Basic Math

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Capitolo 1

First Lesson

1.1 Notations

1. The set of natural numbers (positive integers) is denoted by $\mathbb{N}$. The elements of $\mathbb{N}$ are $0, 1, 2, \cdots$ etc..
2. The set of integer numbers is denoted by $\mathbb{Z}$. The elements of $\mathbb{Z}$ are $0, 1, -1, -2, -2 \cdots$ etc..
3. The set of rationals (numeric fractions) is denoted by $\mathbb{Q}$. The elements of $\mathbb{Q}$ are $-3, 0, 2, \frac{4}{5}$ etc..
4. The set of reals is denoted by $\mathbb{R}$. Elements of R are $-3, 0, 2, \frac{4}{5}, \sqrt{2}, \pi$ etc..
5. Note that all these sets are nested $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$
6. The set of polynomials over $\mathbb{R}$ (with real coefficients) is denoted by $\mathbb{R}[x]$. Elements of $\mathbb{R}[x]$ are $\pi x + 1, x^3 - \frac{2}{5}x^2 + 1, x^2 - \sqrt{2}$ etc..
7. The sets $\mathbb{N}[x], \mathbb{Z}[x], \mathbb{Q}[x]$ are defined similarly.
8. Note that these sets are nested $\mathbb{N}[x] \subset \mathbb{Z}[x] \subset \mathbb{Q}[x] \subset \mathbb{R}[x]$.

Definition 1. *Let $n \in \mathbb{N}$. The set of divisors of n, denoted* $\mathrm{DIV}(n)$*, is the set of all natural numbers that divide n. For example,*

$$\mathrm{DIV}(12) = \{1, 2, 3, 4, 6, 12\}.$$

1.2 Theorems and Propositions

Proposition 2. *For all $a, b \in \mathbb{R}[x]$, the following hold:*

- $a^2 - b^2 = (a + b)(a - b)$.

- $a^3 - b^3 = (a-b)(a^2+ab+b^2)$.
- $a^3 + b^3 = (a+b)(a^2-ab+b^2)$.
- a^2+b^2 *cannot be factored further over* $\mathbb{R}$.

Theorem 3 (Ruffini's Theorem)**.** *Let* $f(x) \in \mathbb{R}[x]$ *and* $a \in \mathbb{R}$ *such that* $f(a) = 0$ *(i.e.,* a *is a root of* $f(x)$*). Then,*

$$(x-a) \mid f(x).$$

Proposition 4. *Let* $f(x), g(x), h(x) \in \mathbb{R}[x]$ *such that* $f(x) \mid h(x)$*,* $g(x) \mid h(x)$*, and* $f(x), g(x)$ *are coprime. Then,*

$$f(x)g(x) \mid h(x).$$

Theorem 5 (Rational Root Theorem)**.** *Let* $f(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_0 \in \mathbb{Z}[x]$ *and* $a \in \mathbb{Q}$ *such that* $f(a) = 0$*. Then,*

$$a \in \left\{ \pm\frac{p}{q} \mid p \in \mathrm{DIV}(a_0),\ q \in \mathrm{DIV}(a_d) \right\}.$$

1.3 Exercises

1. $P(x) = x^5 + x^3 + x^2 + 1 = 0$. Note that $P(-1) = 0$. Use polynomial division.
   - Factorization $x^5 + x^3 + x^2 + 1 = (x^2+1)(x^3+1) = (x+1)(x^2+1)(x^2-x+1)$.
   - The solution is $x = -1$.
2. $P(x) = x^5 - 9x^3 - 8x^2 + 72 = 0$. Note that $P(\pm 3) = P(2) = 0$. Use division.
   - Factorization $x^5 - 9x^3 - 8x^2 + 72 = (x+3)(x-3)(x-2)(x^2+2x+4)$.
   - The solutions are $x = \pm 3, 2$.
3. $P(x) = x^4 - 3x^3 + 2x - 6 = 0$. Note that $P(3) = 0$. Use division.
   - Factorization $x^4 - 3x^3 + 2x - 6 = (x-3)(x^3+2)$.
   - The solutions are $x = 3,\ \pm\sqrt[3]{-2}$.

1.4 Proposed exercises

Solutions will be given in the next installment of these notes. Regrouping will be more difficult here. Use of the root rule and Ruffini is recommended.

1. For the parameter $a \in \mathbb{R}$, $P(x) = x^3 - ax^2 - 2x + 2a = 0$. Note that $P(a) = 0$. Use division.
   - Factorization $x^3 - ax^2 - 2x + 2a = (x-a)(x^2-2)$.
   - The solutions are $x = a,\ \pm\sqrt{2}$.

2. For the parameter $a \in \mathbb{R}$, $P(x) = x^3 - ax^2 - 2x + 2a = 0$. Note that $P(a) = 0$. Use division.
   - Factorization $x^3 - ax^2 - ax + a^2 = (x-a)(x^2-a)$.
   - The solutions are $x = a$ always, if $a \geq 0$ also $\pm\sqrt{a}$.
3. $x^{32} - 1 = 0$
4. $x^8 - 4 = 0$
5. $x^3 + (1-a)x^2 - (a+6)x + 6a = 0$
6. $2x^3 - 17x^2 + 38x - 15 = 0$
7. $3x^4 - 22x^3 - 2x^2 + 66x - 21 = 0$
8. $x^3 - 17x^2 + 92x - 160 = 0$
9. $x^6 - 12x^4 + 47x^2 - 60 = 0$
10. $x^3 - xa^2 - 2xab - xb^2 - x^2 + a^2 + 2ab + b^2 = 0$
11. $x^3 + 3x^2y + 3xy^2 + y^3 - 2x^2 - 4xy - 2y^2 - x - y + 2 = 0$. Hint: try to detect powers.

1.5 Greater Common Divisor - $\mathbb{N}$

Definition 6. *If $a, b \in \mathbb{N}$, the greater common divisor of a, b is the biggest $p \in \mathbb{N}$ such that $p|a$ and $p|b$. Since $1|a$ and $1|b$, if there are no other common divisor,* $\gcd(a,b) = 1$.

Remark 7. *If $a, b \in \mathbb{N}$ a divides $b \Leftrightarrow$ exists $c \in \mathbb{N}$ such that $b = c \cdot a$. We write*

$$a|b \Leftrightarrow \exists\ c \in \mathbb{N} \text{ such that } b = c \cdot a$$

Since for every $a \in \mathbb{N}$ $0 = 0 \cdot a$, we have that 0 is divisible by any natural number. Hence, $\gcd(a,0) = a$

Computing GCD's using factorizations.

Proposition 8. *If we have $a, b \in \mathbb{N}$ and their prime factorization*

$$a = p_1^{\alpha_1} \cdots p_n^{\alpha_n} q_1^{\gamma_1} \cdots q_m^{\gamma_m} \quad \text{and} \quad a = p_1^{\beta_1} \cdots p_n^{\beta_n} s_1^{\theta_1} \cdots s_t^{\theta_t}$$

(the p_i are the common prime factors) then

$$\gcd(a,b) = p_1^{\min(\alpha_1,\beta_1)} \cdots p_n^{\min(\alpha_1,\beta_1)}$$

We can say that the greatest common divisor of a and b, if their prime factorizations are known, is the product of the common prime factors, taken with the minimum exponent.

Example 9.

1. *Since* $600 = 2^3 \cdot 3 \cdot 5^2$ *and* $252 = 2^2 \cdot 3^2 \cdot 7$ *we have that* $\gcd(600, 252) = 2^2 \cdot 3 = 12$.

2. *Since* $70 = 2 \cdot 5 \cdot 7$ *and* $429 = 3 \cdot 11 \cdot 3$ *we have that* $\gcd(70, 429) = 1$.

The greatest common divisor has the following properties

Proposition 10. *If* $a, b, c \in \mathbb{N}$

1. $\gcd(a, b) = \gcd(b, a)$.

2. $\gcd(a, a) = a$.

3. $\gcd(a, 0) = a$.

4. $\gcd(0, 0)$ *is undefined. Why?*

5. $\gcd(ac, bc) = c \gcd(a, b)$.

6. *If* $a = cb+r$, *with* r *the remainder of the division of* a *by* b *we have* $\gcd(a, b) = \gcd(cb+r, b) = \gcd(r, b) = \gcd(b, r)$.

Computing GCD's using Euclid's Algorithm.

Example 11. *Using the rule* $\boxed{\gcd(a, b) = \gcd(b, r)}$ *with* r *the remainder of* a *divided by* b. *We compute some* gcd *using the* `EuclidVerbose` *procedure of CoCoA.*

1. `EuclidVerbose(15,12);`
```
[15, 12]
[12, 3]
[3, 0]
GCD(15,12)=3
3
```

2. `EuclidVerbose(2343,432);`
```
[2343, 432]
[432, 183]
[183, 66]
[66, 51]
[51, 15]
[15, 6]
[6, 3]
[3, 0]
GCD(2343,432)=3
3
```

3. `EuclidVerbose(347,237);`
```
[347, 237]
[237, 110]
[110, 17]
[17, 8]
[8, 1]
```

```
[1, 0]
GCD(347,237)=1
1
```

Definition 12. *If* $a, b \in \mathbb{N}$ *and* $\gcd(a, b) = 1$ *we say that* a, b *are* coprime*. Coprime natural numbers have no common divisors other than* 1*. A prime number* p *is coprime with every natural number except its multiples, i.e., numbers of the form* p^n*.*

Remark 13. *We remark that if* $c \in \mathbb{N}$ *is coprime with* $b \in \mathbb{N}$ *then* $\gcd(ac, b) = \gcd(a, b)$*. We can discard coprime factors.*

1. $\gcd(32 \cdot 5, 27) = \gcd(32, 27)$ *since* $5, 27$ *are coprime.*

1.6 Greater Common Divisor - Polynomials

Definition 14. *An polynomial* $p(x) \in \mathbb{R}[x]$ *is* irreducible *if there is no other polynomial* $f(x) \in \mathbb{R}[x]$ *of degree bigger or equal to* 1 *that divides* $p(x)$*. A polynomial is* reducible *if it is not irreducible. itself. Irreducible polynomials play the role of prime numbers.*

Example 15.

1. *All degree one polynomials are irreducible.*
2. *A polynomial* $ax^2 + bx + c$ *is irreducible if and only if* $\Delta = b^2 - 4ac > 0$*.*
3. $x^2 + x + 1$ *is irreducible since* $\Delta = 1 - 4 < 0$*.*
4. $x^2 + 1$ *is irreducible since* $\Delta = 0 - 4 < 0.$*.*
5. *The polynomial* x^2-5x+6 *is reducible because* $(x-2)|(x^2-5x+6)$*. Also* $\Delta = 25-24 = 1 > 0$*.*
6. *The polynomial* $4x^2-12x+9$ *is reducible because* $(2x-3)|(4x^2-12x+9)$*. Also* $\Delta = 144-144 = 0$*.*
7. *The polynomial* $p(x) = x^4-3x^3+5x^2-9x+6$ *is reducible since* $p(x) = (x-1)(x-2)(x^2+3)$*.*
8. *Find if a polynomial of degree* ≥ 3 *is reducible or not can be quite difficult.*

Remark 16. *All the properties of the GCD over the natural numbers hold for the polynomials. Moreover by convention, the GCD of polynomials is defined not taking into consideration purely numeric factors. Hence, we can take out of the computations any pure number, not only coprime factors.*

$$F(x), G(x) \in \mathbb{R}[x],\ a \in \mathbb{R}, \quad \gcd(F(x), aG(x)) = \gcd(F(x), G(x))$$

We have $\gcd(2x^2, 4x) = x$*, and*

$$\begin{aligned}
\gcd((x-2)(3x-3), x^2-1) &= \gcd(3(x-2)(x-1), x^2-1) \\
&= \gcd((x-2)(x-1), x^2-1) \\
&= \gcd((x-2)(x-1), (x+1)(x-1) \\
&= x-1
\end{aligned}$$

We can compute Polynomial GCD's easily if we know the irreducible factorization, at least of one factor

Example 17. *The polynomials $x-2, x-7, x-1$ are irreducible since they have degree one. The polynomial x^2+2 is irreducible since it has negative Δ.*

$$\gcd((x-2)^2(x^2+2)^3(x-7), (x-2)(x^2+2)^4(x-1)) = (x-2)^{\min(2,1)}(x^2+2)^{\min(3,4)} = (x-2)(x^2+2)^3$$

Example 18. *We have to compute $\gcd(x^4+x-7, x^2-1)$. We define $p(x) = x^4+x-7$ and we note that irreducible factorization $x^2-1 = (x+1)(x-1)$, so*

$$\gcd(x^4+x-7, x^2-1) = \gcd(x^4+x-7, (x+1)(x-1))$$

the GCD has to have the common factors, but there are none, since

$$p(1) = -5 \Rightarrow (x-1) \nmid p(x) \quad \textit{and} \quad p(-1) = -7 \Rightarrow (x+1) \nmid p(x)$$

Hence, $\gcd(x^4+x-7, x^2-1) = 1$ and $x^4+x-7,\ x^2-1$ are coprime.

We can use Euclid's Algorithm for the GCD in the polynomial case, using polynomial divisions.

The computations are done using the `GCDPolyVerbose` command of the CoCoA system. The remainder sequence for $f(x),\ g(x)$ is given by $f(x),\ g(x)$ and the remainders, if suitable regrouping and taking out numeric factors

Example 19.

```
GCD(x^2+x+1,x^2+2)=
We divide x^2+x+1 by x^2+2, the remainder is x - 1
(x^2 + x + 1)=(1)*(x^2 + 2)+(x - 1)

=GCD(x^2+2,x - 1)=
(x^2 + 2)=(x + 1)*(x - 1)+(3)
We divide x^2+2 by x-1, the remainder is 3

=GCD(x - 1,3)=GCD(x - 1,1) (we took out the number 3)

=GCD(x - 1,1)=1 there is no common factor

The remainder sequence is x^2 + x + 1, x^2 + 2, x - 1, 1
```

Example 20.

```
GCD(x^4+x^3-1,x^3+x-2)=
(x^4 + x^3 - 1)=(x + 1)*(x^3 + x - 2)+(-x^2 + x + 1)
We divide x^4+x^3-1 by x^3+x-2 the remainder is -x^2 + x + 1

=GCD(x^3+x-2,-x^2 + x + 1)=
```

```
(x^3 + x - 2)=(-x - 1)*(-x^2 + x + 1)+(3x - 1)
We divide x^3+x-2 by -x^2 + x + 1 the remainder is 3x - 1

=GCD(-x^2 + x + 1,3x - 1)=
(-x^2 + x + 1)=(-1/3x + 2/9)*(3x - 1)+(11/9)
We divide -x^2 + x + 1 by 3x - 1 the remainder is 11/9

=GCD(3x - 1,11/9)=GCD(3x - 1,1)=1 We took out the numeric factor 11/9

The remainder sequence is x^4 + x^3 - 1, x^3 + x - 2, -x^2 + x + 1, 3x - 1, 1
```

Example 21.

```
GCD(x^4 - 6x^3 + 7x^2 + 12x - 18,x^3 + x^2 - 2x - 2)=
(x^4 - 6x^3 + 7x^2 + 12x - 18)=(x - 7)*(x^3 + x^2 - 2x - 2)+(16x^2 - 32)
We divide x^4 - 6x^3 + 7x^2 + 12x - 18 by x^3 + x^2 - 2x - 2,
the remainder is 16x^2 - 32=16(x^2-2)
We take out the number 16, the remainder is now x^2-2

=GCD(x^3 + x^2 - 2x - 2,x^2-2)=
(3x^3 + 2x^2 - 6x - 4)=(-3x - 2)*(-x^2 + 2)+(0)
We divide x^3 + x^2 - 2x - 2 by x^2-2, the remainder is 0

=GCD(x^2-2,0)=x^2-2

The remainder sequence is x^4 - 6x^3 + 7x^2 + 12x - 18, x^3 + x^2 - 2x - 2, x^2 - 2
```

Capitolo 2

Second Lesson

2.1 Derivative for polynomials

Definition 22. *If* $f(x) = a_n x^n + a_{n-1}a^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{R}[x]$ *the* derivative *of* $f(x)$ *or* $D(f(x))$ *or* $f'(x)$ *is*

$$f'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1 + 0$$

Note that since for $a \in \mathbb{R}$, $a = a \cdot x^0$, $(a)' = 0$

Example 23.

$$(x^5 - 3x^2 + x - 2)' = 5x^4 - 6x + 1$$

$$(3x^7 - 3x^5 + x^3 + 1)' = 21x^5 - 15x^4 + 3x^2$$

Proposition 24. *It is easy to prove that, for* $f(x), g(x) \in \mathbb{R}[x]$

1. $(f(x) + g(x))' = f'(x) + g'(x)$.
2. $(f(x) \cdot g(x))' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$.
3. $(f(x)^n)' = n \cdot f(x)^{n-1} \cdot f'(x)$.

Example 25.

$$\begin{aligned} ((3x^2-2)(x^2+x+1))' &= (3x^2-2)'(x^2+x+1) + (3x^2-2)(x^2+x+1)' \\ &= (6x)(x^2+x+1) + (3x^2-2)(2x+1) \\ &= 12x^3 + 9x^2 + 2x - 2 \end{aligned}$$

If we do the computations and then derive, we have

$$((3x^2-2)(x^2+x+1))' = (3x^4 + 3x^3 + x^2 - 2x - 2)' = 12x^3 + 9x^2 + 2x - 2$$

Example 26.

$$\begin{aligned}((x^2+2x-1)^3)' &= 3(x^2+2x-1)^2(x^2+2x-1)' \\ &= 3(x^2+2x-1)^2(2x+2) \\ &= 6x^5+30x^4+36x^3-12x^2-18x+6\end{aligned}$$

If we do the computations and then derive, we have

$$((x^2+2x-1)^3)' = (x^6+6x^5+9x^4-4x^3-9x^2+6x-1)' = 6x^5+30x^4+36x^3-12x^2-18x+6$$

2.2 Multiple factors

Proposition 27. *If $f(x) \in \mathbb{R}[x]$ and $\gcd(f(x), f'(x)) = p(x)$ then $p(x) \mid f(x)$. If $p(x)$ is irreducible, then $p(x)^2 \mid f(x)$.*

Definition 28. *If $\gcd(f(x), f'(x)) = 1$ we say that $f(x)$ is* squarefree*, e.g. there are no multiple factors in its irreducible factorization.*

Remark 29. *If $f(x) \in \mathbb{R}[x]$, we have that $\frac{f(x)}{\gcd(f(x),f'(x))}$ is squarefree.*

Example 30. *Let $p(x) = x^4-4x^3+6x^2-8x+8$; We have that (do the GCD as an exercise)*

$$\gcd(p(x), p'(x)) = x-2$$

That means that, since $x-2$ is irreducible, $(x-2)^2 \mid (x^4-4x^3+6x^2-8x+8)$. Indeed, we have

$$\frac{x^4-4x^3+6x^2-8x+8}{x^2-4x+4} = x^2+2$$

Hence, $x^4-4x^3+6x^2-8x+8 = (x^2+2)(x-2)^2$.

Example 31. *Let $p(x) = x^6+8x^5+2x^4-36x^3+x^2+52x-28$; We have that (do the GCD as an exercise)*

$$\gcd(p(x), p'(x)) = x^3-3x+2$$

That means that $q(x) = x^3-3x+2$ divides $x^6+8x^5+2x^4-36x^3+x^2+52x-28$. Since $q(1) = 0$, $(x-1) \mid q(x)$ and $q(x)$ is not irreducible. We have

$$\frac{x^6+8x^5+2x^4-36x^3+x^2+52x-28}{x^3-3x+2} = x^3+8x^2+5x-14$$

Hence

$$x^6+8x^5+2x^4-36x^3+x^2+52x-28 = (x^3-3x+2)(x^3+8x^2+5x-14)$$

We can further factorize x^3-3x+2 and $x^3+8x^2+5x-14$ using this rule or Ruffini's Rule, and in the end we get

$$x^6+8x^5+2x^4-36x^3+x^2+52x-28 = (x-1)^3(x+2)^2(x+7)$$

Note that using the GCD and derivative rule on $x^3+8x^2+5x-14$ is useless, since it is squarefree by Proposition 29.

2.3 Number of real roots for a polynomial

Proposition 32 (Sturm Algorithm)**.** *Let $f(x) \in \mathbb{R}[x]$. We apply the following modifications to Euclid's Algorithm in the computations of* $\gcd(f(x), f'(x))$

- *We plug back into the computations not the remainder but the remainder with changed sign*
- *We don't take out negative numeric factors, but only positive ones*

The remainder sequence is $f(x), f'(x), r_1(x), \ldots, r_n(x)$. We use the convention that the sign of a polynomial is the sign of the coefficient of its largest degree term. Then let

- *p is number of sign variation in the sequence $f(x), f'(x), r_1(x), \ldots, r_n(x)$.*
- *q is number of sign variation in the sequence $f(x), f'(x), r_1(x), \ldots, r_n(x)$* changing the sign of the polynomials with odd degree.

Then the number of real roots of $f(x)$ is $q - p$.

Remark 33. *When we compute the number of sign variations in a sequence, we skip a 0 as non-existent. For example*

- *the number of sign variations of* $[+ + 0 + +]$ *is* 0*;*
- *the number of sign variations of* $[+ + 0 - +]$ *is* 2*;*

The examples are computed using the CoCoA procedure `GCDPolySTURMVerbose`.

Example 34. *We want to know the number of real roots of the polynomial $x^3 + x^2 + x + 1$.*

```
GCDSTURM(x^3+x^2+x+1,3x^2 + 2x + 1)=
(x^3 + x^2 + x + 1)=(1/3x + 1/9)*(3x^2 + 2x + 1)+(4/9x + 8/9)
We divide x^3+x^2+x+1 by 3x^2 + 2x + 1, remainder is 4/9x + 8/9=4/9(x+2)
We take out 4/9 and change sign to the remainder

=GCDSTURM(3x^2 + 2x + 1,-x - 2)=
(3x^2 + 2x + 1)=(-3x + 4)*(-x - 2)+(9)
We divide x^2 + 2x + 1 by -x - 2, remainder is 9=9*1
We take out 9 and change sign to the remainder

=GCDSTURM(-x - 2,-1)=1

The GCD is 1
The remainder sequence is = x^3 + x^2 + x + 1,
                            3x^2 + 2x + 1,
                            -x - 2,
                            -1
p= # sign change of + + - -   =1
```

```
q= # sign change of - + + -   =2
   we changed the sign of x^3+... from + to - because its degree is 3, odd
   we changed the sign of -x-2 from - to + because its degree is 1, odd
Number of real roots 2-1=1
```

Example 35. *We want to know the number of real roots of the polynomial* $x^4 - 10x^3 + 10$.

```
GCDSTURM(x^4-10x^3+10,2x^3 - 15x^2)=
(x^4 - 10x^3 + 10)=(1/2x - 5/4)*(2x^3 - 15x^2)+(-75/4x^2 + 10)
We divide x^4 - 10x^3 + 10 by 2x^3 - 15x^2, remainder is -75/4x^2 + 10=5/4(-15x^2 + 8)
We take out 5/4 and change sign to the remainder

=GCDSTURM(2x^3 - 15x^2,15x^2 - 8)=
(2x^3 - 15x^2)=(2/15x - 1)*(15x^2 - 8)+(16/15x - 8)
We divide 2x^3 - 15x^2 by 15x^2 - 8, remainder is 16/15x - 8=8/15(2x - 15)
We take out 8/15 and change sign to the remainder

=GCDSTURM(15x^2 - 8,-2x + 15)=
(15x^2 - 8)=(-15/2x - 225/4)*(-2x + 15)+(3343/4)
We divide 15x^2 - 8 by -2x + 15, remainder is 3343/4 =3343/4*1
We take out 3343/4 and change sign to the remainder

=GCDSTURM(-2x + 15,-1)=1

The GCD is 1

The remainder sequence is = x^4 - 10x^3 + 10,
                            4x^3 - 30x^2,
                            15x^2 - 8,
                            -2x + 15,
                            -1
p= # sign change of + + + - -   =1
q= # sign change of + - + - -   =3
   we changed the sign of 4x^3+... from + to - because its degree is 3, odd
   we changed the sign of -2x+15 from - to + because its degree is 1, odd
Number of real roots 3-1=2
```

Example 36. *We want to know the number of real roots of the polynomial* $3x^4+10x^3+13x^2+8x+2$.

```
GCDSTURM(3x^4 + 10x^3 + 13x^2 + 8x + 2,6x^3 + 15x^2 + 13x + 4)=
(3x^4 + 10x^3 + 13x^2 + 8x + 2)=(1/2x + 5/12)*(6x^3 + 15x^2 + 13x + 4)+(1/4x^2 + 7/12x + 1/3)
We divide 3x^4 + 10x^3 + 13x^2 + 8x + 2 by 6x^3 + 15x^2 + 13x + 4,
remainder is 1/4x^2 + 7/12x + 1/3=1/12(3x^2 + 7x + 4)
We take out 1/12 and change sign to the remainder

=GCDSTURM(6x^3 + 15x^2 + 13x + 4,-3x^2 - 7x - 4)=
(6x^3 + 15x^2 + 13x + 4)=(-2x - 1/3)*(-3x^2 - 7x - 4)+(8/3x + 8/3)
```

```
We divide 6x^3 + 15x^2 + 13x + 4 by -3x^2 - 7x - 4, remainder is 8/3x + 8/3=8/3(x+1)
We take out 8/3 and change sign to the remainder


=GCDSTURM(-3x^2 - 7x - 4,-x-1)=
(-3x^2 - 7x - 4)=(3x + 4)*(-x - 1)+(0)
We divide -3x^2 - 7x - 4 by -x-1, remainder is 0


=GCDSTURM(-x-1,0)=-x-1

The GCD is x+1 (we can change the sign at will, just NOT in the sequence)

The remainder sequence is = 3x^4 + 10x^3 + 13x^2 + 8x + 2,
                          12x^3 + 30x^2 + 26x + 8,
                          -3x^2 - 7x - 4,
                          -x - 1
p= # sign change of + + - -   =1
q= # sign change of + - - +   =2
   we changed the sign of 12x^3+... from + to - because its degree is 3, odd
   we changed the sign of -x-1 from - to + because its degree is 1, odd
Number of real roots 2-1=1
```

2.4 Exercises

We want to know the number of real roots of

1. $p_1(x) = x^3 + x + 3$ [Answer:1]
2. $p_2(x) = 2x^3 - x^2 - 2$ [Answer:1]
3. $p_3(x) = x^4 - x - 2$ [Answer:2]
4. $p_4(x) = x^4 - x^3 - 1$ [Answer:2]
5. $p_5(x) = x^5 - x - 1$ [Answer:1]
6. $p_6(x) = x^{16} - 1$ [Answer:2]
7. $p_6(x) = x^{17} - 1$ [Answer:1]

2.5 Some more exercises

Compute the following GCD's with the minimum of computations:

1. $\gcd(x^9 + x^7 - x, x^2 - 1)$ [Solution 1]

2. $\gcd(x^8-1, x^{16}-1)$ [Solution x^8-1]

Compute the following derivatives:

1. $((x^4-2)(x^2-1))'$ [Solution $6x^5-4x^3-4x$]
2. $((x^4+3x-2)(x^2+x+1))'$ [Solution $6x^5+5x^4+4x^3+9x^2+2x+1$]
3. $((x^3-1)^2)'$ =[Solution $6x^5-6x^2$]
4. $((3x^5-x^2)^3)'$ =[Solution $405x^{14}-324x^{11}+81x^8-6x^5$]

Find the real roots of the following polynomials:

1. $x^7+6x^5-2x^4+9x^3-12x^2-18$ [Solution $x=\sqrt[3]{2}$]
2. $x^7-4x^5-5x^4+4x^3+20x^2-20$ [Solution $x=\pm\sqrt{2}, \sqrt[3]{5}$]

2.6 Example of poly GCD computations

Example 37 (First Example). *We want to determine the number of roots for $f(x)=x^3-6x-1$.*

```
F:= x^3 - 6x - 1;
GCDPolySTURMVerbose(F,3x^2 - 6);


GCD(x^3 - 6x - 1,3x^2 - 6)=GCD(x^3 - 6x - 1,x^2 - 2)
DivPoly(x^3 - 6x - 1,x^2 - 2);

Passo 1 I have        x^3 - 6x - 1 moltiplico per x
Passo 1 I subtract x^3 - 2x
Passo 1 get       -4x - 1

Resto=-4x - 1
Quoto=x

(x^3 - 6x - 1)=(x)*(x^2 - 2)+(-4x - 1)

Change the remainder's sign

GCD(x^2 - 2,4x + 1)
DivPoly(x^2 - 2,4x + 1);

Passo 1 ho        x^2 - 2 moltiplico per 1/4x
Passo 1 sottraggo x^2 + 1/4x
Passo 1 ottengo       -1/4x - 2
```

```
Passo 2 ho        -1/4x - 2 moltiplico per -1/16
Passo 2 sottraggo -1/4x - 1/16
Passo 2 ottengo         -31/16

Resto=-31/16
Quoto=1/4x - 1/16

Change the remainder's sign

GCD(4x + 1,31/16)=GCD(4x + 1,1)=1

List of pairs  [[x^3 - 6x - 1, 3x^2 - 6], [x^2 - 2, 4x + 1], [4x + 1, 1]]
remainder Sequence = [x^3 - 6x - 1, 3x^2 - 6, 4x + 1, 1]

p=++++ 0
q=-+-+ 3
#roots=3
-------------------------------
```

Exercise 38 (Second Example)**.** *We want to determine the number of roots of* $f(x) = x^3 - 3x^2 + 3$*.*

```
F:= x^3-3x^2+3;
GCDPolySTURMVerbose(F,3x^2 - 6x);

GCD(x^3-3x^2+3,3x^2 - 6x)=GCD(x^3-3x^2+3,x^2 - 2x)
-- we can see the GCD will be 1, but we need the remainder sequence

DivPoly(x^3-3x^2+3,x^2 - 2x);

DivPoly(x^3-3x^2+3,x^2 - 2x);
Passo 1 ho        x^3 - 3x^2 + 3 moltiplico per x
Passo 1 sottraggo x^3 - 2x^2
Passo 1 ottengo       -x^2 + 3

Passo 2 I have        -x^2 + 3 moltiplico per -1
Passo 2 I subtract   -x^2 + 2x
Passo 2 I get        -2x + 3

Resto=-2x + 3
Quoto=x - 1

(x^3 - 3x^2 + 3)=(x - 1)*(x^2 - 2x)+(-2x + 3)
Change the remainder's sign

GCD(x^2 - 2x,2x - 3)
```

```
DivPoly(x^2 - 2x,2x - 3);

Passo 1 I have        x^2 - 2x moltiplico per 1/2x
Passo 1 I subtract x^2 - 3/2x
Passo 1 I get       -1/2x

Passo 2 I have        -1/2x moltiplico per -1/4
Passo 2 I subtract -1/2x + 3/4
Passo 2 I get         -3/4

Resto=-3/4
Quoto=1/2x - 1/4

(x^2 - 2x)=(1/2x - 1/4)*(2x - 3)+(-3/4)
Change the remainder's sign

GCD(2x - 3,x^2 - 2x,3/4) = GCD(2x - 3,x^2 - 2x,1)=1

(2x - 3)=(2x - 3)*(1)+(0)
List of pairs  [[x^3 - 3x^2 + 3, 3x^2 - 6x], [x^2 - 2x, 2x - 3], [2x - 3, 1]]
remainder Sequence = [x^3 - 3x^2 + 3, 3x^2 - 6x, 2x - 3, 1]
p=++++
q=+-+-
Numero radici 3
```

2.7 Root intervals

Proposition 39. *Let* $f(x) = a_d x^d + \cdots + a_1 x + a_0 \in \mathbb{R}[x]$ *. Then*

$$| x_0 |> d\frac{\max(| a_d |, \ldots, | a_0 |)}{| a_d |} = C \Rightarrow f(x_0) \neq 0$$

This means that all the roots lie within interval $(-C,\ C)$

Example 40. *Consider the polynomial* $f(x) = 5x^5 + 2x^3 - 6$*. Here* $d = 5$ *and*

$$\max(| 5 |, | 2 |, | -6 |) = 6$$

Then the real roots of the polynomial $f(x)$ *lie in the interval*

$$\left(-5 \cdot \frac{6}{5}, 5 \cdot \frac{6}{5}\right) = (-6, 6)$$

Example 41. *Let* $f(x) = 7x^6 - 12x^3 - 5$ *and hence* $d = 6$ *and* $\max(| 7 |, | -12 |, | -5 |) = 12$. *Then the real roots of the polynomial* $f(x)$ *lie in the interval*

$$\left(-6 \cdot \frac{12}{7}, 6 \cdot \frac{12}{7}\right) = \left(-\frac{72}{7}, \frac{72}{7}\right)$$

Since $\frac{72}{7} \simeq 10.28$*, to simplify the computations, we can choose a slighty larger interval,* $(-11, 11)$.

Proposition 42 (Sturm Algorithm for intervals)**.** *Let* $f(x) \in \mathbb{R}[x]$*, and* $f(x), f'(x), r_1(x), \ldots, r_n(x)$ *the sequence of remainders produced by the Sturm algorithm. Let* $a, b \in \mathbb{R}$*. Let*

- V_a *be the number of sign variation in the sequence* $f(a), f'(a), r_1(a), \ldots, r_n(a)$.
- V_b *be the number of sign variation in the sequence* $f(b), f'(b), r_1(b), \ldots, r_n(b)$.

Then the number of roots of $f(x)$ *in the interval* (a, b) *is* $\mid V_b - V_a \mid$.

Remark 43. *When we compute the number of sign variations in a sequence, we skip a* 0 *as non-existent. For example*

- *the number of sign variations of* $[+ + 0 + +]$ *is* 0*;*
- *the number of sign variations of* $[+ + 0 - +]$ *is* 2*;*

Remark 44. *We want to determine the number of roots and, for every root* x_0*, an interval* (a, b) *in which* x_0 *is the only root. We compute the number of roots with the Sturm Algorithm. As a byproduct, we have the remainder sequence. We then use the remainder sequence to determine an interval for every root by splitting the interval* $(-C, C)$.

Remark 45. *Let* $f(x) =\in \mathbb{R}[x]$*. With the notation of Proposition 39 above,*

- *the signs of* $f(C), f'(C), r_1(c), \ldots, r_n(C)$ *are the signs of the leading coefficients of* $f(x), f'(x), r_1(x), \ldots, r_n(x)$
- *the signs of* $f(-C), f'(-C), r_1(-C), \ldots, r_n(-C)$ *are the signs of the leading coefficients of* $f(x), f'(x), r_1(x), \ldots, r_n(x)$ *changing the sign for the odd degree polynomials.*

Example 46 (First Example)**.** *We want to determine the numer of roots and the root intervals for* $f(x) = x^3 - 6x - 1$*. The details of the STURM Algorithm computations are shown in Example 37.*

```
F:= x^3 - 6x - 1;
GCDPolySTURMVerbose(F,Der(F,x));

(x^3 - 6x - 1)=(x)*(x^2 - 2)+(-4x - 1)
(x^2 - 2)=(1/4x - 1/16)*(4x + 1)+(-31/16)
(4x + 1)=(4x + 1)*(1)+(0)
Lista delle coppie  [[x^3 - 6x - 1, 3x^2 - 6], [x^2 - 2, 4x + 1], [4x + 1, 1]]
Record[GCD = 1, Sequence = [x^3 - 6x - 1, 3x^2 - 6, 4x + 1, 1]]
-------------------------------
```

The 3 *roots are in the interval* $(-18, 18)$ *To separate the roots, we find the number of roots in* $(-18, 0)$ *and* $(0, 18)$*. The remainder sequence is* $f_1(x) = x^3-6x-1, f_2(x) = 3x^2-6, f_3(x) = 4x+1, f_4(x) = 1$
We split $(-18, 18)$

- $(-18,0)$. *We have*

$$f_1(-18) = -5725, f_2(-18) = 966, f_3(-18) = -71, f_4(-18) = 1 \quad \Rightarrow p = 3$$

and

$$f_1(0) = -1, f_2(0) = -6, f_3(0) = 1, f_4(0) = 1 \quad \Rightarrow q = 1$$

and the number of roots in $(-18,0)$ *is* $p-q=2$.

- *Since there are* 3 *roots,* $\boxed{\text{the number of roots in } (0,18) \text{ is } 1}$. *OK*

We split $(-18,0)$

- $(-18,-9)$. *We have*

$$f_1(-18) = -5725, f_2(-18) = 966, f_3(-18) = -71, f_4(-18) = 1 \quad \Rightarrow p_{-18} = 3$$

and

$$f_1(-9) = -676, f_2(-9) = 237, f_3(-9) = -35, f_4(-9) = 1 \quad \Rightarrow q_{-9} = 3$$

and the number of roots in $(-18,0)$ *is* $p-q=0$.

- *Since there are* 2 *roots in* $(-18,0)$, *the number of roots in* $(-9,0)$ *is* 2, *and* .

We split $(-18,0)$

- $(-9,0)$. *We have*

$$f_1(-9) = -676, f_2(-9) = 237, f_3(-9) = -35, f_4(-9) = 1 \quad \Rightarrow q_{-9} = 3$$

and

$$f_1(-4) = -676, f_2(-4) = 42, f_3(-4) = -15, f_4(-4) = 1 \quad \Rightarrow q_{-9} = 3$$

and the number of roots in $(-9,4)$ *is* $p-q=0$.

- *Since there are* 2 *roots in* $(-9,0)$, *the number of roots in* $(-4,0)$ *is* 2.

We split $(-18,0)$

- $(-4,-2)$. *We have*

$$f_1(-4) = -676, f_2(-4) = 42, f_3(-4) = -15, f_4(-4) = 1 \quad \Rightarrow q_{-9} = 3$$

and

$$f_1(-2) = 3, f_2(-2) = 6, f_3(-2) = -7, f_4(-2) = 1 \quad \Rightarrow q_{-9} = 2$$

and $\boxed{\text{the number of roots in } (-4,2) \text{ is } p-q=1}$.

- *Since there are* 2 *roots in* $(-4,0)$, $\boxed{\text{the number of roots in } (-2,0) \text{ is } 1}$.

So we have one root each in the intervals $(-4,2)$, $(-2,0)$, $(0,18)$.

2.8 Homework

Exercise 47 (Second Example)**.** *We want to determine the numer of roots and the root intervals for* $P_1(x) = x^3 - 3x^2 + 3$*. The details of the STURM Algorithm computations are shown in Example 52.*

```
F:= x^3-3x^2+3;
GCDPolySTURMVerbose(F,Der(F,x));

(x^3 - 3x^2 + 3)=(x - 1)*(x^2 - 2x)+(-2x + 3)
(x^2 - 2x)=(1/2x - 1/4)*(2x - 3)+(-3/4)
(2x - 3)=(2x - 3)*(1)+(0)
Lista delle coppie  [[x^3 - 3x^2 + 3, 3x^2 - 6x], [x^2 - 2x, 2x - 3], [2x - 3, 1]]
Record[GCD = 1, Sequence = [x^3 - 3x^2 + 3, 3x^2 - 6x, 2x - 3, 1]]
p=++++ 0
q=-+-+ 3
Numero radici 3

Big Interval(-3*3,3*3)=(-9,9)


Subst(V,x,-9); Equal to the STURM p sequence  3
Subst(V,x,0); [3, 0, -3, 1] ++-+ 2
In (-9,0) one root, **************** INTERVAL FOUND
so in (0,9) two roots

In (0,4) two roots
Subst(V,x,0); [3, 0, -3, 1] +0-+ 2
Subst(V,x,4); [19, 24, 5, 1] ++++ 0

In (0,2) one root
Subst(V,x,0); [3, 0, -3, 1] +0-+ 2
Subst(V,x,2); [-1, 0, 1, 1] -+++ 1

So one root  in (2,4)  ********INTERVAL FOUND
and the other root in (0,2) *****INTERVAL FOUND
```

Example 48. *We want to determine the root intervals for* $P_2(x) = 2x^3 - 6x - 3$*.*

```
F:= 2x^3-6x-3;
GCDPolySTURMVerbose(F,Der(F,x));

(2x^3 - 6x - 3)=(2x)*(x^2 - 1)+(-4x - 3)
(x^2 - 1)=(1/4x - 3/16)*(4x + 3)+(-7/16)
(4x + 3)=(4x + 3)*(1)+(0)
```

```
Lista delle coppie  [[2x^3 - 6x - 3, 6x^2 - 6], [x^2 - 1, 4x + 3], [4x + 3, 1]]
remainder Sequence = [2x^3 - 6x - 3, 6x^2 - 6, 4x + 3, 1]
```

Solutions: 3 *real roots,* $x \simeq -1.38, -0.55, 1.94$.

Exercise 49. *We want to determine the root intervals for* $P_3(x) = x^3 - 3x - 1$.

```
F:= x^3-3x-1;
GCDPolySTURMVerbose(F,Der(F,x));

(x^3 - 3x - 1)=(x)*(x^2 - 1)+(-2x - 1)
(x^2 - 1)=(1/2x - 1/4)*(2x + 1)+(-3/4)
(2x + 1)=(2x + 1)*(1)+(0)
Lista delle coppie  [[x^3 - 3x - 1, 3x^2 - 3], [x^2 - 1, 2x + 1], [2x + 1, 1]]
remainder Sequence = [x^3 - 3x - 1, 3x^2 - 3, 2x + 1, 1]
```

Solutions: 3 *real roots,* $x \simeq -1.53, -0.34, 1.87$.

Exercise 50. *We want to determine the root intervals for* $P_4(x) = x^5 - 3x^4 + 1$.

```
F:= x^5-3x^4+1;
GCDPolySTURMVerbose(F,Der(F,x));

(x^5 - 3x^4 + 1)=(1/5x - 3/25)*(5x^4 - 12x^3)+(-36/25x^3 + 1)
(5x^4 - 12x^3)=(5/36x - 1/3)*(36x^3 - 25)+(125/36x - 25/3)
(36x^3 - 25)=(-36/5x^2 - 432/25x - 5184/125)*(-5x + 12)+(59083/125)
(-5x + 12)=(5x - 12)*(-1)+(0)
Lista delle coppie  [[x^5 - 3x^4 + 1, 5x^4 - 12x^3], [5x^4 - 12x^3, 36x^3 - 25], [36x^3 - 25, -5x
remainder Sequence = [x^5 - 3x^4 + 1, 5x^4 - 12x^3, 36x^3 - 25, -5x + 12, -1]
```

Solutions: 3 *real roots,* $x \simeq -0.72, 0.82, 2.98$.

Exercise 51. *We want to determine the root intervals for* $P_5(x) = x^4 - x^3 - 1$.

```
F:= x^4-x^3-1;
GCDPolySTURMVerbose(F,Der(F,x));

(x^4 - x^3 - 1)=(1/4x - 1/16)*(4x^3 - 3x^2)+(-3/16x^2 - 1)
(4x^3 - 3x^2)=(4/3x - 1)*(3x^2 + 16)+(-64/3x + 16)
(3x^2 + 16)=(3/4x + 9/16)*(4x - 3)+(283/16)
(4x - 3)=(-4x + 3)*(-1)+(0)
Lista delle coppie  [[x^4 - x^3 - 1, 4x^3 - 3x^2], [4x^3 - 3x^2, 3x^2 + 16], [3x^2 + 16, 4x - 3],
remainder Sequence = [x^4 - x^3 - 1, 4x^3 - 3x^2, 3x^2 + 16, 4x - 3, -1]
```

Solutions: 2 *real roots,* $x \simeq -0.81, 1.38$.

2.9 Examples: poly root and interval computations

Example 52 (First Example)**.** *We want to determine the number of roots for* $f(x) = x^4 - x^3 - 1$*.*

```
F:= x^4-x^3-1; **************** First element for the sequence
Der(F,x);
4x^3 - 3x^2 **************** Second element for the sequence

-----------------
GCD(x^4-x^3-1,4x^3 - 3x^2);

DivPoly(x^4-x^3-1,4x^3 - 3x^2);

Passo 1 ho        x^4 - x^3 - 1 moltiplico per 1/4x
Passo 1 sottraggo x^4 - 3/4x^3
Passo 1 ottengo       -1/4x^3 - 1

Passo 2 ho        -1/4x^3 - 1 moltiplico per -1/16
Passo 2 sottraggo -1/4x^3 + 3/16x^2
Passo 2 ottengo           -3/16x^2 - 1

Remainder=-3/16x^2 - 1
Quoto=1/4x - 1/16

(x^4 - x^3 - 1)=(1/4x - 1/16)*(4x^3 - 3x^2)+(-3/16x^2 - 1)

Multiplying the remainder by 16 to avoid fractions
Remainder=-3x^2 - 16
Change the remainder's sign
Remainder=3x^2 + 16 ************* Remainder - third element for the sequence

----------------
GCD(4x^3 - 3x^2,3x^2 + 16);

DivPoly(x^3 - 3x^2,3x^2 + 16);
Passo 1 ho        x^3 - 3x^2 moltiplico per 1/3x
Passo 1 sottraggo x^3 + 16/3x
Passo 1 ottengo       -3x^2 - 16/3x

Passo 2 ho        -3x^2 - 16/3x moltiplico per -1
Passo 2 sottraggo -3x^2 - 16
Passo 2 ottengo         -16/3x + 16

Remainder=-16/3x + 16
Quoto=1/3x - 1
```

```
Multiplying the remainder by 3/16 to avoid fractions and simplify
Remainder=-x + 3
Change the remainder's sign
Remainder=x - 3 ***************** Remainder - fourth element for the sequence

---------------
GCD(3x^2 + 16,x - 3);
DivPoly(3x^2 + 16,x - 3);

Passo 1 ho        3x^2 + 16 moltiplico per 3x
Passo 1 sottraggo 3x^2 - 9x
Passo 1 ottengo        9x + 16

Passo 2 ho        9x + 16 moltiplico per 9
Passo 2 sottraggo 9x - 27
Passo 2 ottengo      43

Remainder=43
Quoto=3x + 9

Multiplying the remainder by 1/43 to  simplify
Remainder=1
Change the remainder's sign
Remainder=-1 ******************* Remainder - fifth element for the sequence


Remainder Sequence = [x^4 - x^3 - 1, 4x^3 - 3x^2, 3x^2 + 16, 4x - 3, -1]

p=++++- 1
q=+-+-- 3
#roots=3-1=2
-------------------------------
There are two roots
```

The maximal interval:

$$C = Poly\ degree \cdot \frac{Max\ of\ the\ coeffs}{Coeff\ of\ x^4} = 4 \cdot \frac{1}{1} \Rightarrow\ interval\ is\ (-4, 4)$$

Remember that the cofficients are considered as absolute values (without sign).

- *We try the interval* $(-4, 0)$*. The Remainder Sequence is*

$$[p_0(x), p_1(x), p_2(x), p_3(x), p_4(x)] = [x^4 - x^3 - 1, 4x^3 - 3x^2, 3x^2 + 16, 4x - 3, -1]$$

$++++-$ *as in the maximal interval determined before. This is the same of computing the signs of the sequence*

$$[p_0(-4), p_1(-4), p_2(-4), p_3(-4), p_4(-4)] = [319, -304, 64, -19, -1]$$

and we have $p = 3$

$-0+--$. *We compute the signs of the sequence*

$$[p_0(0), p_1(0), p_2(0), p_3(0), p_4(0)] = [-1, 0, 16, -3, -1]$$

remembering that 0 *is caceled out in the sign evaluation, we have* $q = 2$

We have $\mid p - q \mid = \mid 3 - 2 \mid = 1$, *and so we have one root in* $(-4, 0)$.

- *We have two roots in* $(-4, 4)$, *one is in* $(-4, 0)$ *and the other, by necessity, is in* $(0, 4)$.

Exercise 53. *Find an approximation up to the third digit of the two roots above. That means, find two intervals of lenght* $10^{-3} = 0.001$ *to which a roots belongs using the shrinking method.*

As you see in this exercise, this methods works surely but slowly - at every step, you shrink the lenght of the interval by half, and so to shrink it by a factor of 1000 you have to repeat the procedure 10 times, since $2^{10} = 1024$. We need a faster method.

2.10 Root finding: Newton's method

Definition 54. *Let* $f\colon (a, b) \to \mathbb{R}$ *be a squarefree polynomial function where* $x_0 \in \mathbb{R}$ *is the* starting point. *The sequence* $[x_0, x_1, x_2, \ldots]$ *defined by*

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \quad (54)$$

is the Newton sequence *and the formula (54) above is the* Newton's Formula.

Remark 55. *For the sequence to converge quickly to the roots, it is important to ensure the following:*

1. *For the root to be NOT a multiple root.*
2. *To have only ONE root near* x_0.
3. *Choose a starting point* x_0 *near to the actual root.*

We can make this work by

1. *Shrinking the interval suitably.*
2. *Start with a squarefree polynomial.*
3. *Shrinking the interval suitably.*

We show some easy examples: first we compute $\sqrt{2}$ starting with $x_0 = 2$, reasonably near the actual root.

Example 56. *Let $f(x) = x^2 - 2$ and $x_0 = 2$. We have $f'(x) = 2x$, the formaul is*

$$x_{k+1} = x_k - \frac{x_k^2 - 2}{2x_k}$$

and so

$$\begin{aligned}
x_0 &= 2 \\
x_1 &= 2 - \frac{(2)^2 - 2}{2 \cdot 2} = \frac{3}{2} = 1.5 \\
x_2 &= 1.5 - \frac{(1.5)^2 - 2}{2 \cdot 1.5} = 1.5 - \frac{(1.5)^2 - 2}{2 \cdot 1.5} = \frac{17}{12} = 1.4166666666 \\
x_3 &= \frac{17}{12} - \frac{(\frac{17}{12})^2 - 2}{2 \cdot \frac{17}{12}} = \frac{577}{408} = 1.4142156862 \\
x_4 &= 1.4142156862 - \frac{(1.4142156862)^2 - 2}{2 \cdot 1.4142156862} = 1.41421356231
\end{aligned}$$

We remember that $\sqrt{2} = 1.414213562$, and after four steps we have an 8 digit precision.

The same but we start with $x_0 = 8$, further away from the actual root.

Example 57. *Let $f(x) = x^2 - 2$ and $x_0 = 8$. We have $f'(x) = 2x$ and*

$$\begin{aligned}
x_0 &= 8 \\
x_1 &= 8 - \frac{(8)^2 - 2}{2 \cdot 8} = 4.125 \\
x_2 &= 4.125 - \frac{(4.125)^2 - 2}{2 \cdot 4.125} = 2.3049242424 \\
x_3 &= 2.3049242424 - \frac{(2.3049242424)^2 - 2}{2 \cdot 2.3049242424} = 1.5863158599
\end{aligned}$$

$x_4 = 1.4235494082$
$x_5 = 1.4142441752$
$x_6 = 1.4142135627$
$x_7 = 1.4142135623$
$x_7 = 1.4142135623$

We remember that $\sqrt{2} = 1.414213562$, as computed using a pocket calculator, and we need seven steps we have an 8 digit precision. .

Example 58. *Let $f(x) = x^2 - 2$ and $x_0 = 1000$.*

```
F:=x^2-2;
L:=NewtonMod(F,100,30,10,15);
100
500.001
250.0024999960
125.0052499580
62.5106246430
31.2713096020
15.6676329948
7.8976423478
4.0754412405
2.2830928243
1.5795487524
1.4228665795
1.4142398735
1.4142135626
1.4142135623
1.4142135623
```

Example 59. *Let* $f(x) = x^5 - 3$ *and* $x_0 = 2$.

```
F:=x^2-2;
L:=NewtonMod(F,100,30,10,15);
2
1.6375
1.393449937495
1.273902300672
1.246949664737
1.245733319567
1.245730939624
1.245730939615
1.245730939615
```

We remember that $\sqrt[5]{3} = 1.24573094$, *as computed using a pocket calculator, and after six steps we have an* 8 *digit precision. .*

Example 60. *Let* $f(x) = x^5 - 3$ *and* $x_0 = 3$.

```
F:=x^5-3;
L:=NewtonMod(F,3,15,10,15);
3
2.4074074074
1.9437888634
1.5970606554
1.3698771224
1.2662841244
1.2463873994
```

```
1.2457316307
1.2457309396
1.2457309396
```

We remember that $\sqrt[5]{3} = 1.24573094$*, as computed using a pocket calculator.*

Example 61. *Let* $f(x) = x^5 - 3$ *and* $x_0 = 20$*.*

```
F:=x^5-3;
L:=NewtonMod(F,20,20,10,15);
20
16.00000375
12.8000121552
10.2400320758
8.1920802296
6.5537974052
5.2433631456
4.1954843172
3.3583239775
2.6913761278
2.1645363629
1.7589623675
1.4698493269
1.3044258794
1.250780723
1.2457715503
1.2457309422
1.2457309396
1.2457309396
1.2457309396
```

We remember that $\sqrt[5]{3} = 1.24573094$*, as computed using a pocket calculator.*

Ths method is sensitive to the selection of the strating point, but expecially to the polynomial degree. For lower degress, it is very efficient.

Example 62. *Let* $f(x) = x^{15} - 3$ *and* $x_0 = 2$*.*

```
F:=x^15-3;
L:=NewtonMod(F,4,30,30,15);
4
3.733333334078391  1.631440852743791
3.484444447097212  1.522889493378988
3.252148155766399  1.421917734820804
3.035338292224998  1.32857128198326
2.832982441568673  1.243746503024962
2.644117038706965  1.170265979236223
```

```
2.467842814422866  1.114381532360503
2.303320603682124  1.083998608244157
2.149767587477567  1.076390865753772
2.006454190008554  1.075990670016803
1.872702245988654  1.075989624732454
1.747886082220382  1.075989624725345
 1.075989624725345
```

We remember that $\sqrt[15]{3} = 1.075989625$*, as computed using a pocket calculator.*

Example 63. *Let* $f(x) = x^{115} - 3$ *and* $x_0 = 1$*.*

```
F:=x^115-3;
L:=NewtonMod(F,1,100,20,20);
1
1.01739130434782608695
1.01219859535714983979
1.00994519245838467584
1.00960560774948143004
1.00959892985126879499
1.00959892733230472403
1.0095989273323043658
1.0095989273323043658
1.0095989273323043658
```

We remember that $\sqrt[115]{3} = 1.0009598927$*, as computed using a pocket calculator.*

Example 64. *With a worse starting point: Let* $f(x) = x^{115} - 3$ *and* $x_0 = 2$*. Notice how slow we get near* 1*.*

```
F:=x^115-3;
L:=NewtonMod(F,2,100,20,20);
2
1.98260869565217391304
1.96536862003780718336
1.94827845812443494698
1.93133690631465725178
1.91454267234670371046
1.89789447519586280862
1.88139104497676834941
1.86503112284653558115
1.84881346090873961957
1.83273682211822884026
1.81679998018676598077
1.80100171948948975485
1.78534083497218984393
```

1.76981613205938819311
1.75442642656321960012
1.73917054459310464707
1.72404732246620808492
1.7090556066186758407
1.69419425351764387686
1.67946212957401219097
1.66485811105597730235
1.65038108400331663015
1.63602994414241822467
1.62180359680204937054
1.60770095682985763688
1.59372094850959800525
1.57986250547907976172
1.56612457064882689422
1.55250609612144579079
1.53900604311169408826
1.52562338186724457444
1.51235709159013809987
1.49920616035891950769
1.4861695850514506424
1.47324637126839454985
1.46043553325736503202
1.44773609383773577087
1.43514708432610328591
1.42266754446239803997
1.4102965223366380571
1.39803307431631946554
1.38587626497443842738
1.37382516701813896461
1.3618788612179812394
1.35003643633782489412
1.33829698906532210501
1.32665962394301505386
1.3151234533000325784
1.30368759718438083608
1.29235118329582293754
1.28111334691934275719
1.26997323085918873165
1.25892998537349501256
1.24798276810948147566
1.23713074403924520531
1.22637308539618608692
1.21570897161219029686
1.20513758925591505286

```
1.19465813197311218246
1.18426980043153615473
1.17397180227733436573
1.16376335212159628028
1.15364367160761677396
1.14361198969570725361
1.13366754353589572397
1.12380958093083114529
1.11403736710159131119
1.10435020309798154396
1.09474747572139616918
1.08522879272040130077
1.07579434867067792314
1.06644591444088371728
1.05718950882919385901
1.04804258214247679977
1.03905311240234824126
1.0303487857929650193
1.02225264170034570905
1.01548567641901128278
1.01118027662180827392
1.00973193715521649452
1.0095999210978453851
1.00959892738805853039
1.00959892733230436597
1.0095989273323043658
1.0095989273323043658
1.0095989273323043658
1.0095989273323043658
1.0095989273323043658
1.0095989273323043658
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1.0095989273323043658
1.0095989273323043658
1.0095989273323043658
1.0095989273323043658
1.0095989273323043658
1.0095989273323043658
1.0095989273323043658
```

We remember that $\sqrt[115]{3} = 1.0009598927$*, as computed using a pocket calculator.*

If you are interested, this is a simplified version of the theorem that guarantees you the algorithm convergence.

Theorem 65 (Newton's Approximation)**.** *Let $(a, b) \subset \mathbb{R}$ be an interval and let $f\colon (a, b) \to \mathbb{R}$ a squarefree polynomial function. If $\overline{x} \in (a, b)$ is a root, then there exists an $\epsilon > 0$ such that Newton's method with starting point x_0 with $\mid x_0 - \overline{x} \mid < \epsilon$ quickly converges to the root $\overline{x}$*

Example 66 (First Example)**.** *Let us return to Example 37*

We have to determine the real roots of the polynomial $f(x) = x^4 - x^3 - 1$. We know that there are two roots, one in the interval $(-4, 0)$ and one in the interval $(0, 4)$. We use Newton's method. The general formula is

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

and thus in our case, for a point x_k, the formula is

$$x_{k+1} = x_k - \frac{x_k^4 - x_k^3 - 1}{4x_k^3 - 3x_k^2}$$

- *Interval $(-4, 0)$. Middle point $x_0 = -2$.*

```
F:=x^4 - x^3 - 1;
L:=NewtonMod(F,-2,12,12,20);
-2
-1.477272727272
-1.117933865584
-0.908134322031
-0.829688956906
-0.819339664556
-0.819172556393
-0.819172513396
-0.819172513396
```

  We see that the root is $x_1 \simeq -0.819172513396$

- *Interval $(0, 4)$. Middle point $x_0 = 2$.*

```
F:=x^4 - x^3 - 1;
L:=NewtonMod(F,2,12,12,20);
2
1.65
1.454113738394
1.387577585471
1.380357405588
1.38027757877
1.380277569097
1.380277569097
```

We see that the root is $x_2 \simeq 1.380277569097$

Exercise 67. *We want to determine the numer of roots and the root intervals for* $P_1(x) = x^3 - 6x - 1$. *The details are shown in Second Example of the Third Lesson.*

```
F:= x^3-3x^2+3;
GCDPolySTURMVerbose(F,Der(F,x));

One root in (-9,0)
One root in (0,2)
One root in (2,4)
```

We have the polynomial function $f(x) = x^3 - 3x^2 + 3$, abd $f'(x) = 3x^2 - 6x$. The formula is

$$x_{k+1} = x_k - \frac{x_k^3 - 3x_k^2 + 3}{3x_k^2 - 6x_k}$$

- Interval $(0, 2)$. Middle point $x_0 = 0 + \frac{2-0}{2} = 1$.

```
F:=x^3-3x^2+3;
L:=NewtonMod(F,1,5,10,10);
1
1.3333333333
1.3472222222
1.3472963531
1.3472963553
1.3472963553
```

  We say that $x_1 \simeq 1.3472963553$.

- Interval $(2, 4)$. Middle point $x_0 = 2 + \frac{4-2}{2} = 3$.

```
F:=x^3-3x^2+3;
L:=NewtonMod(F,3,7,12,12);

3
2.666666666666
2.54861111111
2.532390161865
2.532088989397
2.532088886237
2.532088886237
```

  We say that $x_2 \simeq 1.532088886237$.

- Interval $(-9, 0)$. Middle point $x_0 = -9\frac{-9-0}{2} = -4.5$.

```
F:=x^3-3x^2+3;
L:=NewtonMod(F,-4.5,12,10,10);
-4.5
-2.8034188034
-1.748663703
-1.1628353797
-0.9245690643
-0.8808269788
-0.8793867818
-0.8793852415
-0.8793852415
```

We say that $x_3 \simeq -0.8793852415$.

2.11 Root's finding with approximation

A review of the metohd we have learned about finding the real roots of a polynomial, if possibile precise, if not approximate.

We have to determine all the real roots of a polynomial $f(x)$ of degree d with "good" approximation

1. We try to find the exact roots $x_1, \ldots, x_k$ by regrouping, or root's rule+Ruffini's rule.

2. We divide them out $g(x) = \frac{f(x)}{(x-x_1)\cdots(x-x_k)}$. The polynomial $g(x)$ ha now degree $n = d-k$.

3. We perform the Sturm variant of Euclid's Algorithm, finding $\gcd(g(x), g'(x)) = h(x)$. If $\gcd(g(x), g'(x)) \neq 1$ then

   - We find the roots of $\gcd(g(x), g'(x))$ using this method.
   - We set $h(x) = \frac{g(x)}{\gcd(g(x),g'(x))}$ and proceed to determine the roots of $h(x)$, that is squarefree.

4. Using the Sturm sequence that we have built before, we compute the number of real roots of $h(x)$. Let this number be p.

5. We compute the maximum root interval $(-C, C)$ as in Lesson 3.

6. We split this interval $(-M, M)$ using the method outlined in Lesson 3. We find intervals $(a_1, b_1), \ldots, (a_p, b_p)$ such that for every interval there is only one root.

7. For every interval (a_i, b_i) we use Newton's method, starting with the middle point $a_i + \frac{b_i - a_i}{2}$. If the sequence quickly converges up to the precision we are interested in, we have a root. If the sequence does not converges, we split the interval and try again.

2.12 Exercises

The remainder sequence, and sometimes the gcd process, is given so you can check your results. For the roots, we refer to Wolphram Alpha `https://www.wolframalpha.com/` e.g. $2x^3 - 6x - 3 = 0$.

You have to solve the two roots of Exercise 68 and two of the following exercises. All the computational steps have to be shown.

Exercise 68. *Find approximations of* $\sqrt[3]{5}$ *and* $\sqrt[5]{7}$ *with precision of* 10^{-3} *(the first three digits after the comma have to be stable in the Newton's sequence.)*

Exercise 69. *We want to determine the root intervals for* $G_1(x) = 2x^3 - 6x - 3$ *with 4 digits approximation.*

```
F:= 2x^3-6x-3;
GCDPolySTURMVerbose(F,Der(F,x));

(2x^3 - 6x - 3)=(2x)*(x^2 - 1)+(-4x - 3)
(x^2 - 1)=(1/4x - 3/16)*(4x + 3)+(-7/16)
(4x + 3)=(4x + 3)*(1)+(0)
Remainder Sequence = [2x^3 - 6x - 3, 6x^2 - 6, 4x + 3, 1]
```

Exercise 70. *We want to determine the root intervals for* $G_2(x) = x^3 - 3x - 1$ *with 4 digits approximation.*

```
F:= x^3-3x-1;
GCDPolySTURMVerbose(F,Der(F,x));

(x^3 - 3x - 1)=(x)*(x^2 - 1)+(-2x - 1)
(x^2 - 1)=(1/2x - 1/4)*(2x + 1)+(-3/4)
(2x + 1)=(2x + 1)*(1)+(0)
Remainder Sequence = [x^3 - 3x - 1, 3x^2 - 3, 2x + 1, 1]
```

Exercise 71. *We want to determine the root intervals for* $G_3(x) = x^5 - 3x^4 + 1$.

```
F:= x^5-3x^4+1;
GCDPolySTURMVerbose(F,Der(F,x));

(x^5 - 3x^4 + 1)=(1/5x - 3/25)*(5x^4 - 12x^3)+(-36/25x^3 + 1)
(5x^4 - 12x^3)=(5/36x - 1/3)*(36x^3 - 25)+(125/36x - 25/3)
(36x^3 - 25)=(-36/5x^2 - 432/25x - 5184/125)*(-5x + 12)+(59083/125)
(-5x + 12)=(5x - 12)*(-1)+(0)
Remainder Sequence = [x^5 - 3x^4 + 1, 5x^4 - 12x^3, 36x^3 - 25, -5x + 12, -1]
```

Exercise 72. *We want to determine the root intervals for* $G_3(x) = x^5 - 3x^4 + 1$.

```
F:= x^5-3x^4+1;
GCDPolySTURMVerbose(F,Der(F,x));
```

```
(x^5 - 3x^4 + 1)=(1/5x - 3/25)*(5x^4 - 12x^3)+(-36/25x^3 + 1)
(5x^4 - 12x^3)=(5/36x - 1/3)*(36x^3 - 25)+(125/36x - 25/3)
(36x^3 - 25)=(-36/5x^2 - 432/25x - 5184/125)*(-5x + 12)+(59083/125)
(-5x + 12)=(5x - 12)*(-1)+(0)
Remainder Sequence = [x^5 - 3x^4 + 1, 5x^4 - 12x^3, 36x^3 - 25, -5x + 12, -1]
```

Exercise 73. *We want to determine the root intervals for* $G_4(x) = x^3 - 4x - 1$.

```
F:=x^3-4x-1;
GCDPolySTURMVerbose(F,Der(F,x));
Remainder Sequence = [x^3 - 4x - 1, 3x^2 - 4, 8x + 3, 1]
```

Exercise 74. *We want to determine the root intervals for* $G_5(x) = x^4 - 4x - 1$.

```
F:=x^4-4x-1;
GCDPolySTURMVerbose(F,Der(F,x));
Remainder Sequence  [x^4 - 4x - 1, 4x^3 - 4, 3x + 1, 1]
```

Exercise 75. *We want to determine the root intervals for* $G_6(x) = x^5 - x - 1$.

```
F:=x^5-x-1;
GCDPolySTURMVerbose(F,Der(F,x));
Remainder Sequence  [x^5 - x - 1, 5x^4 - 1, 4x + 5, -1]
```