Basic Math - Fourth and Last lesson

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1 Examples: poly root and interval computations

Example 1 (First Example)**.** *We want to determine the number of roots for* $f(x) = x^4 - x^3 - 1$.

```
F:= x^4-x^3-1; **************** First element for the sequence
Der(F,x);
4x^3 - 3x^2 **************** Second element for the sequence

-----------------
GCD(x^4-x^3-1,4x^3 - 3x^2);

DivPoly(x^4-x^3-1,4x^3 - 3x^2);

Passo 1 ho        x^4 - x^3 - 1 moltiplico per 1/4x
Passo 1 sottraggo x^4 - 3/4x^3
Passo 1 ottengo       -1/4x^3 - 1

Passo 2 ho        -1/4x^3 - 1 moltiplico per -1/16
Passo 2 sottraggo -1/4x^3 + 3/16x^2
Passo 2 ottengo           -3/16x^2 - 1

Remainder=-3/16x^2 - 1
Quoto=1/4x - 1/16

(x^4 - x^3 - 1)=(1/4x - 1/16)*(4x^3 - 3x^2)+(-3/16x^2 - 1)

Multiplying the remainder by 16 to avoid fractions
Remainder=-3x^2 - 16
Change the remainder's sign
Remainder=3x^2 + 16 ************* Remainder - third element for the sequence

----------------
GCD(4x^3 - 3x^2,3x^2 + 16);

DivPoly(x^3 - 3x^2,3x^2 + 16);
```

```
Passo 1 ho        x^3 - 3x^2 moltiplico per 1/3x
Passo 1 sottraggo x^3 + 16/3x
Passo 1 ottengo       -3x^2 - 16/3x

Passo 2 ho        -3x^2 - 16/3x moltiplico per -1
Passo 2 sottraggo -3x^2 - 16
Passo 2 ottengo         -16/3x + 16

Remainder=-16/3x + 16
Quoto=1/3x - 1

Multiplying the remainder by 3/16 to avoid fractions and simplify
Remainder=-x + 3
Change the remainder's sign
Remainder=x - 3 ***************** Remainder - fourth element for the sequence

---------------
GCD(3x^2 + 16,x - 3);
DivPoly(3x^2 + 16,x - 3);

Passo 1 ho        3x^2 + 16 moltiplico per 3x
Passo 1 sottraggo 3x^2 - 9x
Passo 1 ottengo        9x + 16

Passo 2 ho        9x + 16 moltiplico per 9
Passo 2 sottraggo 9x - 27
Passo 2 ottengo      43

Remainder=43
Quoto=3x + 9

Multiplying the remainder by 1/43 to  simplify
Remainder=1
Change the remainder's sign
Remainder=-1 ******************* Remainder - fifth element for the sequence


Remainder Sequence = [x^4 - x^3 - 1, 4x^3 - 3x^2, 3x^2 + 16, 4x - 3, -1]

p=++++- 1
q=+-+-- 3
#roots=3-1=2
-------------------------------
There are two roots
```

The maximal interval:

$$M = \textit{Poly degree} \cdot \frac{\textit{Max of the coeffs}}{\textit{Coeff of } x^4} = 4 \cdot \frac{1}{1} \Rightarrow \textit{ interval is } (-4, 4)$$

Remember that the cofficients are considered as absolute values (without sign).

- *We try the interval* $(-4, 0)$. *The Remainder Sequence is*

  $$[p_0(x), p_1(x), p_2(x), p_3(x), p_4(x)] = [x^4 - x^3 - 1, 4x^3 - 3x^2, 3x^2 + 16, 4x - 3, -1]$$

  $++++-$ *as in the maximal interval determined before. This is the same of computing the signs of the sequence*

  $$[p_0(-4), p_1(-4), p_2(-4), p_3(-4), p_4(-4)] = [319, -304, 64, -19, -1]$$

  and we have $p = 3$

  $-0+--$. *We compute the signs of the sequence*

  $$[p_0(0), p_1(0), p_2(0), p_3(0), p_4(0)] = [-1, 0, 16, -3, -1]$$

  remembering that 0 *is caceled out in the sign evaluation, we have* $q = 2$

  We have $\mid p - q \mid = \mid 3 - 2 \mid = 1$, *and so we have one root in* $(-4, 0)$.

- *We have two roots in* $(-4, 4)$, *one is in* $(-4, 0)$ *and the other, by necessity, is in* $(0, 4)$.

Exercise 2. *Find an approximation up to the third digit of the two roots above. That means, find two intervals of lenght* $10^{-3} = 0.001$ *to which a roots belongs using the shrinking method.*

As you see in this exercise, this methods works surely but slowly - at every step, you shrink the lenght of the interval by half, and so to shrink it by a factor of 1000 you have to repeat the procedure 10 times, since $2^{10} = 1024$. We need a faster method.

2 Root finding: Newton's method

Definition 3. *Let* $f\colon (a, b) \to \mathbb{R}$ *be a squarefree polynomial function where* $x_0 \in \mathbb{R}$ *is the* starting point. *The sequence* $[x_0, x_1, x_2, \ldots]$ *defined by*

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \quad (3)$$

is the Newton sequence *and the formula (3) above is the* Newton's Formula.

Remark 4. *For the sequence to converge quickly to the roots, it is important to ensure the following:*

1. *For the root to be NOT a multiple root.*
2. *To have only ONE root near* x_0.

3. *Choose a starting point x_0 near to the actual root.*

We can make this work by

1. *Shrinking the interval suitably.*
2. *Start with a squarefree polynomial.*
3. *Shrinking the interval suitably.*

We show some easy examples: first we compute $\sqrt{2}$ starting with $x_0 = 2$, reasonably near the actual root.

Example 5. *Let $f(x) = x^2 - 2$ and $x_0 = 2$. We have $f'(x) = 2x$, the formaul is*

$$x_{k+1} = x_k - \frac{x_k^2 - 2}{2x_k}$$

and so

$$\begin{aligned}
x_0 &= 2 \\
x_1 &= 2 - \frac{(2)^2 - 2}{2 \cdot 2} = \frac{3}{2} = 1.5 \\
x_2 &= 1.5 - \frac{(1.5)^2 - 2}{2 \cdot 1.5} = 1.5 - \frac{(1.5)^2 - 2}{2 \cdot 1.5} = \frac{17}{12} = 1.4166666666 \\
x_3 &= \frac{17}{12} - \frac{(\frac{17}{12})^2 - 2}{2 \cdot \frac{17}{12}} = \frac{577}{408} = 1.4142156862 \\
x_4 &= 1.4142156862 - \frac{(1.4142156862)^2 - 2}{2 \cdot 1.4142156862} = 1.41421356231
\end{aligned}$$

We remember that $\sqrt{2} = 1.414213562$, and after four steps we have an 8 digit precision.

The same but we start with $x_0 = 8$, further away from the actual root.

Example 6. *Let $f(x) = x^2 - 2$ and $x_0 = 8$. We have $f'(x) = 2x$ and*

$$\begin{aligned}
x_0 &= 8 \\
x_1 &= 8 - \frac{(8)^2 - 2}{2 \cdot 8} = 4.125 \\
x_2 &= 4.125 - \frac{(4.125)^2 - 2}{2 \cdot 4.125} = 2.3049242424 \\
x_3 &= 2.3049242424 - \frac{(2.3049242424)^2 - 2}{2 \cdot 2.3049242424} = 1.5863158599
\end{aligned}$$

$x_4 = 1.4235494082$

$x_5 = 1.4142441752$

$x_6 = 1.4142135627$

$x_7 = 1.4142135623$

$x_7 = 1.4142135623$

We remember that $\sqrt{2} = 1.414213562$, as computed using a pocket calculator, and we need seven steps we have an 8 digit precision. .

Example 7. *Let $f(x) = x^2 - 2$ and $x_0 = 1000$.*

```
F:=x^2-2;
L:=NewtonMod(F,100,30,10,15);
1000
500.001
250.0024999960
125.0052499580
62.5106246430
31.2713096020
15.6676329948
7.8976423478
4.0754412405
2.2830928243
1.5795487524
1.4228665795
1.4142398735
1.4142135626
1.4142135623
1.4142135623
```

Example 8. *Let $f(x) = x^5 - 3$ and $x_0 = 2$.*

```
F:=x^2-2;
L:=NewtonMod(F,100,30,10,15);
2
1.6375
1.393449937495
1.273902300672
1.246949664737
1.245733319567
1.245730939624
1.245730939615
1.245730939615
```

We remember that $\sqrt[5]{3} = 1.24573094$, as computed using a pocket calculator, and after six steps we have an 8 digit precision. .

Example 9. *Let $f(x) = x^5 - 3$ and $x_0 = 3$.*

```
F:=x^5-3;
L:=NewtonMod(F,3,15,10,15);
3
2.4074074074
1.9437888634
```

```
1.5970606554
1.3698771224
1.2662841244
1.2463873994
1.2457316307
1.2457309396
1.2457309396
```

We remember that $\sqrt[5]{3} = 1.24573094$*, as computed using a pocket calculator.*

Example 10. *Let* $f(x) = x^5 - 3$ *and* $x_0 = 20$.

```
F:=x^5-3;
L:=NewtonMod(F,20,20,10,15);
20
16.00000375
12.8000121552
10.2400320758
8.1920802296
6.5537974052
5.2433631456
4.1954843172
3.3583239775
2.6913761278
2.1645363629
1.7589623675
1.4698493269
1.3044258794
1.250780723
1.2457715503
1.2457309422
1.2457309396
1.2457309396
1.2457309396
```

We remember that $\sqrt[5]{3} = 1.24573094$*, as computed using a pocket calculator.*

Ths method is sensitive to the selection of the strating point, but expecially to the polynomial degree. For lower degress, it is very efficient.

Example 11. *Let* $f(x) = x^{15} - 3$ *and* $x_0 = 2$.

```
F:=x^15-3;
L:=NewtonMod(F,4,30,30,15);
4
3.733333334078391   1.631440852743791
3.484444447097212   1.522889493378988
```

```
3.252148155766399  1.421917734820804
3.035338292224998  1.32857128198326
2.832982441568673  1.243746503024962
2.644117038706965  1.170265979236223
2.467842814422866  1.114381532360503
2.303320603682124  1.083998608244157
2.149767587477567  1.076390865753772
2.006454190008554  1.075990670016803
1.872702245988654  1.075989624732454
1.747886082220382  1.075989624725345
 1.075989624725345
```

We remember that $\sqrt[15]{3} = 1.075989625$*, as computed using a pocket calculator.*

Example 12. *Let* $f(x) = x^{115} - 3$ *and* $x_0 = 1$*.*

```
F:=x^115-3;
L:=NewtonMod(F,1,100,20,20);
1
1.01739130434782608695
1.01219859535714983979
1.00994519245838467584
1.00960560774948143004
1.00959892985126879499
1.00959892733230472403
1.0095989273323043658
1.0095989273323043658
1.0095989273323043658
```

We remember that $\sqrt[115]{3} = 1.0009598927$*, as computed using a pocket calculator.*

Example 13. *With a worse starting point: Let* $f(x) = x^{115} - 3$ *and* $x_0 = 2$*. Notice how slow we get near* 1*.*

```
F:=x^115-3;
L:=NewtonMod(F,2,100,20,20);
2
1.98260869565217391304
1.96536862003780718336
1.94827845812443494698
1.93133690631465725178
1.91454267234670371046
1.89789447519586280862
1.88139104497676834941
1.86503112284653558115
1.84881346090873961957
```

1.83273682211822884026
1.81679998018676598077
1.80100171948948975485
1.78534083497218984393
1.76981613205938819311
1.75442642656321960012
1.73917054459310464707
1.72404732246620808492
1.7090556066186758407
1.69419425351764387686
1.67946212957401219097
1.66485811105597730235
1.65038108400331663015
1.63602994414241822467
1.62180359680204937054
1.60770095682985763688
1.59372094850959800525
1.57986250547907976172
1.56612457064882689422
1.55250609612144579079
1.53900604311169408826
1.52562338186724457444
1.51235709159013809987
1.49920616035891950769
1.4861695850514506424
1.47324637126839454985
1.46043553325736503202
1.44773609383773577087
1.43514708432610328591
1.42266754446239803997
1.4102965223366380571
1.39803307431631946554
1.38587626497443842738
1.37382516701813896461
1.3618788612179812394
1.35003643633782489412
1.33829698906532210501
1.32665962394301505386
1.3151234533000325784
1.30368759718438083608
1.29235118329582293754
1.28111334691934275719
1.26997323085918873165
1.25892998537349501256
1.24798276810948147566

1.23713074403924520531
1.22637308539618608692
1.21570897161219029686
1.20513758925591505286
1.19465813197311218246
1.18426980043153615473
1.17397180227733436573
1.16376335212159628028
1.15364367160761677396
1.14361198969570725361
1.13366754353589572397
1.12380958093083114529
1.11403736710159131119
1.10435020309798154396
1.09474747572139616918
1.08522879272040130077
1.07579434867067792314
1.06644591444088371728
1.05718950882919385901
1.04804258214247679977
1.03905311240234824126
1.0303487857929650193
1.02225264170034570905
1.01548567641901128278
1.01118027662180827392
1.00973193715521649452
1.0095999210978453851
1.00959892738805853039
1.00959892733230436597
1.0095989273323043658
1.0095989273323043658
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1.0095989273323043658

```
1.0095989273323043658
```

We remember that $\sqrt[115]{3} = 1.0009598927$*, as computed using a pocket calculator.*

If you are interested, this is a simplified version of the theorem that guarantees you the algorithm convergence.

Theorem 14 (Newton's Approximation)**.** *Let* $(a, b) \subset \mathbb{R}$ *be an interval and let* $f : (a, b) \to \mathbb{R}$ *a squarefree polynomial function. If* $\overline{x} \in (a, b)$ *is a root, then there exists an* $\epsilon > 0$ *such that Newton's method with starting point* x_0 *with* $\mid x_0 - \overline{x} \mid < \epsilon$ *quickly converges to the root* $\overline{x}$

Example 15 (First Example)**.** *Let us return to Example 1*

We have to determine the real roots of the polynomial $f(x) = x^4 - x^3 - 1$*. We know that there are two roots, one in the interval* $(-4, 0)$ *and one in the interval* $(0, 4)$*. We use Newton's method. The general formula is*

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

and thus in our case, for a point x_k*, the formula is*

$$x_{k+1} = x_k - \frac{x_k^4 - x_k^3 - 1}{4x_k^3 - 3x_k^2}$$

- *Interval* $(-4, 0)$*. Middle point* $x_0 = -2$*.*

  ```
  F:=x^4 - x^3 - 1;
  L:=NewtonMod(F,-2,12,12,20);
  -2
  -1.477272727272
  -1.117933865584
  -0.908134322031
  -0.829688956906
  -0.819339664556
  -0.819172556393
  -0.819172513396
  -0.819172513396
  ```

  We see that the root is $x_1 \simeq -0.819172513396$

- *Interval* $(0, 4)$*. Middle point* $x_0 = 2$*.*

  ```
  F:=x^4 - x^3 - 1;
  L:=NewtonMod(F,2,12,12,20);
  2
  1.65
  1.454113738394
  1.387577585471
  ```

```
1.380357405588
1.38027757877
1.380277569097
1.380277569097
```

We see that the root is $x_2 \simeq 1.380277569097$

Exercise 16. *We want to determine the numer of roots and the root intervals for* $P_1(x) = x^3-6x-1$. *The details are shown in Second Example of the Third Lesson.*

```
F:= x^3-3x^2+3;
GCDPolySTURMVerbose(F,Der(F,x));

One root in (-9,0)
One root in (0,2)
One root in (2,4)
```

We have the polynomial function $f(x) = x^3 - 3x^2 + 3$, abd $f'(x) = 3x^2 - 6x$. The formula is

$$x_{k+1} = x_k - \frac{x_k^3 - 3x_k^2 + 3}{3x_k^2 - 6x_k}$$

- Interval $(0, 2)$. Middle point $x_0 = 0 + \frac{2-0}{2} = 1$.

  ```
  F:=x^3-3x^2+3;
  L:=NewtonMod(F,1,5,10,10);
  1
  1.3333333333
  1.3472222222
  1.3472963531
  1.3472963553
  1.3472963553
  ```

  We say that $x_1 \simeq 1.3472963553$.

- Interval $(2, 4)$. Middle point $x_0 = 2 + \frac{4-2}{2} = 3$.

  ```
  F:=x^3-3x^2+3;
  L:=NewtonMod(F,3,7,12,12);

  3
  2.666666666666
  2.54861111111
  2.532390161865
  2.532088989397
  2.532088886237
  2.532088886237
  ```

We say that $x_2 \simeq 1.532088886237$.

- Interval $(-9, 0)$. Middle point $x_0 = -9\frac{-9-0}{2} = -4.5$.

```
F:=x^3-3x^2+3;
L:=NewtonMod(F,-4.5,12,10,10);
-4.5
-2.8034188034
-1.748663703
-1.1628353797
-0.9245690643
-0.8808269788
-0.8793867818
-0.8793852415
-0.8793852415
```

We say that $x_3 \simeq -0.8793852415$.

3 Root's finding with approximation

A review of the metohd we have learned about finding the real roots of a polynomial, if possibile precise, if not approximate.

We have to determine all the real roots of a polynomial $f(x)$ of degree d with "good" approximation

1. We try to find the exact roots $x_1, \ldots, x_k$ by regrouping, or root's rule+Ruffini's rule.
2. We divide them out $g(x) = \frac{f(x)}{(x-x_1)\cdots(x-x_k)}$. The polynomial $g(x)$ ha now degree $n = d - k$.
3. We perform the Sturm variant of Euclid's Algorithm, finding $\gcd(g(x), g'(x)) = h(x)$. If $\gcd(g(x), g'(x)) \neq 1$ then
   - We find the roots of $\gcd(g(x), g'(x))$ using this method.
   - We set $h(x) = \frac{g(x)}{\gcd(g(x), g'(x))}$ and proceed to determine the roots of $h(x)$, that is squarefree.
4. Using the Sturm sequence that we have built before, we compute the number of real roots of $h(x)$. Let this number be p.
5. We compute the maximum root interval $(-M, M)$ as in Lesson 3.
6. We split this interval $(-M, M)$ using the method outlined in Lesson 3. We find intervals $(a_1, b_1), \ldots, (a_p, b_p)$ such that for every interval there is only one root.
7. For every interval (a_i, b_i) we use Newton's method, starting with the middle point $a_i + \frac{b_i - a_i}{2}$. If the sequence quickly converges up to the precision we are interested in, we have a root. If the sequence does not converges, we split the interval and try again.

4 Classwork

The remainder sequence, and sometimes the gcd process, is given so you can check your results. For the roots, pleas ask Wolphram Alpha `https://www.wolframalpha.com/` e.g. $2x^3 - 6x - 3 = 0$.

You have to solve the two roots of Exercise 17 and two of the following exercises. All the computational steps have to be shown.

Exercise 17. *Find approximations of* $\sqrt[3]{5}$ *and* $\sqrt[5]{7}$ *with precision of* 10^{-3} *(the first thrre digits after the comma have to be stabke in the Newton's sequence.)*

Exercise 18. *We want to determine the root intervals for* $G_1(x) = 2x^3 - 6x - 3$ *with 4 digits approximation.*

```
F:= 2x^3-6x-3;
GCDPolySTURMVerbose(F,Der(F,x));

(2x^3 - 6x - 3)=(2x)*(x^2 - 1)+(-4x - 3)
(x^2 - 1)=(1/4x - 3/16)*(4x + 3)+(-7/16)
(4x + 3)=(4x + 3)*(1)+(0)
Remainder Sequence = [2x^3 - 6x - 3, 6x^2 - 6, 4x + 3, 1]
```

Exercise 19. *We want to determine the root intervals for* $G_2(x) = x^3 - 3x - 1$ *with 4 digits approximation.*

```
F:= x^3-3x-1;
GCDPolySTURMVerbose(F,Der(F,x));

(x^3 - 3x - 1)=(x)*(x^2 - 1)+(-2x - 1)
(x^2 - 1)=(1/2x - 1/4)*(2x + 1)+(-3/4)
(2x + 1)=(2x + 1)*(1)+(0)
Remainder Sequence = [x^3 - 3x - 1, 3x^2 - 3, 2x + 1, 1]
```

Exercise 20. *We want to determine the root intervals for* $G_3(x) = x^5 - 3x^4 + 1$.

```
F:= x^5-3x^4+1;
GCDPolySTURMVerbose(F,Der(F,x));

(x^5 - 3x^4 + 1)=(1/5x - 3/25)*(5x^4 - 12x^3)+(-36/25x^3 + 1)
(5x^4 - 12x^3)=(5/36x - 1/3)*(36x^3 - 25)+(125/36x - 25/3)
(36x^3 - 25)=(-36/5x^2 - 432/25x - 5184/125)*(-5x + 12)+(59083/125)
(-5x + 12)=(5x - 12)*(-1)+(0)
Remainder Sequence = [x^5 - 3x^4 + 1, 5x^4 - 12x^3, 36x^3 - 25, -5x + 12, -1]
```

Exercise 21. *We want to determine the root intervals for* $G_3(x) = x^5 - 3x^4 + 1$.

```
F:= x^5-3x^4+1;
GCDPolySTURMVerbose(F,Der(F,x));
```

```
(x^5 - 3x^4 + 1)=(1/5x - 3/25)*(5x^4 - 12x^3)+(-36/25x^3 + 1)
(5x^4 - 12x^3)=(5/36x - 1/3)*(36x^3 - 25)+(125/36x - 25/3)
(36x^3 - 25)=(-36/5x^2 - 432/25x - 5184/125)*(-5x + 12)+(59083/125)
(-5x + 12)=(5x - 12)*(-1)+(0)
Remainder Sequence = [x^5 - 3x^4 + 1, 5x^4 - 12x^3, 36x^3 - 25, -5x + 12, -1]
```

Exercise 22. *We want to determine the root intervals for* $G_4(x) = x^3 - 4x - 1$.

```
F:=x^3-4x-1;
GCDPolySTURMVerbose(F,Der(F,x));
Remainder Sequence = [x^3 - 4x - 1, 3x^2 - 4, 8x + 3, 1]
```

Exercise 23. *We want to determine the root intervals for* $G_5(x) = x^4 - 4x - 1$.

```
F:=x^4-4x-1;
GCDPolySTURMVerbose(F,Der(F,x));
Remainder Sequence  [x^4 - 4x - 1, 4x^3 - 4, 3x + 1, 1]
```

Exercise 24. *We want to determine the root intervals for* $G_6(x) = x^5 - x - 1$.

```
F:=x^5-x-1;
GCDPolySTURMVerbose(F,Der(F,x));
Remainder Sequence  [x^5 - x - 1, 5x^4 - 1, 4x + 5, -1]
```