Esercizio 3

b)

Poiché $0 < \operatorname{arctg}(\lg^2 x) < \pi/2$, e poiché $\ell > 1$

$$(*) \quad \int_a^{+\infty} \frac{\operatorname{arctg}(\lg^2 x)}{|\beta - x|^{\ell}}\, dx \quad \text{converge} \quad \forall\, a > \beta$$

Fissiamo ora β, $\beta > 0$, $\beta \neq 1$: $\operatorname{arctg}(\lg^2 \beta) = \lambda > 0$

quindi

$$\operatorname{arctg}(\lg^2 x) > \frac{\lambda}{2} > 0 \quad \text{per } x \in (\beta - \varrho, \beta + \varrho)$$

pertanto

$$\int_{\beta-\varrho}^{\beta+\varrho} \frac{\operatorname{arctg}(\lg^2 x)}{|\beta - x|^{\ell}}\, dx > \frac{\lambda}{2} \int_{\beta-\varrho}^{\beta+\varrho} \frac{1}{|\beta - x|^{\ell}}\, dx = +\infty$$

Sia ora $\beta = 1$

Usando lo sviluppo di Taylor di $\lg x$ intorno a $x_0 = 1$ e di $\operatorname{arctg} y$ intorno a $y_0 = 0$, si ha che

$$\operatorname{arctg}(\lg^2 x) = (x-1)^2 + o(|x-1|^2)$$

$$\int_0^{+\infty} \frac{\operatorname{arctg}(\lg^2 x)}{|1-x|^{\ell}}\, dx = \int_0^{1/2} \cdot \;+\; \int_{1/2}^{3/2} \cdot \;+\; \int_{3/2}^{+\infty} \cdot \;= \mathrm{I} + \mathrm{II} + \mathrm{III}$$

I esiste finito perché $0 < \operatorname{arctg}(\lg^2 x) < \pi/2$

II esiste finito perché confrontato con $\displaystyle\int_{1/2}^{3/2} \frac{1}{|x-1|^{\ell-2}}\, dx \qquad \ell - 2 < 1$

III esiste finito per $(*)$