FCS
Math: Functions

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1 Sets

A quick review of the definition for sets and set operations.

Definition 1. *Given* A, B *sets,*

- $A \subseteq B \iff$ *for all* $a \in A$ *we have* $a \in B$.
- $A = B \iff A \subseteq B$ *and* $A \supseteq B$.

It is easy to see that

- $A \not\subseteq B \iff$ *there is* $a \in A,\ a \notin B$.
- $A \neq B \iff$ (*there is* $a \in A,\ a \notin B$) *or* (*there is* $b \in B,\ a \notin A$).

We can write he previous definition using the symbols $\forall$ for "for all" and $\exists$ for "there is":

Definition 2. *Given* A, B *sets,*

- $A \subseteq B \iff \forall\ a \in A$ *we have* $a \in B$.
- $A = B \iff A \subseteq B$ *and* $A \supseteq B$.

It is easy to see that

- $A \not\subseteq B \iff \exists\ a \in A,\ a \notin B$.
- $A \neq B \iff (\exists\ a \in A,\ a \notin B)$ *or* $(\exists\ b \in B,\ a \notin A)$.

Definition 3. *Given* A, B *subsets of a set* U *we have that*

- $A \cup B = \{c \in U \ \mid\ c \in A \text{ or } c \in B\}$
- $A \cap B = \{c \in U \ \mid\ c \in A \text{ and } c \in B\}$
- $A - B = \{c \in U \ \mid\ c \in A \text{ and } c \notin B\}$

Definition 4. *Given A, B sets $A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}$ is the* cartesian product *of A, B.*

Example 1.

1. $\{1,2,3\} \times \{7,11\} = \{(1,7),(1,7),(2,7),(2,11),(3,11),(3,11)\}$
2. $\mathbb{N} \times \{1,3\} = \{(0,1),(2,1),(7,1),(4,3),(0,3),(11,3),\ldots\}$
3. $\mathbb{R} \times \mathbb{R} = \{(x,y) \mid x,y \in \mathbb{R}\} = \mathbb{R}^2$ *the real plane.*

2 Exercises

Exercise 1. *Given the sets*

$$A = \{1,3,5,7,14\} \text{ and } B = \{-2,3,4,8\}$$

Describe $A \cup B$, $A \cap B$, $A - B$, $A \times B$.

Solution: The first solutions are trivial,

$$\begin{aligned} A \times B \;=\; & \{(\,(1,-2),(1,3),(1,4),(1,8),(3,-2),(3,3),(3,4),(3,8),(5,-2),(5,3),(5,4), \\ & (5,8),(7,-2),(7,3),(7,4),(7,8),(14,-2),(14,3),(14,4),(14,8)\,)\} \end{aligned}$$

Exercise 2. *Given the sets*

$$A = \{a \in \mathbb{N} \mid a \text{ is a multiple of } 12\} \text{ and } B = \{k \in \mathbb{N} \mid k \text{ is a multiple of } 15\}$$

Describe $A \cup B$, $A \cap B$, $A - B$, $A \times B$.

Solution:

$$\begin{aligned} A \cup B &= \{12,15,24,30,36,45,48,60,72,65,\ldots\} \\ A \cap B &= \{k \in \mathbb{N} \mid k \text{ is a multiple of } LCM(12,15) = 60\} \\ A - B &= \{12,24,36,48,72,\ldots\} \\ A \times B &= \{(a,b) \in \mathbb{N}^2 \mid a \text{ is a multiple of } 12 \text{ and } b \text{ is a multiple of } 15\} \end{aligned}$$

Exercise 3. *Given the sets*

$$A = \{(a,a^2) \mid a \in \mathbb{R}\} \text{ and } A = \{(b,b) \mid b \in \mathbb{R}\}$$

Describe $A \cup B$, $A \cap B$, $A - B$, $A \times B$.

Solution:

The set A is the parabola $y = x^2$, the set B the line $y = x$. The set $A \cup B$ is the union of the two drawing, the set $A \cap B$ is just the two points $\{(0,0),(1,1)\}$, the set $A - B$ is the parabola minus the points $\{(0,0),(1,1)\}$ and

$$A \times B = \{(a,a^2,b,b) \in \mathbb{R}^4 \mid a,b \in \mathbb{R}\}$$

Exercise 4. *Draw on the $\mathbb{R}^2$ plane the sets*

$$[1,2] \times [1,1],\ \ [-1,2] \times [2,+\infty],\ \ \{(1,1),(2,3),(3,7)\}$$

.

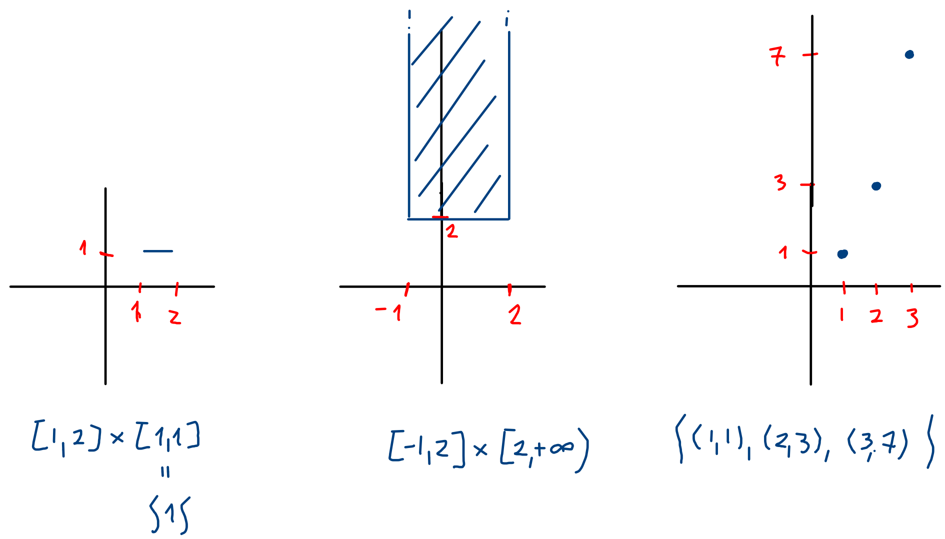


Figure 1:

Exercise 5. *Given the sets*

$$\begin{array}{l} A = \{a \in \mathbb{R} \ \mid \ x^3 - 4x^2 + x + 6 = 0 \in \mathbb{R}\} \\ B = \{a \in \mathbb{R} \ \mid \ x^4 - 5x^2 + 4 = 0 \in \mathbb{R}\} \\ C = \{a \in \mathbb{R} \ \mid \ x^4 - 2x^3 + 4x^2 - 6x + 3 = 0 \in \mathbb{R}\} \end{array}$$

Detail the equalities and inclusions between A, B, C

Solution:
We have that

$$\begin{array}{rcl} x^3 - 4x^2 + x + 6 & = & (x+1)(x-2)(x-3) \\ x^4 - 5x^2 + 4 & = & (x+2)(x-2)(x+1)(x-1) \\ x^4 - 2x^3 + 4x^2 - 6x + 3 & = & (x^2+3)(x-1)^2 \end{array}$$

Hence

$$\begin{array}{rcl} A & = & \{1,2,3\} \\ B & = & \{\pm 1, \pm 2\} \\ C & = & \{1\} \end{array}$$

Hence there are no equalities and the only inclusions are

$$C \subset A, B$$

3 Functions

Definition 5. *For a function*

$$F: \begin{array}{ccc} A & \longrightarrow & B \\ a & \mapsto & F(a) \end{array},$$

we say that A is the domain *of F, B is the* codomain *of F and $F(a)$ is the* rule *or* formula *of F if for all the $a \in A$ the $F(a)$ is well defined. Otherwise, we call F a mere* formula*.*

Example 2.

1. $F: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & \frac{1}{x} \end{array}$, *is a mere formula, because $F(0) = \frac{1}{0}$ does not exists.*
2. $F: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & x^2 - 1 \end{array}$, *is a function, because $F(x) = x^2 - 1$ always exists.*

Definition 6. *Given a formula $F : A \longrightarrow B$, the set*

$$\mathrm{EF}(F) = \{a \in A \ \mid \ F(a) \text{ exists}\}$$

is the definition field *or* existence field *of F.*

Example 3.

1. *Given the formula* $F: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & \frac{1}{x} \end{array}$ *we have* $\mathrm{ExF}(F) = \mathbb{R} - \{0\}$
2. *Given the formula* $F: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & \frac{x}{x^2-4} \end{array}$ *we have* $\mathrm{ExF}(F) = \mathbb{R} - \{\pm 2\}$
3. *Given the formula* $F: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & \sqrt{x} \end{array}$ *we have* $\mathrm{ExF}(F) = \mathbb{R}_0^+ = \{x \in \mathbb{R} \ \mid \ x \geq 0\}$

Exercise 6.

Given the formula

$$F: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & x^3 + 2x^2 - x + 3 \end{array}$$

Find $\mathrm{ExF}(F)$.

Remark 1. *If* $F : A \longrightarrow B$ *is a formula,* $F : \mathrm{ExF}(F) \longrightarrow B$ *is a function.*

Definition 7. *Given the functions*

$$F: \begin{array}{ccc} A & \longrightarrow & B \\ a & \mapsto & F(a) \end{array}, \qquad G: \begin{array}{ccc} C & \longrightarrow & D \\ c & \mapsto & G(c) \end{array}$$

we have

$$F \equiv G$$

if and only if

$$A = C, \ B = D, \ \ \forall \, a \in A, \ F(a) = G(a)$$

We say that F, G are equal as functions.

Example 4.

1. *Given*

$$F: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ a & \mapsto & \sin^2 a + \cos^2 a \end{array}, \qquad G: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ c & \mapsto & 1 \end{array}$$

we have $F \equiv G$ *because for every* $x \in \mathbb{R}$ *we have*

$$F(x) = \sin^2 x + \cos^2 x = 1 = G(x)$$

2. *Given*

$$F: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & \sqrt{x^2} \end{array}, \qquad G: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & x \end{array}$$

we have $F \not\equiv G$ *because for every* $x = -1$ *we have*

$$F(-1) = \sqrt{(-1)^2} = \sqrt{1} = 1 \neq -G(-1)$$

Exercise 7. *We have the function*

$$F: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & 3x^2 - x + 1 \end{array}$$

Compute

1. $F(1) = F(-1)$*? [* $F(-1) = 5 \neq 3 = F(1)$ *]*

2. $F(1), F(0), F(5)$. *[*$F(-1) = 5, \ F(0) = 1, \ F(5) = 71$*]*

3. *Given $a \in \mathbb{R}$, compute $F(a-1), F(3a^2-2), F(\sqrt{a^2+1})$.*

$$\begin{aligned} F(a-1) &= 3(a-1)^2 - (a-1) + 1 = 3a^2 - 7a + 5 \\ F(3a^2-2) &= 3(3a^2-2)^2 - (3a^2-2) + 1 = 27a^4 - 39a^2 + 15 \\ F(\sqrt{a^2+1}) &= 3(\sqrt{a^2+1})^2 - (\sqrt{a^2+1}) + 1 = 3a^2 - \sqrt{a^2+1} + 4 \end{aligned}$$

4. *Given $\clubsuit \in \mathbb{R}$, with $\clubsuit > 0$, compute $F(\clubsuit - 2)$, $F(2\clubsuit)$, $F(\sqrt{\clubsuit})$.*

   $F(\clubsuit-2) = 3\clubsuit^2 - 13\clubsuit + 15$. $F(2\clubsuit) = 12\clubsuit^2 - 2\clubsuit + 1$, $F(\sqrt{\clubsuit}) = 3\clubsuit - \sqrt{\clubsuit} + 1$

Exercise 8. *We have the function*

$$F: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & x^2+1 \end{array}$$

Compute

1. *$F(1) = F(-1)$? [YES]*
2. *For which $a \in \mathbb{R}$ we have $F(a) = F(-a)$. [Solutions of $a^2+1 = a^2+1$, so for all $a \in \mathbb{R}$]*
3. *For which $y \in \mathbb{R}$ we have $F(y) = F(y+1)$. [Solutions of $y^2+1 = (y+1)^2$, so $y = 0$]*
4. *For which $b \in \mathbb{R}$ we have $F(b+2) = F(2b+3)$. [Solutions of $(b+2)^2+1 = (2b+3)^2+1$, so $b = -5/3, -1$]*

Exercise 9. *We have the functions*

$$F: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & x^2 \end{array}, \qquad G: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & x \end{array}$$

1. *Is it true that $F \equiv G$? [NO, there is an $a \in \mathbb{R}$ such that $F(a) \neq G(a)$, for exmple $a = 2$, for which $F(2) = 4 \neq 2 = G(2)$]*
2. *Is there an $a \in \mathbb{R}$ such that $F(a) = G(a)$? [YES, for example $a = 1$, since $F(1) = 1 = G(1)$]*

Exercise 10. *We try to describe a function by*

$$F: \begin{array}{ccc} \mathbb{Q} & \longrightarrow & \mathbb{Q} \\ p/q & \mapsto & p^2/q^2 \end{array}$$

Is F a well defined function?

Solution: YES.

Exercise 11. *We try to describe a function by*

$$F: \begin{array}{ccc} \mathbb{Q} & \longrightarrow & \mathbb{Q} \\ p/q & \mapsto & p+q \end{array}$$

Is F a well defined function? If not, how can we modify the formula of F to have a well defined function?

Solution: NO, because $\frac{6}{3} = \frac{4}{2}$ but $F\left(\frac{6}{3}\right) = 9 \neq 6 = F\left(\frac{4}{2}\right)$

4 Injective and Surjective Functions

Definition 8. *Given the sets A, B, the function*

$$F: \begin{array}{ccc} A & \longrightarrow & B \\ a & \mapsto & F(a) \end{array}$$

is injective *if and only if* $\forall b \in B$ *the equation $F(a) = b$ has at most 1 solution in A or, equivalently, if any element of B it is reached by at most one element of A through F. More precisely, if*

$$\forall\ a, b \in A \quad F(a) = F(b) \Longleftrightarrow a = b$$

Definition 9. *Given the sets A, B, the function*

$$F: \begin{array}{ccc} A & \longrightarrow & B \\ a & \mapsto & F(a) \end{array}$$

is surjective *if and only if* $\forall b \in B$ *the equation $F(a) = b$ has at least 1 solution in A or, equivalently, if any element of B it is reached by at least one element of A through F.*

Definition 10. *Given a function*

$$F: \begin{array}{ccc} A & \longrightarrow & B \\ a & \mapsto & F(a) \end{array}$$

the set

$$Im(F) = \{F(a) \ \mid \ a \in A\} = \{b \in B \ \mid \ \exists\ a \in A \text{ such that } b = F(a)\} \subseteq B$$

is the image *or* range *of F.*

Exercise 12. *Are the following functions injectives?.*

1.

$$F: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & F(x) = 3 - 2x \end{array}$$

Yes, because $\forall\ b \in \mathbb{R}$ the equation $F(x) = b \Longleftrightarrow 3 - 2x = b$ has exactly the solution $x = \frac{3-b}{2}$

2.

$$\begin{array}{rccc} F: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & F(x) = x^4 + 3x^2 + 3 \end{array}$$

No, because $F(1) = F(-1)$

3.

$$\begin{array}{rccc} F: & \mathbb{N} & \longrightarrow & \mathbb{N} \\ & n & \mapsto & F(n) = n^2 \end{array}$$

Yes, because $F(a) = G(b) \iff a^2 = b^2 \iff a = b$ *for positive numbers.*

4.

$$\begin{array}{rccc} F: & \mathbb{Z} & \longrightarrow & \mathbb{Z} \\ & n & \mapsto & F(n) = 3n^4 + n^2 \end{array}$$

No, because $F(-1) = F(1)$

5.

$$\begin{array}{rccc} F: & \{1,2,3,4,5,6\} & \longrightarrow & \{1,2,3,4,5,6\} \\ & 1 & \mapsto & 2 \\ & 2 & \mapsto & 4 \\ & 3 & \mapsto & 1 \\ & 4 & \mapsto & 6 \\ & 5 & \mapsto & 5 \\ & 6 & \mapsto & 3 \end{array}$$

Yes

6.

$$\begin{array}{rccc} F: & \{1,2,3,4,5,6\} & \longrightarrow & \{1,2,3,4,5,6\} \\ & 1 & \mapsto & 2 \\ & 2 & \mapsto & 4 \\ & 3 & \mapsto & 1 \\ & 4 & \mapsto & 2 \\ & 5 & \mapsto & 5 \\ & 6 & \mapsto & 3 \end{array}$$

No

Exercise 13. *Are the following functions surjectives?.*

1.

$$\begin{array}{rccc} F: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & F(x) = 3x + 1 \end{array}$$

Yes, because $\forall\ b \in \mathbb{R}$ *the equation* $F(x) = b \iff 3x + 1 = b$ *has at least the solution* $x = \frac{b-1}{3}$

2.

$$\begin{array}{rccc} F: & \mathbb{N} & \longrightarrow & \mathbb{N} \\ & n & \mapsto & F(n) = x^2 \end{array}$$

No, because 3 *is not hit by anything, or, more precisely, because the equation* $F(x) = 3 \iff n^2 = 3$ *has no solution in the integers.*

3.

$$\begin{array}{rccc} F: & \{1,2,3,4,5,6\} & \longrightarrow & \{1,2,3,4,5,6\} \\ & 1 & \mapsto & 2 \\ & 2 & \mapsto & 4 \\ & 3 & \mapsto & 1 \\ & 4 & \mapsto & 6 \\ & 5 & \mapsto & 5 \\ & 6 & \mapsto & 3 \end{array}$$

Yes

4.

$$\begin{array}{rccc} F: & \{1,2,3,4,5,6\} & \longrightarrow & \{1,2,3,4\} \\ & 1 & \mapsto & 2 \\ & 2 & \mapsto & 1 \\ & 3 & \mapsto & 1 \\ & 4 & \mapsto & 2 \\ & 5 & \mapsto & 2 \\ & 6 & \mapsto & 3 \end{array}$$

No

Exercise 14. *Let A be a set with 10 elements and B a set with 9 elements. Is it possibile for a function $f : A \longrightarrow B$ to be injective ? [No]*

Exercise 15. *Let A be a set with 10 elements and B a set with 11 elements. Is it possibile for a function $f : A \longrightarrow B$ to be surjective ? [No]*

Definition 11. *Given the function*

$$\begin{array}{rccc} F: & A & \longrightarrow & B \\ & a & \mapsto & F(a) \end{array}$$

and $C \subset A$, the function

$$\begin{array}{rccc} G: & C & \longrightarrow & B \\ & a & \mapsto & F(a) \end{array}$$

is called the restriction *of F to C and is written $F_{|C}$*

5 Function Composition. Inverse

Definition 12. *Given two functions*

$$\begin{array}{rccc} F: & A & \longrightarrow & B \\ & a & \mapsto & F(a) \end{array}, \quad \begin{array}{rccc} G: & D & \longrightarrow & C \\ & d & \mapsto & G(d) \end{array}$$

such that $D \subseteq Im(F)$ the function

$$\begin{array}{rccc} G \circ F: & A & \longrightarrow & C \\ & a & \mapsto & G(F(a)) \end{array}$$

is the composition *of G, F.*

Remark 2. *If we have the two functions*

$$F: \begin{array}{ccc} A & \longrightarrow & A \\ a & \mapsto & F(a) \end{array}, \qquad G: \begin{array}{ccc} A & \longrightarrow & A \\ a & \mapsto & G(a) \end{array}$$

it may happen that $F \circ G \not\equiv G \circ F$. *Consider for example*

$$F: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & F(x) = 5x - 2 \end{array}, \qquad G: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & G(x) = x^2 - 1 \end{array}$$

It is easy to verify that for most $x \in \mathbb{R}$

$$\begin{aligned} F \circ G(x) &= F(G(x)) \\ &= F(x^2 - 1) \\ &= 5(x^2 - 1) - 2 \\ &= 5x^2 - 7 \end{aligned}$$

and

$$\begin{aligned} G \circ F(x)(x) &= G(F(x)) \\ &= G(5x - 2) \\ &= (5x - 2)^2 - 1 \\ &= 25x^2 - 20x + 3 \end{aligned}$$

and is is immediate to see that there exists at least one x *(for example* 0*) such that*

$$5x^2 - 7 \neq 25x^2 - 20x + 3$$

Hence $F \circ G(x) \not\equiv G \circ F(x)$

Definition 13. *The function*

$$\mathrm{id}_A: \begin{array}{ccc} A & \longrightarrow & A \\ a & \mapsto & \mathrm{id}_A(a) = a \end{array}$$

is called the identity *of* A.

Definition 14. *Given a function*

$$F: \begin{array}{ccc} A & \longrightarrow & B \\ a & \mapsto & F(a) \end{array}$$

if there is a function

$$G: \begin{array}{ccc} B & \longrightarrow & A \\ b & \mapsto & G(b) \end{array}$$

such that

$$F \circ G \equiv \mathrm{id}_B \textit{ and } G \circ F \equiv id_A$$

we say that F is invertible, G is the inverse of G *and we write* $G \equiv F^{-1}$

Proposition 1. *A function*

$$\begin{array}{rccc} F: & A & \longrightarrow & B \\ & a & \mapsto & F(a) \end{array}$$

is invertible if and only if

$$\forall\ b \in B\ \exists! a \in A \text{ such that } F(a) = b$$

or, rephrased, for every $b \in B$ the equation $F(a) = b$ has exactly one solution in A. In other terms, a function is ivertible if and only if it is both injective and surjective.

Remark 3. *Using another notation, perhaps more familiar, a function*

$$\begin{array}{rccc} F: & A & \longrightarrow & B \\ & a & \mapsto & F(a) \end{array}$$

is invertible if and only if F is a one-to-one correspondence between the sets A and B.

Remark 4. *If we have an invertible function $f : A \longrightarrow B$, it is immediate that $\left(f^{-1}\right)^{-1} = f$, since, by definition, if f^{-1} is the inverse of f, then f is the inverse of f^{-1}.*

Remark 5. *Let us consider the invertible functions $F : A \longrightarrow B$ and $G : B \longrightarrow C$ and the function*

$$\begin{array}{rccc} G \circ F: & A & \longrightarrow & C \\ & a & \mapsto & G \circ F(a) = G(F(a)) \end{array}$$

It is easy to see that $G \circ F$ is invertible and the function

$$\begin{array}{rccc} H: & C & \longrightarrow & A \\ & c & \mapsto & (F^{-1} \circ G^{-1})(c) = F^{-1}(G^{-1}(c)) \end{array}$$

is its inverse.

Since $F : A \longrightarrow B$ is invertible, the function $F^{-1} : B \longrightarrow A$ exists and $F^{-1} \circ F \equiv \mathrm{id}_A$, $F \circ F^{-1} \equiv \mathrm{id}_B$.

Since $G : B \longrightarrow C$ is invertible, the function $G^{-1} : C \longrightarrow B$ exists and $G \circ G^{-1} \equiv \mathrm{id}_C$, $G^{-1} \circ G \equiv \mathrm{id}_B$.

Hence, for all $a \in A$,

$$\begin{aligned} (F^{-1} \circ G^{-1}) \circ (G \circ F)(a) &= (F^{-1} \circ G^{-1} \circ G \circ F)(a) \\ &= (F^{-1} \circ \mathrm{id}_C \circ F)(a) = (F^{-1} \circ F)(a) \\ &= \mathrm{id}_A(a) = a \end{aligned}$$

so $(F^{-1} \circ G^{-1}) \circ (G \circ F) \equiv \mathrm{id}_A$.

For all $c \in C$,

$$\begin{aligned}(G \circ F) \circ (F^{-1} \circ G^{-1})(c) &= (G \circ F \circ F^{-1} \circ G^{-1})(c) \\ &= (G \circ \mathrm{id}_A \circ G^{-1})(c) \\ &= (G \circ G^{-1})(c) = c\end{aligned}$$

so $(G \circ F) \circ (F^{-1} \circ G^{-1}) \equiv \mathrm{id}_C$.

Definition 15. *Let* A, B *be sets. We say that* A has the same cardinalty of B *if and only if there is a one-to-one correspondence between* A *and* B*. We write* $\mid A \mid = \mid B \mid$.

Example 5. *The sets*

$$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots, n, \ldots\} \quad \textit{and} \quad EVEN = \{0, 2, 4, 6, 8, \ldots, 2n, \ldots \ \mid \ n \in \mathbb{N}\}$$

have the same cardinality, because the function

$$\begin{array}{rccc} F: & \mathbb{N} & \longrightarrow & \mathrm{EVEN} \\ & n & \mapsto & 2n \end{array}$$

is invertible (and hence a one-to-one correspondence), since the function

$$\begin{array}{rccc} G: & \mathrm{EVEN} & \longrightarrow & \mathbb{N} \\ & n & \mapsto & n/2 \end{array}$$

is its inverse, as we can see from

$$\forall\ n \in \mathbb{N}\ G \circ F(n) = G(F(n)) = G(2n) = \frac{1}{2}2n = n \Longleftrightarrow G \circ F \equiv \mathrm{id}_{\mathbb{N}}$$

and

$$\forall\ n \in \mathrm{EVEN}\ F \circ G(n) = F(G(n)) = F(\frac{n}{2}) = 2\frac{n}{2} = n \Longleftrightarrow F \circ G \equiv \mathrm{id}_{\mathrm{EVEN}}$$

Exercise 16. *Are to following functions invertible? If the answer is yes, find the inverse if possibile*

1. *if* ODD *is the set of the odd positive numbers* $\begin{array}{rccc} F: & \mathbb{N} & \longrightarrow & \mathrm{ODD} \\ & n & \mapsto & 2n+1 \end{array}$

2. $\begin{array}{rccc} F: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & -3x+4 \end{array}$

3. $\begin{array}{rccc} F: & [-1, +\infty) & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \sqrt{x+1} \end{array}$

4. $$F: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & \mid 2x-1 \mid \end{array}$$

Exercise 17. *Are to following functions invertible? If the answer is no, determine a restriction of the domain and/or codomain that produces and invertible function with the same formula*

1. $$F: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & x^2-4 \end{array}$$

2. $$F: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & x^2+x \end{array}$$

3. $$F: \begin{array}{ccc} [-1,+\infty) & \longrightarrow & \mathbb{R} \\ x & \mapsto & \sqrt{x+1} \end{array}$$

6 Infinity

Example 6 (The salary paradox)**.** *we have two employees, Joe and Jane. Each get* 1000 *euros a month.*

- *Joe every month puts one euro on a pile on his desk and spends the others* 999*. He never draw any money from his pile.*
- *Jane every month puts* 1000 *euros on top of a pile of money on his desk and draws* 500 *euro* from the bottom og the pile.

I argue that "at infinity", Joe has an infinite amount of money, and Jane has nothing, because every single euro that she puts on the pile is, at some moment, spent.

Example 7. *Find a one-to-one correspondence between the sets*

$$A = \{n \in \mathbb{N} \mid n \textit{ is multiple of 3}\} \textit{ and } B = \{n \in \mathbb{N} \mid n \textit{ is multiple of 4}\}$$

We consider the function

$$F: \begin{array}{ccc} A & \longrightarrow & B \\ n & \mapsto & \frac{4}{3}n \end{array}$$

that is well defined because $n \in A$ and hence 3 is a multiple of n, and $n/3 \in \mathbb{N}$.

The function

$$G: \begin{array}{ccc} B & \longrightarrow & A \\ n & \mapsto & \frac{3}{4}n \end{array}$$

is well defined because $n \in B$ and hence 4 is a multiple od n, and $n/4 \in \mathbb{N}$.

The function G is clearly the inverse of F, since

$$G \circ F(n) = G\left(\frac{4}{3}n\right) = \frac{3}{4}\left(\frac{4}{3}n\right) = n \textit{ and } F \circ G(n) = F\left(\frac{3}{4}n\right) = \frac{4}{3}\left(\frac{3}{4}n\right) = n$$

and so F is invertible, a one-to-one correspondence and $\mid A \mid = \mid B \mid$.

Example 8. *Find a one-to-one correspondence between the sets* $A = \{2n+2 \mid n \in \mathbb{N}\}$ *and* $B = \{n^2 \mid n \in \mathbb{N}\}$.

We have $\mid A \mid = \mid \mathbb{N} \mid$ *and* $\mid B \mid = \mid \mathbb{N} \mid$, *so our guess is that* $\mid A \mid = \mid B \mid$ *and thus there is a one-to-one correspondence between* A *and* B, *but we are not sure that the rules for equality apply to the cardinality, so we need to find an explicit invertible function between* A *and* B. *This could be difficult, we know the one-to-one correspondences (invertible functions)*

$$\begin{array}{rccc} F: & \mathbb{N} & \longrightarrow & A \\ & n & \mapsto & 2n+2 \end{array} \quad \text{and} \quad \begin{array}{rccc} G: & \mathbb{N} & \longrightarrow & B \\ & n & \mapsto & n^2 \end{array}$$

we find the inverses solving the equations

$$2n + 2 = a \text{ and } n^2 = b$$

for $n \in \mathbb{N}$, $a \in$ *and* $b \in B$. *We get*

$$n = \frac{a-2}{2} \text{ and } n = \sqrt{b}$$

The inverses are

$$\begin{array}{rccc} F^{-1}: & A & \longrightarrow & \mathbb{N} \\ & n & \mapsto & \frac{n-2}{2} \end{array} \quad \text{and} \quad \begin{array}{rccc} G^{-1}: & B & \longrightarrow & \mathbb{N} \\ & n & \mapsto & \sqrt{n} \end{array}$$

well defined because in the first case, since $n \in A$ *we have* $\frac{n-2}{2} \in \mathbb{N}$ *and in the second case, since* $b \in B$, b *is a perfect square and* $\sqrt{b} \in \mathbb{N}$.

So we need an invertible function

$$A \xrightarrow{H} B$$

while we have

$$\mathbb{N} \underset{F^{-1}}{\overset{F}{\rightleftarrows}} A \quad \text{and} \quad \mathbb{N} \underset{G^{-1}}{\overset{G}{\rightleftarrows}} B$$

The idea is to build the function $A \xrightarrow{H} B$ *using the functions we have*

$$A \xrightarrow{F^{-1}} \mathbb{N} \xrightarrow{G} B$$

so $H \equiv F^{-1} \circ G$ *and so for any* $n \in A$

$$F^{-1} \circ G(n) = F^{-1}(G(n)) = F^{-1}(n^2) = \frac{n^2 - 2}{2}$$

and

$$\begin{array}{rccc} H: & A & \longrightarrow & B \\ & n & \mapsto & \frac{n^2-2}{2} \end{array}$$

is invertible because composition if invertible functions. If we want its explicit inverse,

$$\begin{array}{rccc} H^{-1}: & A & \longrightarrow & B \\ & n & \mapsto & G^{-1}\circ F(n) = \sqrt{2n+2} \end{array}$$

since

$$H \equiv F^{-1}\circ G \Longrightarrow H^{-1} \equiv (F^{-1}\circ G)^{-1} \equiv G^{-1}\circ F$$

Exercise 18. *Find a one-to-one correspondence between the sets* $A = \{n \in \mathbb{N} \mid n \text{ is multiple of } 3\}$ *and* $B = \{n \in \mathbb{N} \mid n \text{ is multiple of } 4\}$

Exercise 19. *Find a one-to-one correspondence between the sets* $A = \{2n+2 \mid n \in \mathbb{N}\}$ *and* $B = \{n^2 \mid n \in \mathbb{N}\}$

Exercise 20. *The sets* $\mathbb{N}, \mathbb{N}^2$ *have the same cardinality?*

Exercise 21. *The sets* $\mathbb{N}, \mathbb{N}^3$ *have the same cardinality? [Difficult]*

Exercise 22 (Hilbert hotel)**.** *We have an hotel with infinite rooms, all occupied. If a new customes comes, can we find a free room for him?*

Example 9. *Do the sets* $\mathbb{N}, \mathbb{Z}$ *have the same cardinality?*

The question is, by definition, equivalent to : there is an invertible function (one-to-one correspondence) between A *and* B*? We build one such function.*

If we rearrange the elements of $\mathbb{N}$ *setting the even numbers before* 0 *and the odd numbers after* 0 *like that* $\mathbb{N} = \{\cdots, 6, 4, 2, 0, 1, 3, 5, \cdots, \}$*, it is natural to define a function like*

$$\begin{array}{ccc} \vdots & \vdots & \vdots \\ 6 & \mapsto & -3 \\ 4 & \mapsto & -2 \\ 2 & \mapsto & -1 \\ 0 & \mapsto & 0 \\ 1 & \mapsto & 1 \\ 3 & \mapsto & 2 \\ 5 & \mapsto & 3 \\ \vdots & \vdots & \vdots \end{array}$$

If we write down the formula for this function we have

$$\begin{array}{rccc} F: & \mathbb{N} & \longrightarrow & \mathbb{Z} \\ & n & \mapsto & \begin{cases} -\frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases} \end{array}$$

that is well defined and invertible. It is well defined because if n *is even,* $-\frac{n}{2} \in \mathbb{Z}$ *and if* n *is odd,* $\frac{n+1}{2} \in \mathbb{Z}$*, so in any case* n *goes to an integer number. It is* *invertibe because the function*

$$G: \quad \mathbb{Z} \longrightarrow \mathbb{N}$$
$$n \mapsto \begin{cases} -2n & \text{if } n \geq 0 \\ 2n-1 & \text{if } n < 0 \end{cases}$$

is its inverse, as we can check. For all $n \in \mathbb{N}$

$$G \circ F(n) = G\left(\begin{cases} -\frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}\right) = \begin{cases} -2(-\frac{n}{2}) & \text{if } n \geq 0 \\ 2(\frac{n+1}{2}) - 1 & \text{if } n < 0 \end{cases} = \begin{cases} n & \text{if } n \geq 0 \\ n & \text{if } n < 0 \end{cases} = n$$

The check $\forall\ n \in \mathbb{Z}\ F \circ G(n) = n$ *is left as an exercise to the reader.*

Definition 16. *Given* A, B, C *sets*

1. $|A| = |B|$ *if and only if there is an invertible function* $F : A \longrightarrow B$.
2. $|A| \neq |B|$ *if and only if there no invertible function* $F : A \longrightarrow B$.
3. $|A| \leq |B|$ *if and only if there is an injective function* $F : A \longrightarrow B$.
4. $|A| < |B|$ *if and only if there is an injective function* $F : A \longrightarrow B$ *but there no invertible function* $F : A \longrightarrow B$.
5. $|A| = |B|$ *and* $|B| = |C|$ *then* $|A| = |C|$. *Since the composition of invertible functions is an invertible function.*
6. *If* A, B *sets* $A \subseteq B$ *then* $|A| \leq |B|$.
7. *(Cantor-Schröder-Bernstein theorem)*

   $$|A| \leq |B| \text{ and } |B| \leq |A| \Longrightarrow |A| = |B|$$

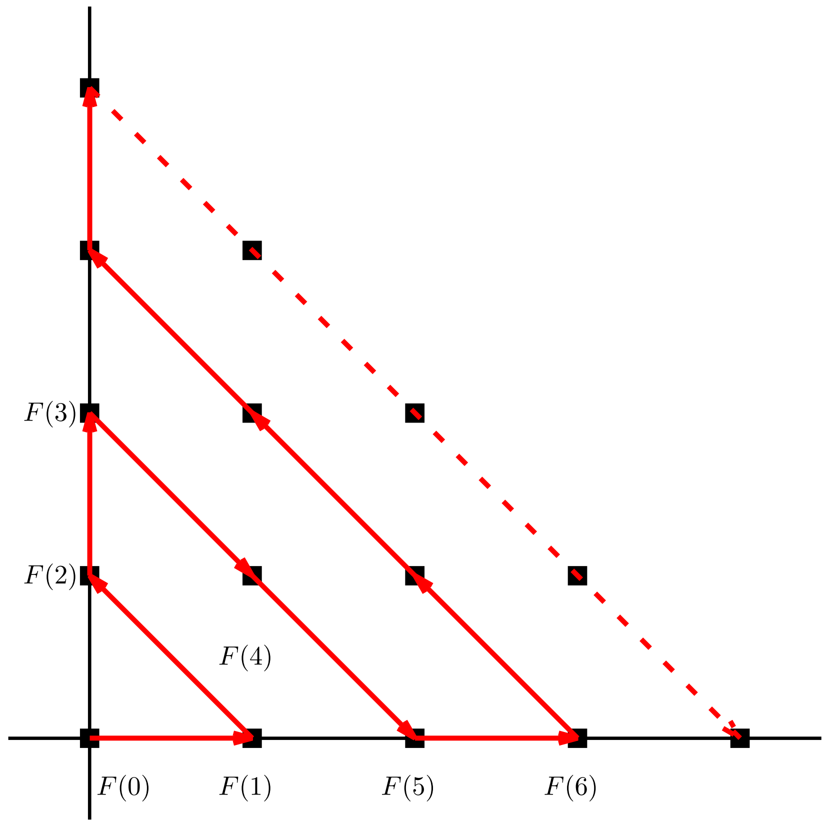
   We can define the sum for infinite cardinalities, but it is weird.
8. *If* A, B *infinite sets and* $|A| = |B|$ *then* $|A| + |B| = |A| = |B|$.
9. *If* A, B *infinite sets and* $|A| > |B|$ *then* $|A| + |B| = |A|$.

   A consequence of Cantor-Schröder-Bernstein theorem:
10. *If* A, B *sets* $|A \cup B| = |A| + |B|$ *and* $|A \cap B| \leq |A|, |B|$. *[This will be shown in class]*

Example 10. *We prove that* $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$ *by building a one-to-one correspondence* $F : \mathbb{N} \longrightarrow \mathbb{N} \times \mathbb{N}$.

We say that $F(0) = (0,0)$, $F(1) = (1,0)$, $F(2) = (0,1)$, $F(3) = (0,2)$, $F(4) = (1,1)$, $F(5) = (2,0)$ *and so on as in the imagine below.*

GNU1 - The one-to-one correspondence $F : \mathbb{N} \longrightarrow \mathbb{N} \times \mathbb{N}$



The function F is a one-to-one correspondence, and so

$$| \mathbb{N} |=| \mathbb{N} \times \mathbb{N} |$$

Example 11. *We prove that*

$$| \mathbb{N} |=| \mathbb{Q} |$$

by building a one-to-one correspondence $F : \mathbb{N} \longrightarrow \mathbb{Q}$. The idea is to mimick the one-to-one correspondence between $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$.

There is a natural function between $\mathbb{Q}$ and $\mathbb{Z} \times \mathbb{N}$,

$$\begin{array}{rccc} T: & \mathbb{Q} & \longrightarrow & \mathbb{N} \times \mathbb{N} \\ & p/q & \mapsto & (p, q) \end{array}$$

with the usual provisos that $q > 0$ and p, q are coprime (no common factor, we don't consider the rationals like $4/2$) to ensure that the definition of p/q is unique.

If we restrict the codomain to the couples (p, q) *such that* $q > 0$ *and* p, q *are coprime we have a one-to one correspondence*

$$T' : \begin{array}{ccc} \mathbb{Q} & \longrightarrow & A \\ p/q & \mapsto & (p,q) \end{array} \quad \text{where } A = \{(p,q) \in \mathbb{N}\times\mathbb{N} \mid q > 0 \text{ and } p, q \text{ are coprime}\}$$

So $|\mathbb{Q}| = |A|$.

We give a description of A*, detailing the elements of* $\mathbb{Z}\times\mathbb{N}$ *that we are taking out. We proceed diagonally*

$$(0,0) \qquad \text{we take it out because denominator is } 0$$

Listed elements of $\mathbb{Q}$*: none.*

$$\begin{array}{rl} (-1,0) & \text{we take it out because denominator is } 0 \\ (0,1) & = 0/1 = 0 \in \mathbb{Q} \ \text{OK} \\ (1,0) & \text{we take it out because denominator is } 0 \end{array}$$

Listed elements of $\mathbb{Q}$*:* 0.

$$\begin{array}{rl} (-2,0) & \text{NO, denominator is } 0 \\ (-1,1) & = -1/1 = -1 \in \mathbb{Q} \ \text{OK} \\ (0,2) & = 0/1 = 0 \in \mathbb{Q} \ \text{we take it out, repeat} \\ (1,1) & = 1/1 = 1 \in \mathbb{Q} \ \text{OK} \\ (2,0) & \text{NO, denominator is } 0 \end{array}$$

Listed elements of $\mathbb{Q}$*:* $0, \pm 1, \pm 2, \pm 1/2$.

$$\begin{array}{rl} (-3,0) & \text{NO, denominator is } 0 \\ (-2,1) & = -2/1 = -2 \in \mathbb{Q} \ \text{OK} \\ (-1,2) & = -1/2 \in \mathbb{Q} \ \text{OK} \\ (0,3) & = 0/1 = 0 \ \text{we take it out, repeat} \\ (1,2) & = 1/2 \in \mathbb{Q} \ \text{OK} \\ (2,1) & = 2/1 = 2 \in \mathbb{Q} \ \text{OK} \\ (3,0) & \text{NO, denominator is } 0 \end{array}$$

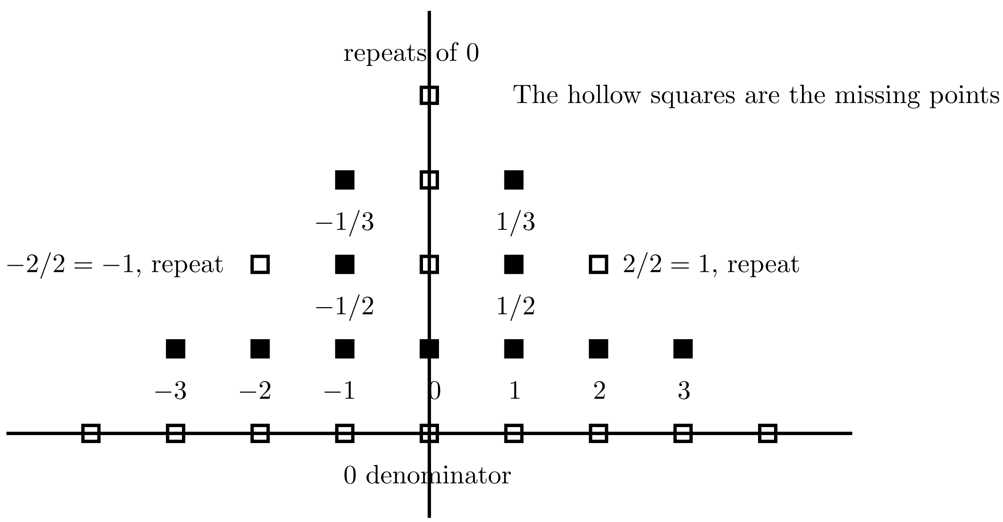
Listed elements of $\mathbb{Q}$*:* $0, \pm 1$.

$$\begin{aligned}
(-4,0) &\quad \textit{NO, denominator is } 0\\
(-3,1) &\quad = -3/1 = -3 \in \mathbb{Q} \ \textit{OK}\\
(-2,2) &\quad = -2/2 = -1 \ \textit{NO, repeat}\\
(-1,3) &\quad = -1/3 \in \mathbb{Q} \ \textit{OK}\\
(0,4) &\quad = 0/1 = 0 \ \textit{NO, repeat}\\
(1,3) &\quad = 1/3 \in \mathbb{Q} \ \textit{OK}\\
(2,2) &\quad = 2/2 = 1 \ \textit{NO, repeat}\\
(3,1) &\quad = 3/1 = 3 \in \mathbb{Q} \ \textit{OK}\\
(4,0) &\quad \textit{NO, denominator is } 0
\end{aligned}$$

Listed elements of $\mathbb{Q}$*:* $0, \pm 1, \pm 2, \pm 1/2, \pm 3, \pm 1/3$.

graphically

GNU2 - The set $A \subseteq \mathbb{N} \times \mathbb{N}$

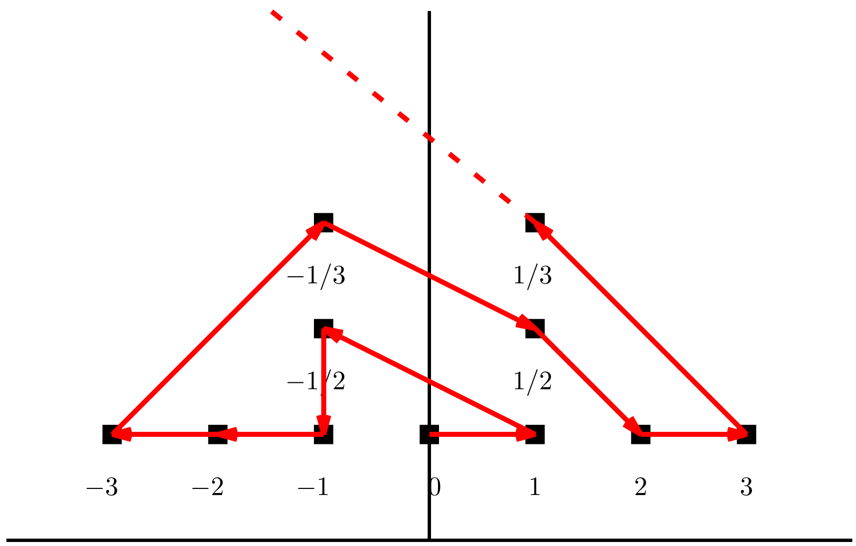


From the description above and the drawing, it is clear that T' is a one-to-one correspondence.

If we prove that there is a one-to-one correspondence $G : \mathbb{N} \longrightarrow A$, and we are done, because then $\mid \mathbb{N} \mid = \mid A \mid$ and $\mid \mathbb{Q} \mid = \mid A \mid$ give $\mid \mathbb{N} \mid = \mid \mathbb{Q} \mid$.

We define G as in the imagine below:

GNU3 - The one-to-one-correspondence $G : \mathbb{N} \longrightarrow A$



The function G is a one-to-one correspondence, and so

$$\mid \mathbb{N} \mid = \mid A \mid$$

Example 12. *Prove $\mid \mathbb{N} \mid \neq \mid \mathbb{R} \mid$.*

We have to prove that there is no one-to-one correspondence $F : \mathbb{N} \longrightarrow \mathbb{R}$. Since $\mid (0,1) \mid = \mid \mathbb{R} \mid$, we can prove that there is no one-to-one correspondence $F : \mathbb{N} \longrightarrow (0,1)$.

By contradiction, we say that there is a one-to-one correspondence $F : \mathbb{N} \longrightarrow (0,1)$. That means that we can enumerate ALL the real numbers *in* $(0,1)$. *That means that we can put them ALL in a list, where the a's, b's etc are digits (numbers in* $0,\ldots,9$*)*

$$\begin{aligned} F(0) &= 0, a_0a_1a_2a_3\cdots \\ F(1) &= 0, b_0b_1b_2b_3\cdots \\ F(2) &= 0, c_0c_1c_2c_3\cdots \\ F(3) &= 0, d_0d_1d_2d_3\cdots \\ \vdots &= \vdots \end{aligned}$$

or, if you prefer, changing a_0 *in* a_0^0, *etc. etc.*

$$\begin{aligned} F(0) &= 0, a_0^0a_1^0a_2^0a_3^0\cdots \\ F(1) &= 0, a_0^1a_1^1a_2^1a_3^1\cdots \\ F(2) &= 0, a_0^1a_1^2a_2^2a_3^2\cdots \\ F(3) &= 0, a_0^3a_1^3a_2^3a_3^3\cdots \\ \vdots &= \vdots \\ F(n) &= 0, a_0^na_1^na_2^na_3^n\cdots \\ \vdots &= \vdots \end{aligned}$$

to find a contradiction, we build a real y *number belonging to* $(0,1)$ *that is NOT in the above list.*

$$y = 0, b_0b_1b_2b_3\cdots b_n\cdots$$

where the b_i*'s are any digit but we have that*

$$b_0 \neq a_0^0,\ b_1 \neq a_1^1,\ b_2 \neq a_2^2,\ b_3 \neq a_3^3, \ldots b_n \neq a_n^n, \ldots$$

And y *is different from* $F(0)$ *since their first digit is different,* y *is different from* $F(1)$ *since their second digit is different,* y *is different from* $F(2)$ *since their third digit is different and so on.*

It is clear that $y \in (0,1)$ *but* y *can't be in the list above, that supposedly contains all the elements of* $(0,1)$, *and here it is our contradiction. The hypotesis that the list above contains all the elements of* $(0,1)$ *is hence false, and it is so impossible to find a one-to-one correspondence between* $\mathbb{N}$ *and* $(0,1)$, *as so it is impossible to find a one-to-one correspondence between* $\mathbb{N}$*and* $\mathbb{R}$. *Hence*

$$\mid \mathbb{N} \mid \neq \mid \mathbb{R} \mid$$

Exercise 23.

1. *Prove that* $\mid \mathbb{N} \mid = \mid \mathbb{Q} \mid$ *using the Cantor-Schröder-Bernstein theorem. [This is quite easy using the right idea].*

2. *Prove that* $| [0,1] |=| \mathbb{R} |$ *using the Cantor-Schröder-Bernstein theorem.*

3. *Prove that* $| [0,1] |=| \mathbb{R} |$ *by finding an one-to-one correspondence. [Difficult]*

4. *Prove that* $| \mathbb{N} |=| \mathbb{Q} |$ *directly by building a function* $F : \mathbb{N} \longrightarrow \mathbb{Q}$.

5. *Find an infinite set whose cardinality is bigger than* $| \mathbb{R} |$.

6. *Prove that* $| \mathbb{N} |=| \mathbb{Z} \times \mathbb{Z} |$.

7. *Prove that* $| \mathbb{N}^3 |=| \mathbb{N} |$.

8. *Prove that* $| \mathbb{Q}^2 |=| \mathbb{N} |$.

9. *Prove that* $\forall\ n \in \mathbb{N},\ | \mathbb{N}^n |=| \mathbb{N} |$.

10. *Prove that* $\forall\ n \in \mathbb{N},\ | \mathbb{Q}^n |=| \mathbb{N} |$.

11. *Let* $\mathbb{Z}[x] = \{$*polynomials with coefficients in* $\mathbb{Z}\}$.
    Prove that $| \mathbb{Z}[x] |=| \mathbb{N} |$. *[Difficult]*

12. *Let* $\mathbb{Q}[x] = \{$*polynomials with coefficients in* $\mathbb{Q}\}$.
    Prove that $| \mathbb{Q}[x] |=| \mathbb{N} |$. *[Difficult]*

13. *Propose as many sets as possible whose cardinality is bigger that the cardinality of* $\mathbb{R}$.