# FCS
# Math: Functions
# Lesson 4 & 5

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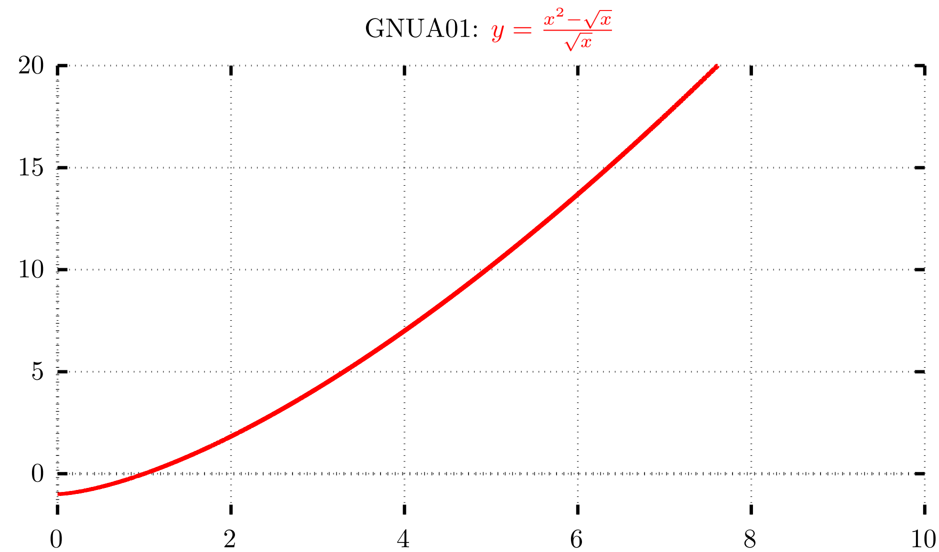
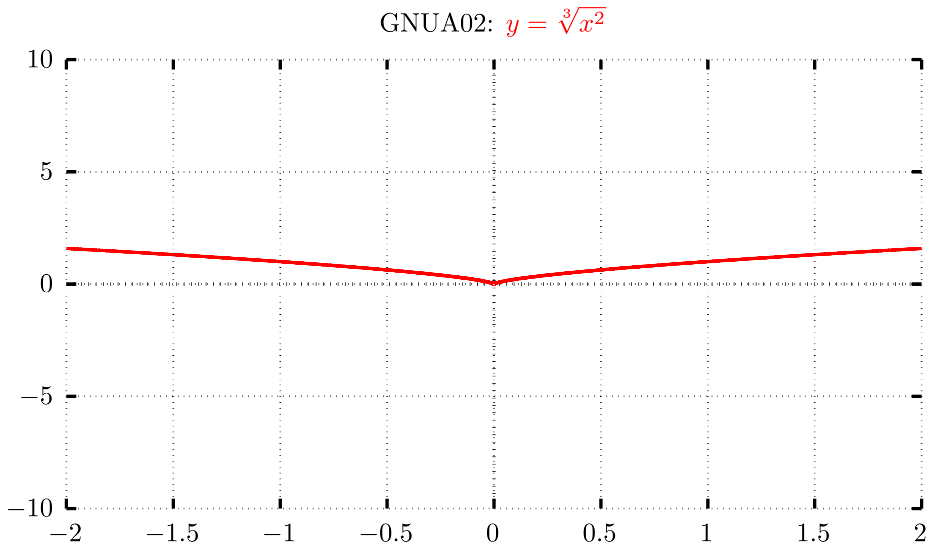
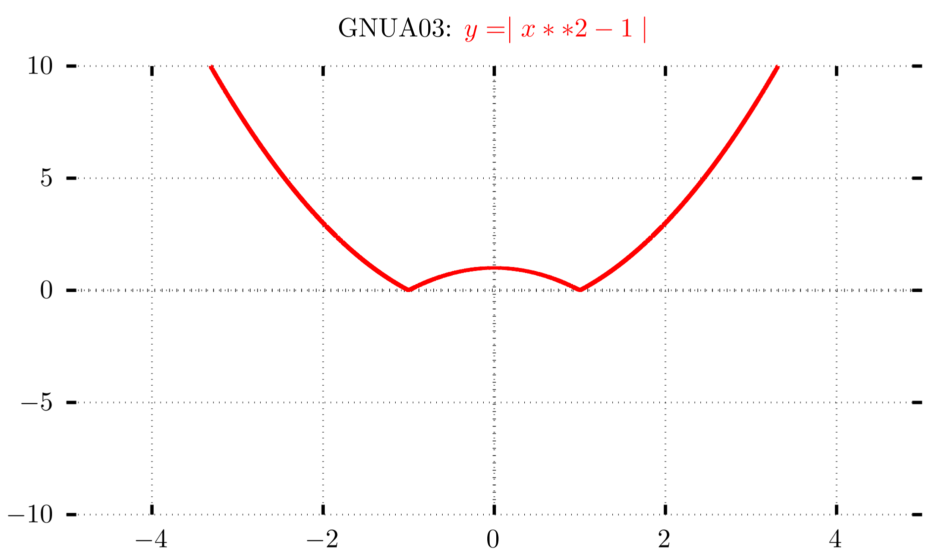
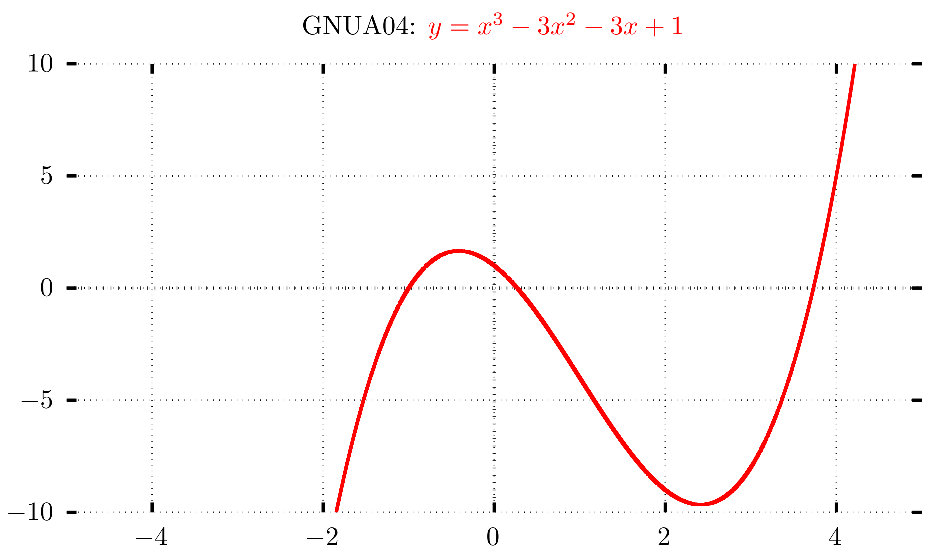
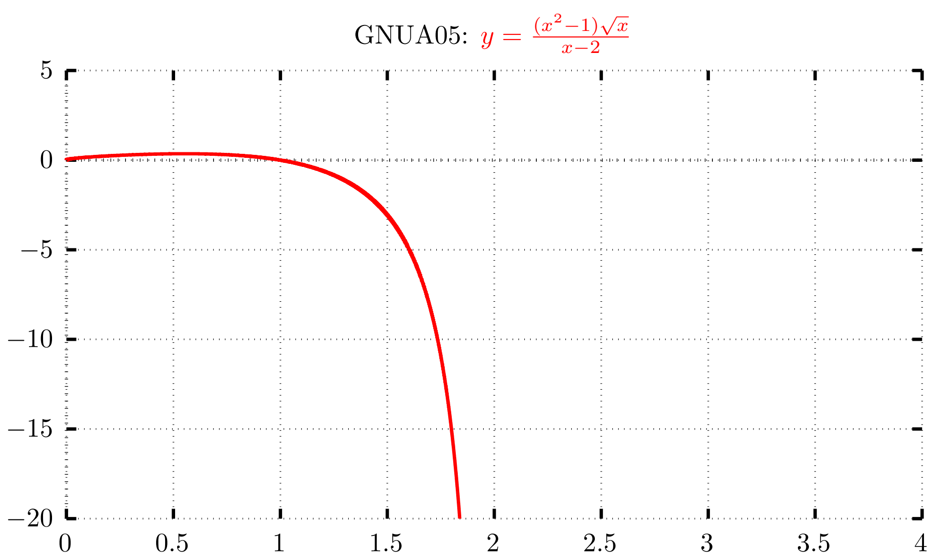
## 1 Lesson 3 Homework solutions




GNUA02: $y = \sqrt[3]{x^2}$
10
5
0
−5
−10
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2


GNUA03: $y = \mid x**2 - 1 \mid$
10
5
0
−5
−10
−4
−2
0
2
4




# 2 Lesson 4

## 2.1 Some definitions

**Definition 1.** *A function $F : \mathbb{R} \longrightarrow \mathbb{R}$ is*

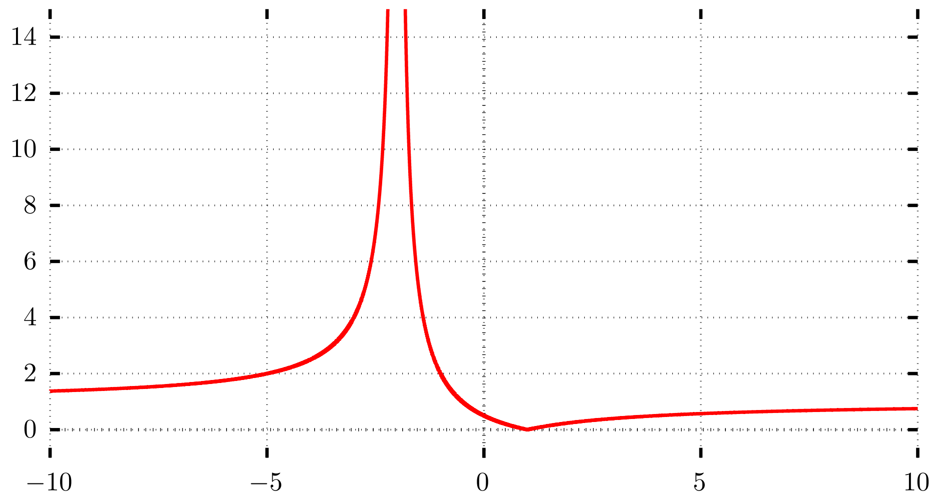
1. *EVEN if and only if for all $x \in \mathbb{R}$ we have $F(x) = F(-x)$*

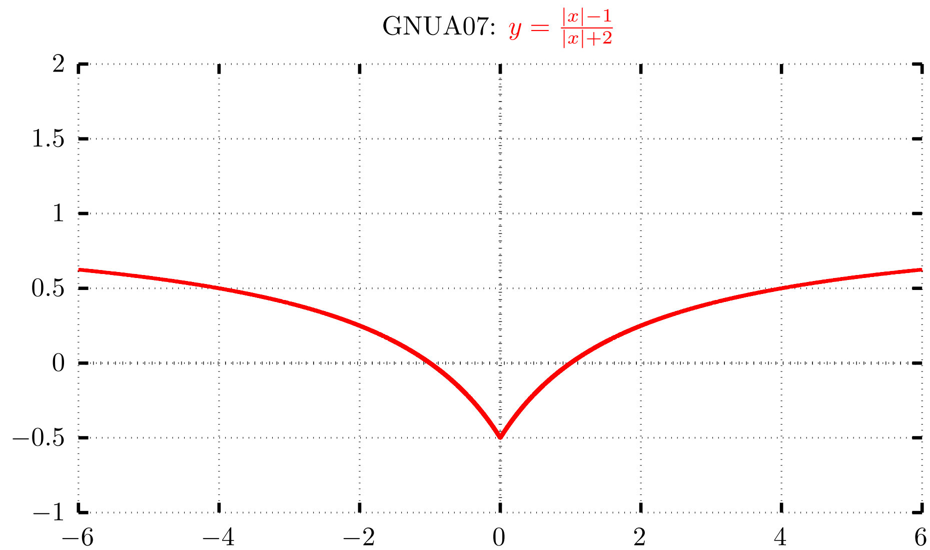
2. *ODD if and only if for all $x \in \mathbb{R}$ we have $F(x) = -F(-x)$, or, equivalently, $F(-x) = -F(x)$.*

**Definition 2.** *A derivable function $F : \mathbb{R} \longrightarrow \mathbb{R}$ with derivable derivative $F'$ is said to have a* positive concavity *in $x_0 \in \mathbb{R}$ if and only if $F(x_0)'' > 0$. A point $\overline{x} \in \mathbb{R}$ such that $F(x_0)'' = 0$ is called a* flex *for $F$ if the sign of $F''$ changes in $\overline{x}$.*

## 2.2 Classwork solutions - First part

GNUA06: $y = \mid \frac{x-1}{x+2} \mid$






- If $F:\mathbb{R}\longrightarrow\mathbb{R},\ G:\mathbb{R}\longrightarrow\mathbb{R}$ are ODD, then $F\cdot G:\mathbb{R}\longrightarrow\mathbb{R}$ is EVEN, because

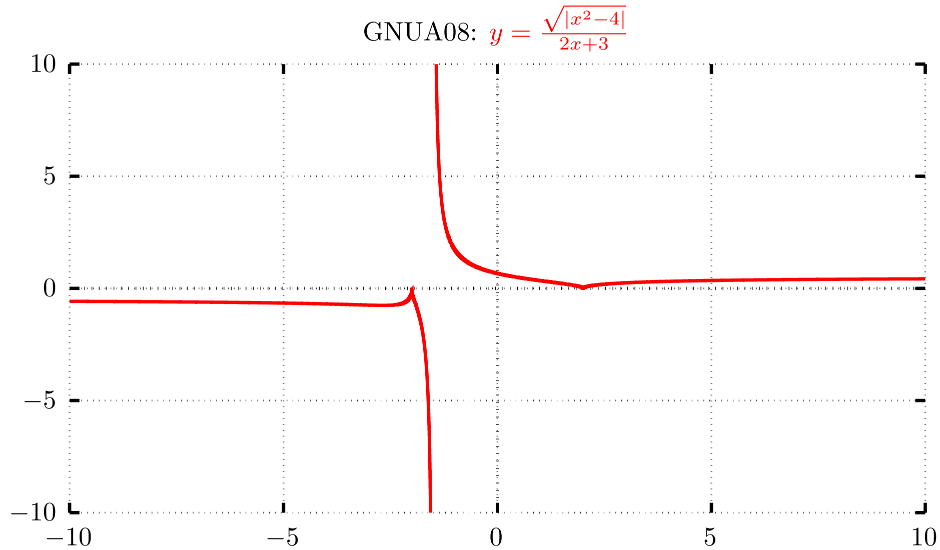
$$\begin{aligned} F\cdot G(x) &= F(x)G(x) \\ &= (-F(-x))(-G(-x)) \\ &= F(-x)G(-x) \\ &= F\cdot G(-x) \end{aligned}$$

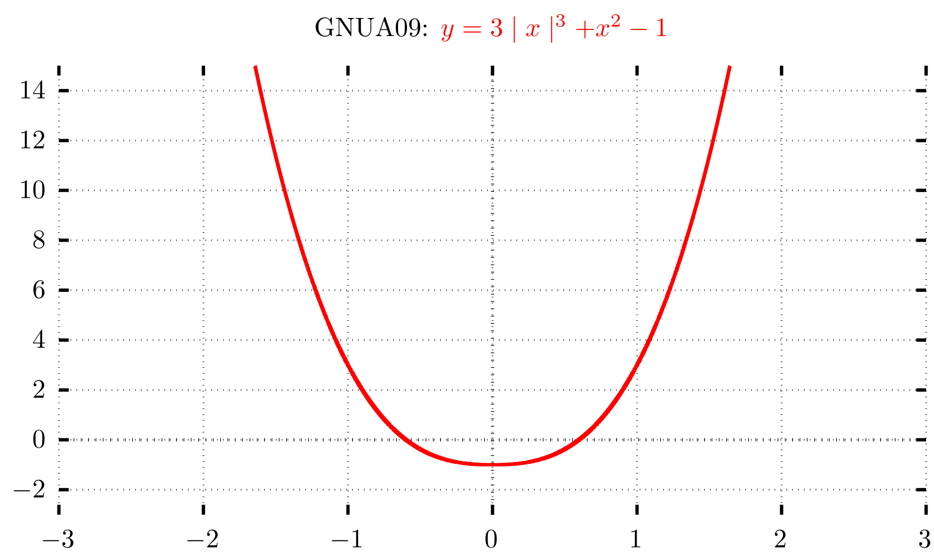
- If $F:\mathbb{R}\longrightarrow\mathbb{R},\ G:\mathbb{R}\longrightarrow\mathbb{R}$ are ODD, then $F\circ G:\mathbb{R}\longrightarrow\mathbb{R}$ is ODD, because

$$\begin{aligned} F\circ G(x) &= F(G(x)) \\ &= F(-G(-x)) \\ &= -F(G(-x)) \\ &= -(F\circ G)(-x) \end{aligned}$$

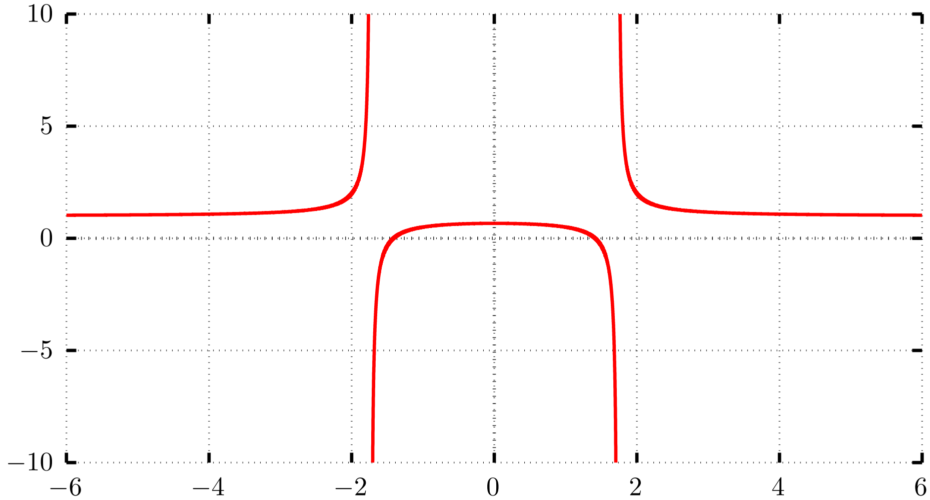
- $\left(\frac{x^2-2}{\sqrt{x}}\right)'' = \frac{3(x^2-2)}{4\sqrt{x^5}}$
- $\left(\mid \frac{x+3}{x-1} \mid\right)'' = \begin{cases} \frac{8}{x^3-3x^2+3x-1} & x\in[-3,1) \\ -\frac{8}{x^3-3x^2+3x-1} & x\notin[-3,1) \end{cases}$

### 2.3 Classwork solutions - Second part

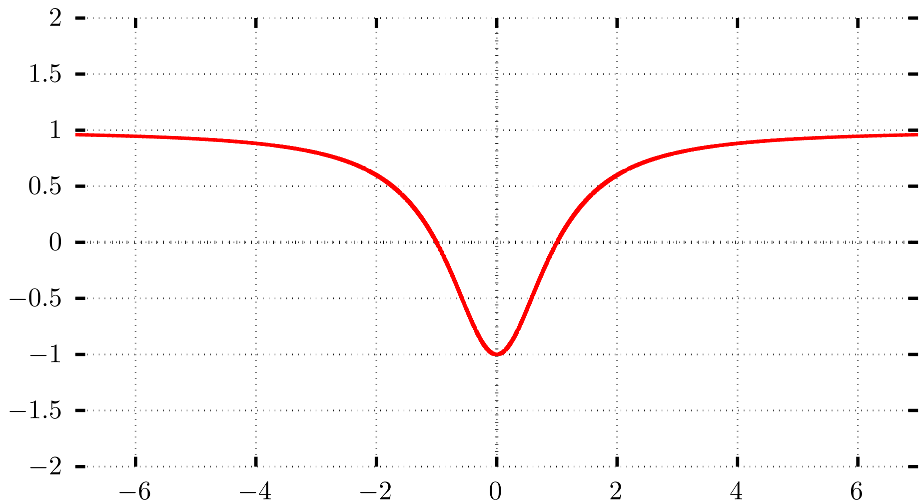


GNUA10: $y = \frac{x^2-2}{x^2-3}$




GNUA11: $y = \frac{x^3+x}{x^3-x}$



## 3 Lesson 5

### 3.1 Oblique asymptotes

**Definition 3.** *Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function for $x >> 0$*

$$r: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & mx+n \end{array} \qquad m \neq 0$$

*an oblique line. Then we say that $r$ is an* oblique asymptote *for $F$, or that $r$ is the limit of $F$ at infinity if*

$$\lim_{x\to+\infty} F(x) - r(x) = 0$$

*That is equivalent to*

$$\lim_{x\to+\infty} \frac{F(x)}{x} = m \qquad \lim_{x\to+\infty} F(x) - mx = n$$

**Example 1.** *Determine the oblique asymptote of*

$$f: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & \sqrt{4x^2-2x} \end{array}$$

*The function $f$ is clearly continuous for $x >> 0$. We have that*

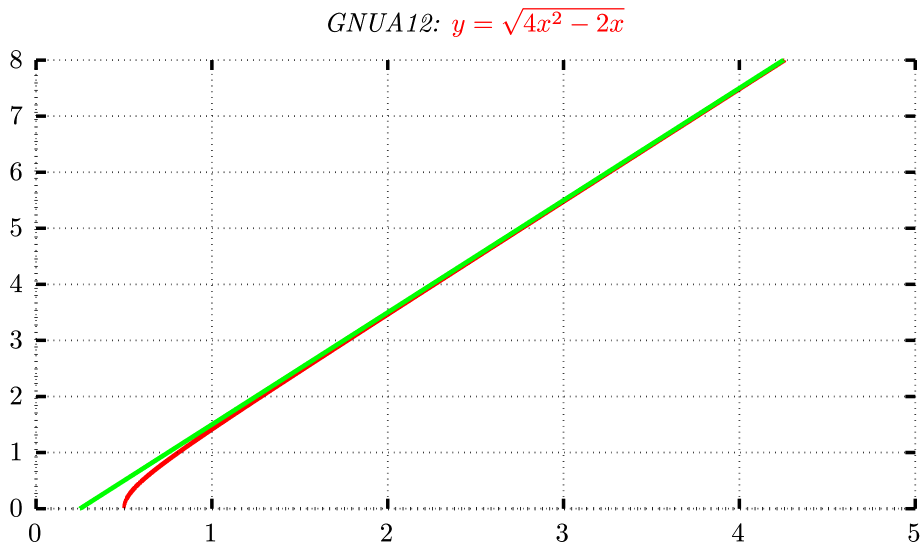
$$\begin{aligned} \lim_{x\to+\infty} \frac{\sqrt{4x^2-2x}}{x} &= \lim_{x\to+\infty} \sqrt{\frac{4x^2}{x^2} - 2\frac{x}{x^2}} \\ &= \lim_{x\to+\infty} \sqrt{4+0} \\ &= 2 \end{aligned}$$

*and*

$$\begin{aligned} \lim_{x\to+\infty} \sqrt{4x^2-2x} - 2x &= \lim_{x\to+\infty} \frac{(\sqrt{4x^2-2x}-2x)(\sqrt{4x^2-2x}+2x)}{\sqrt{4x^2-2x}+2x} \\ &= \lim_{x\to+\infty} \frac{4x^2-2x-4x^2}{\sqrt{4x^2-2x}+2x} \\ &= \lim_{x\to+\infty} \frac{-2x}{\sqrt{4x^2-2x}+2x} \\ &= \lim_{x\to+\infty} \frac{-\frac{2x}{x}}{\sqrt{\frac{4x^2}{x^2}+\frac{2x}{x^2}}+\frac{2x}{x}} \\ &= \lim_{x\to+\infty} \frac{-2}{\sqrt{4+0}+2} \\ &= -\frac{1}{2} \end{aligned}$$

*The oblique asymptote of f is*

$$
\begin{array}{rccc}
r: & \mathbb{R} & \longrightarrow & \mathbb{R} \\
 & x & \mapsto & 2x - \frac{1}{2}
\end{array}
$$




**Example 2.** *Determine the oblique asymptote of*

$$
\begin{array}{rccc}
f: & \mathbb{R} & \longrightarrow & \mathbb{R} \\
 & x & \mapsto & 3x + \frac{1}{x}
\end{array}
$$

*The function f is clearly continuous for* $x >> 0$. *We have that*

$$
\begin{aligned}
\lim_{x \to +\infty} \frac{3x + \frac{1}{x}}{x} &= \lim_{x \to +\infty} \frac{3x}{x} + \frac{1}{x^2} \\
&= \lim_{x \to +\infty} 3 + 0 = 3
\end{aligned}
$$

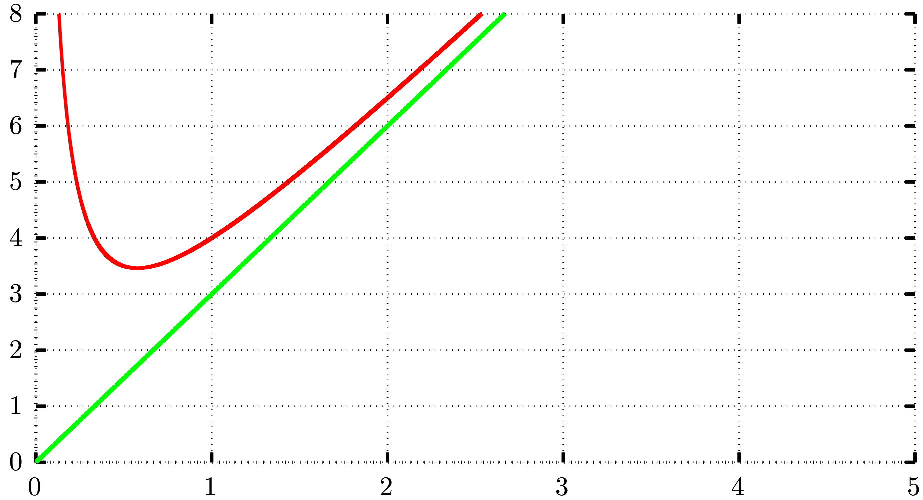
*and*

$$
\lim_{x \to +\infty} 3x + \frac{1}{x} - 3x = \lim_{x \to +\infty} \frac{1}{x} = 0
$$

*The oblique asymptote of f is*

$$
\begin{array}{rccc}
r: & \mathbb{R} & \longrightarrow & \mathbb{R} \\
 & x & \mapsto & 3x
\end{array}
$$

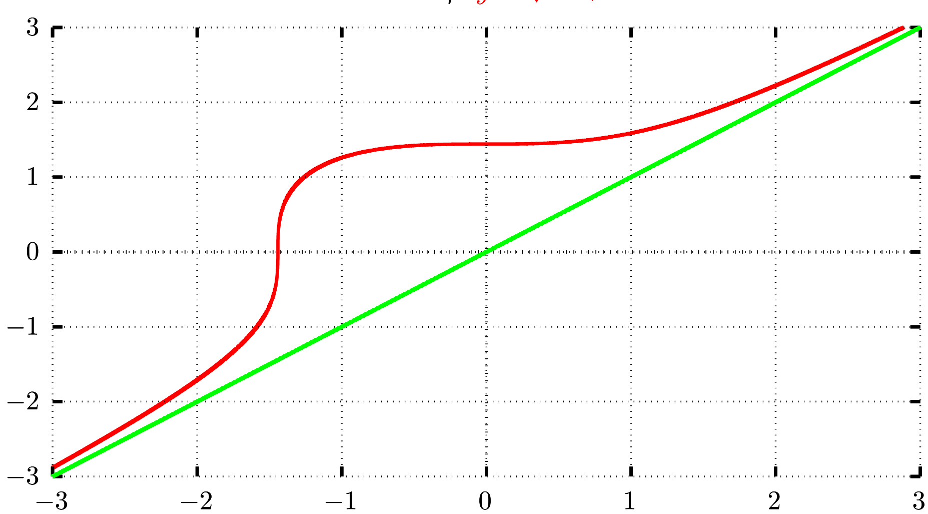
*GNUA13:* $y = 3x + \frac{1}{x}$




**Example 3.** *Draw the graph of the function*

$$\begin{array}{rccc} f: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \sqrt[3]{x^3+3} \end{array}$$

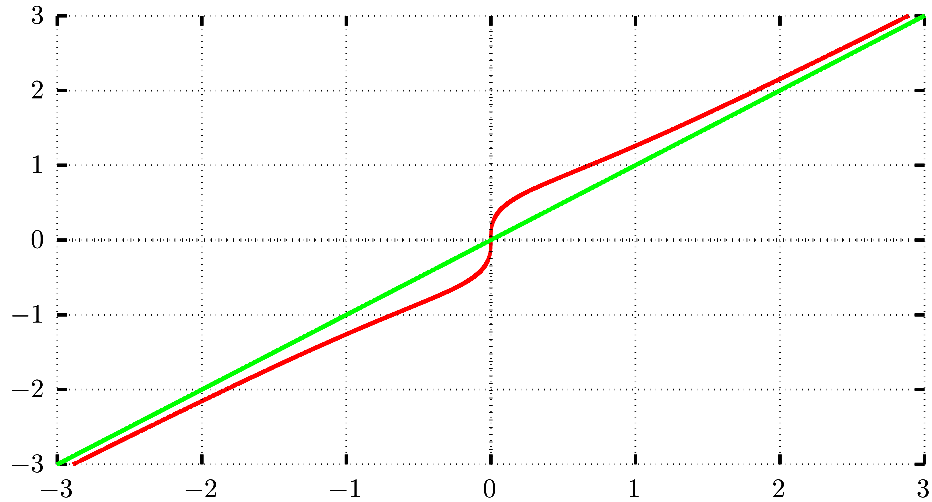
*GNUA14:* $y = \sqrt[3]{x^3+3}$




**Example 4.** *Draw the graph of the function*

$$\begin{array}{rccc} f: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \sqrt[3]{x^3+x} \end{array}$$

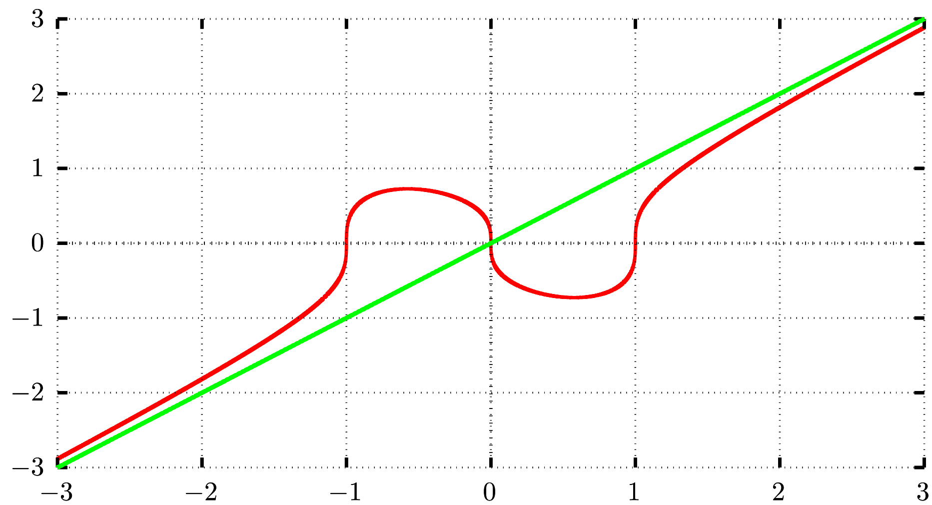
GNUA15: $y = \sqrt[3]{x^3 + x}$



**Example 5.** *Draw the graph of the function*

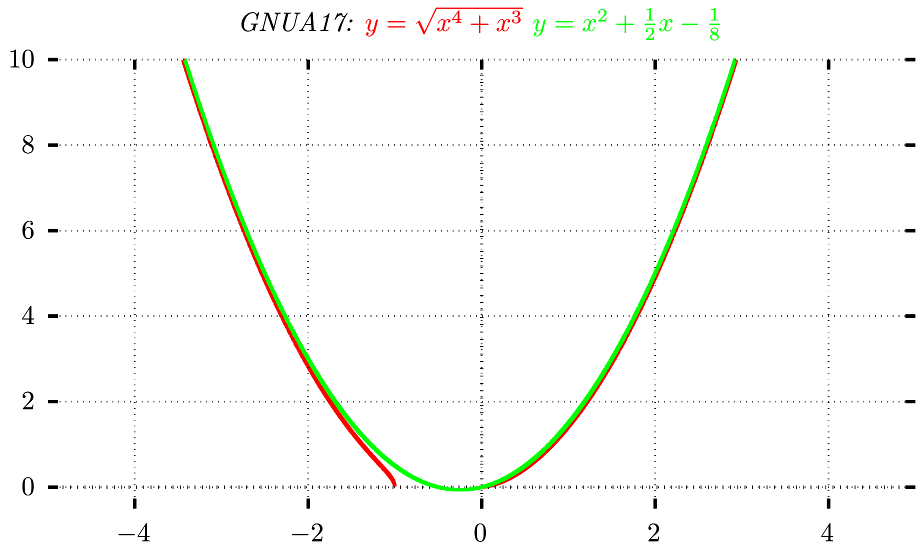
$$f: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & \sqrt[3]{x^3 - x} \end{array}$$

GNUA16: $y = \sqrt[3]{x^3 - x}$



**Example 6.** *Draw the graph of the function*

$$f: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & \sqrt{x^4 + x^3} \end{array}$$




**Example 7.** *Determine the oblique asymptote of*

$$f:\begin{array}{ccc}\mathbb{R} & \longrightarrow & \mathbb{R}\\ x & \mapsto & \frac{x^2-1}{x+3}\end{array}$$

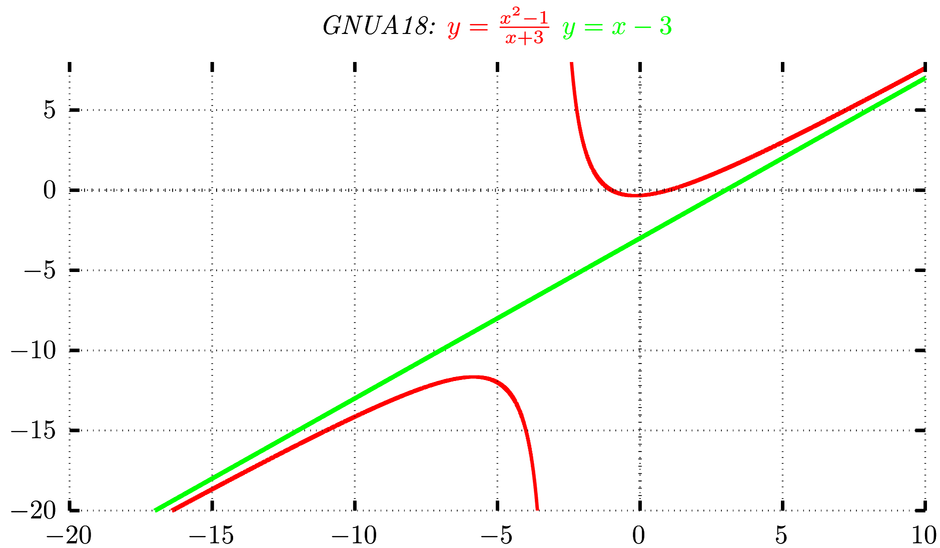
*The function $f$ is clearly continuous for $x >> 0$. Let us proceed with the polynomial division*

$$\begin{array}{rr|l} x^2 & -1 & x+3 \\ \cline{3-3} -x^2-3x & & x-3 \\ \cline{1-1} -3x & -1 & \\ 3x & +9 & \\ \cline{1-2} & 8 & \end{array}$$

*We have that $x^2-1=(x+3)(x-3)+8$ and so*

$$\frac{x^2-1}{x+3}=\frac{(x+3)(x-3)+8}{x+3}=\frac{(x+3)(x-3)}{x+3}+\frac{8}{x+3}=(x-3)+\frac{8}{x+3}$$

*It is clear that the asymptote of $\frac{x^2-1}{x+3}$ is $x-3$*




**Example 8.** *Determine the oblique asymptote of*

$$f : \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & \frac{x^4-2x}{x^2+1} \end{array}$$

*The function* $f$ *is clearly continuous for* $x >> 0$*. Let us proceed with the polynomial division*

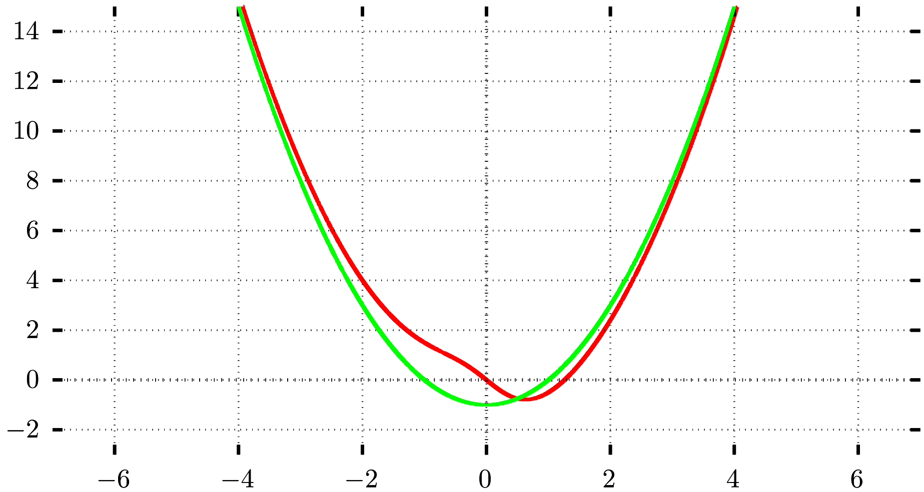
$$\begin{array}{rrr|l} x^4 & & -2x & x^2+1 \\ \cline{4-4} -x^4 & -x^2 & & x^2-1 \\ \cline{1-2} & -x^2 & -2x & \\ & x^2 & +1 & \\ \cline{2-3} & & -2x+1 & \end{array}$$

*We have that* $x^4 - 2x = (x^2+1)(x^2-1) - 2x + 1$ *and so*

$$\frac{x^4-2x}{x^2+1} = \frac{(x^2+1)(x^2-1) - 2x + 1}{x^2+1} = x^2 - 1 + \frac{-2x+1}{x^2+1}$$

*It is clear that the asymptote of* $\frac{x^4-2x}{x^2+1}$ *is* $x^2 - 1$

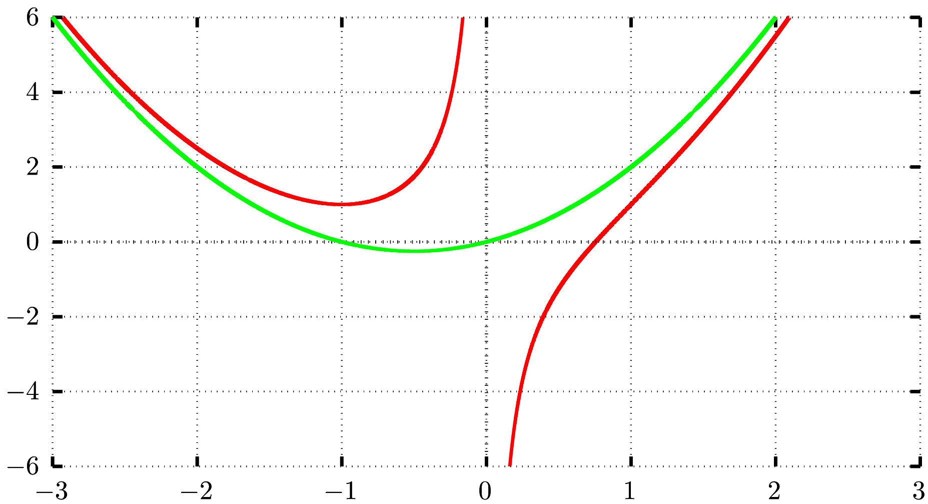
*GNUA18:* $y = \frac{x^4-2x}{x^2+1}$ $y = x^2 - 1$



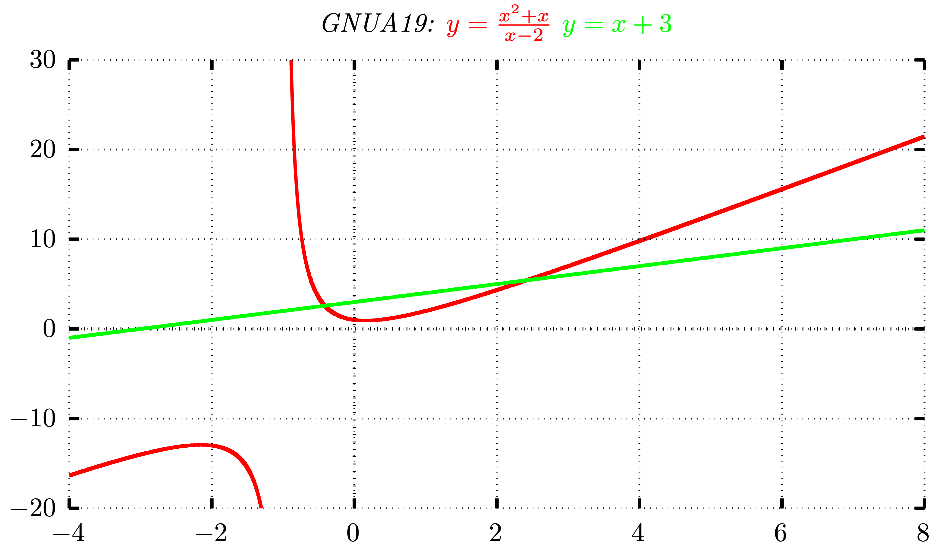

## 4 Fifth lesson classwork

**Example 9.** *Determine the oblique asymptote of*

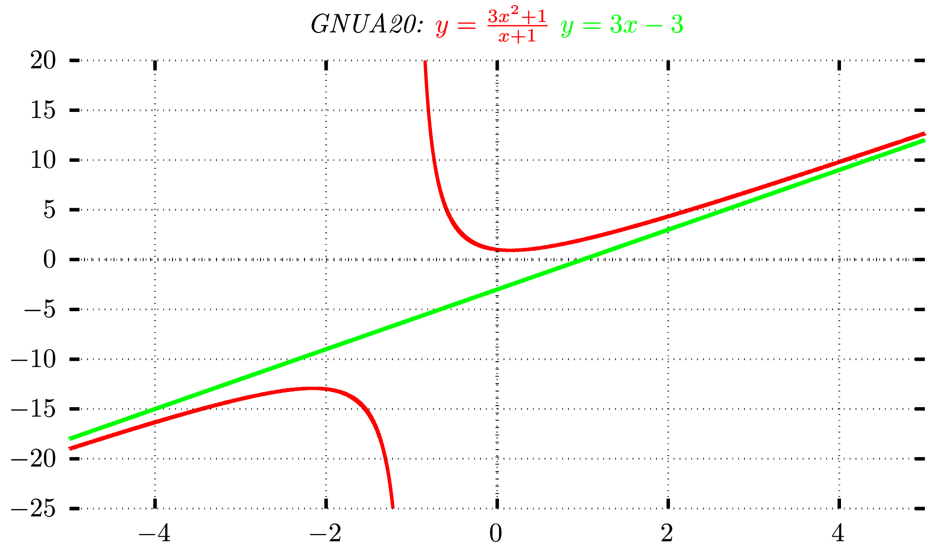
*GNUA19:* $y = x^2 + x - 1/x$ $y = x^2 + x$



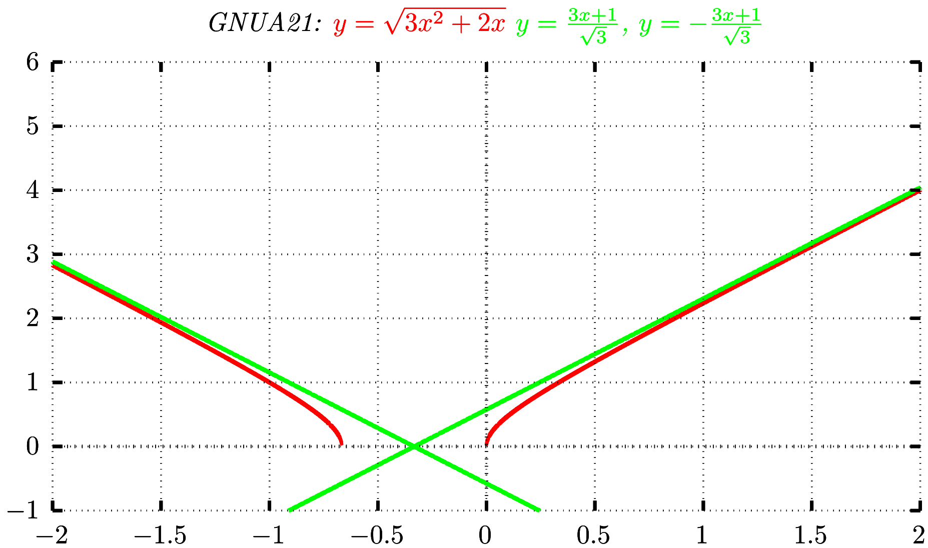

**Example 10.** *Determine the oblique asymptote of*



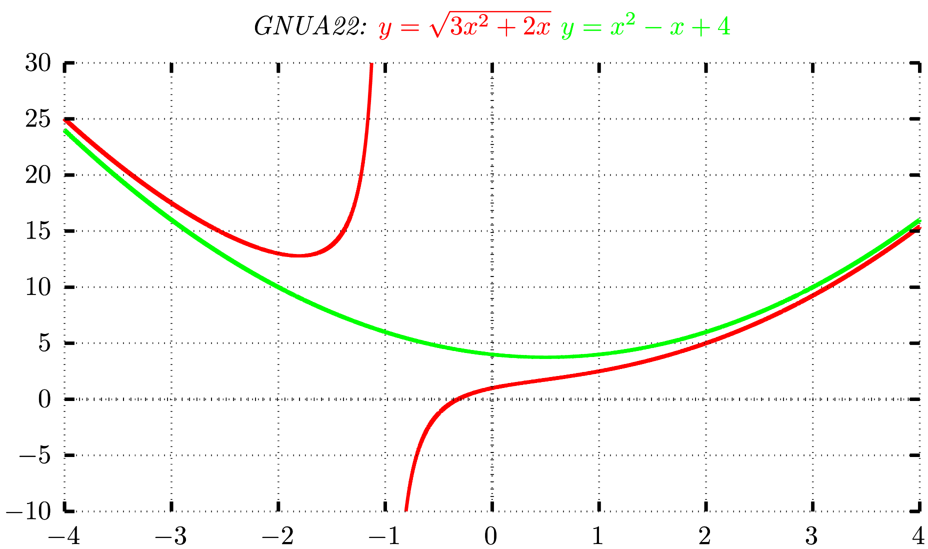

**Example 11.** *Determine the oblique asymptote of*




**Example 12.** *Determine the oblique asymptote of*




**Example 13.** *Determine the oblique asymptote of*




## 5 Fifth lesson homework

Due April $20^{th}$

**Exercise 1.** *Find the asymptotes, if any exist, of the following formulas:*

1. $$F: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & \sqrt{x^2+2x-1} \end{array}.$$

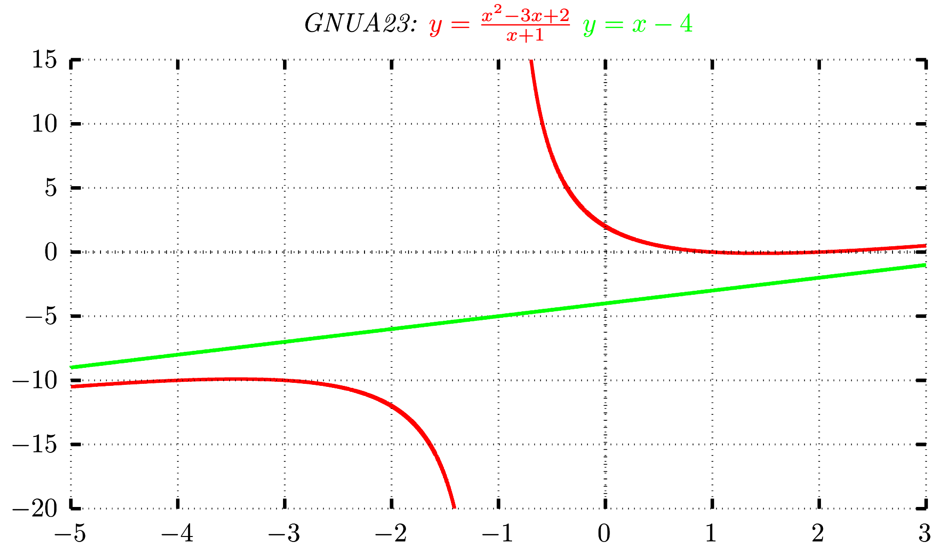
2. $$F : \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & \frac{x^2-x+3}{x+3} \end{array}.$$

3. $$F : \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & \frac{\sqrt{x^4+2}}{x-2} \end{array}.$$

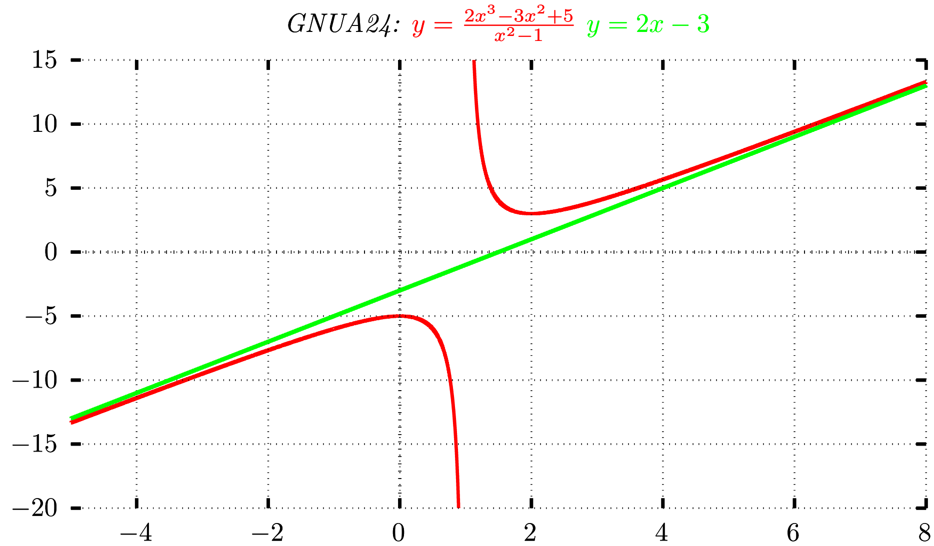
**Exercise 2.** *Draw the graphs of the formulas in the previous exercise:*

## 6 Sixt lesson classwork and more

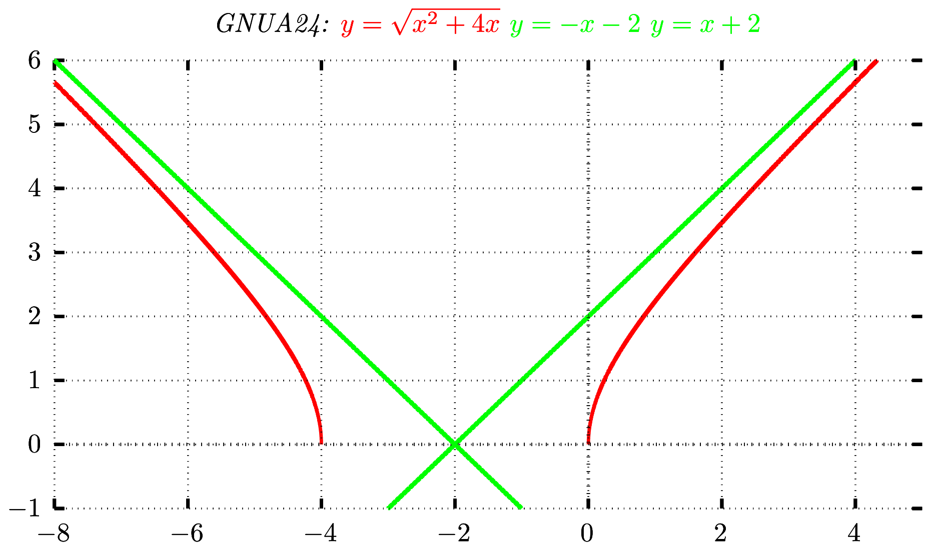
**Example 14.** *Draw*



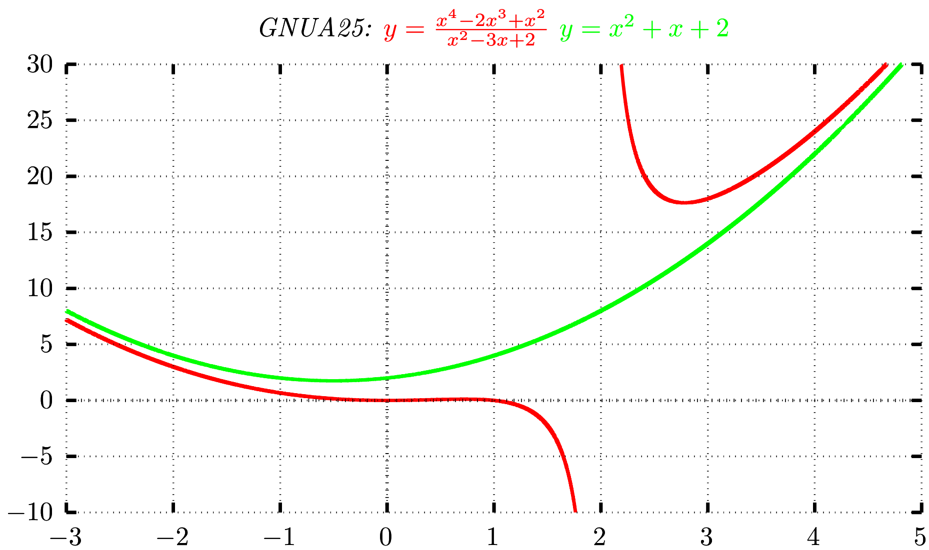

**Example 15.** *Draw*



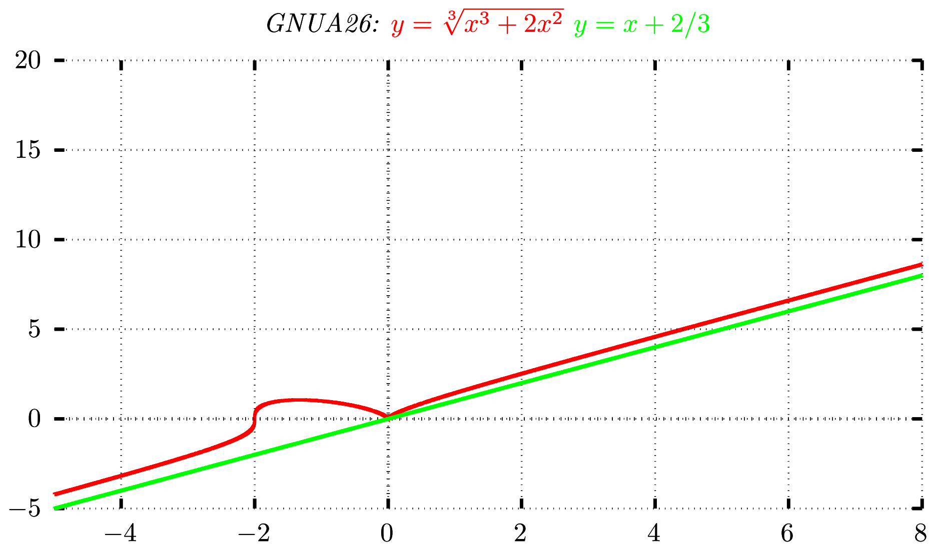
**Example 16.** *Draw*



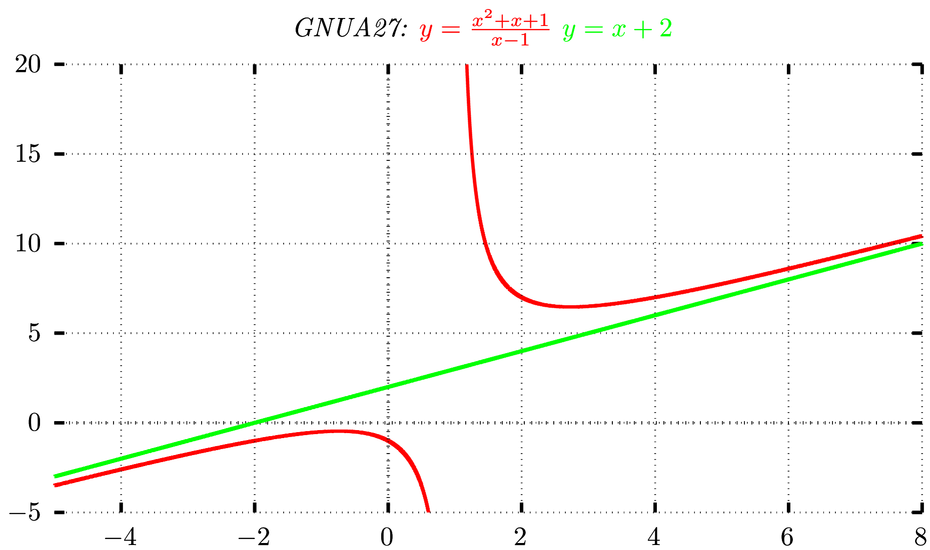

**Example 17.** *Draw*



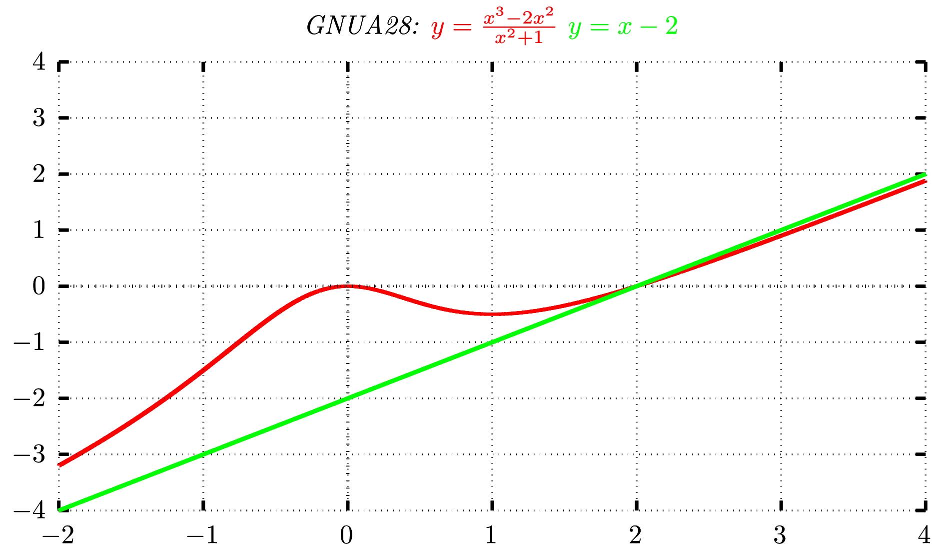
**Example 18.** *Draw*




**Example 19.** *Draw*



**Example 20.** *Draw*




**Example 21.** *Draw*