $T_{!*}(V)$ is isomorphic to the positive part of the quantized enveloping algebra of $\mathfrak{g}$.

3 An important example: Fomin-Kirillov algebras

For $n \geq 2$, let $G = \mathbb{S}_n$, let $g = (12)$, so $H := C_G(g) = \langle(12)\rangle \times \mathbb{S}_{n-2}$. We consider the 1-dimensional representation $\rho = \mathrm{sgn} \boxtimes \mathrm{triv}$ of H and the corresponding simple Yetter-Drinfeld module

$$V_n = \mathrm{Ind}_H^{\mathbb{S}_n} \mathbb{C}_\rho = \mathbb{C}\mathbb{S}_n \otimes_{\mathbb{C}H} \mathbb{C}_\rho = \bigoplus_{\sigma \in \mathbb{S}_n/H} \mathbb{C}\sigma \otimes 1.$$

where $\sigma \otimes 1$ is in degree $\sigma \cdot (12) \in \mathcal{O}_g^{\mathbb{S}_n}$. For any $(lj) \in \mathcal{O}_g^{\mathbb{S}_n}$, we choose $\sigma_{(lj)} \in \mathbb{S}_n$ such that $\sigma_{(lj)} \cdot g = (lj)$ and set $x_{lj} := \sigma_{(lj)} \otimes 1$, so $V_n = \mathrm{span}_{\mathbb{C}}(x_{lj}, 1 \leq l < j \leq n)$.

The relations of $T_{!*}(V_n)$ in degree 2 read as follows, [48]:

$$\begin{aligned} x_{(ij)}^2 &= 0, & & 1 \leq i < j \leq n, \\ x_{(lj)}x_{(jk)} &= x_{(jk)}x_{(lk)} + x_{(lk)}x_{(lj)}, & & 1 \leq l < j < k \leq n, \\ x_{(jk)}x_{(lj)} &= x_{(lk)}x_{(jk)} + x_{(lj)}x_{(lk)}, & & 1 \leq l < j < k \leq n, \\ x_{(mj)}x_{(kl)} &= x_{(kl)}x_{(mj)}, & & 1 \leq m, j, k, l \leq n, \quad |\{m, j, l, k\}| = 4. \end{aligned} \tag{3.1}$$

The quotient of $T_!(V_n)$ subject to the relations (3.1) is the *Fomink-Kirillov* algebra FK_n, introduced in [30] in order to give a combinatorial framework for Schubert calculus. because it contains a finite-dimensional, commutative subalgebra that is isomorphic to the cohomology algebra $H^*(\mathrm{GL}_n(\mathbb{C})/B_n, \mathbb{C}^*)$ of the flag variety $\mathrm{GL}_n(\mathbb{C})/B_n$. Here B_n, is the subgroup of upper-triangular matrices in $\mathrm{GL}_n(\mathbb{C})$. In more concrete terms, $H^*(\mathrm{GL}_n(\mathbb{C})/B_n, \mathbb{C})$ is isomorphic to the coinvariant algebra $\mathbb{C}[X_1, \ldots, X_n]/I_n$, where I_n is the ideal generated by the non-constant, symmetric polynomials. It was shown in [31, 48] that FK_n contains and is free over a subalgebra isomorphic to FK_{n-1}.

For $n \leq 5$ is has been shown in [30, 32, 48], with contributions of M. Graña and Roos, that FK_n is finite-dimensional and that the natural surjection $\mathsf{FK}_n \to T_{!*}(V_n)$ is an isomorphism. A proof of infinite-dimensionality of FK_6 appeared in [15] and it is conjectured that $T_{!*}(V_n) \simeq \mathsf{FK}_n$ for any n. The coinvariant algebra is also contained in $T_{!*}(V_n)$.

The numerology of FK_n, for $n \geq 2$ is intriguing: let d_n be the dimension of the top degree component in the grading of FK_n. Relying on the result in [15], the sequence reads: $1, 4, 12, 40, ?, ?$. The sequence $1, 4, 12, 40, \infty, \infty, \ldots$ gives also the number of indecomposable representations of the pre-projective algebra of type A_{n-1}, [45], the number of clusters of the cluster algebra structure on $\mathbb{C}[U_n]$, where U_n is the group of upper triangular unipotent matrices in $\mathrm{GL}_n(\mathbb{C})$, [41]. In addition, the sequence $1, 4, 12, 40$ describes the sequence of dimensions of prehomogeneous vector spaces with finite generic stabilizer, [24]. In this case, the generic stabilizers are precisely the symmetric groups $\mathbb{S}_n$ for $n = 2, 3, 4, 5$. This suggests that there might be interesting connections among these apparently different families of objects.

If we consider instead the representations $\rho' = \mathrm{sgn} \boxtimes \mathrm{sgn}$ of H and the corresponding irreducible Yetter-Drinfeld module

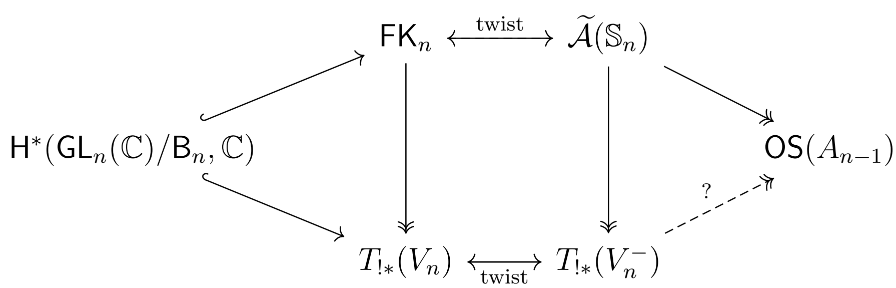
$$V_n^- = \mathrm{Ind}_H^{\mathbb{S}_n} \mathbb{C}_{\rho'} = \mathbb{C}\mathbb{S}_n \otimes_{\mathbb{C}H} \mathbb{C}_{\rho'} = \bigoplus_{\sigma \in \mathbb{S}_n/H} \mathbb{C}\sigma \otimes 1$$

then V_- has the same underlying vector space as V but with a simpler braiding: $c(x_{(mj)} \otimes x_{(kl)}) = -x_{(mj)(kl)(mj)} \otimes x_{(mj)}$ for $1 \leq m, j, k, l \leq n$. The corresponding algebra $T_{!*}(V_n^-)$ has the following relations in degree 2:

$$\begin{aligned} x_{(ij)}^2 &= 0, & & 1 \leq i < j \leq n, \\ x_{(lj)}x_{(jk)} + x_{(jk)}x_{(lk)} + x_{(lk)}x_{(lj)} &= 0, & & 1 \leq l < j < k \leq n, \\ x_{(jk)}x_{(lj)} + x_{(lk)}x_{(jk)} + x_{(lj)}x_{(lk)} &= 0, & & 1 \leq l < j < k \leq n, \\ x_{(mj)}x_{(kl)} + x_{(kl)}x_{(mj)} &= 0, & & 1 \leq m, j, k, l \leq n, \quad |\{m, j, l, k\}| = 4. \end{aligned} \tag{3.2}$$

has the same Hilbert series as $T_{!*}(V)$, and the same holds for the corresponding quadratic approximations, $\widetilde{\mathcal{A}}(\mathbb{S}_n)$ and FK_n. This holds because the Yetter-Drinfeld modules V_n and V_n^- can be obtained from one another twisting by a 2-cocycle of $\mathbb{S}_n$, [54].

The algebra $\widetilde{\mathcal{A}}(\mathbb{S}_n)$ occurred in [43, Section 5], as a cover of an algebra $\mathcal{A}_n$ used to compute the cohomology of the non-reduced Milnor fiber for the hyperplane arrangement $\mathcal{H}_{n-1}$ of type A_{n-1}. It was observed there that the Orlik-Solomon algebra $\mathsf{OS}(A_{n-1}) \simeq H^*(\mathbb{C}^{n-1} \setminus \bigcup_{H \in \mathcal{H}_{n-1}} \mathcal{H}, \mathbb{C})$ is a finite-dimensional quotient of $\widetilde{\mathcal{A}}(\mathbb{S}_n)$. Summarizing, we have the following picture:



where the existence of the dashed arrow is under investigation.

A family of Nichols algebras with similar properties as those of $T_{!*}(V_n)$ for an arbitrary finite Coxeter group W was obtained by Bazlov, [16]. These algebras also contain a subalgebra isomorphic to the corresponding coinvariant algebra of W. For $W \neq \mathbb{S}_n$, they are all infinite-dimensional, [23]. One can also define the analogue of $T_{!*}(V_n^-)$ for arbitrary W. It is again twist-equivalent to Bazlov's algebra, [23]. If the Coxeter graph has no even-labeled edges, the quadratic approximation of $T_{!*}(V_n^-)$ is the algebra $\widetilde{\mathcal{A}}(W)$ defined in [43] and under the same assumptions on the graph of W, the Orlik-Solomon algebra for W is again a quotient of $\widetilde{\mathcal{A}}(W)$. It would be interesting to know if the Orlik-Solomon algebra is also a quotient of the Nichols algebra.

Exercise 3.1. *1. For $j = 1, \ldots, n$, let $\theta_j := -\sum_{1\le i<j} x_{(ij)} + \sum_{j<k\le n} x_{(jk)} \in \mathsf{FK}_n$. Show that the subalgebra $\mathcal{H}_n$ of FK_n generated by the θ_j is commuative.*

2. Let now $n = 3$. Show that in the subalgebra $\mathcal{H}_3$ of FK_3 generated by θ_1, θ_2 and θ_3 there hold the relations:

$$\theta_1 + \theta_2 + \theta_3 = 0, \qquad \theta_1^2 + \theta_2^2 + \theta_3^2 = 0, \qquad \theta_1\theta_2\theta_3 = 0.$$

Deduce that all symmetric functions in θ_1, θ_2 and θ_3 without constant term vanish. ie., $\mathcal{H}_3$ is a quotient of $\mathbb{C}[\theta_1,\theta_2,\theta_3]/(\mathbb{C}[\theta_1,\theta_2,\theta_3]_+^{\mathbb{S}_3})$, where $\mathbb{C}[\theta_1,\theta_2,\theta_3]_+^{\mathbb{S}_3}$ denotes the terms of positive degree. For a proof that the two algebras are isomorphic, we refer to [30].

4 Questions

The main questions we are interested in are the following:

- For which braided vector spaces (V, c) is $T_{!*}(V)$ finite-dimensional?
- Can we classify finite-dimensional Nichols algebras?
- For which braided vector spaces (V, c) is $T_{!*}(V)$ finitely presented?
- For which braided vector spaces (V, c) are the relations of $T_{!*}(V)$ generated up to degree d? Special case, $d = 2$: for which braided vector spaces (V, c) is $T_{!*}(V)$ quadratic?
- Same questions as above but restricting the possibilities for V: for instance, assuming that V a Yetter-Drinfeld module for a fixed group G or a fixed family of groups G, and that c is defined as in (1.2).

4.1 Reduction arguments

1. The construction of $T_{!*}(V)$ is functorial, that is, if (V, c_V) and (U, c_U) are braided vector spaces and $f : V \to U$ is a linear map satisfying $c_U \circ (f \otimes f) = (f \otimes f) \circ c_V$, then f induces an algebra morphism $T_{!*}(f) : T_{!*}(V) \to T_{!*}(U)$. If f is injective, then $T_{!*}(f)$ is injective. Hence, if a braided vector space (U, c) contains a subspace V such that $c(V \otimes V) \subset V \otimes V$ (i.e., a braided vector subspace) and $T_{!*}(V)$ is infinite-dimensional, then the same holds for $T_{!*}(U)$, [36, Proposition 1.10.12].

2. A special case of inclusion of braided vector spaces can be obtained as follows: let $V = \bigoplus_{g\in G} V_g$ be a Yetter-Drinfeld module for G, and let $H \leq G$. Then, the restriction $V_H := \bigoplus_{h\in H} V_h$ is a Yetter-Drinfeld module for H whose associated braiding is the restriction to $V_H \otimes V_H$ of the braiding of V. Hence, we have a natural algebra inclusion $T_{!*}(V_H) \subset T_{!*}(V)$. This shows that simple Yetter-Drinfeld modules might have braided vector subspaces.

3. Let (V, c) be a braided vector space and assume that $V = U_1 \oplus U_2$ for some subspaces $U_1, U_2 \subset V$ such that $c(U_l \otimes U_j) \subseteq U_j \otimes U_l$ for $l,\, j \in \{1, 2\}$. For instance, this happens if V is a Yetter-Drinfeld module and $V = U_1 \oplus U_1$ is a decomposition as a sum of Yetter-Drinfeld submodules. If $c \circ c|_{U_1\otimes U_2} = \mathrm{id}_{U_1\otimes U_2}$, then as vector spaces $T_{!*}(U_1) \otimes T_{!*}(U_2) \simeq T_{!*}(V)$. More precisely, the two algebras are isomorphic if we twist the tensor product algebra multiplication using c where we would usually apply the standard flip τ. If $c \circ c|_{U_1\otimes U_2} \neq \mathrm{id}_{U_1\otimes U_2}$, then one has a proper inclusion $T_{!*}(U_1) \otimes T_{!*}(U_2) \subset T_{!*}(V)$, [36, Proposition 1.10.12].

Exercise 4.1. Let $G = \mathbb{Z}_2$ and $V = V_0 \oplus V_1$, with G acting trivially on V_0 and by -1 on V_1. Show that $T_{!*}(V)$ is isomorphic to $S(V_0) \otimes \wedge(V_1)$.

5 Nichols algebras associated with Yetter-Drinfeld modules

We focus on Nichols algebras for which the braided vector space comes from a Yetter-Drinfeld module of a group G. We call them "Nichols algebras over G". The situation is addressed differently according to whether G is abelian or not.

5.1 Abelian groups and diagonal braidings

In this section G is an abelian group. Then, if $V = \bigoplus_{g\in G} V_g$ is a Yetter-Drinfeld module, each homogeneous component is a G-submodule, and as such it splits as a sum of 1-dimensional G-stable subspaces that are again Yetter-Drinfeld modules. The braiding is then diagonal with respect to a basis of V compatible with this decomposition.

The question becomes then: let $V = \mathrm{span}(v_1, \ldots, v_n)$, let $q = (q_{lj}) \in M_n(\mathbb{C})$ with $q_{lj} \in \mathbb{C}^*$ and let $c_q \in \mathrm{GL}(V \otimes V)$ be defined by $c_q(v_l \otimes v_j) = q_{lj} v_j \otimes v_l$. Under which assumptions on q is $T_{!*}(V)$ finite dimensional? For such q, can we describe $T_{!*}(V)$ by generators and relations?

Example 3 shows that if $T_{!*}(V)$ is finite-dimensional, then q_{ll} is necessarily a non-trivial root of unity for any $l = 1, \ldots, n$, so in this Section we restrict to this situation.

We attach to the matrix q a generalized Dynkin diagram with n nodes, labeled by $q_{11}, \ldots, q_{nn}$. There is an edge between the l-th node and the j-th node if and only if $q_{lj}q_{jl} \neq 1$, and in this case we label the edge by $q_{lj}q_{jl}$.

Remark 5.1. 1. If the generalized Dynkin diagram is not connected, then the reduction argument 3 shows that the Nichols algebra is the (twisted) tensor product of the Nichols algebras whose associated Dynkin diagrams are the connected components of the initial diagram.

2. There is a loss of information when passing from c_q to the generalized Dynkin diagram. However, Nichols algebras with the same diagram differ by a twist via a cocycle, hence they have the same Hilbert series, [1, Theorem 4], [9, Proposition 3.9].

The classification of generalized Dynkin diagrams associated with finite dimensional Nichols algebras with diagonal braiding has been obtained by I. Heckenberger in [34]. They key tool is the Weyl groupoid he introduced in [33] and the idea of a root system given by the degrees in a (restricted) Poincaré-Birkhoff-Witt (PBW) type basis of a Nichols algebra. Finite-dimension is then related to the root system being finite and this in turn corresponds to the Weyl groupoid being finite. The latter can be classified combinatorially with little use of the complicated relations in the Nichols algebra.

The explicit presentations of the corresponding Nichols algebras were then given by I. Angiono in [12, 13]. Here, convex orderings for this generalized notion of root systems are introduced, and it is shown that the defining ideals of Nichols algebras with a finite root system are finitely-generated. A generating set of the ideal can be chosen to contain only elements of two types: powers of generators of a PBW basis, and elements leading to variations of quantum Serre-relations.

Remark 5.2. Further connections with Lie theory occur for the special family of *Cartan braidings*: these are diagonal braidings such that there exist $a_{lj} \in \mathbb{Z}_{\leq 0}$, and $N_l \in \mathbb{Z}_{\leq 0}$, for $l \neq j$ such that

$$q_{ll}^{N_l} = 1, \quad \forall l, \qquad\qquad q_{lj}q_{jl} = q_{ll}^{a_{lj}}, \quad ,\forall l, j, \quad l \neq j$$

The name is due to the fact that we can assign to this braiding a generalized Cartan matrix A by setting $a_{ll} = 2$ for all l and choosing $a_{lj} \in \{-N_l, \ldots, 0\}$ for all $j \neq l$. It was proved in [10, 33] that the corresponding Nichols algebra is finite-dimensional if and only if the matrix A is a Cartan matrix of finite type.

5.2 Non-abelian groups

In this section G is a finite non-abelian group. The reduction techniques in Section 4.1 show that one can proceed inductively, by looking at subgroups or at braided vector subspaces. Restriction to abelian subgroups allows to invoke the classification of finite-dimensional Nichols algebras corresponding to diagonal braidings. However, this is in general not enough to state general results about all Yetter-Drinfeld modules over G.

General results have been obtained setting either conditions on the Yetter-Drinfeld module or on the group, the main lines being: the Yetter-Drinfeld module is not simple; or G is solvable; or G is a non-abelian simple group. The latter strongly relies on the first.

5.2.1 Non simple Yetter-Drinfeld modules

Let V be a non simple Yetter-Drinfeld module. If for some Yetter-Drinfeld module decomposition $V = U_1 \oplus U_2$ the restriction of $c \circ c$ to $U_1 \oplus U_2$ is the identity then we may invoke the Reduction (1) and reduce the problem to an analysis of the Nichols algebras of U_1 and U_2 respectively. Therefore, it is not restrictive to assume that V is *braid-indecomposable*, that is, the restriction of $c \circ c$ to $U_1 \oplus U_2$ is not the identity for any non-trivial decomposition of V. In addition, if the support of V generates a proper subgroup H of G, then V is also a Yetter-Drinfeld module for H, with same braiding. Therefore, there is no loss of generality in assuming that G is generated by the support of V.

A complete classification of the Yetter-Drinfeld modules satisfying these assumptions and whose corresponding Nichols algebra is finite-dimensional is given in [37–39]. They are again classified in terms of variations of Dynkin diagrams whose rank is given by the number of simple summands, encoding information about the modules (dimension, support,...). The dimensions of the corresponding Nichols algebras are explicitly given. Also in this situation the key tool is a Weyl groupoid. The classification shows that there are very strong restrictions on the group G, the supports of the module, and its dimension.

More precisely, for $j = 1, 2, 3, 4$, let φ_j be the 3-cycle in $\{(243),(134),(142),(123)\}$ fixing j, let $n \in \mathbb{N}_{\geq 2}$ and let

$$\Gamma_n := \langle g, h, \epsilon \mid hg = \epsilon gh,\ g\epsilon = \epsilon^{-1}g,\ h\epsilon = \epsilon h,\ \epsilon^n = 1\rangle,$$
$$T := \langle z\rangle \times \langle g_j,\ j = 1, 2, 3, 4,\ \mid g_j g_l = g_{\varphi_j(l)} g_j,\ j = 1, 2, 3, 4\rangle.$$

Theorem 5.3. *([37,38]) Let G be a non-abelian group let U_1, U_2 be finite-dimensional simple Yetter-Drinfeld modules of G, with*

$$\langle \mathrm{supp}(U_1 \oplus U_2)\rangle = G, \quad (\mathrm{id} - c \circ c)|_{U_1 \otimes U_2} \neq 0, \quad \dim T_{!*}(U_1 \oplus U_2) < \infty.$$

Then G is an epimorphic image of $\Gamma_2, \Gamma_3, \Gamma_4$ or T. The possibilities for the cover of G, the dimension of $T_{!*}(U_1 \oplus U_2)$, the dimension of U_1 and U_2, and the sizes of their support are collected in Table 1 and all of them occur.

cover of G	$\dim T_{!*}(U_1 \oplus U_2)$	size supports	$(\dim U_1, \dim U_2)$
Γ_2	64	(2,2)	(2,2)
Γ_3	10368	(3,1)	(3,1)
Γ_3	10368	(3,2)	(3,2)
Γ_3	2304	(3,2)	(3,2)
Γ_3	2304	(3,1)	(3,2)
T	80621568	(4,1)	(4,1)
Γ_4	262144	(4,2)	(4,2)

Table 1: Finite-dimensional Nichols algebras $T_{!*}(U_1 \oplus U_2)$ for G non-abelian

In addition, the conjugacy structure of the supports of U_1 and U_2 can be explicitly described, as well as the Yetter-Drinfeld modules.

The case of modules with more than two simple components is more involved. The simple components turn out to be all induced by 1-dimensional representations of centralizers, and the classification is given in terms of labeled Dynkin-type diagrams called skeleta encoding the size of the supports of the simple components, the commutation among supports, and some information about the Yetter-Drinfeld modules. For further information we refer to [39].

5.2.2 Simple Yetter-Drinfeld modules for non-abelian simple groups

Here one focuses on simple Yetter-Drinfeld modules for G. Each such module is associated to a pair $(\mathcal{O}, \rho)$ where $\mathcal{O} = \mathcal{O}_g$, the support of V, is a conjugacy class in G and ρ is an irreducible representation of $C_G(g)$. One aims at finding criteria on $\mathcal{O}$ ensuring that the associated Nichols algebra is infinite-dimensional *for all choices of* ρ. The strong restrictions on non-simple Yetter-Drinfeld modules described in Section 5.2.1 allow to translate the analysis into purely group theoretic terms. The basic idea is that if G has a subgroup H such that $\mathcal{O} \cap H$ is not a single conjugacy class for H, then for any simple Yetter-Drinfeld module V supported on $\mathcal{O}$ the Yetter-Drinfeld module V_H is not simple for H. The condition $c \circ c \neq \mathrm{id}$ translates in group theoretic terms in the existence of suitable non-commuting, non-H-conjugate elements in $\mathcal{O} \cap H$. The remaining conditions can be translated into extra conditions on $\mathcal{O} \cap H$, or on the size of the H-conjugacy classes therein. These considerations lead to the following

Theorem 5.4. *([2, 3, 6]) Let $\mathcal{O}$ be a conjugacy class in G. Assume that one of these conditions is verified:*

1. *There exist r, $s \in \mathcal{O}$ such that $(rs)^2 \neq (sr)^2$ and r, s are not conjugate in $\langle r, s \rangle$.*
2. *There exist pairwise non-commuting r_1, r_2, r_3, $r_4 \in \mathcal{O}$ such that r_l and r_j are not conjugate in $H = \langle r_1, r_2, r_3, r_4 \rangle$ if $l \neq j$.*
3. *There exists a subgroup H of G and non-commuting r, $s \in H \cap \mathcal{O}$ such that the H-conjugacy classes $\mathcal{O}_r^H$ and $\mathcal{O}_s^H$ of r and s respectively are disjoint, generate H and satisfy* $\min(|\mathcal{O}_r^H|, |\mathcal{O}_s^H|) > 2$ *or* $\max(|\mathcal{O}_r^H|, |\mathcal{O}_s^H|) > 4$.

Then $\dim T_{!*}(V) = \infty$ *for any Yetter-Drinfeld module of G supported on $\mathcal{O}$.*

The conditions in Theorem 5.4 propagate, in the sense that if a class in G satisfies one of these three conditions, then the same holds for its saturation in any overgroup of G and its lift to any covering group of G. For this reason, the first natural family of groups on which they are being tested is the family of non-abelian simple groups.

In this case one has the folklore conjecture:

Conjecture 5.4.1. *If G is a non-abelian finite simple group then there exists no non-zero finite-dimensional Nichols algebra over G.*

Conjecture 5.4.1 holds in the following situations:

1. G is an alternating group, [6]

2. G is a sporadic group with the possible exceptions of Fi_{22}, B and M, see [7, 18] and [28, Remarks 3.6 and 3.7].

3. G is a finite simple group of Lie type $G(q)$ and

   (a) q is even and G is $\mathbf{PSL}_2(q)$ with $q > 2$; $\mathbf{P\Omega}^+_{4n}(q)$, $\mathbf{P\Omega}^-_{4n}(q)$, for $n \geq 1$; ${}^3D_4(q)$, $E_7(q)$, $E_8(q)$, $F_4(q)$, or $G_2(q)$, [4, Theorem 1.3].

   (b) $G = \mathbf{PSp}_{2n}(q)$ for and $n > 3$ any q, or else for $n = 2$ and $q > 7$ [5, Theorem II] and [4, Theorem 6.3].

   (c) $G = \mathbf{PSL}_n(q)$ and $n \geq 4$ for any q or else for $n = 3$ and $q > 2$ [5, Theorem III];

   (d) G is a Suzuki or Ree group, [19].

If G is a finite simple group of Lie type and is none of the above groups, then a class $\mathcal{O}$ in G supporting a Yetter-Drinfeld-module V for which $\dim T_{!*}(V) < \infty$ may occur only if $G \simeq \mathbf{PSL}_2(3)$ and $\mathcal{O}$ is the class of an element of order 4; or $G \simeq \mathbf{PSp}_4(q)$ with $q = 3, 5, 7$ and $\mathcal{O}$ is a class of involutions, or else G is any further group, and $\mathcal{O}$ is semisimple, that is, it is the class of an element of order coprime to q, see [4, Theorem 1.2] and following remarks.

Exercise 5.5. *1. Let $n \in \mathbb{N}_{\geq 5}$ be an odd number. Show that the conjugacy class of n-cycles satisfies condition 3 in Theorem 5.4.*

2. Let $n \in \mathbb{N}_{\geq 5}$. Show that the conjugacy class of 3-cycles satisfies condition 3 in Theorem 5.4 (Hint: use $\mathbb{A}_4$).

3. Verify that the class consisting of transpositions in $\mathbb{S}_n$ for $n \geq 3$ does not satisfy any of the assumptions from Theorem 5.4.

5.2.3 Simple Yetter-Drinfeld modules over solvable groups

The combinatorial analysis of the supports of Yetter-Drinfeld modules has proved to be extremely effective. The case of supports and Yetter-Drinfeld modules of prime dimension has been addressed in [35], using techniques involving deformation and reduction mod p. This lead to important constraints.

Theorem 5.6. *Let G be a non-abelian group and let V be a simple Yetter-Drinfeld module whose support generates G. Assume in addition that $\dim V$ is a prime number and that $\dim T_{!*}(V) < \infty$. Then $\dim V \in \{3, 5, 7\}$ and V occurs in a list of 6 precise examples,* [35, Section 1].

This approach has been exploited further and lead to a complete answer in the case in which G is solvable.

Theorem 5.7. *([8, Theorem 6.14]) Let G be a finite solvable non-cyclic group, and let V be a Yetter-Drinfeld module such that its support is a conjugacy class of G generating it. Assume in addition that $T_{!*}(V) < \infty$. Then V is simple and it has either prime size, or $T_{!*}(V)$ is (twist equivalent to) FK_4, or $V = \mathrm{Ind}_{C_{\mathbb{S}_4}(1234)}^{\mathbb{S}_4}(\mathbb{C}_{-1})$ and $\dim T_{!*}(V) = 576$. In particular, $|G|$ is even and not coprime with* 105.

Combining with the strategies from Section 5.2.2 one can state the following

Theorem 5.8. *([8, Theorem 6.18]) Let V be a Yetter-Drinfeld module for G, with $|G|$ odd. If $\dim T_{!*}(V) < \infty$, then the braiding on V is of diagonal type.*

6 Hopf Algebras in braided categories

Nichols algebras are not just algebras: they have a richer structure that we now introduce. All categories here will be categories of vector spaces with additional structure, Hom sets will be $\mathbb{C}$-vector spaces with bilinear composition ($\mathbb{C}$-linear), and direct sums are allowed (additive).

Definition 6.1. *A (strict) braided monoidal category is a category $\mathcal{C}$ equipped with*

- *a bifunctor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called tensor product*
- *a unit object $\mathbf{1}_{\mathcal{C}}$, with an isomorphism $\iota\colon \mathbf{1}_{\mathcal{C}} \otimes \mathbf{1}_{\mathcal{C}} \to \mathbf{1}_{\mathcal{C}}$*
- *a natural transformation $c_{-,-}\colon \otimes \to \otimes^{\mathrm{op}}$ (the tensor product in opposite order) called braiding*

satisfying compatibility axioms, called pentagon, triangle, and hexagonal axioms, [27, Definition 2.1.1, Definition 8.1.1].

A consequence of the hexagonal axiom is that for all triplets of objects U, V, W there holds

$$(c_{V,W} \otimes \mathrm{id}_U) \circ (\mathrm{id}_V \otimes c_{U,W}) \circ (c_{U,V} \otimes \mathrm{id}_W) = (\mathrm{id}_W \otimes c_{U,V}) \circ (c_{U,W} \otimes \mathrm{id}_V) \circ (\mathrm{id}_U \otimes c_{V,W}).$$

Example 6.2. *The following are examples of braided monoidal categories*

1. *The category* **Vec** *of vector spaces with usual tensor product, $\mathbf{1}_{\mathbf{Vec}} = \mathbb{C}$, and braiding given by the usual flip τ.*
2. *The category* **sVec** *of $\mathbb{Z}_2$-graded vector spaces with usual tensor product, $\mathbf{1}_{\mathbf{sVec}} = \mathbb{C}$, with trivial grading, and braiding given on $V \otimes W$ by $v \otimes w \mapsto (-1)^{|v||w|} w \otimes v$ for homogeneous $v \in V$ and $w \in W$.*
3. *The category of representations of a group G, with usual tensor product of representations, unit object given by the trivial representation and usual flip.*

4. *The category of representations of a semisimple Lie algebra, with usual tensor product or representations, unit object given by the trivial representation and usual flip.*

5. *The category* ${}^G_G\mathcal{YD}$ *of Yetter-Drinfeld modules over* G*. The tensor product of two objects* $V = \bigoplus_{g \in G} V$ *and* $W = \bigoplus_{g \in G} W$ *is the* G*-module* $V \otimes W$ *with grading* $(V \otimes W)_g := \bigoplus_{h,l \in G, hl=g} V_h \otimes W_l$*. The unit object is the trivial representation on* $\mathbb{C}$ *with trivial grading. The braiding* $V \otimes W \to W \otimes V$ *is given by* $c_{V,W}(v \otimes w) = g.w \otimes v$ *for* $v \in V_g$ *and* $w \in W$*.*

Exercise 6.3. *Verify the statements in Example 6.2.*

An algebra in a braided monoidal category $\mathcal{C}$ is an associative algebra A (our objects are vector spaces!) such that the multiplication map m_A and the inclusion $\mathbb{C} \to A$ induced by the unit are morphisms in $\mathcal{C}$. For any two algebras in $\mathcal{C}$, the tensor product $A \otimes B$ is naturally equipped with an algebra structure by setting $m_{A \otimes B} := (m_A \otimes m_B) \circ (\mathrm{id}_A \otimes c_{B,A} \otimes \mathrm{id}_B)$.

Dually, a coalgebra in $\mathcal{C}$ is an object C in $\mathcal{C}$ together with two morphisms $\Delta\colon C \to C \otimes C$ (called comultiplication) and $\varepsilon\colon C \to \mathbf{1}_{\mathcal{C}}$ (called counit) in $\mathcal{C}$ satisfying $(\Delta \otimes \mathrm{id}_C) \circ \Delta = (\mathrm{id}_C \otimes \Delta) \circ \Delta$ and $\mathrm{id}_C = (\varepsilon \otimes \mathrm{id}_C) \circ \Delta = (\mathrm{id}_C \otimes \varepsilon) \circ \Delta$.

A bialgebra B in $\mathcal{C}$ is an object B that is simultaneously an algebra, a coalgebra, and such that the comultiplication $\Delta\colon B \to B \otimes B$ and the counit $\varepsilon\colon B \to \mathbb{C}$ are algebra morphisms.

Example 6.4. *Let* $V \in {}^G_G\mathcal{Y}D$*. Then,*

1. $T_!(V)$ *is a bialgebra in* ${}^G_G\mathcal{Y}D$*, where* ε *is the projection on the zero degree term and* $\Delta(v) = v \otimes 1 + 1 \otimes v$ *for all* $v \in V$*, extended to* $T_!(V)$ *as an algebra morphism.*

2. $T_*(V)$ *is a bialgebra in* ${}^G_G\mathcal{Y}D$*, where* ε *is the projection on the zero degree term and* $\Delta(v) = v \otimes 1 + 1 \otimes v$ *for all* $v \in V$*, extended to* $T_*(V)$ *as an algebra morphism.*

3. *It can be verified that* $Q\colon T_!(V) \to T_*(V)$ *is a morphism in* ${}^G_G\mathcal{Y}D$ *and a coalgebra morphism, that is, it intertwines the comultiplications in* $T_!(V)$ *and* $T_*(V)$ *and is compatible with the counits. Hence,* $T_{!*}(V)$ *is a bialgebra in* ${}^G_G\mathcal{Y}D$*. It can be shown that the degree* 1 *component satisfies*

$$V = \{x \in T_{!*}(V) \mid \Delta(x) = x \otimes 1 + 1 \otimes x\}.$$

Exercise 6.5. *Verify the statement in Example 6.4 1.*

7 Applications

We review here two applications of Nichols algebras in mathematics. Nichols algebras have applications also in physics, for instance through the action of Nichols algebras on vertex algebras via screening operators, [42, 53], but this goes beyond the scope of these lecture notes.

7.1 Classification of pointed Hopf algebras

A Hopf algebra in a braided monoidal category $\mathcal{C}$ is a bialgebra H in $\mathcal{C}$ with a morphism $S\colon H \to H$ in $\mathcal{C}$, called antipode, satisfying

$$m_H(S \otimes \mathrm{id})\Delta(h) = m_H(S \otimes \mathrm{id}) \circ \Delta(h) = \varepsilon(h)1, \quad \forall h \in H.$$

One can show that S is an algebra antimorphism and that if H has an antipode, then it is unique.

Example 7.1. *1. The group algebra $\mathbb{C}G$ of a group G is a Hopf algebra in* **Vec** *with comultiplication, counit, and antipode defined on $g \in G$ by $\Delta(g) = g \otimes g$, $\varepsilon(g) = 1$, and $S(g) = g^{-1}$.*

2. The universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is a Hopf algebra in **Vec** *with comultiplication, counit, and antipode defined on the generators $x \in \mathfrak{g}$ by $\Delta(x) = x \otimes 1 + 1 \otimes x$, $\varepsilon(x) = 0$, and $S(x) = -x$.*

3. If $V \in {}^G_G\mathcal{YD}$, then $T_!(V)$, $T_*(V)$, and $T_{!*}(V)$ are Hopf algebras in ${}^G_G\mathcal{YD}$, with antipode given by $S(v) = -v$ for all $v \in V$.

If H is a Hopf algebra in Vec satisfying $\tau \circ \Delta = \Delta$, then results by Cartier, Milnor, Moore and Kostant from the 60's show that H is a (sort of) semi-direct product of a group algebra and a universal enveloping algebra. However, if we drop the τ-invariance assumption on Δ, the situation becomes wild. The classification of Hopf algebras in full generality is an untractable problem, but significant progress has been achieved for special families, identified for example by finite-dimensionality, and/or properties of the algebra structure, or in terms of other invariants.

There are two natural invariants one can attach to a Hopf algebra H in Vec: the group $G(H) = \{g \in H \mid \Delta(g) = g \otimes g\}$ of the so-called grouplike elements, and the coradical H_0 given by the sum of all subcoalgebras of H that contain no proper subcoalgebra. One has always the inclusion $\mathbb{C}G(H) \subseteq H_0$. A Hopf algebra is *pointed* if $\mathbb{C}G(H) = H_0$.

Exercise 7.2. *Show that if H is a Hopf algebra in Vec, then $G(H)$ is a group.*

Example 7.3. *The group algebra $\mathbb{C}G$, the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra, the quantized enveloping algebras of a semisimple Lie algebra, and Lusztig's finite-dimensional specialization of it (the so-called small quantum group) are examples of pointed Hopf algebras.*

The classification of finite-dimensional pointed Hopf algebras is an open problem in general. The most effective strategy so far has been the lifting method introduced in [11] that we now recall.

Let H be a pointed Hopf algebra. The dual version of the Jacobson filtration induces a filtration on H whose associated graded turns out to be a (sort of) semi-direct product

$\mathbb{C}G(H)\#R$ where R is a bialgebra in ${}^{G(H)}_{G(H)}\mathcal{Y}D$ with an induced grading. Its degree 1 component R_1 is an object in ${}^{G(H)}_{G(H)}\mathcal{Y}D$ and it is shown that R contains $T_{!*}(R_1)$ as a subalgebra. So the problem becomes:

- Classify the finite-dimensional Nichols algebras in ${}^{G(H)}_{G(H)}\mathcal{Y}D$;
- Classify all pointed Hopf algebras whose associated graded is $\mathbb{C}G(H)\#T_{!*}(V)$ where $T_{!*}(V)$ is finite-dimensional;
- Show that finite-dimensional pointed Hopf algebras are generated in degree ≤ 1, so that $R = T_{!*}(R_1)$ in the construction above.

The above strategy, together with the results in [12, 13, 33, 34] lead to a complete classification of finite-dimensional pointed Hopf algebras with abelian coradical, [14].

7.2 Malle's conjecture

Methods in algebraic topology have been combined with Hopf algebraic methods in [26] in order to prove a function field version of a conjecture of Malle [46] on the distribution of Galois groups. For a transitive subgroup G of $\mathbb{S}_n$ the conjecture predicts that the number $N_{G,K}(X)$ of separable field extensions of degree n of a given field K, with Galois group G and discriminant bounded by X behaves as $cX^{a_M}\log(X)^{b_M-1}$, for constants a_M, b_M depending on the group theory of G and the action of $\mathrm{Gal}(\overline{K}^{\text{separable}}/K)$ on G.

An upper bound for $N_{G,K}(X)$ in the case in which K is a function field was given in [26]. More precisely, the authors show that for any n and any transitive subgroup G of $\mathbb{S}_n$ there are constants $c(G)$, $Q(G)$, $e(G)$ and $a(G)$ depending on G such that for all $X > 0$ and all $q > Q(G)$ coprime with $|G|$ there holds

$$N_{G,\mathbb{F}_q(t)}(X) \leq C(G)X^{a(G)}\log(X)^{e(G)}, \qquad \text{where } e(G) \geq b_M - 1.$$

We sketch here the approach, which surprisingly uses Nichols and shuffle algebras, following the account in [55].

First of all, extensions $L/\mathbb{F}_q(t)$ correspond to curves $\Sigma = \mathrm{Spec}(\mathcal{O}_L)$ defined over $\mathbb{F}_q$ and maps $\Sigma \to \mathrm{Spec}(\mathbb{F}_q[t])$. In addition, the set of extensions one wishes to estimate can be viewed as the set of isomorphism classes of the (branched) covers $\Sigma \to \mathrm{Spec}(\mathbb{F}_q[t])$, and also as the set of $\mathbb{F}_q$-points of suitable components in a moduli space of branched covers, called the Hurwitz moduli space $\mathcal{H}_{G,n}$. Then the Grothendieck-Lefschetz formula and Deligne's bounds on eigenvalues of Frobenius are invoked to reduce the initial problem into the algebro-topological problem of controlling the Betti numbers $r(j,n) = \mathrm{rk}_{\mathbb{Q}}(H^j_{\text{sing}}(\mathcal{H}_{G,n}(\mathbb{C}), \mathbb{Q}))$. Now, to estimate the growth of the $r(j,n)$, it is observed that the function $\mathcal{H}_{G,n}(\mathbb{C}) \to \mathrm{Sym}_{\neq}(\mathbb{C})$ which associates to a covering its branch locus is a covering space. This property is used to translate the computation of $H^j_{\text{sing}}(\mathcal{H}_{G,n}(\mathbb{C}), \mathbb{Q})$ into the computation of group cohomology for the fundamental group $\mathbb{B}_n$ with coefficients in suitable subrepresentations of the braid group representation. Via a cellular stratification of $\mathrm{Sym}^n_{\neq}(\mathbb{C})$, the calculation can be rephrased in terms of the cohomology of a shuffle algebra $T_*(V)$, where V is, up to a sign in the braiding, the dual of the Yetter-Drinfeld module $\mathbb{Q}[G \setminus 1]$, with conjugation action of G. This algebra is filtered and the associated graded is a twisted tensor product of $T_{!*}(V)$ with a complementary subalgebra. Then the sought bounds are obtained through an analysis of the Koszul complex of the Nichols algebra and a control on the number of orbits for the action of the braid group.

8 Geometric methods

In these sections we show two different geometric approaches to Nichols algebras. In both cases, it emerges that Nichols algebras are canonical objects among bialgebras in braided categories. We introduce some further notation.

A connected bialgebra (respectively Hopf algebra) in a braided category $\mathcal{C}$ is a bialgebra (respectively, Hopf algebra) B in $\mathcal{C}$ such that both its quotient by the radical, and its coradical are the unit objects in $\mathcal{C}$. We set $\mathsf{CH}(\mathcal{C})$ to be the category of connected Hopf algebras, where morphisms are the morphisms in $\mathcal{C}$ preserving the Hopf algebra operations.

A graded connected bialgebra in a braided category $\mathcal{C}$ is a bialgebra B with a grading $B = \bigoplus_{n\in\mathbb{N}} B_n$ in $\mathcal{C}$ such that

$$B_0 = 1_{\mathcal{C}} = \mathbb{C}, \qquad m_B(B_j \otimes B_l) \subset B_{j+l}, \quad j, l \in \mathbb{N} \qquad \Delta(B_j) \subseteq \oplus_{k=0}^{j} B_k \otimes B_{j-k}.$$

We set $\mathsf{GCB}(\mathcal{C})$ to be the category of graded connected bialgebras, where morphisms are the morphisms in $\mathcal{C}$ preserving the grading and the bialgebra operations. Graded connected bialgebras are connected Hopf algebras, [36, Proposition 6.4.2].

Example 8.1. *For any $V \in {}^G_G\mathcal{Y}D$, the bialgebras $T_!(V)$, $T_{!*}(V)$ and $T_*(V)$ are objects in $\mathsf{GCB}({}^G_G\mathcal{Y}D)$ and in $\mathsf{CH}({}^G_G\mathcal{Y}D)$*

8.1 Nichols algebras as closed orbits

Nichols algebras can be studied by invariant theoretic and deformation theoretic methods. A possible way is the following, due to E. Meir, [47]. His results are for more general categories but we restrict to $\mathcal{C} = {}^G_G\mathcal{Y}D$, for a finite group G for simplicity.

Fixing an object B in $\mathcal{C}$, one can study the moduli space X_B of connected Hopf algebra structures on B. The points in X_B represent the possible structure constants of objects in $\mathsf{CH}(\mathcal{C})$ with underlying object B. More precisely, we consider the affine space

$$\mathbb{A}^N = \mathrm{Hom}_{\mathcal{C}}(B \otimes B, B) \oplus \mathrm{Hom}_{\mathcal{C}}(\mathbb{C}, B) \oplus \mathrm{Hom}_{\mathcal{C}}(B, \mathbb{C}) \oplus \mathrm{Hom}_{\mathcal{C}}(B, B \otimes B) \oplus \mathrm{Hom}_{\mathcal{C}}(B, B).$$

A point in $\mathbb{A}^N$ is a 5-uple $(m, u, \varepsilon, \Delta, S)$. We set X_B as the set of points in $\mathbb{A}^N$ that satisfy the connected Hopf algebra conditions. This way we view the points in X_B as objects in $\mathsf{CH}(\mathcal{C})$. Writing $B = \bigoplus_{S_j \text{ simple object in } \mathcal{C}} M_j \otimes S_j$ where M_j is a $\mathbb{C}$-vector space indicating the multiplicity of S_j in B, one can translate the connected Hopf algebra conditions into polynomial equalities between structure constants, showing that X_B is an affine variety.

The space $\mathbb{A}^N$ has a natural algebraic action of the automorphism group Γ_B of B in $\mathcal{C}$. This action preserves each of the direct summands and X_B, and the orbits in X_B are the isomorphism classes of connected Hopf algebra structures on B. Using the decomposition of B given above, one sees that $\Gamma_B \simeq \prod_j \mathrm{GL}(M_j)$, so it is reductive, and one can use all invariant theoretic machinery, together with the natural properties of Nichols algebras, to deduce the following cheracterization of Nichols algebras.

Theorem 8.2. *([47, Theorem 1.2]) Let $A \in X_B$ and assume it is finite-dimensional. Then, the Γ_B-orbit of A in X_B is closed if and only if A is isomorphic to a Nichols algebra. Hence, all the Γ_B-orbits in X_B are closed if and only if all connected Hopf algebras with underlying object B are Nichols algebras.*

We can also introduce deformation ideas: for $A, H \in X_B$, we say that A specializes to H, or that A is a deformation of H, if H lies in $\overline{\Gamma_B \cdot A}$. Since the closure of every Γ_B-orbit contains a closed orbit, Theorem 8.2 states that every algebra in $\mathsf{CH}(\mathcal{C})$ in X_B is a deformation of a Nichols algebra with underlying object B. In fact, Nichols algebras satisfy a rigidity property.

Theorem 8.3. *([47, Theorem 1.4]) Let $V \in {}^G_G\mathcal{Y}D$. If V is simple and $T_{!*}(V)$ is finite-dimensional, then $T_{!*}(V)$ is rigid, that is, its Γ_B-orbit is not contained in the closure of any orbit in X_B.*

In case $T_{!*}(V)$ is finite-dimensional but not rigid, a description of the possible deformations of $T_{!*}(V)$ can also be given, in terms of quotients of $T_!(V')$ for $V' \subsetneq V$.

8.2 Kapranov and Schechtman's equivalence

We turn now to an interpretation, due to Kapranov and Schechtman, of Nichols algebras in terms of perverse sheaves. For basic notions on perverse sheaves we refer the reader to [17] and to the formulary and crash course in [25, §1.5, 2.7, 5.8].

We consider the infinite-dimensional variety $\mathrm{Sym}(\mathbb{C}) = \coprod_{n\in\mathbb{N}} \mathrm{Sym}^n(\mathbb{C})$, where $\mathrm{Sym}^n(\mathbb{C}) = \mathbb{C}^n/\mathbb{S}_n$. It can be seen as the space of monic polynomials with coefficients in $\mathbb{C}$. This way, for $n \in \mathbb{N}$ the variety $\mathrm{Sym}^n(\mathbb{C})$ is seen as the affine space of polynomials of degree n. It is stratified by setting in the same stratum all polynomials of degree n for which the roots have multiplicities according to the same partition of n. The subset $\mathrm{Sym}_{\neq}(\mathbb{C}) = \coprod_{n\in\mathbb{N}} \mathrm{Sym}^n_{\neq}(\mathbb{C})$ corresponding to multiplicity-free polynomials is open, and each of its connected components $\mathrm{Sym}^n_{\neq}(\mathbb{C})$ is open dense in $\mathrm{Sym}^n(\mathbb{C})$.

The category $\mathsf{FPS}(\mathrm{Sym}(\mathbb{C}), \mathcal{C})$ of factorizable perverse sheaves on $\mathrm{Sym}(\mathbb{C})$ with values in a braided monoidal category $\mathcal{C}$ was introduced in [40]. Objects are pairs $(\mathcal{F}, \mu)$ where $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{N}}$ is a perverse sheaf on $\mathrm{Sym}(\mathbb{C})$ constructible with respect to the stratification given by the multiplicities, and with values in $\mathcal{C}$, and μ is a family of isomorphisms that takes into account the monoid structure on $\mathrm{Sym}(\mathbb{C})$. The description is technical and we omit it. Morphisms are morphisms of perverse sheaves that are compatible with the families of isomorphisms μ.

Each object in $\mathcal{C}$ induces naturally three objects in $\mathsf{FPS}(\mathrm{Sym}(\mathbb{C}),\mathcal{C})$, that are constructed extending to $\mathrm{Sym}(\mathbb{C})$ suitable perverse sheaves on the open dense subset $\mathrm{Sym}_{\neq}(\mathbb{C})$ as we now explain. Perverse sheaves on each $\mathrm{Sym}^n_{\neq}(\mathbb{C})$ are locally constant (i.e., local systems), because $\mathrm{Sym}^n_{\neq}(\mathbb{C})$ consists a unique stratum. Hence, they correspond to representations of the fundamental group $\pi_1(\mathrm{Sym}^n_{\neq}(\mathbb{C})) = \mathbb{B}_n$. Let V be an object in $\mathcal{C}$. The braiding $c_{V,V}\colon V\otimes V\to V\otimes V$ coming from the braided category structure in $\mathcal{C}$ gives a representation of $\mathbb{B}_n$ on $V^{\otimes n}$ for any $n\in\mathbb{N}$, as explained in Section 1.2. The family of representations of this form, where n runs through $\mathbb{N}$ gives a perverse sheaf $\mathcal{L}(V)$ on $\mathrm{Sym}_{\neq}(\mathbb{C})$ that is compatible with the monoid structure of $\mathrm{Sym}_{\neq}(\mathbb{C})$. Let $\mathbf{j}\colon \mathrm{Sym}_{\neq}(\mathbb{C})\to\mathrm{Sym}(\mathbb{C})$ be the natural inclusion. It is an open embedding and it can be shown that the perverse sheaves obtained as extensions $\mathbf{j}_!(\mathcal{L}(V))$, $\mathbf{j}_{!*}(\mathcal{L}(V))$, and $\mathbf{j}_*(\mathcal{L}(V))$ can be equipped with canonical families of isomorphisms $\mu_!$, $\mu_{!*}$ and μ_*, respectively, to become objects in $\mathsf{FPS}(\mathrm{Sym}(\mathbb{C}),\mathcal{C})$.

The category $\mathsf{FPS}(\mathrm{Sym}(\mathbb{C}),\mathcal{C})$ is related to the category of connected bialgebras $\mathsf{CB}(\mathcal{C})$. For an object V in $\mathcal{C}$, the symbol $V[1]$ denotes the braided vector space with underlying space V and braiding $-c_{V,V}$.

Theorem 8.4. *([40]) Let $\mathcal{C}$ be a braided monoidal category. There is an equivalence of categories*

$$L\colon \mathsf{CB}(\mathcal{C})\to\mathsf{FPS}(\mathrm{Sym}(\mathbb{C}),\mathcal{C})$$

such that for every object V in $\mathcal{C}$ there holds

$$L(T_!(V)) = (\mathbf{j}_!(\mathcal{L}(V[1])),\mu_!)\quad L(T_{!*}(V)) = (\mathbf{j}_{!*}(\mathcal{L}(V[1])),\mu_{!*})\quad L(T_*(V)) = (\mathbf{j}_*(\mathcal{L}(V[1])),\mu_*).$$

If $\mathcal{C}$ has a duality, then $\mathsf{CB}(\mathcal{C})$ inherits a duality and L intertwines the duality in $\mathsf{CB}(\mathcal{C})$ with Verdier duality.

In [20] approximation functors generalizing the quadratic approximation of a graded bialgebra quotient of $T_!(V)$ have been defined for all degrees and to all objects in $\mathsf{CB}(\mathcal{C})$. The notions of factorized perverse sheaves can be generalized as to be defined over suitable open subsets of $\mathrm{Sym}(\mathbb{C})$ in [21]. Then, the dictionary in Theorem 8.4 is expanded in [22] as to give a geometric counterpart to the approximation functors. This allows for a geometric translation of the property of a bialgebra to have relations generated up to a given degree, or to be finitely presented.

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