

$$T: V \rightarrow V \quad \dim V = n \quad \text{ENDOM}$$

$$\lambda_0 \in \text{Sp}(T)$$

$$\text{mg}_T(\lambda_0) = \dim V_{\lambda_0}$$

$$\dim (\text{Sol} ([M_T]_B^B - \lambda_0 I) \cdot \underline{v} = \underline{0})$$

$$\dim \quad n - \text{rk} \left( (M_T)_B^B - \lambda_0 I \right) \quad \begin{array}{l} \text{R-C.} \\ \text{T.O.M} \end{array}$$

$$\text{rk} \left[ (M_T)_B^B - \lambda I_n \right]_{\lambda = \lambda_0}$$

PROP  $T: V \rightarrow V$  con  $V = m$   $\lambda_0 \in Sp(T)$

$$\begin{aligned} \text{mg}(\lambda_0) &\leq \text{ma}(\lambda_0) = \bullet \text{ mult di } \lambda - \lambda_0 \text{ in } P_T(\lambda) \\ &\quad \Downarrow \text{UFF.} \\ &\bullet \text{ mult di } \lambda_0 \text{ come RADICE DI} \\ &\quad P_T(\lambda) \\ &\parallel \\ &\dim V_{\lambda_0} \\ &\parallel \\ &K \end{aligned}$$

ABBIAIMO UNA BASE  $v_1, \dots, v_k$  di  $V_{\lambda_0} \subseteq V$

$$B = v_1, \dots, v_k, w_1, \dots, w_{m-k} \text{ BASE DI } V$$

$$\det \left( (M_T)_B - \lambda I_m \right) = P_T(\lambda)$$

$$\begin{bmatrix} \overset{k}{\lambda_0 - \lambda} & & & & & \\ & \overset{m-k}{\lambda_0 - \lambda} & & & & \\ & & & A & & \\ & & & & & \\ & & & & & \\ \hline & & & & & C \end{bmatrix} \begin{bmatrix} \lambda & & & & & \\ & \ddots & & & & \\ & & \lambda & & & \\ & & & \ddots & & \\ & & & & \lambda & \\ & 0 & & & & \lambda \end{bmatrix} = \begin{bmatrix} \overset{\text{det}(J)}{\lambda_0 - \lambda} & & & & & \\ & \lambda_0 - \lambda & & & & \\ & & \ddots & & & \\ & & & \lambda_0 - \lambda & & \\ & 0 & & & & \lambda_0 - \lambda \\ \hline & & & & & C - \lambda I_{m-k} \end{bmatrix}$$

$$v_1, \dots, v_k, w_1, \dots, w_{m-k}$$

$$T(v_1) = \lambda_0 v_1 \quad \lambda_0 \Leftrightarrow v_1$$

$$(\lambda_0 - \lambda)^k \cdot \det(C - \lambda I_{m-k})$$

$$\downarrow (\lambda_0 - \lambda)^k \cdot P_T(\lambda)$$

$$\text{ma}(\lambda_0) \leq k = \text{mg}(\lambda_0)$$

$$R-L : A\underline{x} = \underline{0} \quad A \in \text{Mat}_{m \times n}(\mathbb{K})$$

$$\dim(\text{Sol}(A\underline{x} = \underline{0})) = m - \text{rk}(A)$$

$$A \xrightarrow{\text{GAUSS}} \left[ \begin{array}{c|c} I_r & A \\ \hline 0 & 0 \end{array} \right]$$

$x_1, \dots, x_r$

$$x_1 + a_{11}x_{r+1} + \dots + a_{1, m-r}x_m = 0$$

$\vdots$

$$x_r + a_{r1}$$

$$\text{Vekt. generiert in } \mathbb{K}^m (x_1, \dots, x_r, x_{r+1}, \dots, x_m) \\ + \text{ Null}$$

$$(p_1(x_{r+1}, \dots, x_m), \dots, p_r(x_{r+1}, \dots, x_m), x_{r+1}, \dots, x_m) \quad \begin{array}{l} \text{Vekt.} \\ \text{gen.} \\ \text{in} \\ \text{Sol}(A\underline{x} = \underline{0}) \end{array}$$

$$x_{r+1} (\cdot, \cdot, \cdot, \cdot, 1, 0, \dots, 0)$$

+

$$x_{r+2} (\cdot, \cdot, \cdot, \cdot, 0, 1, 0, \dots, 0)$$

+

$$x_m (\cdot, \cdot, \cdot, \cdot, 0, \dots, 0, 1)$$

↑ Spaltenmatrix aus Spg.

ABWAND R PIVOT

$$\left[ \begin{array}{c|ccc} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$x_1 + 2x_3 + x_4 + 3x_5 = 0$$

$$x_2 + x_3 + x_4 + x_5 = 0$$

$$(x_1, x_2, x_3, x_4, x_5)$$

↓ sog. Cuv

$$(-2x_3 - x_4 - 3x_5,$$

$$-x_3 - x_4 - x_5,$$

$$x_3, x_4, x_5)$$

$$= x_3 (-2, -1, 1, 0, 0)$$

+

$$x_4 (-1, -1, 0, 1, 0)$$

+

$$x_5 (-3, -1, 0, 0, 1)$$

$m-r$  Vekt. Spg

Bas. in

$$\text{Sol}(A\underline{x} = \underline{0})$$

||

$$\dim(\text{Sol}(A\underline{x} = \underline{0}))$$

||

$$m-r$$

$$\left[ \begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

Y

$$2) \quad \text{rk}([A|b]) = \text{rk}(A) = r \Rightarrow \exists \infty^{m-r} \text{ sol}$$

CLAIM:  $\underline{a} \in S_d(A \underline{x} = \underline{b})$

$$\forall \underline{d} \in \text{Sol}(A\underline{x} = \underline{b}) \quad \exists! \underline{c} \in \text{Sol}(A\underline{x} = \underline{0})$$

t.c  $\underline{d} = \underline{c} + \underline{a}$

$$\underline{d} \in \mathcal{S}(\underline{A}\underline{x} = \underline{b}) \Leftrightarrow \underline{A} \cdot \underline{d} = \underline{0} \Rightarrow \underline{A} \cdot \underline{d} - \underline{A} \cdot \underline{d} = 0$$

$$\underline{d} \in \text{Sol}(A\underline{x} = \underline{b}) \Leftrightarrow A \cdot \underline{d} = \underline{b}$$

$$\Leftarrow A(\underline{a} - \underline{d}) = \underline{0}$$

$$\underline{q} - \underline{d} \in \text{Sol}(A\underline{x} = \underline{0})$$

$$V = \text{Span}((1, 3, 4, 4), (2, 1, 1, 2)) \quad \dim V = 2$$

$$W = \text{span}((3, 4, 3, 5), (0, 5, 5, 5)) \quad \dim W = 2$$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 3 & 1 & 4 & 5 \\ 4 & 1 & 3 & 5 \\ 4 & 2 & 5 & 5 \end{bmatrix} \xrightarrow{\text{GAUSS}} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -5 & -5 & -5 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xleftarrow{\text{CVD}}$$

$\dim V + W = 3$   
 $\Downarrow$  GRASS.  
 $\dim V \cap W = 1$

$$a(3,4,3,5) + b(0,5,5,5) \quad \text{Voll von di V}$$

$$-2a - 2b = 0$$

$$a = -b$$

VETT con di  $V$   
+

CONDIZIONE DI APP  
A VVV

$$-b(3, 4, 3, 5) + b(0, 5, 5, 5) \quad \text{VETT GEN di } V_{NW}$$

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$$\begin{matrix} (-3b, b, 2b, 0) \\ v & w \end{matrix} = b \underbrace{(-3, 1, 2, 0)}_{\text{Basis of } V \cap W}$$

$$A \rightarrow \left[ \begin{array}{c|c} I & B \\ \hline 0 & S \end{array} \right]$$

$\sum x = 0$   $\leftarrow$  RELAZIONI

$$T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$(1,0)_{E_2} \rightarrow (1,1)_{E_2}$$

$$(0,1)_{E_2} \rightarrow (0,0)_{E_2}$$

$$(M_T)_{E_2}^{E_2} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\equiv T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$(x,y) \rightarrow \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\uparrow$$

$$\begin{bmatrix} T(\underline{e}_1) & T(\underline{e}_2) \\ 1 & 1 \end{bmatrix}$$

$$T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$\underline{v}_1 = (1,1) \rightarrow (2,2)$$

$$\underline{v}_2 = (1,3) \rightarrow (1,2)$$

B

$$(M_T)_{E_2}^{B} = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}$$

$T(\underline{v}_1)$   
 $\parallel$   $T(\underline{v}_2)$

$$T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$\underline{e}_1 \rightsquigarrow ?$$

$$\underline{e}_2 \rightsquigarrow ?$$

$$(M_T)_{E_2}^{E_2} = ?$$

$$(M_T)_{E_2}^{E_2} = (M_T)_{E_2}^B \cdot M_B^{E_2}$$

$$V = \text{span}(\underline{v}_1, \dots, \underline{v}_r) \subseteq \mathbb{K}^m$$

$$\left[ \begin{array}{c|c} \underline{v}_1 & \\ \vdots & \\ \underline{v}_r & \end{array} \right] \xrightarrow{\sim} \left[ \begin{array}{c|c} \hline \hline & \\ \hline 0 & 0 \\ 0 & 0 \end{array} \right] \begin{array}{l} 3 \\ r-3 \end{array}$$

BASIS DI  $V$  :  $\underline{v}_1, \underline{v}_2, \underline{v}_3$

DI  $\mathbb{K}^m$  :  $\underline{v}_1, \underline{v}_2, \underline{v}_3$  + COMPLEMENTO

PROP... [MMM17]...  $A = (M_T)_B^B$

$P_T \in K[\lambda]$ ,  $\deg P_T(\lambda) = m$   $P_T(\lambda) = (-1)^m \lambda^m$

$A = \begin{bmatrix} a_{11} & & \\ & \ddots & \\ & & a_{mm} \end{bmatrix}$ 
 $P_T(\lambda) = \det \begin{pmatrix} a_{11} - \lambda & & \\ & \ddots & \\ & & a_{mm} - \lambda \end{pmatrix}$ 
 $(a_{11} - \lambda)(a_{22} - \lambda) \dots (a_{mm} - \lambda)$ 
 $(-1)^m \lambda^m$

$(-1)^{m-1} \text{Tr}(A) \lambda^{m-1}$   
 $\lambda \cdot m-1 \text{ terms}$   
 $\lambda \cdot m-2 \text{ terms}$

$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \rightarrow \det$ 
 $\begin{bmatrix} 1-\lambda & 2 & 1 \\ 3 & 1-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{bmatrix}$

$(1-\lambda) \det \begin{bmatrix} 1-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} +$ 
 $(\lambda^3 - 3\lambda + \dots)$



$$T: V \rightarrow V \quad \dim V = n$$

$$0 \in \text{A.V.} \quad \Leftrightarrow \quad \underline{v} \in V_0 \Leftrightarrow T(\underline{v}) = 0 \cdot \underline{v} = \underline{0}$$

$$\Leftrightarrow T(\underline{v}) = \underline{0}$$

$$\Leftrightarrow \underline{v} \in \ker T$$

$$0 \in \text{A.V.} \Leftrightarrow P_T(\lambda) = \lambda^p f(\lambda) = \lambda^p \cdot \lambda^m + \dots \lambda^{m-p}$$

$$0 \in \text{RADIKAL DI } P_T(\lambda)$$

TEU COMPLETAMENTO

$V$   $K$ -sp

$u_1, \dots, u_p$  LIN. IND.  
" "

$B = u_1, \dots, u_m$  BASE DI  $V$   
" "

$$\begin{bmatrix} - & \frac{u_1}{P_B} & - \\ & \vdots & \\ - & \frac{u_p}{P_B} & - \\ 2, 0, \dots, 0 \\ & \vdots & \\ 0, 0, \dots, 0, 1 \end{bmatrix}$$

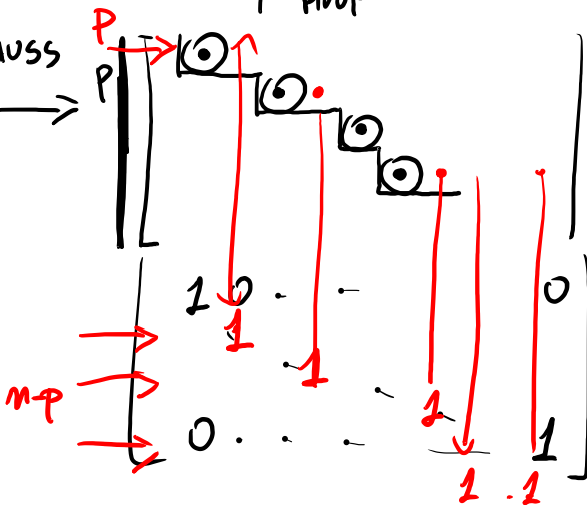
$P$

GAUSS

$n-p$

$(1, 0, \dots, 0)_B, \dots, (0, \dots, 0, 1)_B$

$P$  PIVOT



$$\exists x \quad V_1, V_2 \subseteq W \text{ K-sp}$$

$$V_1 \cup V_2 \subseteq W \Leftrightarrow V_1 \supseteq V_2 \text{ or } V_2 \subseteq V_1$$

$$\text{"}\Leftarrow\text{" omno } V_1 \cup V_2 = V_1 \cup V_2$$

SMO OM' SSP  
DI W

" $\Rightarrow$ "

$$\exists v_1 \in V_1 \text{ t.c. } v_1 \notin V_2$$

$$\exists v_2 \in V_2 \text{ t.c. } v_2 \notin V_1$$

$$\Downarrow \text{ VOGLIO } V_1 \cup V_2 \text{ NON } \in \text{ SSP}$$

$$v_1 + v_2 \in V_1 \cup V_2 \Rightarrow v_1 + v_2 \in V_1 \text{ or } v_1 + v_2 \in V_2$$

$$\Downarrow \\ v_2 \in V_1 \\ \text{ASS}$$

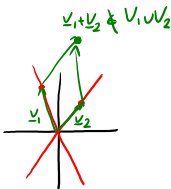
$$\Downarrow \\ v_1 \in V_2 \\ \text{ASS}$$

$$v_1 + v_2 \in V_1 \Leftrightarrow \exists v \in V_1 \text{ t.c.}$$

$$\begin{matrix} v_1 + v_2 = v \\ \Downarrow \\ v_1 \end{matrix} \quad \begin{matrix} \Downarrow \\ v_1 \end{matrix}$$

$$v_2 = v - v_1 \in V_1$$

$$\Downarrow \\ v_2 \in V_1$$



Ex  $M = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}_q \Rightarrow \det(M) = \det(A) \cdot \det(C)$

$M, A, C$  QUADRATE  $p+q=n$

e  $B$  DI DIM. OPPORTUNE

$$M = \left[ \begin{array}{c|c} \begin{matrix} a_{11} & \dots & a_{1p} \\ \vdots & & \vdots \\ a_{p1} & \dots & a_{pp} \end{matrix} & \begin{matrix} B \\ B \end{matrix} \\ \hline 0 & \begin{matrix} c_{11} & \dots & c_{1q} \\ \vdots & & \vdots \\ c_{q1} & \dots & c_{qq} \end{matrix} \end{array} \right] \xrightarrow{\text{GAUSS}}$$

$$\left[ \begin{array}{c|c} \begin{matrix} a_{11} & \dots & a_{1p} \\ \vdots & & \vdots \\ a_{p1} & \dots & a_{pp} \end{matrix} & \begin{matrix} B' \\ B' \end{matrix} \\ \hline 0 & \begin{matrix} c_{11} & \dots & c_{1q} \\ \vdots & & \vdots \\ c_{q1} & \dots & c_{qq} \end{matrix} \end{array} \right]$$